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This is an author-prepared version of the paper, reprinted from the Proceedings of the IEEE Data Compression Conference, DCC 2011, p. 253-262.

http://dx.doi.org/10.1109/DCC.2011.32

The Two-Way Relay Network with Arbitrarily Correlated Sources and an Orthogonal MAC

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Abstract

The problem of lossless joint source-channel coding for the two-way relay network with an orthogonal multiple access channel is studied. Necessary and sufficient conditions for reliable communication are given, and a separation theorem for source and channel coding is proved.

I Introduction

The two-way relay network is of significant importance to modern communication systems. This network models two-way communication in cellular, satellite, wireless ad hoc and peer-to-peer settings. A substantial body of research has comprehensively studied the two-way relay network from the perspective of source coding [1–3]; channel capacity [4–9], and joint channel-network coding [10, 11]. However, despite this intense effort, the fundamental communication limits of the two-way relay network remain largely unknown.

The two-way relay network is shown in Figure 1. Two nodes need to reliably exchange data $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$. The nodes are physically separated, and no direct communication is permitted. Instead, the nodes indirectly communicate via a relay using a two-phase protocol. In phase 1 (uplink), both nodes encode and transmit their data over a multiple access channel (MAC) to the relay. In phase 2 (downlink), the relay re-encodes and transmits this data over a broadcast channel. Each node is required to recover the data of the other node from this broadcast transmission.

The main result of this paper, Theorem 1, gives necessary and sufficient conditions for reliable communication under two specific assumptions. The first assumption is that $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ are generated by a discrete memoryless source. That is, the data is arbitrarily correlated and the communication problem involves joint source-channel coding. Correlated data, for example, might take the form of measurements in a

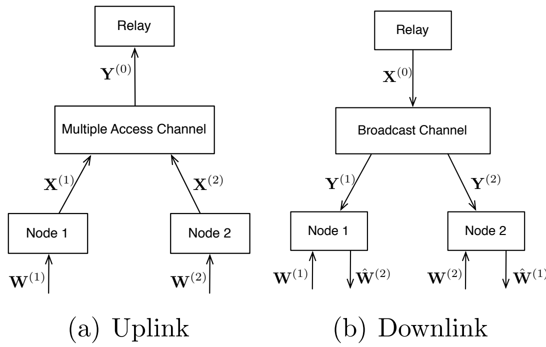

Figure 1: The two-way relay network.

sensor network [12], voice data in a cellular network, and data files in a peer-to-peer network. The second assumption is that the MAC is orthogonal. This assumption has some motivation from practical communication systems; however, in this paper it is primarily motivated by the source-channel separation theorem.

Shannon's classic source-channel separation theorem [13] (see also [14, Sec. 7.3]) states the following: the problem of losslessly transmitting a discrete memoryless source $\mathbf{W} = W_1, W_2, \ldots$ over a discrete memoryless channel can be divided into two independent problems – source coding and channel coding. The source coding problem concerns the determination of the minimum average number of bits per source symbol that is needed to reliably describe $\mathbf{W}$. This minimum is given by the entropy $H(W)$. Conversely, the channel coding problem concerns the determination of the maximum average number of bits per channel symbol that can be reliably transmitted over the channel. This maximum is given by the channel capacity C. The separation theorem states that $\mathbf{W}$ can be reliably transmitted over the channel if and only if $H(W) \leq C$. A consequence of this result is that the individual optimisation of stand-alone source and channel codes is optimal for the overall joint source-channel coding problem.

It is well know that the source-channel separation theorem does not extend to networks, and the classic counter-example is that of communicating correlated sources over a MAC [15], [14, Pg. 592]. In saying this, however, it is also well known that the source-channel separation theorem does hold for communicating correlated sources over an orthogonal MAC [16]. The second main result of this paper, Corollary 2.1, shows that the source-channel separation theorem holds for communicating correlated sources over the two-way relay network with an orthogonal MAC. We emphasise that this result is not trivial for at least two reasons. First, the downlink involves broadcasting with side-information – a setting where joint source-channel codes can outperform stand-alone source and channel codes [17]. Second, Corollary 2.1 does not translate to the lossy (rate-distortion) setting. This last property is in stark contrast to Shannon's separation theorem [13] and the orthogonal MAC separation theorem [16]; see, for example, Gallager [18, Thm. 9.2.2] as well as Xiao and Luo [19].

Paper Outline: Definitions are given in Section II, and our main result for joint source-channel coding is given in Section III. We present the source-channel separation principle in Section IV. Proof details are omitted due to page constraints.

Notation: Random variables are written as uppercase letters, e.g. W. Finite sets and alphabets are written as scripted letters, e.g. $\mathcal{W}$. A generic element of a finite set is identified by the matching lowercase letter, e.g. $w \in \mathcal{W}$. The two-dimensional Euclidean space is denoted by $\mathbb{R}^2$. The m-fold Cartesian product of a set is denoted by $\bigotimes_m$, e.g

$$\bigotimes_m \mathcal{W} \triangleq \underbrace{\mathcal{W} \times \mathcal{W} \times \cdots \times \mathcal{W}}_{m} . \tag{1}$$

A random vector on such a product set is denote by boldfaced uppercase letters, e.g. $\mathbf{W}$ is defined on $\bigotimes_m \mathcal{W}$. Similarly, a generic element of a product set is identified with the bolded lowercase letters, e.g. $\mathbf{w} \in \bigotimes_m \mathcal{W}$. It will be necessary to associate random variables, sets and functions with a particular node in the network. This association is achieved using braced superscripts, e.g. $W^{(1)}$ is a random variable associated with node 1.

II Definitions

Let $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ be finite alphabets and $q(w^{(1)}, w^{(2)}) = \Pr[W^{(1)} = w^{(1)}, W^{(2)} = w^{(2)}]$ be a generic joint pmf on $\mathcal{W}^{(1)} \times \mathcal{W}^{(2)}$. The source generates a random sequence

$$(W_1^{(1)}, W_1^{(2)}),\ (W_2^{(1)}, W_2^{(2)}),\ (W_3^{(1)}, W_3^{(2)}),\ \ldots, \tag{2}$$

by repeatedly selecting symbols from $\mathcal{W}^{(1)} \times \mathcal{W}^{(2)}$ using $q(w^{(1)}, w^{(2)})$ in an independent and identically distributed (iid) manner. The outcomes of $W_1^{(1)}$, $W_2^{(1)}$, … are given to node 1 symbol-by-symbol at the rate of κ_s symbols per second. Similarly, the outcomes of $W_1^{(2)}$, $W_2^{(2)}$, … are given symbol-by-symbol to node 2.

Consider the channel connecting node j to the relay. (Here, and in the sequel, we will use "node j" as a generic label for node 1 or 2.) Assume that this channel operates (accepts and emits symbols) at κ_c symbols per second. Let $\mathcal{X}^{(j)}$ and $\mathcal{Y}^{(0,j)}$ respectively denote the channel input and output alphabets, and let the transitions from $\mathcal{X}^{(j)}$ to $\mathcal{Y}^{(0,j)}$ be governed by a generic conditional pmf[1] $q(y^{(0,j)}|x^{(j)}) = \Pr[Y^{(0,j)} = y^{(0,j)}|X^{(j)} = x^{(j)}]$. Finally, consider the broadcast channel connecting the relay to both nodes. As before, assume that this channel operates at κ_c symbols per second. Let $\mathcal{X}^{(0)}$ denote the input alphabet, let $\mathcal{Y}^{(1)} \times \mathcal{Y}^{(2)}$ denote the product of the output alphabets, and let the transitions from $\mathcal{X}^{(0)}$ to $\mathcal{Y}^{(1)} \times \mathcal{Y}^{(2)}$ be governed by the conditional pmf $q(y^{(1)}, y^{(2)}|x^{(0)}) = \Pr[Y^{(1)} = y^{(1)}, Y^{(2)} = y^{(2)}|X^{(0)} = x^{(0)}]$.

A joint source-channel code of length t, with $\kappa_s t$ and $\kappa_c t$ being integers, is a tuple of "encoding" and "decoding" functions ($f^{(0)}$, $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, $g^{(2)}$). Here

$$f^{(j)} \ : \ \bigotimes_{\kappa_s t} \mathcal{W}^{(j)} \to \bigotimes_{\kappa_c t} \mathcal{X}^{(j)} \ , \quad j = 1, 2 \ , \tag{3a}$$

[1]For brevity, the same letter q has been used to denote the different source and channels pmfs. The particular pmf under consideration will always be clear from context, so no confusion should arise from this slight abuse of notation.

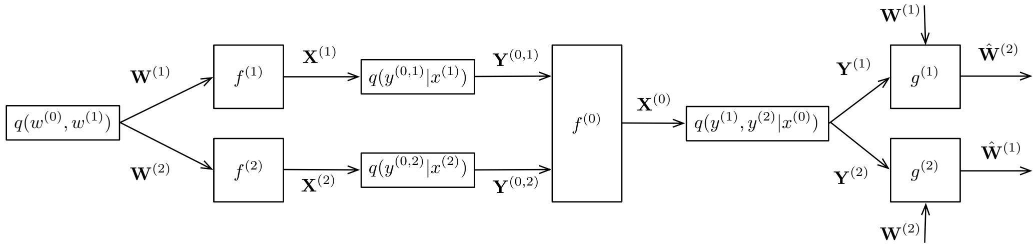

Figure 2: A joint source-channel code for the two-way relay network with an orthogonal MAC.

defines the encoder at node j,

$$f^{(0)} \ : \ \bigotimes_{\kappa_c t} \mathcal{Y}^{(0,1)} \times \bigotimes_{\kappa_c t} \mathcal{Y}^{(0,2)} \to \bigotimes_{\kappa_c t} \mathcal{X}^{(0)} \tag{3b}$$

defines the encoder $f^{(0)}$ at the relay, and

$$g^{(j)} : \bigotimes_{\kappa_s t} \mathcal{W}^{(j)} \times \bigotimes_{\kappa_c t} \mathcal{Y}^{(j)} \to \bigotimes_{\kappa_s t} \mathcal{W}^{(\sim j)} \ , \quad j = 1, 2 \ , \tag{3c}$$

defines the decoder at node j. Note, $\sim j$ in (3c) is short for $\sim 1 = 2$ and $\sim 2 = 1$. The ratio of channel symbols to source symbols, which we abbreviate as $\kappa \triangleq \kappa_c / \kappa_s$, is called the bandwidth expansion. In the sequel, κ_c and κ_s are fixed constants. That is, these rates are fixed by the physical limitations of the communication system.

A joint source-channel code ($f^{(0)}$, $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, $g^{(2)}$) is used in the following manner. Let $(\mathbf{W}^{(1)}, \mathbf{W}^{(2)}) \triangleq (W_1^{(1)}, W_1^{(2)}), (W_2^{(1)}, W_2^{(2)}), \ldots, (W_{\kappa_s t}^{(1)}, W_{\kappa_s t}^{(2)})$ denote the first $\kappa_s t$ symbols of (2). Node j transmits $\mathbf{X}^{(j)} \triangleq f^{(j)}(\mathbf{W}^{(j)})$ over the orthogonal MAC. The relay observes $\mathbf{Y}^{(0,j)}$, which has a pmf defined by

$$\mathbf{q}(\mathbf{y}^{(0,j)}|\mathbf{x}^{(j)}) \triangleq \prod_{i=1}^{\kappa_c t} q(y_i^{(0,j)}|x_i^{(j)}) \ , \quad j = 1, 2 \ .$$

The relay transmits $\mathbf{X}^{(0)} \triangleq f^{(0)}(\mathbf{Y}^{(0,1)}, \mathbf{Y}^{(0,2)})$ over the broadcast channel. Nodes 1 and 2 respectively observe $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$, which have a joint pmf defined by

$$\mathbf{q}(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}|\mathbf{x}^{(0)}) \triangleq \prod_{i=1}^{\kappa_c t} q(y_i^{(1)}, y_i^{(2)}|x_i^{(0)}) \ .$$

Node j decodes $\hat{\mathbf{W}}^{(\sim j)} \triangleq g^{(j)}(\mathbf{W}^{(j)}, \mathbf{Y}^{(j)})$. Figure 2 depicts these encoding and decoding operations. Let $P_e^{(j)} \triangleq \Pr[\hat{\mathbf{W}}^{(\sim j)} \neq \mathbf{W}^{(\sim j)}]$ denote the average probability that node j reconstructs $\mathbf{W}^{(\sim j)}$ in error, and define $P_e \triangleq \max\{P_e^{(1)}, P_e^{(2)}\}$. The next definition formalises the notion of reliable communication.

Definition 1. The bandwidth expansion κ is said to be achievable if the following holds: for every $\epsilon > 0$ there exists a joint source-channel code ($f^{(0)}$, $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, $g^{(2)}$) of the form (3) for some sufficiently large t with probability of error $P_e \leq \epsilon$.

III Joint Source-Channel Coding

We now give a single-letter expression for Definition 1. Let $C^{(j)}$ denote the capacity [14, Chap. 8] in bits per channel symbol of the (orthogonal) uplink channel from node j to the relay.

Theorem 1. *The bandwidth expansion κ is achievable if and only if there exists a pmf $p(x^{(0)})$ on $\mathcal{X}^{(0)}$ such that*

$$H(W^{(1)}|W^{(2)}) \le \kappa \min\left\{C^{(1)}, I(X^{(0)};Y^{(2)})\right\} \tag{4a}$$

$$H(W^{(2)}|W^{(1)}) \le \kappa \min\left\{C^{(2)}, I(X^{(0)};Y^{(1)})\right\} . \tag{4b}$$

Proof outline. The forward coding part ("if") follows from the results of Section IV. The reverse converse ("only if") part is omitted. □

Theorem 1 is useful because, in principle, it can be numerically evaluated for any combination of source and channel pmfs. Indeed, we will show in the next section that the existence of a pmf $p(x^{(0)})$ satisfying (4) is equivalent to the point $(H(W^{(1)}|W^{(2)}), H(W^{(2)}|W^{(1)}))$ lying within a certain closed convex subset of $\mathbb{R}^2$.

We conclude this section by showing that Theorem 1 includes the source network of Wyner, Wolf and Willems [1] as a special case. Specifically, consider noiseless "dummy" communication channels where $Y^{(0,j)} = X^{(j)}$ and $Y^{(j)} = X^{(0)}$. Then, Theorem 1 collapses to

$$H(W^{(j)}|W^{(\sim j)}) \le \kappa \min\left\{H(X^{(j)}), H(X^{(0)})\right\} , \quad j = 1, 2 , \tag{5}$$

which is equivalent to

$$\kappa H(X^{(0)}) \ge \min\left\{H(W^{(1)}|W^{(2)}), H(W^{(2)}|W^{(1)})\right\} \tag{6a}$$

$$\kappa H(X^{(1)}) \ge H(W^{(1)}|W^{(2)}) \tag{6b}$$

$$\kappa H(X^{(2)}) \ge H(W^{(2)}|W^{(1)}) , \tag{6c}$$

To obtain the result [1, Sec. III.3], note that $\kappa H(X^{(j)})$ is the average number of bits per source symbol that node j sends to the relay; similarly, $\kappa H(X^{(0)})$ is the average number of bits per source symbol that the relay sends to both users.

IV Separation of Source and Channel Coding

We now prove the forward "coding" part of Theorem 1. In doing this, we will actually obtain a stronger result; namely, the individual optimisation of stand-alone source and channel codes is optimal for the overall joint source-channel coding problem. In Subsections A and B, we give single-letter expressions for the capacity and source-coding regions of the two-way relay network, respectively. In Subsection C, we give necessary and sufficient conditions for the bandwidth expansion κ to be achievable using stand-alone source and channel codes. These conditions match those of Theorem 1 and lead to the source-channel separation principle.

A Channel Capacity Region

The capacity region of interest is informally defined as follows. A rate pair $(r_c^{(1)}, r_c^{(2)})$ in $\mathbb{R}^2$ is said to be achievable if node j can reliably communicate a message of rate

$r_c^{(j)}$ bits per channel symbol to node $\sim j$. The closure of the set of all achievable rate pairs, $\mathcal{C}$, is called the capacity region.

We now give a single-letter expression for $\mathcal{C}$. Let $\mathcal{C}^*$ denote the set of all ($r_c^{(1)}$, $r_c^{(2)}$) in $\mathbb{R}^2$ for which there exists a pmf $p(x^{(0)})$ on $\mathcal{X}^{(0)}$ such that

$$0 \leq r_c^{(1)} \leq \min\left\{C^{(1)}, I(X^{(0)}; Y^{(2)})\right\} \tag{7a}$$

$$0 \leq r_c^{(2)} \leq \min\left\{C^{(2)}, I(X^{(0)}; Y^{(1)})\right\} . \tag{7b}$$

It is worth noting that $\mathcal{C}^*$ is related to the trivial outer bound for the capacity region of the broadcast channel [17, 20]. This outer bound is defined next.

Definition 2. Let $\mathcal{B}_{out}$ denote the set of all rate pairs $(r_c^{(1)}, r_c^{(2)})$ in $\mathbb{R}^2$ for which there exists a pmf $p(x^{(0)})$ on $\mathcal{X}^{(0)}$ such that $0 \leq r^{(j)} \leq I(X^{(0)}; Y^{(\sim j)})$, for $j = 1, 2$.

Tuncel [17] showed that $\mathcal{B}_{out}$ is convex. We now show that $\mathcal{C}^*$ is also convex.

Lemma 1. *$\mathcal{C}^*$ is a closed convex subset of $\mathbb{R}^2$.*

Proof outline. The proof is similar to [17, Lem. 3]; in particular, the lemma follows from the concavity and continuity of $I(X^{(0)}; Y^{(j)})$ in $p(x^{(0)})$. □

The convexity of $\mathcal{C}^*$ ensures that it can be numerically approximated as a convex optimisation problem. This property is particularly useful because $\mathcal{C}^*$ is in fact equal to the capacity region $\mathcal{C}$. This result is summarised next.

Lemma 2. *The capacity region $\mathcal{C}$ (in bits per channel symbol) of the two-way relay network with an orthogonal MAC is given by $\mathcal{C} = \mathcal{C}^*$.*

Proof outline. The coding theorem is an extension of [21, Thm. 1] or [17, Thm 6]. The converse is omitted. □

Lemma 2 shows that the capacity region $\mathcal{C}$ is obtained by adding the uplink capacity constraints (i.e. $C^{(1)}$ and $C^{(2)}$) to $\mathcal{B}_{out}$. These additional capacity constraints are a consequence of the uplink transmission phase. Similarly, the presence of $\mathcal{B}_{out}$ is a consequence of the downlink transmission phase.

The capacity region $\mathcal{C}$ has a particularly simple form for the following set of broadcast channels[2].

Definition 3. Let $\mathcal{Q}(y^{(1)}, y^{(2)}|x^{(0)})$ denote the set of conditional pmfs $q(y^{(1)}, y^{(2)}|x^{(0)})$ mapping $\mathcal{X}^{(0)}$ to $\mathcal{Y}^{(1)} \times \mathcal{Y}^{(2)}$ for which there exists a pmf $p(x^{(0)})$ on $\mathcal{X}^{(0)}$ that simultaneously achieves capacity for both marginal channels $q(y^{(1)}|x^{(0)})$ and $q(y^{(2)}|x^{(0)})$. Denote these capacities by $C^{(0,1)} = \max_{p(x^{(0)})} I(X^{(0)}; Y^{(1)})$ and $C^{(0,2)} = \max_{p(x^{(0)})} I(X^{(0)}; Y^{(2)})$.

The next corollary follows directly from Definition 3 and Lemma 1.

Corollary 1.1. *If $q(y^{(1)}, y^{(2)}|x^{(0)}) \in \mathcal{Q}(y^{(1)}, y^{(2)}|x^{(0)})$, then*

$$\mathcal{C} = \left\{ (r_c^{(1)}, r_c^{(2)}) \in \mathbb{R}^2 \ : \begin{array}{ccll} 0 & \leq & r_c^{(1)} & \leq \min\left\{C^{(1)}, C^{(0,2)}\right\} \\ 0 & \leq & r_c^{(2)} & \leq \min\left\{C^{(2)}, C^{(0,1)}\right\} \end{array} \right\} . \tag{8}$$

[2]E.g., the binary symmetric broadcast channel [14, Ex.15.6.5] belongs to this set.

B Source Coding Region

The source coding region of interest is informally defined as follows. A rate pair $(r_s^{(1)}, r_s^{(2)}) \in \mathbb{R}^2$ is said to be achievable if node j can compress its source sequence $\mathbf{W}^{(j)}$ to a message $M^{(j)}$ using $r_s^{(j)}$ bits per source symbol such that node $\sim j$ can reconstruct $\mathbf{W}^{(j)}$ using $M^{(j)}$ and its own source $\mathbf{W}^{(\sim j)}$ as side-information. The closure of the set of all achievable rate pairs, $\mathcal{R}$, is called the source coding region. This source coding problem is shown in Figure 3. Note that the relay is not included in the source code.

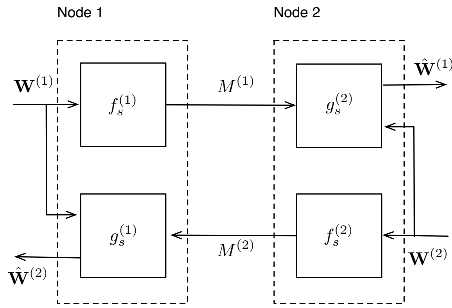

Figure 3: Source Coding.

Lemma 3. *The source coding region $\mathcal{R}$ (in bits per source symbol) is given by*

$$\mathcal{R} \triangleq \left\{ (r_s^{(1)}, r_s^{(2)}) \in \mathbb{R}^2 : \begin{array}{ll} r_s^{(1)} & \geq H(W^{(1)}|W^{(2)}) \\ r_s^{(2)} & \geq H(W^{(2)}|W^{(1)}) \end{array} \right\} . \tag{9}$$

Proof outline. The lemma follows from the Slepian-Wolf theorem [22]. □

C Source-Channel Separation

We now give necessary and sufficient conditions for reliable communication using stand-alone source and channel codes. Suppose that there exists source rates $(r_s^{(1)}, r_s^{(2)}) \in \mathcal{R}$ with $(r_s^{(1)}/\kappa,\ r_s^{(2)}/\kappa) \in \mathcal{C}$. Then, by definition, there exists the following: a source code mapping $\mathbf{W}^{(j)}$ to a message $M^{(j)}$ using $r_s^{(j)}$-bits per source symbol; and a channel code that can reliably transport $M^{(j)}$ using $(r_s^{(j)}/\kappa)$-bits per channel symbol. That is, it is possible to reliably communicate $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ over the network using stand-alone source and channel codes. Conversely, if there are no such source rates $(r_s^{(1)}, r_s^{(2)})$, then it is not possible to reliably communicate $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ using stand-alone source and channel codes. We now formalise this idea.

Definition 4. The bandwidth expansion factor κ is said to be achievable with stand-alone source and channel codes if $\mathcal{R} \cap \kappa\mathcal{C} \neq \emptyset$, where

$$\kappa\mathcal{C} \triangleq \left\{ (r^{(1)}, r^{(2)}) \in \mathbb{R}^2 \ : \ \left(r^{(1)}/\kappa,\ r^{(2)}/\kappa\right) \in \mathcal{C} \right\} . \tag{10}$$

Theorem 2. *The bandwidth expansion κ is achievable with stand-alone source and channel codes if and only if there exists a pmf $p(x^{(0)})$ on $\mathcal{X}^{(0)}$ with*

$$H(W^{(1)}|W^{(2)}) \leq \kappa \min\left\{C^{(1)}, I(X^{(0)}; Y^{(2)})\right\} \tag{11a}$$

$$H(W^{(2)}|W^{(1)}) \leq \kappa \min\left\{C^{(2)}, I(X^{(0)}; Y^{(1)})\right\} . \tag{11b}$$

Proof outline. The equivalence of $\mathcal{R} \cap \kappa\mathcal{C} \neq \emptyset$ and (11) follows from Lemmas 2 and 3, and the fact that $(r^{(1)}, r^{(2)}) \in \mathcal{C}$ implies that

$$\left\{ (\tilde{r}^{(1)}, \tilde{r}^{(2)}) \in \mathbb{R}^2 \ : \begin{array}{ll} 0 & \leq \tilde{r}^{(1)} \leq r^{(1)} \\ 0 & \leq \tilde{r}^{(2)} \leq r^{(2)} \end{array} \right\} \subseteq \mathcal{C} . \tag{12}$$

$\square$

Theorems 1 and 2 together yield the desired source-channel separation principle.

Corollary 2.1. *The bandwidth expansion κ is achievable (with joint source-channel codes) if and only if κ is achievable with stand-alone source and channel codes.*

Corollary 2.1 demonstrates that there is no rate penalty by first encoding each source with an optimal Slepian-Wolf source code, and then using an optimal channel code for the two-way relay network. The following points are worth noting.

Remark 1. If the broadcast channel belongs to $\mathcal{Q}(y^{(1)}, y^{(2)}|x^{(0)})$, then $\mathcal{C}$ is completely defined by the "two-hop" channel capacities $\min\{C^{(1)}, C^{(0,2)}\}$ and $\min\{C^{(2)}, C^{(0,1)}\}$. Moreover, $\mathcal{R} \cap \kappa\mathcal{C} \neq \emptyset$ if and only if

$$H(W^{(j)}|W^{(\sim j)}) \leq \kappa \min\{C^{(j)}, C^{(0,\sim j)}\}\ , \quad j = 1, 2\ . \tag{13}$$

From (13), it is clear that a channel code operating adequately close to the optimal rate point $(\min\{C^{(1)}, C^{(0,2)}\}, \min\{C^{(2)}, C^{(0,1)}\})$ will be sufficient for all source pmfs $q(w^{(1)}, w^{(2)})$ that can be reliably communicated over the two-way relay network. However, if the broadcast channel does not belong to $\mathcal{Q}(y^{(1)}, y^{(2)}|x^{(0)})$, then there may be a non-trivial trade-off between the channel coding rates $r_c^{(1)}$ and $r_c^{(2)}$.

Remark 2. Note that the downlink transmission implicitly involves broadcasting with side-information. For example, the terms $I(X^{(0)}; Y^{(j)})$ stemming from $\mathcal{B}_{out}$ are reminiscent of Tuncel's result [17, Thm.6], where joint source-channel codes were shown to outperform stand-alone source and channel codes. The separation theorem holds for the two-way relay network – in contrast to the set up of [17] – because the channel code can exploit the special complimentary nature of the side-information.

Remark 3. Shannon's point-to-point separation theorem [14, Thm. 8.13.1] also holds in the lossy (rate-distortion) setting [18, Thm 9.2.2]. Similarly, Han's [16] separation theorem for the orthogonal MAC also holds in the lossy setting [19]. Given this, one might conjecture that Corollary 2.1 also holds for lossy reconstructions. Unfortunately, this is not the case; we give a counter-example in the next section.

D Failure of Separation for Lossy Reconstructions

In the separate source-channel coding architecture, $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ are independently compressed to indices $M^{(1)}$ and $M^{(2)}$ at their respective nodes. These indices are passed to the channel code for reliable transportation over the network. Recall, the relay is part of the channel code and therefore plays no role in compressing $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ to $M^{(1)}$ and $M^{(2)}$. The main purpose of this subsection is to show that such separation can be suboptimal for lossy reconstructions. The basic idea is that the relay will need to perform some "source coding" to ensure efficient use of the downlink broadcast channel.

Let $\delta^{(j)} : \mathcal{W}^{(j)} \times \mathcal{W}^{(j)} \to [0, \infty)$ denote generic per-letter distortion measures. Let $(d^{(1)}, d^{(2)}) \in \mathbb{R}^2$. We now define the lossy source coding region, which will replace the lossless source coding region $\mathcal{R}$ in Definition 4. A rate pair $(r_s^{(1)}, r_s^{(2)}) \in \mathbb{R}^2$ is said to be $(d^{(1)}, d^{(2)})$-achievable if the following holds: node j can compress its source sequence $\mathbf{W}^{(j)}$ to a message $M^{(j)}$, using $r_s^{(j)}$ bits per source symbol, such that node $\sim j$ can reconstruct $\hat{\mathbf{W}}^{(j)} = g_s^{(\sim j)}(M^{(j)}, \mathbf{W}^{(\sim j)})$ (see Fig. 3) with an average distortion

$$\Delta^{(j)} \triangleq \frac{1}{\kappa_s t} \sum_{i=1}^{\kappa_s t} \delta^{(j)}(W_i^{(j)}, \hat{W}_i^{(j)}) \leq d^{(j)} . \tag{14}$$

The lossy source coding rate region $\mathcal{R}(d^{(1)}, d^{(2)})$ is defined as the closure of the set of all $(d^{(1)}, d^{(2)})$-achievable rate pairs. Since feedback is not permitted, we have that [23]

$$\mathcal{R}(d^{(1)}, d^{(2)}) = \left\{ (r^{(1)}, r^{(2)}) \in \mathbb{R}^2 \ : \ r^{(j)} \geq R_{WZ}^{(j|\sim j)}(d^{(j)}) \ , j = 1, 2 \right\} \ , \tag{15}$$

where $R_{WZ}^{(1|2)}(d^{(1)})$ and $R_{WZ}^{(2|1)}(d^{(2)})$ are the Wyner-Ziv rate-distortion functions for source coding with side-information at the decoder [24].

Lemma 4. *The bandwidth expansion κ is $(d^{(1)}, d^{(2)})$-achievable with stand-alone source and channel codes if and only if there exists a pmf $p(x^{(0)})$ on $\mathcal{X}^{(0)}$ with*

$$R_{WZ}^{(j|\sim j)}(d^{(j)}) \leq \kappa \min \left\{ C^{(j)}, I(X^{(0)}; Y^{(\sim j)}) \right\} \ , \quad j = 1, 2 \ . \tag{16}$$

Now consider the joint source-channel coding setting. The bandwidth expansion κ is said to be $(d^{(1)}, d^{(2)})$-achievable (with joint source-channel codes) if for every $\epsilon > 0$ there exists a joint source-channel code ($f^{(0)}$, $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, $g^{(2)}$) of the form (3) with an average distortion $\Delta^{(j)} \leq d^{(j)} + \epsilon$. The next example describes a situation where the conditions of Lemma (4) are not needed for achievability in this sense; that is, a joint source-channel code can outperform stand-alone source and channel codes.

Example 1. Suppose that $\kappa_c = \kappa_s = 1$, $\mathcal{W}^{(j)} = \{0, 1\}$, and $(W^{(1)}, W^{(2)})$ is a doubly symmetric binary source with cross-over probability ρ; i.e., $0 < \rho < 1/2$,

$$q(w^{(1)}, w^{(2)}) = \begin{cases} \frac{1}{2}(1-\rho) & \text{if } w^{(1)} = w^{(2)} \ , \\ \frac{1}{2}\rho & \text{otherwise.} \end{cases} \tag{17}$$

Note, we have that $R_{WZ}^{(1|2)}(d) = R_{WZ}^{(2|1)}(d)$ for all $0 \leq d \leq 1/2$.

Suppose that both uplink channels are noiseless with unit capacity; i.e., $\mathcal{X}^{(j)} = \mathcal{Y}^{(0,j)} = \{0, 1\}$ and $q(y^{(0,j)}|x^{(j)}) = 1$ if $y^{(0,j)} = x^{(j)}$ and zero otherwise. Choose $d^{(1)} = d^{(2)}$ and the downlink to be a single channel with capacity c such that

$$h(\rho) - h(d) < c < R_{WZ}^{(1|2)}(d) \ , \tag{18}$$

where $h(\cdot)$ is the binary entropy function (such d and c can always be found [24, Sec. 2]). Since $c < 1$, the capacity region is given by

$$\mathcal{C} = \left\{ (r^{(1)}, r^{(2)}) \in \mathbb{R}^2 \ : 0 \leq r^{(j)} \leq c \ , \quad j = 1, 2 \right\} \ . \tag{19}$$

Note, $\mathcal{R}(d, d) \cap \mathcal{C} = \emptyset$; thus, κ is not achievable with stand-alone source and channel codes.

Consider the following joint source-channel code. Node j transmits $\mathbf{W}^{(j)}$ uncoded to the relay, and relay computes the modulo-two sum $\mathbf{Z} = \mathbf{W}^{(1)} \oplus \mathbf{W}^{(2)}$. The symbols in $\mathbf{Z}$ are iid with pmf $\Pr[Z = 0] = 1 - \rho$ and $\Pr[Z = 1] = \rho$. The relay compresses $\mathbf{Z}$

to $\hat{\mathbf{Z}}$ with distortion d. The minimum rate needed for this compression is $h(\rho) - h(d)$ bits per source symbol. The relay transmits $\hat{\mathbf{Z}}$ over the downlink channel to both nodes. Node j sets $\hat{W}_i^{(\sim j)} = \hat{Z}_i \oplus W_i^{(j)}$ for $i = 1, 2, \ldots, \kappa_s t$. It can be verified that these reconstructions achieve an average distortion d (see also [3, Example 1]).

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