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Achievable Rates and Optimal Schedules for Half Duplex Multiple-Relay Networks

Lawrence Ong, Wei Wang, Mehul Motani
Department of Electrical and Computer Engineering
National University of Singapore
Email: lawrence.ong@cantab.net, {wang.wei, motani}@nus.edu.sg

Abstract— **We study the half duplex multiple-relay channel (MRC) where every node can either transmit or listen but not both at the same time. We derive a capacity upper bound based on a max-flow min-cut argument and achievable transmission rates based on the decode-forward coding strategy (DF), for both the half duplex discrete memoryless MRC and the half duplex phase fading Gaussian MRC. The upper bound and achievable rates are functions of the transmit state vector (a description of which nodes transmit and which receive). More precisely, they are functions of the time fraction of different transmit state vectors, which we term a schedule. We formulate the optimal scheduling problem as a max-min optimization to find the schedule that maximizes the DF rate for the half duplex MRC. We use a technique based on minimax hypothesis testing to solve this problem and demonstrate it on a four-node MRC, getting closed form solutions in certain scenarios. For the phase fading Gaussian channel, surprisingly, we discover that optimal schedules can be solved using linear programming.**

I. Introduction

In this paper, we study the half duplex wireless multiple-relay channel (MRC) [1], [2], [3], in which every node can either listen or transmit but not both at the same time. The motivation for this is that most radio frequency transceivers available today either operate in half duplex mode or in full duplex mode with the transmitter and the receiver operating on separate frequency bands (the latter can be modeled as orthogonal channels [4]). Information-theoretic studies of half duplex networks provide a framework for analyzing issues in half duplex networks that do not appear in the full duplex counterpart.

For the half duplex MRC, we first introduce the idea of *transmit state vector* to describe which nodes are transmitting and which nodes are receiving at any time. Take a D-node MRC for example, where the source is always transmitting and the destination is always listening. The total number of transmit state vectors is 2^{D-2}, capturing if each of the $D-2$ relays is transmitting or listening.

We derive an upper bound on the capacity and achievable rates based on the decode-forward coding strategy (DF) [5], [1], [2] for the half duplex MRC. We show that the capacity upper bound and achievable DF rates for the half duplex MRC depend on the time fractions of the transmit state vectors (or the schedule), but do not depend on the sequence or order of the states. This means we do not need to coordinate the transmit-listen sequence among the nodes to maximize the achievable DF rate or to find the capacity upper bound. We view the combination of time fractions of different transmit state vectors as a *schedule* for the network.

We then formulate the optimal scheduling problem to find an optimal schedule, i.e., a schedule that maximizes the DF rate for the half duplex MRC. This optimization turns out to be a max-min problem, which is not easily solved as the number of transmit state vectors grows exponentially with the number of nodes. We propose a technique to solve for optimal DF schedules based on an approach used to solve for the minimax detection rule in hypothesis testing [6, Ch. II.C]. Using this technique, we are able to obtain closed form solutions for certain channel topologies and gain insight into operating the half duplex network, e.g., some nodes do not need to transmit at all. For the half duplex phase fading Gaussian MRC, surprisingly, we discover that optimal schedules can be solved by linear programming. This allows us to find optimal schedules for MRCs with several tens of nodes (e.g., it takes less than an hour to solve for an MRC with 20 nodes).

II. Channel Models

Though the half duplex single-relay channel (SRC) has been studied [7], [8], [9], [10], the half duplex MRC has not been studied except for the case where the relays only receive signals from the source and the destination only receives signals from the relays [11], [12]. Here, we investigate the general half duplex MRC where all nodes can potentially hear all other nodes.

A. Half Duplex Discrete Memoryless MRC

Consider a D-node half duplex MRC with nodes $\{1, 2, \dots, D\} \triangleq \mathcal{D}$. Node 1 is the source, node D the destination, and nodes 2 to $(D-1)$ relays. Message w is generated at node 1 and is to be sent to node D. A node can only transmit (T) or listen (L), but not both simultaneously. We assume that the source, node 1, is always transmitting, and the destination, node D, is always listening. We define $\mathcal{R}$ as the set of all relays. As not all relays are always needed, we define an *active relay set* $\mathcal{A} \subseteq \mathcal{R}$ that consists of relays that help the source-destination pair in data transmission. We define the set of all *unused relays* (relays not in the active set) as $\mathcal{U} \triangleq \mathcal{R} \setminus \mathcal{A}$.

Now, we define transmit state vector to describe which nodes (in $\mathcal{A}$) are transmitting and which nodes are listening, and a few definitions pertaining to the half duplex MRC.

Definition 1: Consider the half duplex MRC and an active relay set $\mathcal{A} = \{a_1, a_2, \ldots, a_{|\mathcal{A}|}\}$. The transmit state vector can be expressed as $\mathbf{s} = (s_1, s_2, \ldots, s_{|\mathcal{A}|}) \in \{L, T\}^{|\mathcal{A}|}$, where $s_i = T$ if node a_i transmits, and is L otherwise (i.e., if node a_i listens). For $a_i, a_j \in \mathcal{A}, a_i \neq a_j$, we assume $a_i > a_j$ if $i > j$.

Definition 2: Consider an active relay set $\mathcal{A}$. We define $\mathcal{T}(\mathbf{s})$ as the set of all *active relays* that are transmitting, i.e., $\{a_i :\ \text{all } a_i \in \mathcal{A} \text{ where } s_i = T\}$. Similarly, we define $\mathcal{L}(\mathbf{s})$ as the set of all active relays that are listening, i.e., $\{a_j :\ \text{all } a_j \in \mathcal{A} \text{ where } s_j = L\}$. Note that $\mathcal{T}(\mathbf{s}) \cup \mathcal{L}(\mathbf{s}) = \mathcal{A}$.

We set $\tilde{x}_i \in \mathcal{X}_i$ to be the "transmit" message of node i when it is in the listening state, i.e., if $i \in \mathcal{L}(\mathbf{s})$, then $x_i = \tilde{x}_i$. Similarly, we set $\tilde{y}_j \in \mathcal{Y}_j$ to be the "received" signal of node j when it is in the transmitting state, i.e., if $t \in \mathcal{T}(\mathbf{s})$, then $y_t = \tilde{y}_t$. For unused relays, we set them to the listening mode, i.e., if $i \in \mathcal{U}$, then $x_i = \tilde{x}_i$. We assume that $\mathcal{A}$, $\mathbf{s}$, and $\{\tilde{x}_i, \tilde{y}_j\}$ of all nodes are fixed and known *a priori* to all nodes.

The channel distribution for the D-node half duplex MRC with active relay set $\mathcal{A}$ at state $\mathbf{s}$ is given by

$$\begin{aligned} &p(y_2, y_3, \ldots, y_D | x_1, x_2, \ldots, x_{D-1}, \mathbf{s}) \\ &= p^*\big(y_{\mathcal{L}(\mathbf{s})}, y_{\mathcal{U}}, y_D | x_1, x_2, \ldots, x_{D-1}, \{y_j = \tilde{y}_j\}_{j \in \mathcal{T}(\mathbf{s})}\big) \\ &\qquad\qquad \times \mathbf{1}(y_j = \tilde{y}_j, \forall j \in \mathcal{T}(\mathbf{s})), \quad (1) \end{aligned}$$

on $\mathcal{Y}_2 \times \mathcal{Y}_3 \times \cdots \times \mathcal{Y}_D$, for each $(x_1, x_2, \ldots, x_{D-1}) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_{D-1}$. $\mathbf{1}(x)$ is the indicator function that is 1 if x is true, and is 0 otherwise. In this paper, we only consider memoryless and time invariant channels [2].

We use the following notations. x_i denotes an input from node i into the channel, $x_{i,j}$ denotes the j-th input from node i into the channel, y_i denotes an output from the channel to node i, and $y_{i,j}$ denotes the j-th output from the channel to node i.

B. Block Codes and Achievable Rates

In the MRC, the information source at node 1 emits random messages w, each taking on values from a finite set of size M, that is $w \in \{1, ..., M\} \triangleq \mathcal{W}$. The messages are to be sent to the destination, node D.

We consider block coding and define each n uses of the channel as a block. We define block codes, average error probability, achievable rate, and capacity as follows.

Definition 3: An (M, n) code of a D-node MRC comprises:

- An encoding function at node 1, $f_1 : \mathcal{W} \to \mathcal{X}_1^n$, which maps a source letter to a codeword of length n.
- n encoding functions at node $t, t = 2, 3, \ldots, D-1$, $f_{t,i} : \mathcal{Y}_t^{i-1} \to \mathcal{X}_t, i = 1, 2, \ldots, n$, such that $x_{t,i} = f_{t,i}(y_{t,1}, y_{t,2}, \ldots, y_{t,(i-1)})$, which map past received signals to the signal to be transmitted into the channel.
- A decoding function at the destination, $g_D : \mathcal{Y}_D^n \to \mathcal{W}$, such that $\hat{w} = g_D(y_D^n)$, which maps n received signals to a source letter estimate.

Definition 4: On the assumption that the source letter w is uniformly distributed over $\{1, ..., M\}$, the average error probability is defined as $P_e = \Pr\{\hat{w} \neq w\}$.

Definition 5: The rate $R \leq \frac{1}{n} \log M$ is achievable if, for any $\epsilon > 0$, there is at least one (M, n) code such that $P_e < \epsilon$.

Definition 6: The capacity is defined as the closure of the set of all achievable rates.

For a set of nodes $\mathcal{B} = \{b_1, b_2, \ldots, b_{|\mathcal{B}|}\} \subseteq \mathcal{D}$, we define $X_{\mathcal{B}} = (X_{b_1}, X_{b_2}, \ldots, X_{b_{|\mathcal{B}|}})$.

C. Half Duplex Phase Fading Gaussian MRC

Now, we define the half duplex phase fading D-node Gaussian MRC. For relays that are transmitting, i.e., $j \in \mathcal{T}(\mathbf{s})$, we set $y_j = \tilde{y}_j = 0$. For relays that are listening, i.e., $i \in \mathcal{L}(\mathbf{s}) \cup \mathcal{U}$, we set $x_i = \tilde{x}_i = 0$. In transmit state vector $\mathbf{s}$, the received signal at node t is given by

$$Y_t = \begin{cases} \sum_{i \in \{1\} \cup \mathcal{T}(\mathbf{s})} \sqrt{\lambda_{i,t}} e^{j\theta(i,t)} X_i + Z_t \\ \qquad\qquad\qquad\qquad , t \in \mathcal{L}(\mathbf{s}) \cup \mathcal{U} \cup \{D\} \\ \tilde{y}_t = 0 \qquad\qquad\quad , \text{otherwise} \end{cases}, \quad (2)$$

where X_i, input to the channel form node i, is a zero-mean, complex random variable. Z_t, the receiver noise at node t, is an i.i.d., zero-mean, complex, Gaussian random variable with variance $E[Z_t Z_t^\dagger] = N_t$. $Z_t^\dagger$ is the complex-conjugate transpose of Z_t. $\lambda_{i,t}$, capturing the path loss from node i to node t, is $\kappa d_{i,t}^{-\eta}$ for $d_{i,t} \geq 1$, and is κ otherwise. $d_{i,t} \geq 0$ is the distance between nodes i and t, $\eta \geq 2$ the attenuation exponent (with $\eta = 2$ for free space transmission), and κ a positive constant. $e^{j\theta(i,t)}$ is the phase fading random variable, where $\theta(i,t)$ is uniformly distributed over $[0, 2\pi)$. $\theta(i,t)$ for all i and t are jointly independent of each other.

We assume that all nodes know κ and $d_{i,t}$. We also assume that node t only knows $\theta(i,t), \forall i$ and does not know any $\theta(i,j)$ for $j \neq t$. Hence, the transmitted signals of node i cannot be chosen as a function of $\theta(i,t)$ for any t.

In this paper, we consider the following individual-node *per-symbol* power constraint. Setting the half duplex constraints, $x_j = \tilde{x}_j = 0$ for node j in the listening state, we get

$$E[X_i X_i^\dagger] \leq \begin{cases} P_i & , i \in \mathcal{T}(\mathbf{s}) \cup \{1\} \\ 0 & , \text{otherwise} \end{cases}. \quad (3)$$

III. Capacity Upper Bound

A. Capacity Upper Bound for the Half Duplex Discrete Memoryless MRC

An upper bound on the capacity of the half duplex MRC is given by the following theorem.

Theorem 1: Consider a D-node half duplex discrete memoryless MRC. If rate R is achievable, then there exists some joint distribution in the form $p(x_1, x_2, \ldots, x_{D-1}, \mathbf{s}) = p(\mathbf{s}) p(x_1, x_{\mathcal{T}(\mathbf{s})} | \{x_j = \tilde{x}_j\}_{j \in \mathcal{L}(\mathbf{s})}, \mathbf{s}) \mathbf{1}(x_j = \tilde{x}_j, \forall j \in \mathcal{L}(\mathbf{s}))$, such that

$$R \leq \sum_{\mathbf{s} \in \{L,T\}^{D-2}} p(\mathbf{s}) I(X_1, X_{\mathcal{B} \cap \mathcal{T}(\mathbf{s})}; Y_{\mathcal{B}^c \cap \mathcal{L}(\mathbf{s})}, Y_D | X_{\mathcal{B}^c \cup \mathcal{L}(\mathbf{s})}), \quad (4)$$

for all $\mathcal{B} \subseteq \mathcal{R}$, and $\mathcal{B}^c = \mathcal{R} \setminus \mathcal{B}$. $\mathcal{R}$ is the set of all relays, and $\mathcal{A} = \mathcal{R}$. $\mathbf{s}$ is the transmit state vector. $\mathcal{T}(\mathbf{s})$ is the set of all relays that are transmitting and $\mathcal{L}(\mathbf{s})$ is the set of all relays that are listening in state $\mathbf{s}$.

Proof: [Proof of Theorem 1] The proof follows from the cut-set bound for the full duplex multiple-source multiple-destination network [13, Th. 14.10.1]. Since the half duplex network is a "restricted version" of the full duplex network, we show how the result in [13, Th. 14.10.1] specializes to the cut-set bound for the half duplex MRC.

The full duplex MRC is a special case of the full duplex multiple-source multiple-destination network, in which there is only one source-destination pair. Rates at which the source (node 1, which does not listen) can transmit to the destination (node D, which does not transmit) must be bounded by the cut rates of all cuts separating the source and the destination, i.e., if R is achievable, there must exists some $p(x_1, x_2, \dots, x_{D-1})$, such that

$$R \le I(X_1, X_{\mathcal{B}}; Y_{\mathcal{B}^c}, Y_D | X_{\mathcal{B}^c}), \tag{5}$$

for all $\mathcal{B} \subseteq \mathcal{R}$, and $\mathcal{B}^c = \mathcal{R} \setminus \mathcal{B}$. $\mathcal{R}$ is the set of all relays.

The half duplex MRC is a special case of the full duplex MRC, in which a node can only transmit or listen. As using a relay can never decrease the cut rate compared to not using it, we set $\mathcal{A} = \mathcal{R}$. This means $\mathcal{U} = \varnothing$. Consider the time fraction $p(\mathbf{s}')$ where the channel is in state $\mathbf{s}'$ and consider the cut that partitions $\mathcal{D}$ into $\{1\} \cup \mathcal{B}$ and $\mathcal{B}^c \cup \{D\}$. If "rate fraction" $R(\mathbf{s}')$ in this time fraction is achievable, there must exists some input distribution in the following form (because of the half duplex constraints)

$$p(x_1, x_2, \dots, x_{D-1}|\mathbf{s}') = p(x_1, x_{\mathcal{T}(\mathbf{s}')}|\{x_j = \tilde{x}_j\}_{j \in \mathcal{L}(\mathbf{s}')}, \mathbf{s}') \times \mathbf{1}\left(x_j = \tilde{x}_j, \forall j \in \mathcal{L}(\mathbf{s}')\right), \tag{6}$$

such that

$$R(\mathbf{s}') \le p(\mathbf{s}') I(X_1, X_{\mathcal{B} \cap \mathcal{T}(\mathbf{s}')}; Y_{\mathcal{B}^c \cap \mathcal{L}(\mathbf{s}')}, Y_D | X_{\mathcal{B}^c \cup \mathcal{L}(\mathbf{s}')}). \tag{7}$$

We note that in set $\mathcal{B}$, only nodes in $\mathcal{T}(\mathbf{s}')$ transmit; in set $\mathcal{B}^c$, only nodes in $\mathcal{L}(\mathbf{s}')$ listen.

Still on the same cut, now, we consider the entire period/block of transmissions that consists of different transmit state vectors. From [10, Appendix A], we know that the upper bound for achievable rates across two states is the the sum of that of the individual state. Extending that, for any distribution of the transmit state vectors $p(\mathbf{s})$, if rate R for the half duplex MRC is achievable under the transmit state vector distribution, there must exists some distribution in the form

$$p(\mathbf{s})p(x_1, x_2, \dots, x_{D-1}|\mathbf{s}) = p(\mathbf{s})p(x_1, x_{\mathcal{T}(\mathbf{s})}| \{x_j = \tilde{x}_j\}_{j \in \mathcal{L}(\mathbf{s})}, \mathbf{s}) \mathbf{1}\left(x_j = \tilde{x}_j, \forall j \in \mathcal{L}(\mathbf{s})\right), \tag{8}$$

such that

$$R = \sum_{\mathbf{s} \in \{L,T\}^{D-2}} R(\mathbf{s}) \tag{9a}$$
$$\le \sum_{\mathbf{s} \in \{L,T\}^{D-2}} p(\mathbf{s}) I(X_1, X_{\mathcal{B} \cap \mathcal{T}(\mathbf{s})}; Y_{\mathcal{B}^c \cap \mathcal{L}(\mathbf{s})}, Y_D | X_{\mathcal{B}^c \cup \mathcal{L}(\mathbf{s})}), \tag{9b}$$

where $\sum_{\mathbf{s} \in \{L,T\}^{D-2}} p(\mathbf{s}) = 1$.

As (9b) must be true for all possible cuts separating the source and the destination, we get Theorem 1. ■

Remark 1: The above result can also be obtained from [14, Corollary 2] with the following modifications: (1) there is only one source-destination pair, (2) there are at most 2^{D-2} transmit state vectors, (3) for each state, node i in the listening state transmits $\tilde{x}_i$, and node j in the transmitting state receives $\tilde{y}_j$.

B. Capacity Upper Bound for the Half Duplex Phase Fading Gaussian MRC

For the phase fading Gaussian MRC, we set the inputs from transmitting nodes to the channel to be independent Gaussian, as coherent combining is not possible. We note that $\tilde{x}_j = 0, \forall j \in \mathcal{L}(\mathbf{s})$. So, we get the following capacity upper bound.

Theorem 2: Consider a D-node half duplex phase fading Gaussian MRC. If rate R is achievable, then there exists a $p(\mathbf{s})$, such that

$$R \le \sum_{\mathbf{s} \in \{L,T\}^{D-2}} \left[p(\mathbf{s}) \times L\left(\sum_{j \in (\mathcal{B}^c \cap \mathcal{L}(\mathbf{s})) \cup \{D\}} \frac{\sum_{i \in \{1\} \cup (\mathcal{B} \cap \mathcal{T}(\mathbf{s}))} \lambda_{i,j} P_i}{N_j} \right) \right], \tag{10}$$

for all $\mathcal{B} \subseteq \mathcal{R}$, $\mathcal{B}^c = \mathcal{R} \setminus \mathcal{B}$, $\mathcal{A} = \mathcal{R}$, and $L(x) = \log(1+x)$.

Proof: [Proof of Theorem 2] Theorem 2 follows directly from Theorem 1 by using independent Gaussian inputs for all nodes. See [10, Lemma 1], [2, Theorems 6 & 7] for the optimality of independent Gaussian inputs in phase fading channels. ■

IV. Achievability

A. Achievability of the Half Duplex Discrete Memoryless MRC

A lower bound on the capacity of the half duplex D-node MRC is given by the following theorem.

Theorem 3: Consider a D-node half duplex MRC. Rates up to the following value are achievable.

$$R_{\text{DF}} = \max_{\mathcal{M} \in \Pi(\mathcal{D})} \max_{p(\mathbf{s})} \max_{p(x_1, x_2, \dots, x_{D-1}|\mathbf{s})} \min_{m_t \in \mathcal{M} \setminus \{1\}} \sum_{\substack{\mathbf{s} \in \{L,T\}^{|\mathcal{M}|-2} \\ m_t \in \mathcal{L}(\mathbf{s}) \cup \{D\}}} \left[p(\mathbf{s}) I\Big(X_1, X_{\{m_2, \dots, m_{t-1}\} \cap \mathcal{T}(\mathbf{s})}; Y_{m_t} \Big| X_{\{m_t, \dots, m_{|\mathcal{M}|-1}\} \cup \mathcal{L}(\mathbf{s}) \cup \mathcal{U}}, \mathbf{S} = \mathbf{s} \Big) \right]. \tag{11a}$$

We assume that nodes not in route $\mathcal{M}$ are unused. So, $\mathcal{A} = \{m_2, m_3, \dots, m_{|\mathcal{M}|-1}\}$.

Here, $\mathcal{M} = \{m_1 = 1, m_2, \dots, m_{|\mathcal{M}|} = D\}$ is the route [15], [16], i.e., an ordered set of nodes from the source to the destination. $\Pi(\mathcal{D})$ is the set of all possible routes from the source to the destination. The first maximization is over all possible route selections. The second maximization is over all possible schedule $p(\mathbf{s}) = p(s_1, s_2, \dots, s_{|\mathcal{M}|-2})$. The third maximization is over all input distribution of the form $p(x_1, x_2, \dots, x_{D-1}|\mathbf{s}) = p(x_1, x_{\mathcal{T}(\mathbf{s})}|\{x_i = \tilde{x}_i\}_{i\in\mathcal{L}(\mathbf{s})\cup\mathcal{U}}, \mathbf{s})\mathbf{1}(x_i = \tilde{x}_i, \forall i \in \mathcal{L}\cup\mathcal{U})$. We assume that all relays not in the route do not transmit, i.e., $\mathcal{M}^c \subseteq \mathcal{L}(\mathbf{s}), \forall \mathbf{s}$.

Remark 2: When $D = 3$, our result reduces to that for the half duplex SRC [7].

Proof: [Proof of Theorem 3] Refer to [17]. ■

Remark 3: For a chosen route, the maximum DF rate only depends on the fractions of the transmit state vectors, and does not depend on the sequence of the states.

B. Achievability of the Half Duplex Phase Fading Gaussian MRC

In the half duplex phase fading Gaussian MRC, where the fading phases are only known to the receivers, we can show that R_{DF} in Theorem 3 can be attained by independent Gaussian inputs [2]. Hence, we have the following theorem.

Theorem 4: Consider a D-node half duplex phase fading Gaussian MRC. Rates up to the following value are achievable.

$$R_{\text{DF}} = \max_{\mathcal{M}\in\Pi(\mathcal{D})} \max_{p(\mathbf{s})} \min_{m_t\in\mathcal{M}\setminus\{1\}} \sum_{\substack{\mathbf{s}\in\{L,T\}^{|\mathcal{M}|-2}\\ m_t\in\mathcal{L}(\mathbf{s})\cup\{D\}}} p(\mathbf{s}) L\left(\frac{\sum_{i\in\{1\}\cup(\{m_2,\dots,m_{t-1}\}\cap\mathcal{T}(\mathbf{s}))}\lambda_{i,m_t}P_i}{N_{m_t}}\right), \tag{12}$$

where $L(x) = \log(1+x)$ and $\mathcal{A} = \{m_2, m_3, \dots, m_{|\mathcal{M}|-1}\}$.

Proof: [Proof of Theorem 4] The rate is obtained from Theorem 3 using independent Gaussian inputs. ■

V. The Optimal DF Scheduling Problem

We define a schedule and an optimal DF schedule of the half duplex network as follows.

Definition 7: A schedule for the half duplex network is defined as the probability mass function of all possible transmit state vectors, or $p(\mathbf{s})$.

Definition 8: For a chosen route, an optimal DF schedule is the schedule that gives the maximum DF rate.

Now, we formulate the optimal scheduling problem for the D-node half duplex phase fading Gaussian MRC. On some route $\mathcal{M}\in\Pi(\mathcal{D})$, an optimal schedule is a probability mass function $p^*(\mathbf{s})$ on $\{L,T\}^{|\mathcal{M}|-2}$ such that

$$p^*(\mathbf{s}) \in \operatorname*{argmax}_{p(\mathbf{s})} \min_{m_t\in\mathcal{M}\setminus\{1\}} \sum_{\substack{\mathbf{s}\in\{L,T\}^{|\mathcal{M}|-2}\\ m_t\in\mathcal{L}(\mathbf{s})\cup\{D\}}} \left[p(\mathbf{s}) L\left(\frac{\sum_{i\in\{1\}\cup(\{m_2,\dots,m_{t-1}\}\cap\mathcal{T}(\mathbf{s}))}\lambda_{i,m_t}P_i}{N_{m_t}}\right)\right]. \tag{13}$$

For an active set $\mathcal{A}$, we can define a mapping $f(\mathbf{s})$ from $\{L,T\}^{|\mathcal{A}|}$ to $\mathcal{Z}^+$, as follows.

$$f(\mathbf{s}) = \mathbf{1}(s_1 = T)2^{|\mathcal{A}|-1} + \mathbf{1}(s_2 = T)2^{|\mathcal{A}|-2} + \dots + \mathbf{1}(s_{|\mathcal{A}-1|} = T)2^1 + \mathbf{1}(s_{|\mathcal{A}|} = T)2^0 \tag{14a}$$

$$= \sum_{i=1}^{|\mathcal{A}|} \mathbf{1}(s_i = T)2^{|\mathcal{A}|-i}. \tag{14b}$$

Note that $f(\mathbf{s}) \in \{0, 1, \dots, 2^{|\mathcal{A}|} - 1\}$. With this mapping, we can use an alternate notation for the transmit state vector probability, i.e., $p(\mathbf{s}) = p_{f(\mathbf{s})}$. So, the optimal scheduling problem can be re-written as finding a set of $\{p_0, p_1, \dots, p_{2^{|\mathcal{A}|}-1}\}$ such that $p_i \geq 0$ and $\sum_i p_i = 1$ that satisfies (13).

VI. A Technique to Solve Max-Min Problems

From the previous section, we know that the optimal DF scheduling problem for the half duplex phase fading Gaussian MRC is a max-min optimization problem. In this section, we propose a technique to solve max-min optimization problems. The optimization technique is adapted from a solution approach for minimax hypothesis testing [6, Ch. II.C]. In the next section, we show how we can use this technique to solve the optimal DF scheduling problem for the four-node half duplex MRC.

Consider the following max-min optimization problem.

$$\max_{\mathbf{p}\in\mathcal{G}} \min\{R_2(\mathbf{p}), R_3(\mathbf{p}), R_4(\mathbf{p})\}, \tag{15}$$

where $\mathbf{p} = (p_0, p_1, \dots, p_M)$ is a vector of real numbers, and $R_i(\mathbf{p})$ is a real and continuous function of $\mathbf{p}$ for $i = 2, 3, 4$.

First, we define a new function, which is a weighted average of $R_i(\mathbf{p})$ for $i = 2, 3, 4$.

$$R(\boldsymbol{\alpha}, \mathbf{p}) = \alpha_1 R_2(\mathbf{p}) + \alpha_2 R_3(\mathbf{p}) + (1-\alpha_1-\alpha_2)R_4(\mathbf{p}), \ \boldsymbol{\alpha}\in\mathcal{H}, \tag{16}$$

where $\mathcal{H} \triangleq \{(\alpha_1, \alpha_2) \in \mathcal{R}^2 : \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 \leq 1\}$. For fixed $\mathbf{p}$, $R(\boldsymbol{\alpha}, \mathbf{p})$ is a triangle plane with vertices at $(\alpha_1 = 0, \alpha_2 = 0, R_4(\mathbf{p})), (\alpha_1 = 1, \alpha_2 = 0, R_2(\mathbf{p}))$, and $(\alpha_1 = 0, \alpha_2 = 1, R_3(\mathbf{p}))$.

We define another function

$$V(\boldsymbol{\alpha}) \triangleq \max_{\mathbf{p}\in\mathcal{G}} R(\boldsymbol{\alpha}, \mathbf{p}) = R(\boldsymbol{\alpha}, \mathbf{p}^{\boldsymbol{\alpha}}), \tag{17}$$

where

$$\mathbf{p}^{\boldsymbol{\alpha}} \in \operatorname*{argmax}_{\mathbf{p}} R(\boldsymbol{\alpha}, \mathbf{p}), \tag{18}$$

for some $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$.

We can show that $V(\boldsymbol{\alpha})$ is continuous and convex in $\boldsymbol{\alpha} \in \mathcal{H}$, and all planes $R(\boldsymbol{\alpha}, \mathbf{p}), \forall \mathbf{p} \in \mathcal{G}$, lie below the curve $V(\boldsymbol{\alpha})$. Fig. 1 shows an example of $V(\boldsymbol{\alpha})$ and $R(\boldsymbol{\alpha}, \mathbf{p}')$ for some $\mathbf{p}' \in \mathcal{G}$. For any $\boldsymbol{\alpha}' \in \mathcal{H}$, $R(\boldsymbol{\alpha}', \mathbf{p}) \leq V(\boldsymbol{\alpha}')$, with equality when $\mathbf{p} = \mathbf{p}^{\boldsymbol{\alpha}'}$. Also, for any $\boldsymbol{\alpha}'$, the plane $R(\boldsymbol{\alpha}, \mathbf{p}^{\boldsymbol{\alpha}'})$ is tangential to $V(\boldsymbol{\alpha})$ at $\boldsymbol{\alpha}'$.

Let $\boldsymbol{\alpha}^* = (\alpha_1^*, \alpha_2^*) \in \operatorname{argmin}_{\boldsymbol{\alpha}\in\mathcal{H}} V(\boldsymbol{\alpha})$. Note that $\boldsymbol{\alpha}^*$ is one of the solutions for $\min_{\boldsymbol{\alpha}\in\mathcal{H}} V(\boldsymbol{\alpha})$, and there might be more than one solution. We can show that $\mathbf{p}^{\boldsymbol{\alpha}^*}$ is a

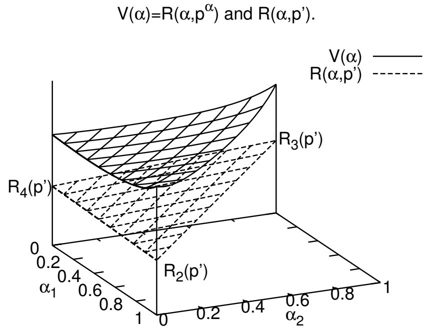


Fig. 1. $V(\boldsymbol{\alpha})$ and $R(\boldsymbol{\alpha}, \mathbf{p}')$.

solution to the optimization problem (15). We define $\mathbf{p}^* = (p_0^*, p_1^*, p_2^*, p_3^*) = \mathbf{p}^{\boldsymbol{\alpha}^*}$.

So, we approach the optimization as follows. We first determine a few subsets in $\mathcal{H}$ where $\boldsymbol{\alpha}^*$ lies. Each subset maps to some relationship among $\{R_i(\mathbf{p}^{\boldsymbol{\alpha}^*})\}$, and it simplifies the search for a solution $\mathbf{p}^{\boldsymbol{\alpha}^*} = \mathbf{p}^*$.

For the max-min optimization problem with three terms, i.e., $R_i(\mathbf{p}), i = 2, 3, 4$, we can divide $\mathcal{H}$ into seven subsets in which $\boldsymbol{\alpha}^*$ can lie, and the corresponding relationship among $\{R_i(\mathbf{p}^{\boldsymbol{\alpha}^*})\}_{i=2,3,4}$. We can prove the following sufficient and necessary conditions for the seven subsets (cases), as follows.

i) $\alpha_1^* = 0, \alpha_2^* = 0$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = R_4(\mathbf{p}) \Leftrightarrow R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_2(\mathbf{p}^{\boldsymbol{\alpha}^*})$, and $\max_{\mathbf{p}} R_4(\mathbf{p}) = R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$.

ii) $\alpha_1^* = 0, \alpha_2^* = 1$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = R_3(\mathbf{p}) \Leftrightarrow R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_2(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$, and $\max_{\mathbf{p}} R_3(\mathbf{p}) = R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$.

iii) $\alpha_1^* = 1, \alpha_2^* = 0$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = R_2(\mathbf{p}) \Leftrightarrow R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$, and $\max_{\mathbf{p}} R_2(\mathbf{p}) = R_2(\mathbf{p}^{\boldsymbol{\alpha}^*})$.

If cases i–iii are false, i.e., the minima of $V(\boldsymbol{\alpha})$ do not occur at $(0, 0), (0, 1)$, or $(1, 0)$, we have the cases below.

iv) $\alpha_1^* = 0, 0 < \alpha_2^* < 1$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = \alpha_2^* R_3(\mathbf{p}) + (1 - \alpha_2^*) R_4(\mathbf{p})$
$\Leftrightarrow R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) = \max_{\mathbf{p}}\{R_3(\mathbf{p}) = R_4(\mathbf{p})\} \leq R_2(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_3(\mathbf{p})$, and $R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_4(\mathbf{p})$.

v) $\alpha_2^* = 0, 0 < \alpha_1^* < 1$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = \alpha_1^* R_2(\mathbf{p}) + (1 - \alpha_1^*) R_4(\mathbf{p})$
$\Leftrightarrow R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) = \max_{\mathbf{p}}\{R_2(\mathbf{p}) = R_4(\mathbf{p})\} \leq R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_2(\mathbf{p})$, and $R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_4(\mathbf{p})$.

vi) $\alpha_1^* + \alpha_2^* = 1, 0 < \alpha_1^* < 1, 0 < \alpha_2^* < 1$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = \alpha_1^* R_2(\mathbf{p}) + \alpha_2^* R_3(\mathbf{p})$
$\Leftrightarrow R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) = \max_{\mathbf{p}}\{R_2(\mathbf{p}) = R_3(\mathbf{p})\} \leq R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_2(\mathbf{p})$, and $R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_3(\mathbf{p})$.

If cases i–vi are false, i.e., the minima of $V(\boldsymbol{\alpha})$ do not occur at $\alpha_1 = 0, \alpha_2 = 0$, or $\alpha_1 + \alpha_2 = 1$, then we have the case below.

vii) $0 < \alpha_1^* < 1, 0 < \alpha_2^* < 1, 0 < \alpha_1^* + \alpha_2^* < 1$:
$\Leftrightarrow R(\boldsymbol{\alpha}^*, \mathbf{p}) = \alpha_1^* R_2(\mathbf{p}) + \alpha_2^* R_3(\mathbf{p}) + (1 - \alpha_1^* - \alpha_2^*) R_4(\mathbf{p})$
$\Leftrightarrow R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_4(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R^*$, $R^* < \max_{\mathbf{p}} R_i(\mathbf{p})$, and $R^* < \max_{\mathbf{p}}\{R_i(\mathbf{p}) = R_j(\mathbf{p})\}_{i \neq j}$.

Now, we propose a procedure to determine in which case an optimal $\boldsymbol{\alpha}^*$ lies.

- If $\exists \mathbf{p}' \in \mathcal{G}$ s.t. $R_4(\mathbf{p}') = \max_{\mathbf{p}} R_4(\mathbf{p}), R_4(\mathbf{p}') \leq R_3(\mathbf{p}')$, $R_4(\mathbf{p}') \leq R_2(\mathbf{p}')$, then $\mathbf{p}'$ is an optimal schedule. Hence, we have case i. Similar arguments can be made for cases ii and iii.
- If $\exists \mathbf{p}' \in \mathcal{G}$ s.t. $R_3(\mathbf{p}') = R_4(\mathbf{p}') = \max_{\mathbf{p}}\{R_3(\mathbf{p}) = R_4(\mathbf{p})\} \leq R_2(\mathbf{p}')$, $R_3(\mathbf{p}') < \max_{\mathbf{p}} R_3(\mathbf{p})$, and $R_4(\mathbf{p}') < \max_{\mathbf{p}} R_4(\mathbf{p})$, then $\mathbf{p}'$ is an optimal schedule. Hence, we have case iv. Similar arguments can be made for cases v and vi.
- Else we have case vii.

After determining where $\boldsymbol{\alpha}^*$ lies, we use the necessary conditions relating $\{R_i(\mathbf{p}^{\boldsymbol{\alpha}^*})\}_{i=2,3,4}$ to solve for $\mathbf{p}^{\boldsymbol{\alpha}^*}$.

Remark 4: A technique adapted from minimax hypothesis testing was used to solve the power allocation problem for the SRC [18], which is a max-min optimization with two terms. In this section, we presented a technique to solve max-min optimizations with three terms. giving necessary and sufficient conditions for optimal $\boldsymbol{\alpha}^*$.

Remark 5: The technique proposed in this section suggests how one can use the same technique to solve a general max-min optimization problem with $K \geq 2$ terms, i.e., $\max_{\mathbf{p} \in \mathcal{G}} \min\{R_1(\mathbf{p}), R_2(\mathbf{p}), \ldots, R_K(\mathbf{p})\}$.

VII. Example: Optimal Schedules and Achievable Rates for the Half Duplex MRC

Now, we derive expressions for achievable rates for the four-node half duplex phase fading Gaussian MRC, and show how to find an optimal schedule for the channel. We assume that the route $\mathcal{M} = \{1, 2, 3, 4\}$ is chosen. For this route, $\mathcal{A} = \{2, 3\}$ and there are four transmit state vectors with the following probabilities: $p(\mathbf{s} = (L, L)) = p_0$, $p(\mathbf{s} = (L, T)) = p_1$, $p(\mathbf{s} = (T, L)) = p_2$, and $p(\mathbf{s} = (T, T)) = p_3$, where $p_i \geq 0$ and $\sum_{i=0}^{3} p_i = 1$.

From (12), the following rate is achievable on the four-node half duplex phase fading Gaussian MRC,

$$R_{\text{DF}} = \max_{p_i \geq 0, \sum p_i = 1} \min\{R_2(\mathbf{p}), R_3(\mathbf{p}), R_4(\mathbf{p})\}, \quad (19)$$

where

$$R_2(\mathbf{p}) = p_0 L(\lambda_{1,2} P_1 / N_2) + p_1 L(\lambda_{1,2} P_1 / N_2) \quad (20a)$$

$$R_3(\mathbf{p}) = p_0 L(\lambda_{1,3} P_1 / N_3) + p_2 L((\lambda_{1,3} P_3 + \lambda_{2,3} P_2) / N_3) \quad (20b)$$

$$\begin{aligned} R_4(\mathbf{p}) = {} & p_0 L(\lambda_{1,4} P_1 / N_4) + p_1 L((\lambda_{1,4} P_1 + \lambda_{3,4} P_3) / N_4) \\ & + p_2 L((\lambda_{1,4} P_1 + \lambda_{2,4} P_2) / N_4) \\ & + p_3 L((\lambda_{1,4} P_1 + \lambda_{2,4} P_2 + \lambda_{3,4} P_3) / N_4), \end{aligned} \quad (20c)$$

where $\mathbf{p} = (p_0, p_1, p_2, p_3)$. An optimal schedule is some $\mathbf{p}^* = (p_0^*, p_1^*, p_2^*, p_3^*)$ that attains R_{DF}.

We provide two methods for solving the optimization in (19).

A. Using the Technique in Section VI

The optimization problem in (19) can be written as

$$\max_{\mathbf{p}\in\mathcal{G}} \min\{R_2(\mathbf{p}), R_3(\mathbf{p}), R_4(\mathbf{p})\}, \tag{21}$$

where $\mathbf{p} = (p_0, p_1, p_2, p_3)$ and $\mathcal{G}$ as the set of all feasible schedules, i.e., $\mathcal{G} \triangleq \{(p_0, p_1, p_2, p_3) \in \mathcal{R}^4 : p_i \geq 0, \sum_i p_i \leq 1, \forall i = 0, 1, 2, 3\}$.

Now, we use the technique presented in Section VI. Since we have three terms in the max-min optimization, there are seven cases. Now, we consider the cases. In case i, $\mathbf{p}^{\boldsymbol{\alpha}^*} \in \text{argmax}_{\mathbf{p}} R_4(\mathbf{p})$. So, $\mathbf{p}^* = (p_0 = 0, p_1 = 0, p_2 = 0, p_3 = 1)$ (see (20c)). However, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = 0 < R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$ and $R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) = 0 < R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$, meaning that the necessary conditions for case i cannot be satisfied. This means that case i will not occur. By a similar argument, we can show that case ii will not occur.

Next, we consider case iii. $\mathbf{p}^{\boldsymbol{\alpha}^*} \in \text{argmax}_{\mathbf{p}} R_2(\mathbf{p})$. Referring to (20a), this means the optimal schedule is such that $p_0^* + p_1^* = 1$, $p_2^* = p_3^* = 0$, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$ and $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$. This means

$$\frac{L(\lambda_{1,2}P_1/N_2)}{L(\lambda_{1,3}P_1/N_3)} \leq p_0^* \leq 1 \tag{22a}$$

$$0 \leq p_0^* \leq \frac{L((\lambda_{1,4}P_1 + \lambda_{3,4}P_3)/N_4) - L(\lambda_{1,2}P_1/N_2)}{L((\lambda_{1,4}P_1 + \lambda_{3,4}P_3)/N_4) - L(\lambda_{1,4}P_1/N_4)} \tag{22b}$$

$$p_1^* = 1 - p_0^*. \tag{22c}$$

Now, we consider case iv. $\mathbf{p}^{\boldsymbol{\alpha}^*} \in \text{argmax}_{\mathbf{p}}(\alpha_2^* R_3(\mathbf{p}) + (1-\alpha_2^*) R_4(\mathbf{p}))$. From (20a)–(20c), we see that the optimal schedules are $\text{argmax}_{\mathbf{p}}\{\alpha_2^* R_3(\mathbf{p}) + (1-\alpha_2^*) R_4(\mathbf{p})\} = \{\mathbf{p} \in \mathcal{G} : p_2^* + p_3^* = 1, p_0^* = p_1^* = 0\}$. However, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = 0 < R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$ and $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = 0 < R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$, meaning that the necessary conditions for case iv cannot be satisfied. Hence, case iv will not occur. By a similar argument, we can show that case v will not occur.

Next, we consider case vi. $\mathbf{p}^{\boldsymbol{\alpha}^*} \in \text{argmax}_{\mathbf{p}}(\alpha_1^* R_2(\mathbf{p}) + \alpha_2^* R_3(\mathbf{p})) = \{\mathbf{p} \in \mathcal{G} : p_0^* + p_2^* = 1, p_1^* = p_3^* = 0\}$, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$. So, $p_0^* L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) = p_0^* L\left(\frac{\lambda_{1,3}P_1}{N_3}\right) + p_2^* L\left(\frac{\lambda_{1,3}P_3 + \lambda_{2,3}P_2}{N_3}\right)$. Solving these equations, we get

$$p_0^* = \frac{L\left(\frac{\lambda_{1,3}P_3+\lambda_{2,3}P_2}{N_3}\right)}{L\left(\frac{\lambda_{1,3}P_3+\lambda_{2,3}P_2}{N_3}\right) + L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) - L\left(\frac{\lambda_{1,3}P_1}{N_3}\right)} \tag{23a}$$

$$p_2^* = \frac{L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) - L\left(\frac{\lambda_{1,3}P_1}{N_3}\right)}{L\left(\frac{\lambda_{1,3}P_3+\lambda_{2,3}P_2}{N_3}\right) + L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) - L\left(\frac{\lambda_{1,3}P_1}{N_3}\right)}. \tag{23b}$$

Also, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_2(\mathbf{p})$, and $R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) < \max_{\mathbf{p}} R_3(\mathbf{p})$, i.e., $0 < p_0^*, p_2^* < 1$; and $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_3(\mathbf{p}^{\boldsymbol{\alpha}^*}) \leq R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$, i.e.,

$$\begin{aligned} &L\left(\frac{\lambda_{1,3}P_3 + \lambda_{2,3}P_2}{N_3}\right)\left[L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) - L\left(\frac{\lambda_{1,4}P_1}{N_4}\right)\right] \\ &\leq L\left(\frac{\lambda_{1,4}P_4 + \lambda_{2,4}P_2}{N_4}\right)\left[L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) - L\left(\frac{\lambda_{1,3}P_1}{N_3}\right)\right]. \end{aligned} \tag{24}$$

Finally, we consider case vii. We have the following linearly independent equations: $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_3(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $R_2(\mathbf{p}^{\boldsymbol{\alpha}^*}) = R_4(\mathbf{p}^{\boldsymbol{\alpha}^*})$, $p_0^* + p_1^* + p_2^* + p_3^* = 1$. This means that we can express p_1, p_2, p_3 only in terms of p_0. Hence, the optimization (21) can be simplified to

$$\max R_2(p_0), \tag{25}$$

subject to $0 \leq p_i \leq 1$, for $i = 0, 1, 2, 3$.

We have shown that only cases iii, vi, and vii are possible. We can use the procedure derived in Section VI to check where $\boldsymbol{\alpha}^*$ lies, and solve for $\mathbf{p}^*$. Summarizing,

- If $\frac{L(\lambda_{1,2}P_1/N_2)}{L(\lambda_{1,3}P_1/N_3)} \leq \frac{L((\lambda_{1,4}P_1+\lambda_{3,4}P_3)/N_4) - L(\lambda_{1,2}P_1/N_2)}{L((\lambda_{1,4}P_1+\lambda_{3,4}P_3)/N_4) - L(\lambda_{1,4}P_1/N_4)}$, $0 \leq \frac{L((\lambda_{1,4}P_1+\lambda_{3,4}P_3)/N_4) - L(\lambda_{1,2}P_1/N_2)}{L((\lambda_{1,4}P_1+\lambda_{3,4}P_3)/N_4) - L(\lambda_{1,4}P_1/N_4)}$, and $\frac{L(\lambda_{1,2}P_1/N_2)}{L(\lambda_{1,3}P_1/N_3)} \leq 1$, then case iii is true. We can find an optimal schedule according to (22a)–(22c).
- If $0 < \frac{L\left(\frac{\lambda_{1,3}P_3+\lambda_{2,3}P_2}{N_3}\right)}{L\left(\frac{\lambda_{1,3}P_3+\lambda_{2,3}P_2}{N_3}\right) + L\left(\frac{\lambda_{1,2}P_1}{N_2}\right) - L\left(\frac{\lambda_{1,3}P_1}{N_3}\right)} < 1$ and (24) are both satisfied, then case vi is true. An optimal schedule is given by (23a) and (23b).
- Else, case vii is true. We need to solve for $\mathbf{p}^* = \text{argmax}_{\mathbf{p}}\{R_2(\mathbf{p}) = R_3(\mathbf{p}) = R_4(\mathbf{p})\}$, i.e., the optimization problem in (25). Note that this is a simpler problem than the original optimization problem (21).

B. Using Linear Programming

We note that the optimal scheduling problem (19) for the half duplex phase fading Gaussian MRC involves constraints that are linear functions of the unknowns (p_0, p_1, p_2, p_3), as all $\{L(\cdot)\}$ are fixed. This suggests that we can transform the max-min optimization problem above into a linear programming optimization problem and solve it using the simplex method [19, Ch. 3].

We introduce an auxiliary variable u and re-write the optimization in (19) as follows.

$$R_{\text{DF}} = \max u, \tag{26}$$

subject to

$$p_0 L(\lambda_{1,2}P_1/N_2) + p_1 L(\lambda_{1,2}P_1/N_2) - u \geq 0 \tag{27a}$$

$$p_0 L(\lambda_{1,3}P_1/N_3) + p_2 L((\lambda_{1,3}P_1 + \lambda_{2,3}P_2)/N_3) - u \geq 0 \tag{27b}$$

$$\begin{aligned} &p_0 L(\lambda_{1,4}P_1/N_4) + p_1 L((\lambda_{1,4}P_1 + \lambda_{3,4}P_3)/N_4) \\ &+ p_2 L((\lambda_{1,4}P_1 + \lambda_{2,4}P_2)/N_4) \\ &+ p_3 L((\lambda_{1,4}P_1 + \lambda_{2,4}P_2 + \lambda_{3,4}P_3)/N_4) - u \geq 0 \end{aligned} \tag{27c}$$

$$p_0 + p_1 + p_2 + p_3 = 1 \tag{27d}$$

$$p_0 \geq 0, \quad p_1 \geq 0, \quad p_2 \geq 0, \quad p_3 \geq 0, \quad u \geq 0. \tag{27e}$$

This transforms the original optimization problem into a linear programming problem. Let the solution be $(p_0^*, p_1^*, p_2^*, p_3^*, u^*)$. $(p_0^*, p_1^*, p_2^*, p_3^*)$ is an optimal DF schedule, and u^* is the highest DF rate over all possible schedules.

C. Computation Results

In this section, we demonstrate that solving optimal schedules provides insights into how one would operate the network. We consider the four-node half duplex phase fading Gaussian MRC where the nodes are places on a straight line. The node coordinates are: node 1 $(0, 0)$, node 2 $(0, y_2)$, node 3 $(0, y_3)$, and node 4 $(0, 100)$. We assume that $\kappa = 1$, $\eta = 2$, $P_i = 10$ for $i = 1, 2, 3$, and $N_j = 0.01$, for $j = 2, 3, 4$.

Fig. 2a shows if case iii, vi, or vii is true using the technique in Section VI, for different relay positions, i.e., $0 \leq y_2, y_3 \leq 100$. From Section. VII-A, we know that case iii only occurs when node 2 is further from node 1 than node 3 is from node 1, or $y_2 \geq y_3$. Furthermore, case vi does not occur at all in these network topologies. In addition, if the four-node MRC is arranged such that the positions of nodes 1–4 are $(0, 0)$, $(0, y_2)$, $(0, y_3)$, $(0, 100)$ respectively, and $0 < y_2 < y_3 < 100$, then only case vii will occur.

Fig. 2b shows optimal schedules for varying node 3's position, i.e., $(0, y_3)$ while fixing the rest of the nodes' positions. From Fig. 2a, we see that for $y_2 = 66$ and $20 \leq y_3 < 53$, $\boldsymbol{\alpha}^*$ lies in case iii, meaning that the optimal schedules are $(p_0^*, 1 - p_0^*, 0, 0)$. We see that in this case, we only need to operate the network in two states: (L, T) and (L, L). This means node 2 does not need to transmit, and only node 3 needs to toggle between the transmitting and the listening states.

D. Networks with More Than Two Relays

The linear programming solution in section VII-B can be used to calculate optimal schedules for capacity upper bounds and achievable DF rates for phase fading MRCs with more than two relays. We consider D-node line networks with node i at $(0, i - 1)$ for $i = 1, \ldots, D$ and route $\mathcal{M} = \{1, 2, \ldots, D\}$. We assume that $\kappa = 1$, $\eta = 2$, $P_i = 10$ for $i = 1, \ldots, D - 1$, and $N_j = 1$, for $j = 2, \ldots, D$.

Fig. 2c shows the rate for a relay chain with different number of relays. As the source and destination become farther apart, the rate for half-duplex strategy slightly reduces. However, the rate will finally converge to some constant when the distance between source and destination goes to infinity.

VIII. Non-Linear Optimal Scheduling

Besides the phase fading Gaussian MRC, the technique in Section VI can be used to solve for optimal schedules in other types of networks. We briefly discuss two types of networks here.

A. Per-Block Power Constraints

In Section. II-C, we consider per-symbol power constraints. A criticism for this definition is that the total transmit energy for a node in a block depends on the fraction of its transmitting state. Hence, for fixed $P_i = P, \forall i$, the total transmit energy per block for different nodes can vary.

To tackle this fairness issue, we may introduce *per-block* energy constraint, $\frac{1}{n}\sum_{k=1}^{n} E[X_{i,k}X_{i,k}^\dagger] \leq P_i$. This definition captures the entire block of transmissions, regardless of fractions of transmit state vectors. Under this definition, the total transmit energy per block for user i is always nP_i regardless of the fraction of the node in the transmit state in a block. If we set $x_j = \tilde{x}_j = 0$ for node j in the listening state, we get the following *instantaneous* power constraint in state $\mathbf{s}$.

$$E[X_i X_i^\dagger] \leq \begin{cases} P_i & , i = 1 \\ \frac{P_i}{f_T(i)} & , i \in \mathcal{T}(\mathbf{s}) \\ 0 & , \text{otherwise} \end{cases}, \tag{28}$$

where $f_T(i)$ is the fraction of channel uses in a block where node i transmits, i.e., $f_T(i) = \sum_{\text{node } i \text{ transmit in state } \mathbf{s}}^{\mathbf{s}} \Pr\{\mathbf{S} = \mathbf{s}\}$.

So, the optimal scheduling problem on route $\mathcal{M}$ for per-block power constraint is to find a schedule $p(\mathbf{s})$ on $\{L, T\}^{|\mathcal{M}-2|}$ for

$$\max_{p(\mathbf{s})} \min_{m_t \in \mathcal{M}\setminus\{1\}} \sum_{\substack{\mathbf{s}\in\{L,T\}^{|\mathcal{M}|-2} \\ m_t \in \mathcal{L}(\mathbf{s})\cup\{D\}}} \Bigg[p(\mathbf{s}) \times L\left(\frac{\sum_{i\in\{1\}\cup(\{m_2,\ldots,m_{t-1}\}\cap\mathcal{T}(\mathbf{s}))} \lambda_{i,m_t} P_i}{f_T(i) N_{m_t}} \right) \Bigg]. \tag{29}$$

Note that this is a max-min optimization which can be solved by using the technique in Section VI. However, the optimization is non-linear as $f_T(i)$ is a function of $p(\mathbf{s})$.

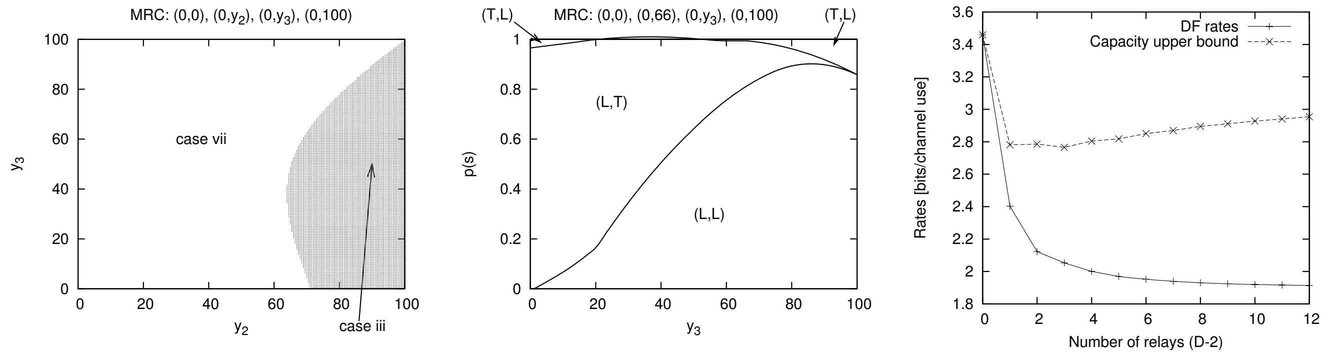
B. Static Gaussian Channels

Next, we consider static Gaussian channels where $\theta(i, t) = 0, \forall i, t$, with per-symbol power constraints. Using DF with Gaussian inputs and route $\mathcal{M}$, node m_i, in state $\mathbf{s}$, transmits $X_{m_i} = \sum_{m_j \in\{1\}\cup(\{m_i,\ldots,m_{|\mathcal{M}|-1}\}\cap\mathcal{T}(\mathbf{s}))} \sqrt{\beta_{m_i,m_j}(\mathbf{s}) P_{m_i}} U_{m_j}$, for $\beta_{m_i,m_j}(\mathbf{s}) \geq 0$, $0 \leq \sum_{m_j} \beta_{m_i,m_j}(\mathbf{s}) \leq 1$. U_{m_j} are independent Gaussian random variables with unit variance. $\beta_{m_i,m_j}(\mathbf{s})$ is the fraction of power of node m_i, used to transmit independent *sub-codewords* U_{m_j} in state $\mathbf{s}$. On route $\mathcal{M}$, rates up to those in (30) are achievable.

The optimal scheduling problem on route $\mathcal{M}$ for static Gaussian MRC with per-symbol power constraint is to find a schedule $p(\mathbf{s})$ on $\{L, T\}^{|\mathcal{M}|-2}$ that attains (30). Note that the optimization involves extra power splits terms $\{\beta_{i,j}(\mathbf{s})\}$ which are functions of $p(\mathbf{s})$. Again, we see that the optimization is a max-min optimization.

IX. Reflections

In this paper, we investigated achievable rates and optimal schedules for DF for the half duplex MRC. The code construction in this paper differs from that of the traditional half duplex network. Traditionally, a node transmits an entire codeword within one transmit cycle. The transmission time for a codeword is shorter than the duration for which the



(a) Graph showing cases in which optimal $\boldsymbol{\alpha}^*$ lies for varying relay positions, for the four-node MRC.

(b) Graph showing optimal schedules for $y_2 = 66$ and varying node 3's position, for the four-node MRC.

(c) Capacity upper bound and DF rates for line networks.

Fig. 2. Optimal schedules and rates.

$$\max_{p(\mathbf{s})} \max_{\beta(\mathbf{s})} \min_{m_t \in \mathcal{M}\setminus\{1\}} \sum_{\substack{\mathbf{s}\in\{L,T\}^{|\mathcal{M}|-2} \\ m_t \in \mathcal{L}(\mathbf{s})\cup\{D\}}} \left[p(\mathbf{s}) L \left(\frac{\sum_{j=1}^{t-1} \left(\sum_{m_i \in \{1\}\cup(\{m_2,\dots,m_j\}\cap\mathcal{T}(\mathbf{s}))} \sqrt{\beta_{m_i,m_j}(\mathbf{s}) \lambda_{m_i,m_t} P_{m_i}} \right)^2}{N_{m_t}} \right) \right]. \quad (30)$$

nodes stay in a transmit/listen state. In this approach, we need to consider how data are routed in different states. Depending on the state, data are sent to and from different sets of nodes. Thus, the order of the states becomes important, and we need to optimize that.

In this paper, we approached the scheduling problem differently. Each codeword spans over all transmit/listen states of the nodes, meaning that the states change during one codeword duration. The route (i.e., how data flow) remains the same throughout all data transmissions. The advantage of this approach is that we do not need to worry about the flow of data when we consider different states. The order of the states is not important and we only need to optimize the time fraction (or the probability) of different states.

Finally, we remark that, unlike the full duplex case where DF achieves the cut-set upper bound and hence the capacity when the relays are "close" to the source [2], DF in the half duplex case does not achieve (though is close to) the cut-set bound if the relay is at a distance ϵ from the source, for any arbitrary $\epsilon > 0$.

REFERENCES

[1] L. Xie and P. R. Kumar, "An achievable rate for the multiple level relay channel," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1348–1358, Apr. 2005.

[2] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.

[3] L. Ong and M. Motani, "Myopic coding in multiterminal networks," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3295–3314, Jul. 2008.

[4] A. A. El Gamal and S. Zahedi, "Capacity of a class of relay channels with orthogonal components," *IEEE Trans. Inf. Theory*, vol. 51, no. 5, pp. 1815–1817, May 2005.

[5] T. M. Cover and A. A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. IT-25, no. 5, pp. 572–584, Sep. 1979.

[6] H. V. Poor, *An introduction to signal detection and estimation*. New York: Springer-Verlag, 1994.

[7] A. Host-Madsen, "On the capacity of wireless relaying," in *Proc. of IEEE Semiannual Vehicular Technology Conference (VTC)*, Birmingham, AL, May 6-9 2002, pp. 1333–1337.

[8] M. A. Khojastepour, A. Sabharwal, and B. Aazhang, "On the capacity of 'cheap' relay networks," in *Proc. Conf. on Inf. Science and Syst. (CISS)*, Baltimore, MD, Mar. 12-14 2003.

[9] G. Kramer, "Models and theory for relay channels with receive constraints," in *Proc. 42nd Allerton Conf. on Commun., Control, and Comput.*, Monticello, IL, Sep. 29-Oct. 1 2004.

[10] A. Host-Madsen and J. Zhang, "Capacity bounds and power allocation for the wireless relay channel," *IEEE Trans. Inf. Theory*, vol. 51, no. 6, pp. 2020–2040, Jun. 2005.

[11] M. Gastpar and M. Vetterli, "On the capacity of large gaussian relay networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 765–, Mar. 2005.

[12] F. Xue and S. Sandhu, "Cooperation in a half-duplex gaussian diamond relay chanenl," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3806–3814, Oct. 2007.

[13] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. John Wiley and Sons, 1991.

[14] M. A. Khojastepour, A. Sabharwal, and B. Aazhang, "Bounds on achievable rates for general multi-terminal networks with practical constraints," in *Proc. 2nd Int. Workshop on Information Processing in Sensor Networks (IPSN)*, Palo Alto, CA, Apr. 22-23 2003, pp. 146–161.

[15] L. Ong and M. Motani, "Optimal routing for the decode-and-forward strategy in the Gaussian multiple relay channel," in *Proc. IEEE Int. Symposium on Inf. Theory (ISIT)*, Nice, France, Jun. 24-29 2007, pp. 1061–1065.

[16] ——, "Optimal routing for decode-and-forward based cooperation in wireless network," in *Proc. 4th Annu. IEEE Conf. on Sensor and Ad Hoc Commun. and Networks (SECON)*, San Diego, CA, Jun. 18-21 2007, pp. 334–343.

[17] L. Ong, W. Wang, and M. Motani, "Achievable rates and schedules for half duplex phase fading multiple-relay networks," in *Proc. 2008 Int. Symposium on Inf. Theory and its Applications (ISITA)*, Auckland, New Zealand, Dec. 7-10 2008.

[18] Y. Liang, V. V. Veeravalli, and H. V. Poor, "Resource allocation for wireless fading relay cahnnels: Max-min solution," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, pp. 3432–3453, Oct. 2007.

[19] D. G. Luenberger, *Linear and Nonlinear Programming*. Addison-Wesley, 1984.