MOMENTS OF PRODUCTS OF ELLIPTIC INTEGRALS

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Abstract. We consider the moments of products of complete elliptic integrals of the first and second kinds. In particular, we derive new results using elementary means, aided by computer experimentation and a theorem of W. Zudilin. Diverse related evaluations, and two conjectures, are also given.

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1. Motivation and General Approach

We study the complete elliptic integral of the first kind, \( K(x) \), and the second kind, \( E(x) \), defined by:

\[
K(x) = \frac{\pi}{2} \, _2F_1 \left( \frac{1}{2}, \frac{1}{2} \bigg| x^2 \right), \quad E(x) = \frac{\pi}{2} \, _2F_1 \left( \frac{-1}{2}, \frac{1}{2} \bigg| x^2 \right). \tag{1}
\]

As usual, \( K'(x) = K(x') \), \( E'(x) = E(x') \), where \( x' = \sqrt{1 - x^2} \). Recall that \( _pF_q \) denotes the generalized hypergeometric series,

\[
_2F_1 \left( a_1, \ldots, a_p \bigg| b_1, \ldots, b_q \bigg| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{2}
\]

The complete elliptic integrals, apart from their theoretical importance in arbitrary precision numerical computations ([8]) and the theory of theta functions, are also of significant interest in applied fields such as electrodynamics ([18]), statistical mechanics, and random walks ([9, 10]). Indeed, they were first used to provide explicit solutions to the perimeter of an ellipse (among other curves) as well as the (exact) period of an ideal pendulum.

The author was first drawn to the study of integral of products of \( K \) and \( E \) in [9], in which it is shown that

\[
2 \int_0^1 K(x)^2 \, dx = \int_0^1 K'(x)^2 \, dx, \tag{3}
\]

by relating both sides to a moment of the distance from the origin in a four step uniform random walk on the plane.
A much less recondite proof was only found later: set \( x = (1 - t)/(1 + t) \) on the left hand side of (3), and apply the quadratic transform (4) below, and the result readily follows.

The four quadratic transforms ([8]), which we will use over and over again, are:

\[
K'(x) = \frac{2}{1 + x} K \left( \frac{1 - x}{1 + x} \right) \tag{4}
\]

\[
K(x) = \frac{1}{1 + x} K \left( \frac{2\sqrt{x}}{1 + x} \right) \tag{5}
\]

\[
E'(x) = (1 + x) E \left( \frac{1 - x}{1 + x} \right) - xK'(x) \tag{6}
\]

\[
E(x) = \frac{1 + x}{2} E \left( \frac{2\sqrt{x}}{1 + x} \right) + \frac{1 - x^2}{2} K(x). \tag{7}
\]

In the following sections we will consider definite integrals involving products of \( K, E, K', E' \), especially the moments of the products. A goal of this paper is to produce closed forms for these integrals whenever possible. When this is not achieved, closed forms for certain linear combinations of integrals are instead obtained. Thus, we are able to prove a large number of experimentally observed identities in [2].

The somewhat rich and unexpected results lend themselves for easy discovery, thanks to the methods of experimental mathematics: for instance, the integer relations algorithm PSLQ [12], the Inverse Symbolic Calculator (ISC, now hosted at CARMA, [14]), the Online Encyclopedia of Integer Sequences (OEIS, [17]), the Maple package gfun, Gosper’s algorithm (which finds closed forms for indefinite sums of hypergeometric terms, [15]), and Sister Celine’s method [15]. Indeed, large scale computer experiments [2] reveal that there is a huge number of identities in the flavour of (3). Once discovered, many results can be routinely established by the following elementary techniques:

1. Connections with and transforms of hypergeometric and Meijer G-functions ([18]), as in the case of random walk integrals (Section 3).
2. Interchange order of summation and integration, which is justified as all terms in the relevant series are positive (Section 4).
3. Change the variable \( x \) to \( x' \), usually followed by a quadratic transform (Section 5).
4. Use a Fourier series (Section 6).
5. Apply Legendre’s relation (Section 7).
6. Differentiate a product and integrate by parts (Section 8).

Note that Section 2 and most of Section 7 are expository. The propositions in Section 4 are well-known, but the arithmetic nature of the moments, Theorem 3 and Lemma 3 in Section 6 do not seem to have featured in previous literature. Section 3 contains new general formulae for the moments of the product of two elliptic integrals, and Section 8 contains many new, though mostly easy, linear relations between the moments. Some useful identities of elliptic integrals are also gathered throughout the paper.
2. One elliptic integral

The moments of a single \( K, E, K', E' \) are well known (e.g. see [8]). For completeness here we state a slightly more general result.

It follows by a straightforward application of the beta integral (see e.g. [6]) that

\[
\int_0^1 x^m x^n K(x) \, dx = \frac{\pi \Gamma \left( \frac{1}{2} (m + 1) \right) \Gamma \left( \frac{1}{2} (n + 2) \right)}{4 \Gamma \left( \frac{1}{2} (m + n + 3) \right)} \, {}_3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{m+1}{2}; 1, \frac{m+n+3}{2}; 1 \right), \tag{8}
\]

\[
\int_0^1 x^m x^n E(x) \, dx = \frac{\pi \Gamma \left( \frac{1}{2} (m + 1) \right) \Gamma \left( \frac{1}{2} (n + 2) \right)}{4 \Gamma \left( \frac{1}{2} (m + n + 3) \right)} \, {}_3F_2 \left( -\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2}; 1, \frac{m+n+3}{2}; 1 \right). \tag{9}
\]

Using the obvious transformation \( x \mapsto x' \), we have

\[
\int_0^1 x^{2n+1} K(x)^a E(x)^b K'(x)^c E'(x)^d \, dx = \int_0^1 x(1-x^2)^n K'(x)^a E'(x)^b K(x)^c E(x)^d \, dx, \tag{10}
\]

an equation which we appeal to often. Thus, using (10), we see that (8, 9) also encapsulate the moments for \( K' \) and \( E' \). We note that for convergence, \( m > -1, n > -2 \). When \( m = 1 \) both formulae reduce to a \( {}_2F_1 \) and can be summed by Gauss’ theorem ([1]).

If in addition \( 2m + n + 1 = 0 \) in (8), then Dixon’s theorem ([1]) applies and we may sum the \( {}_3F_2 \) explicitly in terms of the \( \Gamma \) function. For instance, we may compute \( \int_0^1 K(x)/x' \, dx \) (which also follows from the Fourier series in Section 6). In (9), Dixon’s theorem may only be applied to the single special case \( \int_0^1 x' E(x) \, dx \).

In [6], the corresponding results for the moments of the \textit{generalized elliptic integrals} are derived similarly.

3. Two complementary elliptic integrals

Though the simple cases corresponding to \( n = 0 \) in this section are tabulated in [11], the general results appear to be new.

In [19], Zudilin’s Theorem connects, as a special case, triple integrals of rational functions over the unit cube with generalized hypergeometric functions \( \gamma F_6 \)'s.

We state a restricted form of the theorem which is sufficient for our purposes:

**Theorem 1** (Zudilin). \textit{Given} \( h_0, \ldots, h_5 \) \textit{for which both sides converge},

\[
\int_{[0,1]^3} x^h_2 y^h_3 z^h_4 (1-x)^h_0 (1-y)^h_0 (1-z)^h_0 \, dx \, dy \, dz = \frac{\Gamma(h_0 + 1) \prod_{j=2}^4 \Gamma(h_j) \prod_{j=1}^4 \Gamma(h_0 + 1 - h_j - h_{j+1})}{\prod_{j=1}^5 \Gamma(h_0 + 1 - h_j)} \times \gamma F_6 \left( \frac{h_0}{2}, 1 + h_0 - h_1, 1 + h_0 - h_2, 1 + h_0 - h_3, 1 + h_0 - h_4, h_5 \middle| 1 \right). \tag{11}
\]
In [9], this theorem is applied to derive hypergeometric evaluations for moments of random walks from their triple integral representations which satisfy the left hand side of the theorem.

The idea here is to write a single integral involving products of elliptic integrals as a double, then a triple integral of the required form, and then apply Theorem 1. To do so, we require the following formulae, which are readily verified (see e.g. [16]):

\[
\int_0^1 \frac{dx}{\sqrt{x(1-x)(a-x)}} = \frac{2}{\sqrt{a}} K \left( \frac{1}{\sqrt{a}} \right),
\]

(12)

\[
\int_0^1 \sqrt{\frac{a-x}{x(1-x)}} \ dx = 2\sqrt{a} E \left( \frac{1}{\sqrt{a}} \right),
\]

(13)

\[
\int_a^1 \frac{dy}{\sqrt{y(1-y)(y-a)}} = 2K'(\sqrt{a}),
\]

(14)

\[
\int_a^1 \frac{\sqrt{y}}{\sqrt{(1-y)(y-a)}} \ dy = 2E'(\sqrt{a}).
\]

(15)

Using the above relations, we have, for instance,

\[
\int_0^1 E'(y)^2 \ dy = \frac{1}{2} \int_0^1 \int_0^1 \sqrt{\frac{y}{(1-y)(y-a^2)}} \ E(\sqrt{1-a^2}) \ dy \ da
\]

\[
= \frac{1}{4} \int_0^1 \int_0^1 \sqrt{\frac{y}{(1-y)z(1-z)}} \ E(\sqrt{1-yz}) \ dy \ dz
\]

\[
= \frac{1}{8} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{y(1-yz)}{(1-y)z(1-z)}} \sqrt{\frac{1}{\frac{1-yz}{x(1-x)-(1-y)(1-z)}}} \ dx \ dy \ dz
\]

\[
= \frac{1}{8} \int_{[0,1]^3} \sqrt{\frac{y(1-x(1-y(1-z)))}{x(1-x)(1-y)z(1-z)}} \ dx \ dy \ dz.
\]

The first equality follows from (15), the second from changing \(a^2 \mapsto yz\), the third from (13), and the fourth from \(z \mapsto 1 - z\). Now Theorem 1 applies to the last integral.

Similarly, by building up the \(E'\) integral then \(K'\), we obtain:

\[
\int_0^1 E'(x)K'(x) \ dx = \frac{1}{8} \int_{[0,1]^3} \sqrt{\frac{1-x(1-y(1-z))}{x(1-x)y(1-y)z(1-z)}} \ dx \ dy \ dz,
\]

Alternatively, by building up the \(K'\) integral then \(E'\), we get:

\[
\int_0^1 E'(x)K'(x) \ dx = \frac{1}{8} \int_{[0,1]^3} \sqrt{\frac{\sqrt{y} \ dx \ dy \ dz}{x(1-x)(1-y)z(1-z)(1-x(1-y(1-z)))}}.
\]

Finally, we also have
\[
\int_0^1 K'(x)^2 \, dx = \frac{1}{8} \int_{[0,1]^3} \frac{dx dy dz}{\sqrt{x(1-x)y(1-y)z(1-z)(1-x(1-y(1-z)))}}.
\]

Slightly generalising this strategy, we are led to:

**Proposition 1.** For all real \( n > -1 \),

\[
(1) \quad \int_0^1 x^n E'(x)^2 \, dx = \frac{2^{4n}(n+1)^3(n+3)^2}{16(n+2)^3(n+4)} \frac{\Gamma^8 \left( \frac{n+1}{2} \right)}{\Gamma^4(n+1)} \, \gamma F_6 \left( \begin{array}{c}
-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, n+3, n+\frac{7}{2} \\
1, \frac{n+3}{2}, \frac{n+5}{2}, \frac{n+4}{2}, \frac{n+6}{2}
\end{array} \right) 1,
\]

\[
(2) \quad \int_0^1 x^n E'(x)K'(x) \, dx = \frac{2^{4n}(n+1)^2}{16(n+2)} \frac{\Gamma^8 \left( \frac{n+1}{2} \right)}{\Gamma^4(n+1)} \, \gamma F_6 \left( \begin{array}{c}
-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, n+2, n+\frac{5}{2} \\
1, \frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}
\end{array} \right) 1,
\]

\[
(3) \quad \int_0^1 x^n K'(x)^2 \, dx = \frac{2^{4n}(n+1)}{16} \frac{\Gamma^8 \left( \frac{n+1}{2} \right)}{\Gamma^4(n+1)} \, \gamma F_6 \left( \begin{array}{c}
\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, n+1, n+\frac{5}{2} \\
1, \frac{n+1}{2}, \frac{n+2}{2}, \frac{n+4}{2}
\end{array} \right) 1.
\]

When \( n \) is odd, the \( \gamma F_6 \)'s reduce to known constants, which we prove below.

**Theorem 2.** When \( n \) is odd, the \( n \)th moment of \( K^2, E'^2, K'E', K^2, E^2 \) and \( KE \) is expressible as \( a + b\zeta(3) \), where \( a, b \in \mathbb{Q} \).

**Proof.** We prove the case for the pair \( K'^2 \) and \( K^2 \); the other two pairs are similar.

Firstly, when \( n \) is odd, the summand of the \( \gamma F_6 \) for \( K'^2 \) is a rational function:

\[
\frac{(2k+m+1)(k+1)^2(k+2)^2 \cdots (k+m)^2}{(k+1/2)^4(k+3/2)^4 \cdots (k+m+1/2)^4},
\]

here we ignored rational constants at the front and wrote \( n = 2m + 1 \). We can explicitly sum (19) and verify the statement of the theorem for the first few moment of \( K'^2 \). By using the change of variable \( x \mapsto x' \) as in (10), we can likewise do this for \( K^2 \).

Now it is not hard to show that the moments of \( K^2 \) satisfy a recursion:

\[
(n+1)^3 K_{n+2} - 2n(n^2+1)K_n + (n-1)^3 K_{n-2} = 2.
\]

Results like this are proven in Section 8.2. The recursion shows that the statement holds for all odd moments of \( K^2 \). Then (10) gives the result for \( K^2 \).

Note that by computing the moment of \( E'(x)K'(x) \) in two ways, we obtain a transformation formula for the \( \gamma F_6 \)'s involved. Also, by either one of the two known transformations for non-terminating \( \gamma F_6 \)'s ([4]), we can write each of our \( \gamma F_6 \) as the sum of two \( 4F_3 \)'s, where one series readily simplifies to known constants when \( n \) is odd, while the harder term becomes reducible in light of Theorem 2.
Remark 1. Therefore, all the odd moments of $K'^2, E'^2, K'E'$ have particularly simple forms involving $\zeta(3)$. By using (10), we can iteratively obtain all the odd moments of $K^2, E^2, KE$. For example,
\[ \int_0^1 x^3 K(x)^2 \, dx = \frac{1}{8} (2 + 7 \zeta(3)), \quad \int_0^1 x K'(x)^2 \, dx = \frac{7}{4} \zeta(3). \]

Remark 2. We sketch another proof by expanding (19) into partial fractions.
As each partial fraction has at most a quartic on the denominator, the irrational constants from the sum can only come from \{\zeta(2), \zeta(3), \zeta(4)\}, and possible contribution from the linear denominators. But as the linear terms must converge, their sum must eventually telescope, and hence contribute only a rational number.

We recall that partial fractions can be obtained via a derivative process akin to computing Taylor series coefficients; indeed, if we write
\[ f(x) = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}, \]
then $A_n = f(a), A_{n-1} = f'(a)/1!, \ldots, A_1 = f^{(n-1)}(a)/(n-1)!$.

When applied to (19), it is easy to check that, when $n \equiv 3 \pmod{4}$, the presence of the numerator $2k + m + 1$ makes the terms with quadratic and quartic denominators telescope out, leaving us with rational numbers (these terms occur in pairs related by the transformation $k \mapsto -\frac{n+1}{2} - k$, where said linear numerator switches sign). Similarly, the terms with cubic denominators double.

When $n \equiv 1 \pmod{4}$, $2k + m + 1$ cancels out with one of the factors, making the corresponding denominator a cubic. We check that its partial fraction has no quadratic term: this is equivalent to showing (19) with all powers of $2k + m + 1$ removed has 0 derivative at $k = -\frac{n+1}{4}$, which holds as it is symmetric around that point. So in both cases only the cubic terms remain, giving us $\zeta(3)$.

This type of partial fraction argument is at the heart of the result that infinitely many odd zeta values are irrational (see [5], which, incidentally, is the motivation for Zudilin’s Theorem 1).

4. One elliptic integral and one complementary elliptic integral

Here we take advantage of the closed form for moments of $K', E'$ which follow from (8) and (9):
\[ \int_0^1 x^n K'(x) \, dx = \frac{\pi \Gamma(\frac{1}{2}(n+1))^2}{4 \Gamma(\frac{1}{2}(n+2))^2}, \]
\[ \int_0^1 x^n E'(x) \, dx = \frac{\pi \Gamma(\frac{1}{2}(n+3))^2}{2(n+1) \Gamma(\frac{1}{2}(n+2)) \Gamma(\frac{1}{2}(n+4))}, \]
and the series for $K, E$ equivalent to Definition 1:
\[ K(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)^2 x^{2k}}{\Gamma(k+1)^2} \frac{x^{2k}}{2}, \quad E(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k-1/2)\Gamma(k+1/2) x^{2k}}{\Gamma(k+1)^2} \frac{x^{2k}}{4}. \]

Hence, the proposition below may be simply proved by interchanging the order of summation and integration.
Proposition 2. We have the following moments:

\[ \begin{align*}
(1) & \quad \int_0^1 x^n K(x) K'(x) \, dx = \frac{\pi^2 \Gamma(\frac{1}{2}(n+1))^2}{8 \Gamma(\frac{1}{2}(n+2))^2} \binom{\frac{1}{2}, \frac{n+1}{2}, \frac{n+1}{2}}{1, \frac{n+2}{2}, \frac{n+2}{2}} F_3 \left( \frac{1}{2}, \frac{1}{2}, \frac{n+1}{2} ; 1, \frac{n+2}{2}, \frac{n+2}{2} \right), \\
(2) & \quad \int_0^1 x^n E(x) K'(x) \, dx = \frac{\pi^2 \Gamma(\frac{1}{2}(n+1))^2}{8 \Gamma(\frac{1}{2}(n+2))^2} \binom{\frac{1}{2}, \frac{n+1}{2}, \frac{n+1}{2}}{1, \frac{n+2}{2}, \frac{n+2}{2}} F_3 \left( -\frac{1}{2}, \frac{1}{2}, \frac{n+1}{2} ; 1, \frac{n+2}{2}, \frac{n+2}{2} \right), \\
(3) & \quad \int_0^1 x^n K(x) E'(x) \, dx = \frac{\pi^2 (n+1) \Gamma(\frac{1}{2}(n+1))^2}{8 (n+2) \Gamma(\frac{1}{2}(n+2))^2} \binom{\frac{1}{2}, \frac{1}{2}, \frac{n+3}{2}}{1, \frac{n+2}{2}, \frac{n+4}{2}} F_3 \left( \frac{1}{2}, \frac{1}{2}, \frac{n+1}{2} ; 1, \frac{n+2}{2}, \frac{n+4}{2} \right), \\
(4) & \quad \int_0^1 x^n E(x) E'(x) \, dx = \frac{\pi^2 (n+1) \Gamma(\frac{1}{2}(n+1))^2}{8 (n+2) \Gamma(\frac{1}{2}(n+2))^2} \binom{\frac{1}{2}, \frac{1}{2}, \frac{n+3}{2}}{1, \frac{n+2}{2}, \frac{n+4}{2}} F_3 \left( -\frac{1}{2}, \frac{1}{2}, \frac{n+1}{2} ; 1, \frac{n+2}{2}, \frac{n+4}{2} \right). 
\end{align*} \]

When \( n \) is odd, the moments yield a closed form as a rational multiple of \( \pi^3 \) plus a rational multiple of \( \pi \), as we can expand the summand (a rational function) as partial fractions much like in Remark 2. To prove this observation, we need Legendre’s relation (8):

\[ E(x) K'(x) + E'(x) K(x) - K(x) K'(x) = \frac{\pi}{2}. \] (24)

Note that by using the symmetry between parts 2 and 3, as well as by applying (24), we obtain linear identities connecting these \( \binom{2,2+2,n+3}{1,2+2,n+4} \)’s. Due to the lack of Taylor expansions of \( E', K' \) around the origin as well as sufficiently simple moments for \( E, K \), this method cannot be used to evaluate any more moments.

Lemma 1. For odd \( n \), the \( n \)-th moment of \( K(x) K'(x) \) is a rational multiple of \( \pi^3 \), and the \( n \)-th moment of \( E(x) K'(x), K(x) E'(x) \) and \( E(x) E'(x) \) is \( \frac{\pi}{4(n+1)} \) plus a rational multiple of \( \pi^3 \).

Proof. We experimentally discover that, letting \( g_n := \int_0^1 x^{2n-1} K(x) K'(x) \, dx \), we have the recursion

\[ 2n^3 g_{n+1} - (2n - 1)(2n^2 - 2n + 1)g_n + 2(n - 1)^3 g_{n-1} = 0. \]

This (contiguous) relation, once discovered, can be proven by extracting the summand, simplifying and summing using Gosper’s algorithm. Thus, after computing two starting values, the claim is proven for the moments of \( KK' \). Note that the recursion also holds when \( n \) is not an integer.

For the moments of \( EK' \) or \( KE' \), we take the derivative of \( x^{2n} K(x) K'(x) \) via the product rule, and integrate each piece in the result. By using (24) and the proven claim for the moments of \( KK' \), we deduce that the term involving \( \pi \) is \( \frac{\pi}{4(n+1)} \). For the moments of \( EE' \), we instead consider the derivative of \( (1 - x^2)x^{2n} E(x) E'(x) \) and use the proven results for \( EK' \) and \( KE' \). Note that this trick involving integration by parts is exploited in Section 8. \( \square \)
Experimentally, we find that the sequence $h(n) := 16^{n+1}g_{n+1}$ matches entry A036917 of the Online Encyclopedia of Integer Sequences; indeed, they share the same recursion and initial values. Moreover, the OEIS provides

$$h(n) = \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k}^2 = \frac{16^n \Gamma(n+1/2)^2}{\pi \Gamma(n+1)^2} \pFq{4}{3}{\frac{-n,-n,\frac{1}{2},\frac{1}{2}}{\frac{1}{2}-n,\frac{1}{2}-n,1}}{1}. \quad (25)$$

The first equality is routine as we can produce a recurrence for the binomial sum – for instance, using Sister Celine’s method; the second equality is notational.

The generating function for $h(n)$ is simply

$$\sum_{n=0}^{\infty} h(n)t^n = \frac{4}{\pi^2} K\left(4\sqrt{t}\right)^2,$$

which is again easy to prove using the series for $K(t)$. Recall that $h(n)$ is related to the moments of $K(x)K'(x)$, and thus we have:

**Theorem 3.**

$$\int_0^1 \frac{x}{1-t^2x^2} K(x)K'(x) \, dx = \frac{\pi}{4} K(t)^2. \quad (26)$$

Equation (26) seems to be a remarkable extension of its (much easier) cousins,

$$\int_0^1 \frac{1}{1-t^2x^2} K'(x) \, dx = \frac{\pi}{2} K(t) \quad \text{and} \quad \int_0^1 \frac{1}{1-t^2x^2} E'(x) \, dx = \frac{\pi}{2t^2}(K(t) - E(t)). \quad (27)$$

Note that manipulations of (26, 27) give myriads of integrals, we list some of which below ($G$ denotes *Catalan’s constant*):

$$\int_0^1 \frac{\arctan(x)}{x} K'(x) \, dx = \pi G,$$

$$\int_0^1 \frac{2}{x} K(x)K'(x)(K(x) - E(x)) \, dx = \int_0^1 K(x)^2 E'(x) \, dx,$$

$$\int_0^1 \left(\frac{1}{2} + \frac{n+1}{2n+3}\right) x^2 K(x)K'(x) \, dx = \frac{(n+1)\pi}{4} \int_0^1 t^n K(t)^2 \, dt.$$

The last identity specialises to

$$\int_0^1 \frac{-\log(1-x^2)}{x} K(x)K'(x) \, dx = \frac{7}{8} \pi \zeta(3).$$

**5. Sporadic results**

We list some results found by ad hoc methods; some are not moment evaluations per se but were found interesting by the author, while others are preparatory for later sections.
5.1. **Explicit primitives.** Curiously, a small number of integrals happen to have explicit primitives; we list some here:

\[ x^n K(x), \ x^n E(x), \ \frac{x^n E(x)}{1 - x^2}, \ \frac{E(x)}{1 + x}, \ \frac{x F(x)}{(1 - x^2)^{3/2}}. \]

where \( F \) can be \( K, K', E \) or \( E' \). The primitives are expressible in terms of \( K \) and \( E \) when \( n > 0 \) is odd or when \( n < 0 \) is even in the first three cases (and also when \( n = 0 \) in the third case). The last case, as well as many other integrals, are found in [13].

Trivially, transformations of the above still yield explicit primitives. We note that some CAS, when used naively, struggle to find primitives even for this very short list, one example is given by applying \( x \mapsto x' \) in the last case:

\[ \int K'(x) \frac{dx}{x^2} = \frac{E'(x) - K'(x)}{x}. \]

5.2. **Imaginary argument.** In [16] vol III, some integrals with the argument \( ix \) are considered, e.g.

\[ \int_0^1 x K'(x) K(ix) \, dx = \frac{1}{2} G \pi. \]

This can be proven by expanding \( x K(ix) \) as a series and summing the moments of \( K'(x) \). Other evaluations are done similarly; for instance, we can easily obtain recursions for the moments of \( K(ix) \) and \( E(ix) \).

We also record here that Euler’s hypergeometric transformation gives

\[ E(ix) = \sqrt{x^2 + 1} \ E(x/\sqrt{x^2 + 1}), \quad K(ix) = 1/\sqrt{x^2 + 1} \ K(x/\sqrt{x^2 + 1}). \]

5.3. **Quadratic transforms.** Using the quadratic transforms (4, 5), we obtain

\[
\begin{align*}
\int_0^1 K(x)^n \, dx &= \frac{1}{2} \int_0^1 K'(t)^n \left( \frac{1 + t}{2} \right)^{n-2} \, dt, \\
\int_0^1 K'(x)^n \, dx &= 2 \int K(t)^n (1 + t)^{n-2} \, dt. 
\end{align*}
\]

(28)

Setting \( n = 1 \) we recover the known special case

\[ \int_0^1 \frac{K(x)}{x + 1} \, dx = \frac{\pi^2}{8}. \]

Using a cubic transform of the Borweins ([7]), this identity is generalized in [6]. The appropriate generalization of (3) – itself obtained by setting \( n = 2 \) in (28) – is

\[ \int_0^1 2 F_1 \left( \frac{1}{3}, \frac{2}{1}; 1 - x^3 \right)^2 \, dx = 3 \int_0^1 2 F_1 \left( \frac{1}{3}, \frac{2}{3}; 3 \right) x^2 \, dx. \]

Using (5) on the integrand \( x K(x)^3 \), we get

\[ \int_0^1 2(1 - x) K(x)^3 \, dx = \int_0^1 x K(x)^3 \, dx, \]
when combined with (28), we deduce
\[
\int_0^1 K'(x)^3 \, dx = \frac{10}{3} \int_0^1 K(x)^3 \, dx = 5 \int_0^1 xK(x)^3 \, dx = 5 \int_0^1 xK'(x)^3 \, dx. \tag{29}
\]

Using (6, 7), we have
\[
\int_0^1 E(x)^n \frac{2^{n+1}}{(x+1)^{n+2}} \, dx = \int_0^1 (E'(x) + xK'(x))^n \, dx, \tag{30}
\]
\[
\int_0^1 E'(x)^n \frac{2^{n+1}}{(x+1)^{n+2}} \, dx = \int_0^1 (2E(x) - (1 - x^2)K(x))^n \, dx. \tag{31}
\]

When \( n = 1, 2 \) we obtain closed forms, e.g.
\[
\int_0^1 E(x) \frac{x^2}{(x+1)^3} \, dx = \frac{\pi^2}{32} + \frac{1}{4}, \quad \int_0^1 E'(x) \frac{x}{(x+1)^3} \, dx = \frac{G}{8} + \frac{5}{16}.
\]

5.4. Relation to random walks. In [9], many moment relations are derived while computing \( W_4(n) \), the \( n \)th moment of the distance from the origin of a 4-step uniform random walk on the plane. For instance, we have:
\[
W_4(1) = \frac{16}{\pi^3} \int_0^1 (1 - 3x^2)K'(x)^2 \, dx.
\]

In [10], the following identities are proven via Meijer G-functions:
\[
\frac{\pi^3}{4} W_4(-1) = \int_0^{\pi/2} K(\sin t)^2 \sin t \, dt
\]
\[
= 2 \int_0^{\pi/2} K(\sin t)^2 \cos t \, dt = \int_0^{\pi/2} K(\sin t)K(\cos t) \, dt,
\]
and
\[
\int_0^1 K(x)K'(x)x' \, dx = \int_0^1 K(x)^2 \, dx. \tag{32}
\]

6. Fourier series

As recorded in [3], we have the following Fourier (sine) series valid on \( (0, 2\pi) \):

Lemma 2.
\[
K(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)^2}{\Gamma(n + 1)^2} \sin((4n + 1)t). \tag{33}
\]

For completeness, we sketch a proof here:

Proof. By symmetry we see that only the coefficients of \( \sin((2n+1)t) \) are non-zero. Indeed, by a change of variable \( \cos t \mapsto x \), the coefficients are
\[
\frac{4}{\pi} \int_0^1 K'(x) \sin((2n+1)t) \frac{\sin t}{\sin t} \, dt.
\]
The fraction in the integrand is precisely $U_{2n}(x)$, where $U_n(x)$ denotes the Chebyshev polynomial of the second kind ([18]), given by

$$U_{2n}(x) = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} (2x)^{2n-2k}.$$ 

We now interchange summation and integration, and use the moments of $K'$. The resulting coefficient contains a $3F2$, which after a transformation ([4], section 3.2) becomes amenable to Saalschütz’s theorem ([1]), and we obtain (33). □

The same method gives a Fourier sine series for $E(\sin t)$ valid on $(0, \pi)$, which we have not been able to locate in the literature. In mirroring the last step, the resulting $3F2$ is reduced to the closed form below using Sister Celine’s method:

**Lemma 3.**

$$E(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^2}{2\Gamma(n+1)^2} \sin((4n+1)t) + \sum_{n=0}^{\infty} \frac{(n+1/2)\Gamma(n+1/2)^2}{2(n+1)\Gamma(n+1)^2} \sin((4n+3)t).$$

(34)

Parseval’s formula applied to (33) and (34) gives

$$\int_0^{\pi/2} K(\sin t)^2 \, dt = 2 \int_0^{\pi/2} K(\sin t)E(\sin t) \, dt$$

$$= \int_0^{1} \frac{K(x)^2}{\sqrt{1-x^2}} \, dx = 2 \int_0^{1} \frac{E(x)^2}{\sqrt{1-x^2}} \, dx$$

$$= 2 \int_0^{1} K(x)K'(x) \, dx = \frac{\pi^3}{4} \ 4F3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \left| 1\right.\right).$$

(35)

We also get $\int_0^{\pi/2} K(\sin t)^2 \cos(4t) \, dt$ as a sum of three $4F3$’s, and $\int_0^{\pi/2} E(\sin t)^2 \, dt$ as a sum of four $4F3$’s. Section 3.7 of [3] provides a number of identities of this sort with more exotic arguments, as well as connections with Meijer G-functions.

Experimentally we find the surprisingly simple answer to the integral

$$\int_0^{\pi/2} K(\sin t)^2 \frac{\sin 4t}{4} \, dt = \int_0^{1} K(x)^2(x-2x^3) \, dx = \int_0^{1} xK(x)(x^2K(x) - E(x)) \, dx$$

$$= \int_0^{1} K'(x)^2(2x^3 - x) \, dx = \int_0^{1} xK'(x)(x^2K'(x) - E'(x)) \, dx$$

$$= \int_0^{1} xK^2(x) - 2xE(x)K(x) \, dx = -\frac{1}{2}. $$

(36)

All equalities are routine to check except for the last one, which is equivalent to

$$\int_0^{1} xK(x)^2 + 2xE(x)^2 - 3xK(x)E(x) \, dx = 0,$$

and so the equality holds as we know all the odd moments.
Inserting a factor of \(\cos^2 t\) before squaring the Fourier series (33) and integrating, we are led to

\[
\int_0^1 x'K(x)^2 \, dx = \frac{\pi^3}{16} \left( 24 F_3 \left( \begin{array}{c} \frac{1}{2} \frac{1}{2} \frac{1}{2} \\ 1,1,1 \end{array} \right) - 1 \right) = \int_0^1 \frac{\sqrt{x}}{x+1} K'(x)^2 \, dx.
\]

The Fourier series (33) combined with a quadratic transform gives:

\[
\int_0^{\pi/2} K(\sin t) \, dt = \int_0^1 \frac{K(x)}{\sqrt{1-x^2}} \, dx = \int_0^1 \frac{K'(x)}{\sqrt{1-x^2}} \, dx = \int_0^1 \frac{K(x)}{\sqrt{x}} \, dx = \frac{1}{2} \int_0^1 \frac{K'(x)}{\sqrt{x}} \, dx = K \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{16\pi} \Gamma \left( \frac{1}{4} \right)^4.
\]

A generalization of this result is found in [6].

7. LEGENDRE’S RELATION

Legendre’s relation \(6K' + E'K - KK' = \frac{\pi}{2}\) is related to the Wronskian of \(K\) and \(E\), and shows that the two integrals are closely coupled (we have already seen its role in the proof of Lemma 1).

If we multiply both sides of Legendre’s relation (24) by \(K'(x)\) and integrate, we arrive at

\[
\int_0^1 3E'(x)K(x)K'(x) - K(x)K'(x)^2 \, dx = \frac{\pi^3}{8}.
\]

Similarly, had we multiplied by \(K(x)\), the result would be

\[
\int_0^1 3E(x)K(x)K'(x) - 2K(x)^2K'(x) \, dx = \int_0^1 2E'(x)K(x)^2 - E(x)K(x)K'(x) \, dx = \pi G.
\]

Using closed forms of the moments, we also have:

\[
\int_0^1 2xE'(x)K(x)^2K'(x) - xK(x)^2K'(x)^2 \, dx = \frac{\pi^4}{32};
\]

\[
\int_0^1 2xE'(x)^2K(x)E(x) - xK(x)K'(x)E(x)E'(x) \, dx = \frac{\pi^2}{16} + \frac{\pi^4}{128}.
\]

Of course, we can multiply Legendre’s relation by any function whose integral vanishes on the interval \((0, 1)\) to produce another relation. Suitable candidates for the function include \(x(K(x) - K'(x))\), \(2(2K'(x) - 3E'(x))\), \(2E'(x) - K'(x)\), \(2E(x) - K(x) - 1\), and a vast range of polynomials. For instance one could obtain

\[
\int_0^1 2E'^2(x)K(x) + 2E(x)E'(x)K(x) - 5E'(x)K(x)K'(x) + K(x)K'(x)^2 \, dx = 0.
\]

Unfortunately, we are not able to ‘uncouple’ any of the above sums and differences to obtain a closed form for the integral of a single product.
8. Integration by parts

The following simple but fruitful idea is crucial to this section. We look at the derivative \((1 - x^2)^n \frac{d}{dx} (x^k K(x)^a E(x)^b K'(x)^c E'(x)^d)\), and integrate by parts to yield

\[
\int_0^1 (1 - x^2)^n \frac{d}{dx} \left( x^k K(x)^a E(x)^b K'(x)^c E'(x)^d \right) dx = \int_0^1 2nx(1 - x^2)^{n-1} x^k K(x)^a E(x)^b K'(x)^c E'(x)^d dx + C, \tag{38}
\]

where the constant \(C \neq 0\) if and only if the integrand is a power of \(E\) or \(E'\).

In practice, we take \(n, k \in \{0, 1, 2\}\) to produce the cleanest identities. We also explore when \(n\) is a half-integer, as well as replacing \(1 - x^2\) by \(1 - x\) in (38).

8.1. Bailey’s tables for products of two elliptic integrals. We now systematically analyse the tables kindly provided by D. H. Bailey, the construction of which is described in [2]. The tables contain all known (in fact, almost certainly all) linear relations for integrals of products of up to \(k\) elliptic integrals \((k \leq 6)\) and a polynomial in \(x\) with degree at most 5. In this subsection we exclusively look at the case \(k = 2\) and spell out the details.

We use \(x \frac{d}{dx} E(x)^2 = 2E(x)^2 - 2E(x)K(x)\) and integrate by parts to deduce

\[
\int_0^1 3E(x)^2 - 2E(x)K(x) dx = 1. \tag{39}
\]

More generally,

\[
1 = (n + k + 1) \int_0^1 x^n E(x)^n - nx^k E(x)^{n-1} K(x) dx. \tag{40}
\]

Two more special cases of the above are prominent:

\[
\int_0^1 5x^2 E(x)^2 - 2E(x)K(x) dx = 1, \tag{41}
\]

\[
\int_0^1 (n + 2)x^{n-1} E(x)^2 - 2x^{n-1} E(x)K(x) dx = 1. \tag{42}
\]

The derivative of \(K(x)E(x)\) (via integration by parts) gives

\[
\int_0^1 (1 - 3x^2)E(x)K(x) + E(x)^2 - (1 - x^2)K(x)^2 dx = 0, \tag{43}
\]

which is part of the more general

\[
\int_0^1 nx^{n-1} E(x)K(x) - (n + 2)x^{n+1} E(x)K(x) + x^{n-1} E(x)^2 - x^{n-1}K(x)^2 + x^{n+1}K(x)^2 dx = 0. \tag{44}
\]

The derivative of \(K(x)^2\) produces

\[
\int_0^1 (1 + x^2)K^2(x) dx = 2 \int_0^1 K(x)E(x) dx, \tag{45}
\]
while more generally,
\[
\int_0^1 2x^{n-1}E(x)K(x) + (n - 2)x^{n-1}K(x)^2 - nx^{n+1}K(x)^2 \, dx = 0. \quad (46)
\]

The derivative of \( E'(x)^2 \) gives (using (10) for the first equality)
\[
\int_0^1 2xE'(x)^2 - xE'(x)K'(x) \, dx = \int_0^1 2xE(x)^2 - xE(x)K(x) \, dx = \frac{1}{2}.
\]

The derivative of \( K'(x)^2 \) gives
\[
\int_0^1 2K'(x)E'(x) - (1 - x^2)K'(x)^2 \, dx = 0,
\]
reconfirming a result from random walks ([9]), which is first proven in a much more roundabout way via a non-trivial group action on the integrand.

The derivative of \( E'(x)K'(x) \) gives
\[
\int_0^1 (1 - 3x^2)E(x)K'(x) \, dx = \int_0^1 E'(x)^2 - x^2K'(x)^2 \, dx,
\]
which, when combined with our last result, gives
\[
\int_0^1 (1 + 3x^2)E'(x)K'(x) \, dx = \int_0^1 K'(x)^2 - E'(x)^2 \, dx.
\]

The derivative of \( K(x)K'(x) \) gives
\[
\int_0^1 x^2K(x)K'(x) + K(x)E'(x) - K'(x)E(x) \, dx = 0,
\]
which, when combined with Legendre’s relation (24), results in
\[
\int_0^1 2E'(x)K(x) - (1 - x^2)K(x)K'(x) \, dx = \frac{\pi}{2}.
\]

Our results here and in previous sections actually provide direct proofs of most entries in Bailey’s tables where the polynomial is linear. In fact, it would simply be a matter of tenacity to prove many entries involving polynomial of higher degrees. As an example, we indicate how to prove an entry which requires more work:
\[
\int_0^1 E(x)(3E'(x) - K'(x)) \, dx = \frac{\pi}{2}. \quad (47)
\]

We write the left hand side as two \( 4F_3 \)'s, combine their summands into a single term and simplify; the result can be summed explicitly by Gosper’s algorithm, and the limit on the right hand side follows.

The same method applies to entries involving higher degree polynomials, e.g.
\[
\int_0^1 E'(x)K(x) - E(x)K'(x) + x^2K(x)K'(x) \, dx = 0.
\]

There is only one entry in Bailey’s tables (for linear polynomials) that we cannot prove, though it is true to at least 1500 digits:
Conjecture 1.

\[
\int_0^1 2K(x)^2 - 4E(x)K(x) + 3E(x)^2 - K'(x)E'(x) \, dx \overset{?!}{=} 0. \tag{48}
\]

(Here, the notation \(?!!\), \(?!!?!\) etc denotes the equivalence of conjectural identities, so for instance all equations with the label \(?!!?!\) are equivalent as conjectures.)

We note that, among moments of products of two elliptic integrals, there are only five that we do not possess closed forms of:

\[
E(x)^2, \quad x^2E(x)^2, \quad E(x)K(x), \quad xE(x)K(x), \quad x^2K(x)^2,
\]

as all the odd moments are known, and the other even moments may be obtained from these ignition values. Unfortunately, we can only prove four equations connecting them, namely \((39, 41, 43, 45)\). A proof of \((48)\) would give us enough information to solve for all five moments; for instance, we would have

\[
\frac{32}{\pi^4} \int_0^1 E(x)K(x) \, dx \overset{?!}{=} \frac{16}{\pi^4} + 7F_6 \left( \frac{1}{4}, 1, 1, 1, 1, 1 \right) - \frac{1}{2} F_6 \left( \frac{1}{4}, 1, 1, 1, 1, 2 \right) + \text{etc.}
\]

8.2. Recurrences for the moments. As already hinted in the proof of Lemma 1, the moments enjoy recurrences with polynomial coefficients. For example, by combining \((42, 44, 46)\), we obtain, with \(K_n := \int_0^1 x^nK(x)^2 \, dx\),

\[
(n + 1)^3K_{n+2} - 2n(n^2 + 1)K_n + (n - 1)^3K_{n-2} = 2. \tag{49}
\]

This then shows that \(K_n\) is a rational number plus a rational multiple of \(\zeta(3)\) for odd \(n\), as this approach is used in the proof of Theorem 2.

Similarly, recurrences for other products may be obtained, though the linear algebra becomes more prohibitive. We have, for \(E_n := \int_0^1 x^nE(x)^2 \, dx\),

\[
(n + 1)(n + 3)(n + 5)E_{n+2} - 2(n^3 + 3n^2 + n + 1)E_n + (n - 1)^3E_{n-2} = 8, \tag{50}
\]

while the recursion for the moments of \(EK\) follows from this and \((42)\). The recursion for the moments of \(K^2\) is identical to \((49)\) except the right hand side is 0.

8.3. More results. We discover some results not found in Bailey’s tables by incorporating constants such as \(\pi\) and \(G\) into the search space. Below we highlight some of the prettier formulæ.

Take \((1 - x^2) \frac{d}{dx} (x^2K'(x)^2)\) and integrate by parts, we obtain

\[
\int_0^1 xK(x)K'(x) \, dx = \int_0^1 2x^3K(x)K'(x) \, dx = \int_0^1 \frac{1 - x}{1 + x} K(x)K'(x) \, dx = \frac{\pi^3}{16}. \tag{51}
\]

The derivative of \(x^{2n}K(x)K'(x)\) together with \((24)\) gives

\[
\int_0^1 x^{2n-1}(E'(x)K(x) + n(x^2 - 1)K(x)K'(x)) \, dx = \frac{\pi}{8n}. \tag{52}
\]
We can take $n = \frac{1}{2}$ in (38); for instance, the derivative of $xx'K(x)$ gives

$$
\int_0^1 \frac{E(x)}{x'} \, dx = \int_0^1 \frac{x^2K(x)}{x'} \, dx,
$$

while the derivative of $xx'K(x)^2$ recaptures (35).

The derivative of $x'K(x)$ gives

$$
\int_0^1 \frac{K(x) - E(x)}{xx'} \, dx = \frac{\pi}{2},
$$

note that each part does not converge. In fact,

$$
K(x) - E(x) = \frac{\pi x^2}{4} F_1 \left( \frac{1}{2}, \frac{3}{2} \mid x^2 \right), \tag{53}
$$

therefore

$$
\int_0^1 \frac{K(x) - E(x)}{x} \, dx = \frac{\pi}{2} - 1, \quad \int_0^1 \frac{K(x) - E(x)}{x^2} \, dx = 1, \quad \int_0^1 \frac{K(x) - E(x)}{xx'} \, dx = \frac{\pi}{2}.
$$

The derivative of $x(1 - x)K(x)^2$ gives

$$
\int_0^1 \frac{2K(x)E(x)}{x + 1} \, dx = \int_0^1 K(x)^2 \, dx.
$$

Collecting what we know about the integral of $K(x)^2$, we have the following:

**Theorem 4.** The alternative forms of equation (3) are:

$$
\int_0^1 K(x)^2 \, dx = \frac{1}{2} \int_0^1 K'(x)^2 \, dx
$$

$$
= \int_0^1 K'(x)^2 \frac{x}{x'} \, dx
$$

$$
= \int_0^1 K(x)K'(x)x' \, dx
$$

$$
= \int_0^1 2K(x)E(x) \frac{x}{x + 1} \, dx
$$

$$
= \frac{2}{\pi} \int_0^1 \frac{\arcsin x}{\sqrt{1 - x^2}} K(x)K'(x) \, dx
$$

$$
= \frac{4}{\pi} \int_0^1 \text{arctanh}(x)K(x)K'(x) \, dx.
$$

**Proof.** The last two equalities follow from (26); the rest has been proven elsewhere. \qed
8.4. Bailey’s tables for products of three elliptic integrals. We now consider the linear relations involving the product of three elliptic integrals ($k = 3$ in the tables). As the number of relations found is huge, we restrict most of our attention to a class of integrals that turn out to be pair-wise related by a rational factor.

Below we tabulate all the products for which ‘neat’ integrals may be deduced by differentiating them and integrating by parts:

<table>
<thead>
<tr>
<th>Product:</th>
<th>Integral:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K(x)^3$</td>
<td>$\int_0^1 K(x)^3 - 3K(x)^2 E(x) , dx = 0$</td>
</tr>
<tr>
<td>$K(x)^2 K'(x)$</td>
<td>$\int_0^1 K(x)^2 E'(x) + K(x)^2 K'(x) - 2K(x)K'(x) E(x) , dx = 0$</td>
</tr>
<tr>
<td>$K'(x)^2 K(x)$</td>
<td>$\int_0^1 E(x)K'(x)^2 - 2E'(x)K(x)K'(x) , dx = 0$</td>
</tr>
<tr>
<td>$K'(x)^3$</td>
<td>$\int_0^1 K'(x)^3 - 3K'(x)^2 E'(x) , dx = 0$</td>
</tr>
<tr>
<td>$E'(x)^3$</td>
<td>$\int_0^1 5x E'(x)^3 - 3x E'(x)^2 K'(x) , dx = 1$</td>
</tr>
<tr>
<td>$E(x)^3$</td>
<td>$\int_0^1 4E(x)^3 - 3E(x)^2 K(x) , dx = 1$</td>
</tr>
</tbody>
</table>

We now prove

$$\int_0^1 K(x)^2 K'(x) \, dx = \frac{2}{3} \int_0^1 K(x)K'(x) \, dx,$$

by making the change $x \mapsto \frac{1-x}{1+x}$ to the left hand side, use a quadratic transform (4), then apply $x \mapsto \frac{2\sqrt{x}}{1+x}$ to one piece of the result followed by another quadratic transform (5). We obtain $\int_0^1 3x K(x)K'(x)^2 \, dx = \int_0^1 K(x)K'(x)^2 \, dx$. Finally we combine the pieces to prove the claim.

If we make the change of variable $x \mapsto \frac{1-x}{1+x}$, then apply (6), we have

$$\int_0^1 \frac{K(x)^2 E(x)}{1+x} \, dx = \frac{4}{9} \int_0^1 K(x)^3 \, dx.$$

Integrating $x^2(1-x)K(x)^3$ by parts, we can show that $\int_0^1 xK(x)^2 E(x)/(x+1) \, dx$ is also linearly related to the above integral.

Therefore, gathering the results in this section and equation (29), we have determined:

**Theorem 5.** Any two integrals in each of the following two groups are related by a rational factor:

$$K(x)^3, K'(x)^3, xK(x)^3, xK'(x)^3, K(x)^2 E(x), K'(x)^2 E'(x), \frac{K^2(x)E(x)}{1+x}, \frac{xK^2(x)E(x)}{1+x},$$

$$K(x)K'(x)^2, K(x)^2 K'(x), xK(x)K'(x)^2, xK(x)^2 K'(x).$$ (54)

We cannot yet, however, show that any two integrals, one from each group, are related by a rational factor, though it is true numerically to extremely high precision. In fact, the Inverse Symbolic Calculator gives the remarkable evaluation:
Conjecture 2.

\[
\int_0^1 K'(x)^3 \, dx \quad \overset{?}{=} \quad 2K \left( \frac{1}{\sqrt{2}} \right)^4 = \frac{\Gamma(1/4)^8}{128\pi^2}.
\]

Once proven, this would give explicit closed forms for the integrals of \( E'K'K \), \( EK'K \), and \( E'K^2 \) by the results of Section 7.

In view of Theorem 5, (20) and (25), interchanging the order of summation and integration gives an equivalent form of Conjecture 2:

\[
\sum_{n=0}^{\infty} \frac{8}{(2n+1)^2} \binom{\frac{1}{2}, \frac{1}{2}, n + 1, n + 1}{1, n + \frac{3}{2}, n + \frac{3}{2}, 1} \equiv \frac{\Gamma(n+1/2)^4}{\Gamma(n+1)^4} \binom{\frac{1}{2}, \frac{1}{2}, -n, -n}{1, \frac{1}{2} - n, \frac{1}{2} - n, 1} \overset{?}{=} \frac{\Gamma(1/4)^8}{24\pi^4}.
\]

In fact, the integral in Conjecture 2 has re-expressions as integral over products of theta, or of Dedekind \( \eta \) functions, but it is not clear if these alternative forms make the conjecture any easier to attack.

8.5. Products of four elliptic integrals and conclusion. If we take the derivative for \( K(x)^4 \), use the integral (28) connecting \( K'(x)^4 \) and \( K(x)^4 \), plus a quadratic transform, then we obtain

\[
\int_0^1 24E(x)K(x)^3 - 8K(x)^4 - K'(x)^4 \, dx = 0,
\]

which is one of the first non-trivial identities in Bailey’s tables for \( k = 4 \). Many more tabulated relations for products of three and four elliptic integrals can be proven, albeit the complexity of the proofs increases. As perceptively noted in [2],

“[it] seems to be more and more the case as experimental computational tools improve, our ability to discover outstrips our ability to prove.”

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