Comparison of circular flexure hinge design equations and the derivation of empirical stiffness formulations

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Abstract—Flexure hinges are commonly used in many applications which require precise and smooth motions in the nanometer scale. There were various formulations derived using different methods to calculate the stiffness of circular flexure hinges. This article compares these equations with FEA predictions. The limitation of these equations at different t/R (R is the radius and t is the neck thickness) ratios are revealed. Based on the limitations of these design equations, a guideline to select the most accurate equations for hinge design calculations is presented. In addition to the review and comparisons, general empirical stiffness equations in the x- and y-direction were formulated in this study (with errors less than 3% when compared to FEA simulations) for a wide range of t/R ratios ($0.05 \leq t/R \leq 0.8$).

I. INTRODUCTION

Flexure hinges have been widely used in applications such as nanopositioning scanners for scanning probe microscopes [1]–[4], nano-assembly and nano-machining [5]–[7]. Nanomanipulation has emerged as an important technological advancement that increases the use of flexure hinges. Many micro-motion stages were developed using conventional technologies based on servomotors, ball screws and rigid linkages. However, these conventional technologies encounter problems such as friction, wear, backlash and lubrication, which struggle to achieve high positioning accuracy. Flexure-based mechanisms (also known as compliant mechanisms), which solely move through deformations of flexure hinges, provide smooth motions without encountering these problems. These mechanisms are capable of achieving highly precise positioning with nanometer resolutions.

An accurate analytical or empirical model is needed to obtain a good estimation of workspace or stiffness of a compliant mechanism. A compliant mechanism normally uses a few flexure hinges to provide the desired fine motions in various directions. The accuracy of a compliant stage model relies on the precision of a flexure hinge model. Therefore, compliance (the inverse of stiffness) or stiffness equations of flexure hinges are demanded to be as accurate as possible to reduce the accumulated modeling errors of hinges, subsequently reducing the overall modeling error of compliant mechanisms.

Circular flexure hinges were chosen in this study due to their large applications in compliant micro- and nano-positioning devices [1], [8]–[12]. Circular flexure hinges are

precise in rotation where their center of rotations do not displace as much as other flexure hinges such as the left-type [13] and the corner-fillet [14]. There have been many methods adopted to derive satisfactory design equations of flexure hinges, including the integration of linear differential equations of a beam [15]–[17], Castigliano's second theorem [17], inverse conformal mapping [13] and empirical equations formed from finite-element-analysis (FEA) [18], [19]. However, some of these methods provide better accuracies than the others depending on the t/R ratios of circular flexure hinges (see Fig. 1). Three design equations, i.e. the rotational compliance $\Delta\alpha_z/M_z$, the translational compliance at the x axis $\Delta x/F_x$ and the translational compliance at the y axis $\Delta y/F_y$, are discussed in this paper.

Circular flexure hinges were first introduced in [15]. The full and simplified design equations were formulated. The error of the simplified equation relative to the full equation was within 1% for hinges with t/R in the range 0.02 to 0.1, and within 5-12% for thicker hinges ($0.2 \leq t/R \leq 0.6$) [13]. However, both the full and simplified rotational compliance equations of $\Delta\alpha_z/M_z$ show a large difference of 25% for $t/R = 0.6$ when compared with FEA results in [13]. A comparison of experimental results of $M_z/\Delta\alpha_z$ in [20] with the analytical results in [15] was conducted by the authors. The comparison revealed that the simplified equation shows a difference of up to 10% and the full equation shows a difference of up to 16%. It was also reported in [20] that the full equation of $\Delta\alpha_z/M_z$ in [15] had larger errors than the simplified equation when compared with the FEA simulations. Design equations presented in [16] have the same results as the full equations in [15], except signs of $\Delta\alpha_z/F_z$ and $\Delta y/M_z$ were opposite. Therefore, their equations have the same accuracy (neglecting signs).

Rotational compliance equations ($\Delta\alpha_z/M_z$) developed in [13] were categorized into three groups, i.e. thin ($t/R \leq 0.07$), intermediate ($0.07 < t/R \leq 0.2$) and thick ($0.2 < t/R \leq 0.6$), and an equation was derived to calculate compliances for each category respectively. These equations were within 10% error when comparing with FEA and experimental results. An empirical equation was derived to calculate the rotational compliance of flexure hinges in [18]. This equation is tractable and adequate only for thick hinges with t/R in the range 0.2 to 1.0 as reported in [13]. Closed form compliance equations were derived using the Castigliano's second theorem [21]. The analytical compliance results for hinges, with t/R equal to 0.05, 0.1 and 0.2, were compared with FEA simulations and the differences were within 10%. There is no comparison of results presented for

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Fig. 1. Flexure hinge. t is the neck thickness. R is the radius of the hinge. $\Delta\alpha_z$, Δx and Δy are the rotation, x-deformation and y-deformation of the hinge respectively. M_z , F_x and F_y are the moment and forces acting on the corresponding axes. The three design equations discussed in this paper are $\Delta\alpha_z/M_z$, $\Delta x/F_x$ and $\Delta y/F_y$.

other t/R ratios.

This article presents a review of the accuracy of circular flexure hinge design equations, at different t/R ratios, based on their differences when compared to FEA predictions. By analyzing the limitations of each design equation, suggestions of the most accurate and appropriate equations to be used at any particular t/R range are provided in this article. In addition to the review and comparisons, general empirical equations were developed based on FEA results to estimate compliance/stiffness in the x- and y-axis as there is no accurate equation (within 5%) to predict the corresponding compliance/stiffness of circular flexure hinges for a wide t/R range ($0.05 \leq t/R \leq 0.8$).

This article is organised as follows. The FEA modeling of circular flexure hinges is presented in Sec. II. Experimental validation of the FEA model is presented in Sec. III. Sec. IV presents the comparison the three design equations $\Delta\alpha_z/M_z$, $\Delta x/F_x$ and $\Delta y/F_y$ with FEA simulations. In Sec. V, the derivation of empirical equations of the hinge is presented.

II. FEA MODELING OF CIRCULAR FLEXURE HINGES

There are only three experimental compliance/stiffness results of flexure hinges found [20] for three t/R cases. Due to the limited availability of experimental results, FEA simulations were used as a benchmark for comparison for a wide range of t/R . The FEA model presented in this paper provides an accuracy of within 3% when compared with these three experimental results.

ANSYS was used to conduct FEA of flexure hinges. Flexure hinge models were generated using 8-node, two-dimensional, plane elements (PLANE82) with two degree-of-freedom on each node, which are translations in the nodal x and y directions. This element type is more suitable to model irregular shapes and curved boundaries without much loss of accuracy. The modeled flexure hinge has a thickness of 12.7 mm (aluminium alloy, 7075-T6) with a Young's modulus (E) of 71.7 GPa and a Poisson ratio (ν) of 0.33. A mapped meshing technique was used instead

Fig. 2. FEA mapped meshing

Fig. 3. Flexure hinge geometries of a FEA model

of a “smart” meshing, the latter automatically produces fine meshing at areas that high stress concentrations were most likely to occur. Mapped meshing is advantageous over “smart” meshing because mapped meshing provides better control of the distribution and size of elements in an area (see Fig. 2). It was found that the accuracy of the FEA model was significantly influenced by the way the boundary conditions were assigned on a model. For example, when a point force is applied on a node, it could cause a local stress spike on the node which reduces the accuracy of the FEA results. Therefore, constraints and forces were applied at a distance of at least $3h$ (see Fig. 3) from a node where its displacements will be read, to reduce influences of the constraints and applied forces on the FEA results. Forces in the x- and y-direction and a moment along the z-direction were applied. The corresponding nodal deformations at Point 0 and Point 1 were read. Analytical design equations [13], [15]–[18] were derived to calculate compliances at Point 1. However, nodal deformations read from FEA results at Point 1 were the total deformation contributed by the left-hand side beam section as well as the hinge section (see Fig. 3). Therefore, rotations and deformations caused by the beam section (nodal deformations at Point 0) were subtracted from total deformations read at Point 1 to obtain pure deformations caused by the hinge.

III. EXPERIMENTAL VALIDATION OF THE FEA MODEL

Three flexure hinges from each of the categories of thin ($t/R \leq 0.07$), intermediate ($0.07 < t/R \leq 0.2$) and thick ($0.2 < t/R \leq 0.8$) circular hinges were selected for experimental study to verify the accuracy of the FEA model. Unfortunately, a consistent result of the Δx -compliance ($\Delta x/F_x$) could not be obtained due to the limitation of sensor resolution. However, the order of accuracy of the Δx -compliance is approximately the same as the $\Delta\alpha_z$ - and Δy -compliances. This assumption was made because the FEA modelling concept of determining the Δx -compliance was the same as that of the $\Delta\alpha_z$ - and Δy -compliances.

The experimental setup of measuring the $\Delta\alpha_z$ - and Δy -compliance is shown in Figure 4. The experiment was setup on an anti-vibration table. For the measurement of $\Delta\alpha_z$ -compliance, two eddy-current sensors (3316 and 3317) were used to detect deformations of the hinge. For the measurement of Δy -compliance, a fibre-optic sensor was used instead of the two eddy-current sensors. This is because the fibre-optic sensor has a smaller resolution which is more suited for measuring the small Δy -deformation of flexure hinges. Forces were applied on the flexure hinges by hanging weights at a predetermined position.

Fig. 4. Experimental setup of the (a) $\Delta\alpha_z$ -compliance and (b) Δy -compliance measurement.

The $\Delta\alpha_z$ -deformation of a flexure hinge can be calculated easily from deformations detected by the two sensors. Figure 5(a) shows the schematic of the experimental setup and dimensions.

The $\Delta\alpha$ -compliance ($\Delta\alpha_z/M_z$) is calculated as,

$$\frac{\Delta\alpha_z}{M_z} = \frac{\Delta\alpha_z}{F_1 (L_1 + R)} \quad (1)$$

where $\Delta\alpha_z = \arctan\left(\frac{d_1 - d_2}{L_1 - L_2}\right)$. d_1 and d_2 are the deformation measured by the sensors.

The overall y-axis deformation measured by the fibre-optic sensor consists of a deformation component caused by a force, $F_y = F_2$ and a deformation component caused by a moment, $M_{z2} = F_2 L_2$ (see Figure 5(b)). In order to calculate the Δy -deformation and compliance which are only caused by F_y , the deformation caused by M_{z2} is required to be subtracted from the measured results.

The Δy -deformation caused by M_{z2} is $\Delta y_{M_{z2}} = (\Delta\alpha_z/M_z) \cdot M_{z2} \cdot R$, where $\Delta\alpha_z/M_z$ is determined from Eq. 1. Thus, the Δy -deformation caused by F_y is,

$$\Delta y = \Delta y_{\text{fibre-optics}} - \Delta y_{M_{z2}} \quad (2)$$

The Δy -compliance ($\Delta y/F_y$) of the hinge is then calculated by dividing Equation 2 with F_y .

TABLE I
COMPARISON BETWEEN FEA AND EXPERIMENTAL RESULTS

Flexure hinge (mm)			$\Delta\alpha_z/M_z$ (rad/Nm)			$\Delta y/F_y$ ($\mu\text{m/N}$)		
t	R	t/R	FEA	Exp.	% diff.	FEA	Exp.	% diff.
0.5	3	0.167	0.157	0.159	-1.0	1.518	1.477	2.8
0.7	1.87	0.374	0.056	0.055	1.3	0.227	0.218	4.3
0.84	1.1	0.764	0.029	0.028	2.6	0.047	0.044	6.2

Fig. 5. Schematics of the (a) $\Delta\alpha_z$ -compliance and (b) Δy -compliance measurement of flexure hinges.

Table I shows the comparison of compliances between FEA and experimental results. The FEA determined $\Delta\alpha_z$ -compliances were within 3% of the experimental results while the Δy -compliances were within 6% of the experimental compliances. These results indicated that the developed FEA model was defined with appropriate boundary conditions and mesh sizes, and the model is capable of providing good accuracies of compliance results for flexure hinges with different t/R ratios. The comparison of results also confirmed that the FEA model can be used as a benchmark for comparisons with other analytical results without much loss of accuracy.

IV. COMPARISON OF COMPLIANCE RESULTS WITH FEA

A. Rotational compliance equations, $\Delta\alpha_z/M_z$

Compliances, $\Delta\alpha_z/M_z$ of different circular flexure hinges (with various t and R values), where t/R in a range 0.05 to

Fig. 6. Percentage differences of $\Delta\alpha_z/M_z$ compared to FEA results.

Fig. 7. Percentage differences of $\Delta x/F_x$ compared to FEA results

0.65, were calculated using design equations of a) Paros and Weisbord [15] (full), b) Paros and Weisbord [15] (simplified), c) Lobontiu [17], d) Wu and Zhou [16], e) Tseytlin [13], f) Smith et al. [18] (empirical) and g) Schotborgh et al. [19] (empirical). A concise list of these design equations can be found in [22]. Their results were compared with the FEA simulations obtained in Sec. II. The differences are shown in Fig. 6. It is noted that

- equations of Paros and Weisbord (full), Lobontiu and Wu and Zhou were derived using a similar method (that is the integration of the linear differential equation of a beam); thus they have the same accuracy when compared with FEA simulations. The accuracies of their results decrease over the t/R range. Differences are less than 5% when $0.05 \leq t/R < 0.15$. Differences increase to more than 5% but less than 10% when $0.15 \leq t/R < 0.3$. When $0.3 \leq t/R \leq 0.65$, differences increase from 10% to 19.2%

Fig. 8. Percentage differences of $\Delta y/F_y$ compared to FEA results. (a) without shear compliance. (b) with shear compliance

- Paros and Weisbord's (simplified) results are accurate within 5% difference for $0.05 \leq t/R \leq 0.2$. Differences increase up to 10% when $t/R = 0.5$ and 12% when $t/R = 0.65$
- Tseytlin's results are accurate (within 6% difference) for $t/R \leq 0.23$ and $0.35 \leq t/R < 0.65$. Differences increase to more than 6% when $0.23 < t/R < 0.35$. When $t/R = 0.25$, the difference is 11.2%. Generally, the results are within 10% difference approximately for the entire t/R range. Interestingly, Tseytlin's analytical results differed from his experimental and FEA results by about 10% [13]. This suggests the accuracy of FEA results in this article match the accuracy of Tseytlin's analytical and FEA results.
- Smith et al.'s (empirical) rotational compliance equation is only accurate (within 4% difference) for $0.20 \leq t/R \leq 0.65$. This observation is similar to that of Tseytlin where he claimed that Smith et al.'s empirical equation is accurate only for thick flexure hinges with t/R in the range 0.2 to 1.0
- Schotborgh et al.'s results are the most accurate compared to the rest. Differences in their results are less than 2.5% for the whole $0.05 \leq t/R \leq 0.65$ range

B. Compliance equations in the x- and y-direction, $\Delta x/F_x$ and $\Delta y/F_y$

Differences of $\Delta x/F_x$ and $\Delta y/F_y$, calculated using design equations of a) Paros and Weisbord [15] (full), b) Paros and Weisbord [15] (simplified), c) Lobontiu [17] and d) Wu and Zhou [16] were plotted in Figs. 7 and 8 respectively. For $\Delta x/F_x$,

- Paros and Weisbord (full), Lobontiu, and Wu and Zhou have the same results. Their results are within 6% difference compared to FEA results
- results of Paros and Weisbord (simplified) are not as accurate as that of Paros and Weisbord (full), Lobontiu, and Wu and Zhou. Their results have a minimum difference of 6.6% and a maximum difference of 38%

For $\Delta y/F_y$ (without shear compliance, SC),

- results of Paros and Weisbord (full), Lobontiu, and Wu and Zhou, without considering shear compliance (SC), are the same. Their differences are within 10% for $0.05 \leq t/R \leq 0.2$. differences increase up to 30% when $t/R = 0.65$
- Paros and Weisbord's (simplified) results are within 10% when $0.05 \leq t/R \leq 0.17$. Differences increase to 31% when $t/R = 0.65$

For $\Delta y/F_y$ (with shear compliance, SC),

- Lobontiu's results are the most accurate compared to the others. The results are within 5% difference for $0.05 \leq t/R \leq 0.1$. The differences increase to within 10% for $0.1 < t/R \leq 0.3$ and to within 15% for $0.3 < t/R \leq 0.65$
- results of Paros and Weisbord's (full) and Wu and Zhou are the same. Their results are within 5% difference for $0.05 \leq t/R \leq 0.1$. The differences increase to within 10% for $0.1 < t/R \leq 0.25$ and to within 20% for $0.25 < t/R \leq 0.65$
- Paros and Weisbord's (simplified) results are within 5% difference for $0.05 \leq t/R < 0.1$. The differences increase to within 10% for $0.1 \leq t/R \leq 0.17$. The maximum difference is 25.5% when $t/R = 0.65$

Generally, all compliance equations, $\Delta y/F_y$ are more accurate when shear compliances are considered. Schotborgh's empirical stiffness/compliance equations in the x- and y-direction were not compared in this article because his equations were derived using hinge models with different height (h , see Fig. 3). Furthermore, his empirical equations consider the deformation effect outside the hinge geometry (as discussed via a personal communication with Schotborgh). Other research groups [15]–[17] did not consider this effect in their equations. The aspect of this deformation effect on the hinge compliances is beyond the scope of this paper. The significance of this effect are currently being investigated and will be revealed in the future.

V. EMPIRICAL STIFFNESS EQUATIONS IN THE X- AND Y-DIRECTION

From previous section, it was noted that there is no accurate design equations (within 5% difference) at this stage

TABLE II
COEFFICIENTS OF POLYNOMIAL FUNCTIONS FOR K_x AND K_y

Coefficients	K_x , 5th order	K_y , 6th order
c_0	0.036343	1.92×10^{-5}
c_1	0.98683	-0.00083463
c_2	-1.5469	0.021734
c_3	3.1152	0.064783
c_4	-3.0831	-0.088075
c_5	1.2031	0.062278
c_6	-	-0.018781

to estimate compliances/stiffness in the y-direction for $t/R > 0.15$ and in the x-direction for $0.1 < t/R < 0.25$. Therefore, general empirical equations (in stiffness form, named K_x and K_y) were formed based on FEA results to estimate stiffness in both the x- and y-direction for a wide range of t/R ratios ($0.05 \leq t/R \leq 0.8$). FEA models with various t/R ratios, which was set from 0.05 to 0.8 with an increment of 0.01, were generated in ANSYS. Unit forces, F_x and F_y were applied on each model and the corresponding deformations, Δx and Δy were read. Polynomial functions with 3rd, 4th, 5th and 6th order were fitted through the data points to obtain empirical stiffness equations. The results of these four empirical equations were compared with FEA results and the differences were plotted in Figs. 9 and 10. 5th and 6th order polynomial functions were used for K_x and K_y respectively in order to keep the differences as small as possible. By using the two polynomial functions, maximum differences occurred when $t/R = 0.05$, which are 1.1% and 2.7% for K_x and K_y respectively. Both empirical equations are shown in Eq. 3. Table II exhibits coefficients of these equations.

$$K_x = Eb \left[\sum_{k=0}^n c_k \left(\frac{t}{R} \right)^k \right], K_y = Eb \left[\sum_{k=0}^n c_k \left(\frac{t}{R} \right)^k \right] \quad (3)$$

where c_k are coefficients of polynomial functions, n is the order of a polynomial function.

Finally, Table III summarizes the suggested compliance/stiffness equations to be used for any particular t/R range, and their minimum, maximum and average differences.

VI. CONCLUSIONS

This article reviews the accuracy of previously derived design equations of circular flexure hinges. Based on the review, a guideline on how to select the most suitable compliance/stiffness design equation for a particular t/R value is presented. FEA simulations were used as a benchmark to compare various design equations of flexure hinges for a wide t/R range. In addition to the review and comparisons, general empirical stiffness equations in the x- and y-direction were formulated in this study for $0.05 \leq t/R \leq 0.8$. The differences of these empirical equations were found to be less than 3% when compared to FEA simulations.

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TABLE III
SUGGESTED COMPLIANCE/STIFFNESS EQUATIONS FOR A PARTICULAR t/R RANGE

Research Group	$\Delta\alpha_z/M_z$ t/R range	% difference			$\Delta y/F_y$ (with SC) t/R range	% difference			$\Delta x/F_x$ t/R range	% difference		
		min.	max.	ave.		min.	max.	ave.		min.	max.	ave.
Paros (full) [15]	$0.05 \leq t/R < 0.1$	1.8	5.0	3.5	$0.05 \leq t/R \leq 0.1$	2	4	3.1	$0.25 \leq t/R \leq 0.65$	0.3	4.9	2.4
Paros (simpl.) [15]	$0.05 \leq t/R \leq 0.2$	1.2	4.9	3.1	$0.05 \leq t/R \leq 0.1$	3	5.6	4.3	Not recommended			
Lobontiu [23]	$0.05 \leq t/R < 0.1$	1.8	5.0	3.5	$0.05 \leq t/R \leq 0.1$	2	3.9	2.9	$0.25 \leq t/R \leq 0.65$	0.3	4.9	2.4
Wu and Zhou [16]	$0.05 \leq t/R < 0.1$	1.8	5.0	3.5	$0.05 \leq t/R \leq 0.1$	2	4	3.1	$0.25 \leq t/R \leq 0.65$	0.3	4.9	2.4
Tseytlin [13]	$0.4 \leq t/R \leq 0.6$	0.7	4.5	2.5	NA	NA	NA	NA	NA	NA	NA	NA
Smith [18]	$0.2 \leq t/R \leq 0.65$	0.8	3.7	2.4	NA	NA	NA	NA	NA	NA	NA	NA
Schotborgh [19]	$0.05 \leq t/R \leq 0.65$	0.03	2.5	1.2	NA	NA	NA	NA	NA	NA	NA	NA
This article	NA	NA	NA	NA	$0.05 \leq t/R \leq 0.8$	0	2.7	0.07	$0.05 \leq t/R \leq 0.8$	0	1.1	0.08

Fig. 9. Accuracy of the empirical equations, K_x represented by the 3rd, 4th, 5th and 6th order polynomial functions.

Fig. 10. Accuracy of the empirical equations, K_y represented by the 3rd, 4th, 5th and 6th order polynomial functions.

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