Comments on “Feedback Stabilization Over Signal-to-Noise Ratio Constrained Channels”

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The above paper [1] obtains in closed-form the infimal signal-to-noise ratio (SNR) required for stabilization of finite-dimensional linear time invariant (LTI) feedback loops, in continuous and discrete-time, for state feedback and output feedback control.

The objective of the present note is to prove that the discrete-time output feedback infimal SNR result for stabilization reported in [1, Theorem III.2] is missing a term in δ , the component due to the relative degree of the plant model.

We report below a revised version of Theorem III.2 and its proof containing the missing term $\sum_{k=1}^{r-1} |\mu_k|^2$ together with a simple example to illustrate the point.

Theorem III.2: Consider the feedback system of [1, Fig.5], assume that A_d has $\bar{\mathbb{D}}^C$ eigenvalues $\{\phi_i; i = 1, 2, \dots, m\}$, and define $T_d(z)$ as in [1, eq. (29)]. Let $G_d(z)$ have relative degree $r \geq 1$, denote the nonminimum phase zeros of $G_d(z)$ by $\{\zeta_i; i = 1, \dots, q\}$, and assume that these zeros are distinct. Then

$$\inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 = \left(\prod_{i=1}^m |\phi_i|^2 \right) - 1 + \eta + \delta \quad (1)$$

where

$$\begin{aligned} \eta &= \sum_{l=1}^q \sum_{i=1}^q \frac{\gamma_l \bar{\gamma}_i}{\zeta_l \bar{\zeta}_i - 1}, \\ \delta &= \begin{cases} 0, & \text{if } r = 1 \\ \sum_{k=1}^{r-1} |\beta_k|^2 + \sum_{k=1}^{r-1} |\mu_k|^2, & \text{if } r > 1 \end{cases} \\ \gamma_l &= (1 - |\zeta_l|^2) \left(B_\phi^{-1}(\zeta_l) - \sum_{k=0}^{r-1} \beta_k \zeta_l^{-k} \right) \prod_{\substack{i=1 \\ i \neq l}}^q \frac{1 - \zeta_l \bar{\zeta}_i}{\zeta_l - \zeta_i}, \\ \mu_k &= \sum_{l=1}^q \gamma_l \zeta_l^{k-1} \end{aligned}$$

and $\eta > 0$ unless $G_d(z)$ is minimum phase or A_d has no eigenvalues in $\bar{\mathbb{D}}^C$. It follows that [1, Problem 4] is solvable if and only if the admissible SNR satisfies

$$\frac{\mathcal{P}_d}{\sigma^2} > \left(\prod_{i=1}^m |\phi_i|^2 \right) - 1 + \eta + \delta.$$

Proof: Consider the function spaces $L_2(\mathbb{D})$, $H_2(\mathbb{D})$, $H_2^\perp(\mathbb{D})$, and $H_\infty(\mathbb{D})$, whose stability region is the open unit disk.

Introduce a coprime factorization $G_d(z) = N(z)/M(z)$, and the parametrization of all stabilizing controllers $K_d(z) = (X(z) + M(z)Q(z))/Y(z) - N(z)Q(z)$, where $X(z)$ and $Y(z)$ satisfy the Bezout identity $N(z)X(z) + M(z)Y(z) = 1$. It follows

that $T_d(z)$ is equal to $1 - M(z)(Y(z) - N(z)Q(z))$. Further factorize $M(z) = B_\phi(z)M_0(z)$, where $B_\phi(z)$ is the Blaschke product in [1, eq. (31)]. It follows from the Bezout identity that $B_\phi^{-1}(z)$ and $M_0(z)Y(z)$ have power series expansions at infinity of the form

$$B_\phi^{-1}(z) = \sum_{k=0}^{\infty} \beta_k z^{-k}$$

and

$$M_0(z)Y(z) = \sum_{k=0}^{r-1} \beta_k z^{-k} - \sum_{k=r}^{\infty} \alpha_k z^{-k}$$

where β_k is defined by [1, (32)]. Since $B_\phi(z)$ is all-pass of unit magnitude, $B_\phi^{-1}(z) \in H_2^\perp$, $N(z)$ has relative degree r , and the set $\{z^{-k}; k = 0, \dots, \infty\}$ forms an orthonormal basis for H_2 . Consider the Blaschke product $B_\zeta(z) = \prod_{k=1}^q (z - \zeta_k)/(1 - z\bar{\zeta}_k)$ of nonminimum phase plant zeros, factorize $N(z) = N_m(z)B_\zeta(z)$, and define

$$B_\zeta^{-1}(z) \sum_{k=r}^{\infty} \alpha_k z^{-k} = \Gamma(z) + \Gamma^\perp(z)$$

where $\Gamma(z) \in H_2(\mathbb{D})$ and $\Gamma^\perp(z) \in H_2^\perp(\mathbb{D})$. It follows that¹

$$\begin{aligned} \|T_d\|_{H_2}^2 &= \left(\prod_{i=1}^m |\phi_i|^2 \right) - 1 + \sum_{k=1}^{r-1} |\beta_k|^2 \\ &\quad + \left\| B_\zeta^{-1} \sum_{k=r}^{\infty} \alpha_k z^{-k} + M_0(z)N_m(z)Q(z) \right\|_{L_2}^2 \\ &= \left(\prod_{i=1}^m |\phi_i|^2 \right) - 1 + \|\Gamma^\perp\|_{H_2^\perp}^2 \\ &\quad + \|\Gamma + M_0 N_m Q\|_{H_2}^2. \end{aligned} \quad (2)$$

The issue in the proof in [1] is that the suggested choice of $Q(z) = -\Gamma(z)/(M_0(z)N_m(z))$, minimizing the squared H_2 norm of $\|\Gamma + M_0 N_m Q\|_{H_2}^2$, is not always proper. This is so since $\Gamma(z)$ may be of relative degree 1, whilst $N_m(z)$ has relative degree r by assumption. At this point notice that $\Gamma(z)$ is given by

$$\begin{aligned} \Gamma(z) &= B_\zeta^{-1} \sum_{k=r}^{\infty} \alpha_k z^{-k} - \Gamma^\perp \\ &= B_\zeta^{-1} \sum_{k=r}^{\infty} \alpha_k z^{-k} - \sum_{l=1}^q \frac{\gamma_l}{z - \zeta_l} \\ &= B_\zeta^{-1} \sum_{k=r}^{\infty} \alpha_k z^{-k} \\ &\quad - \sum_{l=1}^q \gamma_l \left(\frac{\zeta_l^{r-1} z^{-(r-1)}}{z - \zeta_l} + \sum_{k=1}^{r-1} \zeta_l^{(k-1)} z^{-k} \right) \\ &= \underbrace{\left(B_\zeta^{-1}(z) \sum_{k=r}^{\infty} \alpha_k z^{-k} - \sum_{l=1}^q \frac{\gamma_l \zeta_l^{r-1} z^{-(r-1)}}{z - \zeta_l} \right)}_{\tilde{\Gamma}} \\ &\quad - \sum_{k=1}^{r-1} \underbrace{\left(\sum_{l=1}^q \gamma_l \zeta_l^{k-1} \right)}_{\mu_k} z^{-k}. \end{aligned}$$

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¹For more details please refer to [1].

Again, orthonormality of the functions $\{z^{-k}\}$ implies that the squared L_2 norm of the second term on the RHS above is equal to $\sum_{k=1}^{r-1} |\mu_k|^2$. Thus (2) becomes

$$\|T_d\|_{H_2}^2 = \left(\prod_{i=1}^m |\phi_i|^2 \right) - 1 + \sum_{k=1}^{r-1} |\beta_k|^2 + \|\Gamma^\perp\|_{H_2^\perp}^2 + \sum_{k=1}^{r-1} |\mu_k|^2 + \|\tilde{\Gamma} + M_0 N_m Q\|_{H_2}^2. \quad (3)$$

Since $N_m(z)$ has relative degree r and $M_0(z)$ has relative degree zero, it follows that the second term on the RHS of (3) may be minimized by defining $Q(z) = -\tilde{\Gamma}(z)/M_0(z)N_m(z)$ in $H_\infty(\mathbb{D})$.

The final step, following the proof in [1], is to compute $\|\Gamma^\perp\|_{H_2^\perp}^2$ in (2) for which we obtain

$$\|\Gamma^\perp\|_{H_2^\perp}^2 = - \sum_{l=1}^q \text{Res} \left\{ z^{-1} \Gamma^\perp(z) \bar{\Gamma}^\perp(z^{-1}), \zeta_l \right\} \quad (4)$$

where

$$\text{Res} \left\{ z^{-1} \Gamma^\perp(z) \bar{\Gamma}^\perp(z^{-1}), \zeta_l \right\} = \sum_{i=1}^q \gamma_l \bar{\gamma}_i / (1 - \zeta_l \bar{\zeta}_i).$$

The last expression together with (3), (4) and the minimizing choice of $Q(z)$ imply the result in (1). ■

Example: In this example we consider two cases,

a) For the first case we consider the plant model given by

$$G_d(z) = \frac{z - \zeta}{z^2(z - \phi)} \quad (5)$$

with $\zeta \in \bar{\mathbb{D}}^C$, $\phi \in \bar{\mathbb{D}}^C$ and relative degree $r = 2$. For the above plant we recognize γ_1 as

$$\begin{aligned} \gamma_1 &= (1 - \zeta^2) \left(\frac{1 - \zeta\phi}{\zeta - \phi} + \phi - \frac{(1 - \phi^2)}{\zeta} \right) \\ &= (1 - \zeta^2) \left(\frac{1 - \phi^2}{\zeta - \phi} \right) \frac{\phi}{\zeta} \end{aligned}$$

and the infimal SNR for stabilization as

$$\inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 = \phi^2 - 1 + \frac{\gamma_1^2}{\zeta^2 - 1} + (\phi^2 - 1)^2 + \gamma_1^2$$

which upon substituting γ_1 reduces to

$$\inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 = \phi^2(\phi^2 - 1) \left(\frac{1 - \zeta\phi}{\phi - \zeta} \right)^2. \quad (6)$$

b) For the second case we consider the plant model given by

$$G_d(z) = \frac{(z - \zeta)(-\varepsilon z + 1)}{z^2(z - \phi)} \quad (7)$$

with $\zeta \in \bar{\mathbb{D}}^C$, $0 < \varepsilon < 1$ that is $1/\varepsilon \in \bar{\mathbb{D}}^C$, $\phi \in \bar{\mathbb{D}}^C$ and relative degree $r = 1$. For the above plant we recognize γ_1 and γ_2 as

$$\begin{aligned} \gamma_1 &= (1 - \zeta^2) \left(\frac{1 - \phi^2}{\zeta - \phi} \right) \left(\frac{\varepsilon - \zeta}{\zeta\varepsilon - 1} \right) \\ \gamma_2 &= \frac{(\varepsilon^2 - 1)}{\varepsilon} \left(\frac{1 - \phi^2}{1 - \phi\varepsilon} \right) \left(\frac{\varepsilon - \zeta}{1 - \zeta\varepsilon} \right) \end{aligned}$$

and the infimal SNR for stabilization as

$$\begin{aligned} \inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 &= \phi^2 - 1 + \frac{\gamma_1^2}{\zeta^2 - 1} \\ &\quad + \frac{2\gamma_1\gamma_2}{\zeta/\varepsilon - 1} + \frac{\gamma_2^2}{1/\varepsilon^2 - 1} \\ &= \phi^2 - 1 + \frac{\gamma_1^2}{\zeta^2 - 1} \\ &\quad + \frac{2\gamma_1\varepsilon\gamma_2}{\zeta - \varepsilon} + \frac{\varepsilon^2\gamma_2^2}{1 - \varepsilon^2}. \end{aligned} \quad (8)$$

From the definition of γ_1 and γ_2 , we can obtain the limit when $\varepsilon \rightarrow 0$ of

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \gamma_1 &= (1 - \zeta^2) \left(\frac{1 - \phi^2}{\zeta - \phi} \right) \zeta \\ \lim_{\varepsilon \rightarrow 0} \varepsilon\gamma_2 &= \zeta(1 - \phi^2). \end{aligned} \quad (9)$$

As $\varepsilon \rightarrow 0$ the plant model in (7) approaches the plant model in (5) and thus in the limit the infimal squared H_2 norm of $T_d(z)$ in (6) and (8) should give the same result. Consider then the limit of $\varepsilon \rightarrow 0$ for the expression in (8)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 &= \\ \lim_{\varepsilon \rightarrow 0} \left[\phi^2 - 1 + \frac{\gamma_1^2}{\zeta^2 - 1} + \frac{2\gamma_1\varepsilon\gamma_2}{\zeta - \varepsilon} + \frac{\varepsilon^2\gamma_2^2}{1 - \varepsilon^2} \right]. \end{aligned}$$

By means of (9) we can see that the above limit is given by

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 &= \\ \phi^2 - 1 + (\zeta^2 - 1) \left(\frac{1 - \phi^2}{\zeta - \phi} \right)^2 \zeta^2 \\ &\quad + 2(1 - \zeta^2) \frac{(1 - \phi^2)^2 \zeta}{\zeta - \phi} + \zeta^2(1 - \phi^2)^2. \end{aligned}$$

Further algebraic manipulation shows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \inf_{K_d(z) \in \mathcal{K}_d} \|T_d\|_{H_2}^2 &= \phi^2 - 1 \\ &\quad + (1 - \phi^2)^2 \times \left[\frac{\zeta^2(\zeta^2 - 1)}{(\zeta - \phi)^2} + 2 \frac{(1 - \zeta^2)\zeta}{\zeta - \phi} + \zeta^2 \right] \\ &= \phi^2(\phi^2 - 1) \left(\frac{1 - \zeta\phi}{\phi - \zeta} \right)^2 \end{aligned} \quad (10)$$

which matches the result in (6). The fact that (10) is equal to (6) shows that for this example the proposed δ term, in particular for case (a), is indeed correct.

Remark: The nature of the correction term $\sum_{k=1}^{r-1} |\mu_k|^2$ required for [1, Theorem III.2] is such that its absence is evident only if the plant model $G_d(z)$ contains simultaneously poles in $\bar{\mathbb{D}}^C$, nonminimum phase zeros and has relative degree $r > 1$. In any other scenario considered by the assumptions introduced in [1], the original result of Theorem III.2 is still correct.

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REFERENCES

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