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Robust Output-Feedback Model Predictive Control for Systems with Unstructured Uncertainty[★]

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Abstract

In this paper, we present novel results that parameterize a broad class of robust output-feedback model predictive control (MPC) policies for discrete-time systems with constraints and unstructured model uncertainty. The MPC policies we consider employ: (i) a linear state estimator, (ii) a pre-determined feedback gain (iii) a set of “tighter constraints” and (iv) a quadratic cost function in the degrees of freedom and the estimated state. Contained within the class, we find both well-known control policies and policies with novel features. The unifying aspect is that all MPC policies within the class satisfy a robust stability test. The robust stability test is suited to synthesis and incorporates a novel linear matrix inequality (LMI) condition which involves the parameters of the cost function. The LMI is shown to always be feasible under an appropriate small-gain condition on the pre-determined feedback gain and the state estimator. Moreover, we show, by means of both theoretical and numerical results, that choosing the cost function parameters subject to the proposed condition often leads to good nominal performance whilst at the same time guaranteeing robust stability.

Key words: Predictive control; Output-feedback; Model uncertainty; Constrained control

1 Introduction

This paper¹ presents a robust stability test and an associated class of robust output-feedback model predictive control (MPC) policies for discrete-time systems with constraints and unstructured model uncertainty. We consider the linear fractional transformation (LFT) problem setup of Fig. 1, in which the open-loop system is described by a feedback interconnection of a known rational transfer function $G(z)$ and an unknown causal dynamical system Δ . We assume that Δ describes *unstructured uncertainty* in the sense that it may comprise nonlinear, time-varying and infinite dimensional dynamics, provided only that it is causal and that its ℓ_2 -gain is bounded. In addition, for the case of state-constrained or open-loop unstable dynamics, we require the ℓ_∞ -gain of Δ to be bounded (i.e., the operator norm of Δ induced by the ℓ_∞ -signal is bounded), at least in certain directions. Whilst this problem setup is typical for unconstrained control design, it has novel aspects for constrained output-feedback MPC design.

Fig. 1. Uncertain system.

We consider output-feedback policies of the form proposed by Lee and Kouvaritakis (2001). Thus, at each time step k , the predicted control policy is parameterized as an N -step open-loop perturbation of a linear “candidate policy” $u_k = -F\hat{x}_k$, that is,

$$u_{k+i} = -F\hat{x}_{k+i} + v_{i|k}, \quad \forall i \geq 0, \quad v_{i|k} = 0, \quad \forall i \geq N, \quad (1)$$

where u_{k+i} is the predicted control input, F is a feedback gain, $v_{0|k}, \dots, v_{N-1|k}$ are the “degrees of freedom” in the optimization problem and $\hat{x}_{k+i}$ is a state estimate produced by a linear estimator. We consider a general class of quadratic MPC cost functions

$$J = \|\hat{x}_k, V\|_P^2, \quad (2)$$

where $V = [v_{0|k}, \dots, v_{N-1|k}]$, $\|a\|_P^2 = a^T P a$, and where $[a, \dots, b]$ denotes $[a^T \dots b^T]^T$. To ensure robust

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constraint satisfaction, we replace the “nominal constraints” on the degrees of freedom V by a set of “tighter constraints”, as proposed by Gossner, Kouvaritakis, and Rossiter (1997); Chisci, Rossiter, and Zappa (2001); Lee and Kouvaritakis (2001). To ensure robust stability, we propose a new linear matrix inequality (LMI) condition for the parameters of the MPC cost function (i.e., the matrix P in (2)). One of the main contributions of the present paper is then to present a general stability test proving the sufficiency of this constructive approach to robust output-feedback MPC.

We also establish a strong result regarding the feasibility of the proposed LMI condition, namely that there always exists a class of cost function matrices P that satisfies the LMI, provided that the closed-loop transfer function obtained by use of the candidate policy $u_k = -F\hat{x}_k$ satisfies a small-gain condition. Thus, given any candidate policy satisfying the latter condition, we propose to determine the MPC cost function off-line as an “optimal” upper bound on a given nominal cost function. The proposed procedure for computing the cost function guarantees robust stability and is nonconservative in the sense that it leads to an “arbitrarily tight” upper bound on the given nominal cost function when the model uncertainty is “sufficiently small”. In contrast to typical min-max MPC methodology (see, e.g., Lee and Yu (1997)), the resulting MPC design is a *mixed objective design*, since it aims for nominal performance (e.g., quadratic regulation of the output y_k) whilst satisfying a given closed-loop ℓ_2 -gain constraint.

The LMI condition of the stability test is closely related to the LMI that appears in the *bounded real lemma* (de Souza and Xie, 1992) when using the system matrices of a specific *prediction system*. The prediction system we employ is obtained by augmenting the system dynamics and the state estimation dynamics with a “controller state” and is similar to the autonomous system formulation originally proposed by Kouvaritakis, Rossiter, and Schuurmans (2000). Following Kouvaritakis et al. (2000), prediction system formulations have been employed by several authors to compute both the cost and the constraints associated with the degrees of freedom in robust MPC algorithms for linear systems with polytopic uncertainty (Imsland, Bar, and Foss, 2005; Cannon and Kouvaritakis, 2005; Imsland, Rossiter, Pluymers, and Suykens, 2006). In the present paper, we use the matrices of a specific prediction system to compute the cost function. To compute the constraints, we first follow the lines of Lee and Kouvaritakis (2001) before we then employ the constraint tightening approach proposed by Chisci et al. (2001). The end result of these steps is a set of constraints which defines a robustly/disturbance invariant set for the prediction system (see Remark 3.4). A further relation to the previous work of Kouvaritakis et al. (2000); Lee and Kouvaritakis (2001); Chisci et al.

(2001) is that their cost function, which is of the form

$$J = \sum_{i=0}^{N-1} \|v_{i|k}\|_{\Psi}^2 = \|V\|_{\text{diag}[\Psi, \dots, \Psi]}^2, \quad \Psi > 0, \quad (3)$$

satisfies the proposed LMI condition whenever the feasible set of the LMI is non-empty. Hence, given any appropriate candidate policy $u_k = -F\hat{x}_k$, one may always choose the cost (3) and then the resulting robust MPC design is equivalent to a version of the output-feedback design of Lee and Kouvaritakis (2001), or, alternatively, a version of the state-feedback design of Chisci et al. (2001) applied to control the state estimation dynamics. We also note that, when using the cost (3), robust stability can alternatively be concluded from the results in Lee and Kouvaritakis (2001), that is, from the fact that the MPC policy is equivalent to the candidate policy once the closed-loop system state enters an attractive positively invariant “target set” where the unconstrained minimizer $V = 0$ is feasible.

In this regard, the novelty and main contribution of the mixed objective design described here lies in the proposed choice of cost function parameters and the associated LMI condition of the stability test. Indeed, we provide a convex parameterization of a class of cost functions that all lead to robust stability whenever the cost (3) leads to robust stability. Such a convex parameterization is of importance since it can be extremely difficult to find a candidate policy $u_k = -F\hat{x}_k$ which is: (A) robustly stabilizing in the sense that it induces a sufficiently small ℓ_2 -gain, (B) constraint admissible in a sufficiently large invariant target set and (C) optimal in some appropriate sense. Accordingly, it is often beneficial to choose a de-tuned candidate policy $u_k = -F\hat{x}_k$ which satisfies only (A) and (B), and in this case, when the candidate policy does not satisfy (C), one is motivated to search for a cost function whose unconstrained minimizer w.r.t. V is non-zero (whenever the state estimate is non-zero) [see (2)-(3)]. The proposed LMI condition enables one to perform a search for the cost function using convex optimization techniques, and as a consequence, our approach enables the design of a well-tuned robust policy which “inherits” properties (A) and (B) from the de-tuned candidate policy on which it is based (c.f. policy improvement, Bertsekas (2005)), but yet does not reduce to the policy $u_k = -F\hat{x}_k$ in some attractive positively invariant target set. This possibility is illustrated in the numerical example of Section 5.2.1. Another important case covered by our approach is the proposed design for input-constrained open-loop stable systems based on the choice $F = 0$, presented in Section 5.1.

The present problem setup is different from the bulk of the literature on robust MPC. For example, many researchers, including Kothare, Balakrishnan, and Morari (1996), Kouvaritakis et al. (2000), Pluymers, Rossiter, Suykens, and De Moor (2005), have considered the state-

feedback case with parametric uncertainty in the form of a polytopic system model. Related contributions on norm-bounded parametric uncertainty include the works of Kothare et al. (1996), Casavola, Famularo, and Franz (2004). Note that the norm-bounded model uncertainty accounted for in, for example, Kothare et al. (1996); Casavola et al. (2004) is parametric/structured, since it is assumed that Δ is memoryless when establishing Lyapunov stability. In particular, to establish invariance of the relevant ellipsoidal set in Kothare et al. (1996), it is assumed that the current output of Δ is bounded by the norm of the current state (e.g., $\|w_k\|^2 \leq \|x_k\|^2$). The same assumption is required in the proofs of an output-feedback design proposed by Wan and Kothare (2002). Whilst such an assumption is appropriate for memoryless nonlinearities, it does not hold for model uncertainty of the generic non-parametric/unstructured type considered in the present paper. Nevertheless, our departure from the state-feedback case with parametric uncertainty is well motivated, since certain types of model errors, especially those that occur at relatively high frequencies (Doyle and Stein, 1981) (e.g., input-output model errors), can be captured in our framework but not by a parametric uncertainty description.

Min-max MPC schemes which can account for dynamic uncertainty, but are based on perfect state information, have been proposed by, for example, Lee and Kouvaritakis (2000); Chen, Gao, and Wang (2006); Goulart and Kerrigan (2006). An interesting early approach to robust output-feedback MPC, which considers parametric errors in the impulse response coefficients of the system, is the work of Vuthandam, Genceli, and Nikolaou (1995). In stark contrast to this previous work, the present paper is concerned with model uncertainty described in the frequency domain via weighting functions. Furthermore, we do not assume that the actual system has a finite impulse response. The stability test proposed by Heath, Li, Wills, and Lennox (2006) applies to MPC policies and uncertain systems of the type considered here. However, the test requires zero to always be feasible in the on-line optimization problem and thus their approach is suited only to the case of input-constrained open-loop stable systems. The robust output-feedback MPC design of Canale and Milanese (2003) also applies to the case of input-constrained open-loop stable systems, but the uncertainty description in Canale and Milanese (2003) is quite different from the LFT setup considered here.

The paper outline is as follows: Section 2 describes the uncertain open-loop system. Section 3 presents a class of robust MPC policies for the uncertain system; each policy within the class satisfies a robust stability test which we present in Section 3.4. Section 4 studies the LMI condition of the stability test. Section 5 brings together the results of the preceding sections and proposes a mixed objective MPC design. Section 6 concludes the paper.

2 Preliminaries

The set of real numbers is denoted $\mathbb{R}$ and the set of (non-negative) integers is denoted $\mathbb{N}$ ($\mathbb{N}^+$). The Hilbert space of all real square-summable (vector) sequences $\mathbf{z} = \{z_k\}_{k=-\infty}^{\infty}$ is denoted ℓ_2 and the associated *extended space* is denoted ℓ_{2e} (Desoer and Vidyasagar, 1975) and defined by $\ell_{2e} \triangleq \{\mathbf{z} = \{z_k\}_{k=-\infty}^{\infty} | \mathbf{P}_T^- \mathbf{z} \in \ell_2, \forall T \in \mathbb{N}\}$, where $\mathbf{P}_T^-$ is a projection/truncation operator such that $\mathbf{P}_T^- \mathbf{z} = \{\{z_k\}_{k=-\infty}^T, 0\}$. We use $z_{i,k}$, $i \in \{1, \dots, n_z\}$, to refer to the scalar components of a vector sequence $\{z_k\}_{k=-\infty}^{\infty}$, $z_k \in \mathbb{R}^{n_z}$. We define the ℓ_2 -gain and the ℓ_∞ -gain of a causal operator Δ , respectively, as follows (see, e.g., Dahleh and Diaz-Bobillo (1995)):

$$\|\Delta\|_\infty \triangleq \sup_{\mathbf{z} \in \ell_2, \mathbf{z} \neq 0} \frac{\|\Delta \mathbf{z}\|_\infty}{\|\mathbf{z}\|_\infty}, \quad \|\Delta\|_{\ell_\infty[\text{ind}]} \triangleq \sup_{\mathbf{z} \in \ell_\infty, \mathbf{z} \neq 0} \frac{\|\Delta \mathbf{z}\|_\infty}{\|\mathbf{z}\|_\infty},$$

where $\Delta \mathbf{z}$ denotes the response of Δ to $\mathbf{z} = \{z_k\}_{k=-\infty}^{\infty}$, $\|\mathbf{z}\|_\infty \triangleq (\sum_{k=-\infty}^{\infty} z_k^T z_k)^{1/2}$, $\|\mathbf{z}\|_\infty \triangleq \sup_k \|z_k\|_\infty$, $\|z_k\|_\infty \triangleq \max_i |z_{i,k}|$ and $\ell_\infty \triangleq \{\mathbf{z} | \|\mathbf{z}\|_\infty < \infty\}$. We say that Δ is stable if $\|\Delta\|_\infty < \infty$. Let $\ell_{2e}(\mathbb{Z})$ (or $\ell_2(\mathbb{Z})$) denote the subset of ℓ_{2e} (or ℓ_2) consisting of sequences $\mathbf{z} = \{z_k\}_{k=-\infty}^{\infty}$ which also satisfy $z_k \in \mathbb{Z}$, $\forall k \in \mathbb{N}$, where $\mathbb{Z} \subseteq \mathbb{R}^{n_w}$. Note that, if we restrict the inputs of an operator Δ on ℓ_{2e} to take values in $\ell_{2e}(\mathbb{Z})$ and the set $\mathbb{Z}$ is bounded (at least in certain directions), then the corresponding responses $\mathbf{w} = \Delta \mathbf{z}$ will belong to some subset $\ell_{2e}(\mathbb{W})$, where the set $\mathbb{W}$ may be taken to be bounded (at least in certain directions) provided that $\|\Delta\|_{\ell_\infty[\text{ind}]} < \infty$. We are interested in such a “restriction” of an operator Δ on ℓ_{2e} . We use $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ to indicate the restricted input and output spaces.

We consider a discrete-time uncertain system whose evolution from time $k = 0$ onwards is described by the feedback interconnection shown in Fig. 1, where

$$G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} \quad (4)$$

is a known rational transfer function, and where $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ is an operator satisfying $\|\Delta\|_\infty \leq 1$. To impose strict causality from the control input $u_k \in \mathbb{R}^{n_u}$ to the system output $y_k \in \mathbb{R}^{n_y}$ for any causal $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$, we require $G_{21}(z)$ and $G_{22}(z)$ to be strictly causal. To ensure that the system is well-defined, we assume, for simplicity, that $G_{11}(z)$ is strictly causal. A state-space representation of the uncertain system then has the form

$$x_{k+1} = Ax_k + B_w w_k + Bu_k, \quad x_0 = x, \quad (5a)$$

$$z_k = C_z x_k + D_z u_k, \quad (5b)$$

$$y_k = Cx_k, \quad (5c)$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state, and where the auxiliary system input $w_k \in \mathbb{W}$ is the response of $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ to the auxiliary system output $z_k \in \mathbb{Z}$. The system is subject to constraints which we express in state space coordinates as follows:

$$q_k \in \mathbb{Q}, \forall k \in \mathbb{N}^+, \quad q_k = C_q x_k + D_q u_k, \quad (6)$$

where C_q and D_q are matrices of appropriate dimension and $\mathbb{Q} \subseteq \mathbb{R}^{n_q}$ is some given (polyhedral) set. Moreover, these constraints are consistent with the domain of $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ in the following sense:

Assumption 2.1 $C_q x + D_q u \in \mathbb{Q} \Rightarrow C_z x + D_z u \in \mathbb{Z}$.

Remark 2.1 The input-output map, H , of the system in Fig. 1 is given by the LFT

$$H = G_{22} + G_{21} \Delta (I - G_{11} \Delta)^{-1} G_{12}.$$

Hence, we interpret $G_{22}(z)$ as the nominal model of the system, whereas the norm-bound $\|\Delta\|_\infty \leq 1$, the sets $\mathbb{Z}$, $\mathbb{W}$ and $G_{11}(z)$, $G_{12}(z)$, $G_{21}(z)$ constitute a description of the model uncertainty associated with $G_{22}(z)$.

Example 2.1 As a simple yet important example of a class of uncertain systems that fit the present framework, consider a system with additive uncertainty of the form

$$H = G_{22}(z) + W(z)z^{-1}\Delta(z), \quad (7)$$

subject to input constraints $u_k \in \mathbb{U}$, where $W(z)$ is a causal weighting function (e.g., a high-pass filter). In this case, we may choose $G_{11}(z) = 0$, $G_{12}(z) = I$, $G_{21}(z) = W(z)z^{-1}$, $C_q = 0$, $D_q = I$ and $\mathbb{Z} = \mathbb{Q} = \mathbb{U}$ to satisfy Assumption 2.1. We may then describe the uncertainty $\Delta(z)$ as a feedback uncertainty $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ using some set $\mathbb{W}$. If $\mathbb{U}$ and $\|\Delta(z)\|_{\ell_\infty[\text{ind}]}$ are bounded we may take $\mathbb{W}$ to be bounded.

In the sequel we require the following definitions:

Definition 2.1 A causal operator $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ is said to be an admissible uncertainty if: (i) its response $\mathbf{w} = \Delta \mathbf{z}$ to any $\mathbf{z} \in \ell_{2e}(\mathbb{Z})$ satisfies $\mathbf{w} \in \ell_{2e}(\mathbb{W})$, (ii) its ℓ_2 -gain satisfies $\|\Delta\|_\infty \leq 1$ and (iii) its “initial condition” $\{z_k\}_{k=-\infty}^{-1}$ satisfies $\{z_k\}_{k=-\infty}^{-1} \in \ell_2(\mathbb{Z})$.

Remark 2.2 Note that the response of an admissible $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$ to any $\mathbf{z}^+ = \{z_k\}_{k=0}^\infty \in \ell_{2e}(\mathbb{Z})$ depends on the initial condition $\mathbf{z}^- = \{z_k\}_{k=-\infty}^{-1} \in \ell_2(\mathbb{Z})$ and is given by $\mathbf{w} = \Delta(\mathbf{z}^- + \mathbf{z}^+) \in \ell_{2e}(\mathbb{W})$. In view of causality and $\|\Delta\|_\infty \leq 1$, the response $\mathbf{w} = \{w_k\}_{k=-\infty}^\infty$ also satisfies: $\sum_{k=0}^L \|w_k\|^2 \leq \text{const} + \sum_{k=0}^L \|z_k\|^2$, for any integer $L \in \mathbb{N}^+$, where $\text{const} = \|\mathbf{z}^-\|^2$.

Definition 2.2 A causal control policy (based on the measured output y_k) is said to robustly stabilize the system (5) if the following holds: $\{x_k\}_{k=0}^\infty \in \ell_2$, $\{u_k\}_{k=0}^\infty \in \ell_2$, $\{q_k\}_{k=0}^\infty \in \ell_2(\mathbb{Q})$, for all admissible uncertainties $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$.

Definition 2.3 A set Ω is robustly invariant for the system $x_{k+1} = f(x_k, w_k)$ with input $w_k \in \mathbb{W}$ (and output $q_k = h(x_k, w_k) \in \mathbb{Q}$) if: $f(x_k, w_k) \in \Omega$, $\forall [x_k, w_k] \in \Omega \times \mathbb{W}$ (and, in addition, $h(x_k, w_k) \in \mathbb{Q}$, $\forall [x_k, w_k] \in \Omega \times \mathbb{W}$).

3 A Class of Robust MPC Policies

In this section we present a class of robustly stabilizing output-feedback MPC policies for the system (5).

3.1 The State Estimator and the Candidate Policy

As mentioned in Section 1, we consider output-feedback policies which employ a linear state estimator. For simplicity, we consider full-order state estimators of the form

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k), \quad \hat{x}_0 = \hat{x}, \quad (8)$$

where u_k and y_k are the input and output of the uncertain system (5), and where $L \in \mathbb{R}^{n_x \times n_y}$ is a design matrix such that $(A - LC)$ is stable (i.e., its eigenvalues are strictly inside the unit disk). The dynamics of the associated estimation error $e_k \triangleq x_k - \hat{x}_k$ are

$$e_{k+1} = (A - LC)e_k + B_w w_k, \quad e_0 = x - \hat{x}. \quad (9)$$

Next we present two conditions on the state estimator and the associated candidate policy $u_k = -F\hat{x}_k$ which ensure that $u_k = -F\hat{x}_k$ robustly stabilizes the system (5) whenever the initial value of the augmented state $x_k^a \triangleq [e_k, \hat{x}_k]$ belongs to some “target set” $\mathbb{X}^a = \mathbb{E} \times \hat{\mathbb{X}}$. These conditions will, in turn, be shown to imply the existence of a class of robust MPC policies which performs no worse than the candidate policy in the sense that they also robustly stabilize the system whenever $x_0^a \in \mathbb{X}^a$. The conditions are expressed using the system

$$x_{k+1}^a = \underbrace{\begin{bmatrix} (A - LC) & 0 \\ LC & (A - BF) \end{bmatrix}}_{A_a} x_k^a + \underbrace{\begin{bmatrix} B_w \\ 0 \end{bmatrix}}_{B_{a,w}} w_k, \quad (10a)$$

$$z_k = \underbrace{\begin{bmatrix} C_z & (C_z - D_z F) \end{bmatrix}}_{C_a} x_k^a, \quad (10b)$$

as follows:

C1 A_a is stable and $\|C_a(zI - A_a)^{-1}B_{a,w}\|_\infty < 1$.

C2 We have available a non-empty (polyhedral) set $\mathbb{X}^a = \mathbb{E} \times \hat{\mathbb{X}}$, $\mathbb{E} \subseteq \mathbb{R}^{n_x}$, $\hat{\mathbb{X}} \subseteq \mathbb{R}^{n_x}$, which is robustly invariant for the system (10a) with input $w_k \in \mathbb{W}$ and output $q_k \in \mathbb{Q}$, where $q_k = C_q e_k + (C_q - D_q F) \hat{x}_k$.

Remark 3.1 *Provided the set $\mathbb{Q}$ is polyhedral with $0 \in \text{int}(\mathbb{Q})$ and the set $\mathbb{W}$ is convex and sufficiently small, it is possible to compute (polyhedral) sets $\mathbb{E}$, $\hat{\mathbb{X}}$, satisfying **C2** using techniques in, for example, Kolmanovsky and Gilbert (1998); Lee and Kouvaritakis (2001); Raković, Kerrigan, Kouvaritakis, and Mayne (2005); Mayne, Raković, Findeisen, and Allgöwer (2006).*

3.2 The Cost Function and the LMI Condition

To guarantee robust closed-loop stability, we propose an LMI condition for the cost function matrix P in (2). To this end, we introduce the following *prediction system* (similar to the autonomous system formulation of Kouvaritakis et al. (2000)) obtained by augmenting the system (10) with a “controller state” $V_k \in \mathbb{R}^{N n_u}$, $N \geq 1$:

$$\begin{bmatrix} x_{k+1}^a \\ V_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A_a & B_{a,u} D \\ 0 & M \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_k^a \\ V_k \end{bmatrix} + \underbrace{\begin{bmatrix} B_{a,w} \\ 0 \end{bmatrix}}_{\bar{B}_w} w_k, \quad (11a)$$

$$z_k = \underbrace{\begin{bmatrix} C_a & D_z D \end{bmatrix}}_{\bar{C}_z} \begin{bmatrix} x_k^a \\ V_k \end{bmatrix}, \quad (11b)$$

where $B_{a,u} \triangleq [0, B]$, $D \triangleq [I_{n_u} \ 0 \ \dots \ 0] \in \mathbb{R}^{n_u \times N n_u}$, and where M is matrix such that, using $V = [v_0, \dots, v_{N-1}]$, we have $MV = [v_1, \dots, v_{N-1}, 0]$, that is,

$$M \triangleq \begin{bmatrix} 0 & I_{(N-1)n_u \times (N-1)n_u} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{N n_u \times N n_u}, \quad (12)$$

where I_n denotes the $n \times n$ identity matrix. The LMI condition that we propose for the cost function matrix will be studied in detail in Section 4 and is as follows:

Condition 3.1 *The cost function matrix P is such that, for some scalars $m_0 > 0$, and $m_1 \geq 0$, and some symmetric matrix $\Omega_0 \in \mathbb{R}^{2n_x \times 2n_x}$, the following holds:*

$$\Phi(P, \Omega_0, m_1) \leq -m_0 \text{diag}[I_n, 0], \quad \Omega(P, \Omega_0) \geq 0, \quad (13)$$

where $n = 2n_x + n_u$, $\Omega(P, \Omega_0) \triangleq \text{diag}[\Omega_0, 0] + \text{diag}[0, P]$,

$$\begin{aligned} \Phi(P, \Omega_0, m_1) &\triangleq \begin{bmatrix} \bar{A} & \bar{B}_w \\ \bar{C}_z & 0 \end{bmatrix}^T \begin{bmatrix} \Omega(P, \Omega_0) & 0 \\ 0 & m_1 I_{n_z} \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B}_w \\ \bar{C}_z & 0 \end{bmatrix} \\ &\quad - \text{diag}[\Omega(P, \Omega_0), m_1 I_{n_w}]. \end{aligned} \quad (14)$$

In Section 4, we will show that there exists a set of cost function matrices P that satisfies Condition 3.1 if and only if **C1** holds. Hence, since (13) is an LMI in the variables m_0, m_1, Ω_0, P , it is straightforward to determine an “optimal” cost function matrix $P^* \geq 0$ subject to Condition 3.1. We also show that, provided $G_{11}(z)$ and $G_{12}(z)$ are “sufficiently small”, such a P^* may be taken to be an “arbitrarily tight” upper bound on the matrix $P_{\{Q,R,F\}} \geq 0$ of the nominal cost:

$$J(x, V) = \|[x, V]\|_{P_{\{Q,R,F\}}}^2 = \sum_{i=0}^{\infty} \|x_i\|_Q^2 + \|u_i\|_R^2, \quad (15)$$

subject to $x_{i+1} = Ax_i + Bu_i$, $x_0 = x$, $u_i = -Fx_i + v_i$, $\forall i \geq 0$, $v_i = 0$, $\forall i \geq N$, where $V = [v_0, \dots, v_{N-1}]$ and where $Q \geq 0$ and $R > 0$ are selectable weighting matrices.

3.3 The Constraints

At each time step k , the MPC policy minimizes the cost function (2) subject to a set of constraints which guarantee that the predicted N -step perturbation of the candidate policy [see (1)] is constraint admissible and results in a “terminal state” $x_{k+N}^a = [e_{k+N}, \hat{x}_{k+N}]$ that belongs to the target set $\mathbb{X}^a = \mathbb{E} \times \hat{\mathbb{X}}$ introduced in **C2**. For simplicity, we will assume that $e_k \in \mathbb{E}$. Thus, since $e_{k+N} \in \mathbb{E}$ follows from **C2** and the structure of the matrix A_a in (10a), the problem of ensuring $x_{k+N}^a \in \mathbb{X}^a$ can be solved by steering the state estimate $\hat{x}_{k+N}$ to the target set $\hat{\mathbb{X}}$. As is well-known, this goal may be achieved by imposing the following tightened constraints on the nominal trajectory with initial state $\hat{x}_k$, as originally proposed by Gossner et al. (1997); Chisci et al. (2001):

$$\hat{x}_{i+1|k} = (A - BF)\hat{x}_{i|k} + Bv_{i|k}, \quad \hat{x}_{0|k} = \hat{x}_k, \quad (16a)$$

$$\hat{q}_{i|k} = (C_q - D_q F)\hat{x}_{i|k} + D_q v_{i|k}, \quad (16b)$$

$$\hat{q}_{i|k} \in \hat{\mathbb{Q}}_i, \forall i \in \{0, \dots, N-1\}, \quad \hat{x}_{N|k} \in \mathbb{X}_F, \quad (16c)$$

where $V = [v_{0|k}, \dots, v_{N-1|k}]$ are the degrees of freedom and the tightened constraint sets are given by

$$\hat{\mathbb{Q}}_i \triangleq \hat{\mathbb{Q}}_0 \sim (C_q - D_q F) \mathbb{R}_i, \quad \hat{\mathbb{Q}}_0 \triangleq \mathbb{Q} \sim C_q \mathbb{E}, \quad (17a)$$

$$\mathbb{X}_F \triangleq \hat{\mathbb{X}} \sim \mathbb{R}_N, \quad \mathbb{R}_i \triangleq \sum_{j=0}^{i-1} (A - BF)^j L C \mathbb{E}. \quad (17b)$$

Here, $U \sim W = \{z \in \mathbb{R}^n \mid z + w \in U, \forall w \in W\}$ and $\mathbb{R}_i$ is a set summation so that $\hat{x}_{k+i} - \hat{x}_{i|k} \in \mathbb{R}_i$, $\forall i \in \{1, \dots, N\}$. For further details, see Chisci et al. (2001).

Remark 3.2 *When $C_q = 0$, $F = 0$ and $\hat{\mathbb{X}} = \mathbb{R}^{n_x}$ there is no “constraint tightening” and (17) leads to the “nominal constraints”, $\hat{\mathbb{Q}}_i = \mathbb{Q}$ and $\mathbb{X}_F = \mathbb{R}^{n_x}$.*

For later reference, we introduce the set $\mathbb{S}$ so that the constraints (16)-(17) can be written as $[\hat{x}_k, V] \in \mathbb{S}$,

$$\mathbb{S} \triangleq \{[\hat{x}_k, V] \mid (16) - (17) \text{ hold}\}. \quad (18)$$

We also recall the following result (see, e.g., Chisci et al. (2001); Richards and How (2006)).

Theorem 3.2 *Suppose that **C2** holds and let $\mathbb{S}_N$ denote the set of feasible state estimates, that is*

$$\mathbb{S}_N \triangleq \{\hat{x} \mid \exists V \in \mathbb{R}^{N n_u} \text{ such that } [\hat{x}, V] \in \mathbb{S}\}. \quad (19)$$

Then, (i) the following set inclusions hold:

$$\mathbb{S}_N \supseteq \cdots \supseteq \mathbb{S}_2 \supseteq \mathbb{S}_1 \supseteq \hat{\mathbb{X}}; \quad (20)$$

Moreover, (ii) for all $w_k \in \mathbb{W}$ and all $[x_k^a, V_k] \in \mathbb{E} \times \mathbb{S}$, we have, using (11a), that $[x_{k+1}^a, V_{k+1}] \in \mathbb{E} \times \mathbb{S}$. Finally, (iii) if $\hat{x}_k \in \hat{\mathbb{X}}$, then $V = 0$ is feasible, that is, $[\hat{x}, 0] \in \mathbb{S}$, $\forall \hat{x} \in \hat{\mathbb{X}}$.

3.4 The Robust MPC Policy

The resulting robust MPC policy is of the following familiar form:

Algorithm 3.1 *At time $k \in \mathbb{N}^+$, solve*

$$V^*(\hat{x}_k) = \arg \min_{V \in \mathbb{R}^{N n_u}} \{J(\hat{x}_k, V) \text{ s. t. } [\hat{x}_k, V] \in \mathbb{S}\},$$

where, as in (2), $J(\hat{x}_k, V) = \|\hat{x}_k, V\|_P^2$ and where the set $\mathbb{S}$ is as in (18). Then apply $u_k = -F\hat{x}_k + v_{0|k}^*$ to the system (5), where $V^*(\hat{x}_k) = [v_{0|k}^*, \dots, v_{N-1|k}^*]$.

Next we show that Algorithm 3.1 robustly stabilizes the system (5). As our first step, we recall the following well-known result which may be proven using (16)-(18) and point (ii) of Theorem 3.2.

Theorem 3.3 (Robust Constraint Satisfaction)

Suppose that $[e_0, \hat{x}_0] \in \mathbb{E} \times \mathbb{S}_N$ and that **C2** holds. Then it follows that Algorithm 3.1 results in $\{q_k\}_{k=0}^\infty \in \ell_{2e}(\mathbb{Q})$, for all admissible uncertainties $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$.

The following stability test shows that **C1**, **C2** and Condition 3.1 are sufficient for the MPC policy to robustly stabilize the system (5).

Theorem 3.4 (Robust Stability Test) *Suppose that $[e_0, \hat{x}_0] \in \mathbb{E} \times \mathbb{S}_N$ and that **C1**, **C2** and Condition 3.1 hold. Then Algorithm 3.1 robustly stabilizes system (5).*

Proof *Robust constraint satisfaction follows from Theorem 3.3. To prove that all closed-loop signals are in ℓ_2 ,*

for any admissible uncertainty $\Delta : \ell_{2e}(\mathbb{Z}) \rightarrow \ell_{2e}(\mathbb{W})$, consider the “storage function” candidate

$$\Pi^*(e, \hat{x}) \triangleq \| [e, \hat{x}, V^*(\hat{x})] \|_{\Omega(P, \Omega_0)}^2 = \| [e, \hat{x}] \|_{\Omega_0}^2 + J^*(\hat{x}),$$

where $J^*(\hat{x}) \triangleq J(\hat{x}, V^*(\hat{x})) = \|\hat{x}, V^*(\hat{x})\|_P^2$. Note from (13) that the function $\Pi^*(e, \hat{x})$ is non-negative and that it is well-defined for all $[e, \hat{x}] \in \mathbb{E} \times \mathbb{S}_N$. For any $k \in \mathbb{N}^+$, we also have from (13) that

$$\| [e_k, \hat{x}_k, V^*(\hat{x}_k), w_k] \|_{\Phi(P, \Omega_0, m_1)}^2 \leq -m_0 \| [e_k, \hat{x}_k, v_{0|k}^*] \|^2.$$

This inequality may be rearranged to obtain

$$\begin{aligned} & \| [e_{k+1}, \hat{x}_{k+1}, MV^*(\hat{x}_k)] \|_{\Omega(P, \Omega_0)}^2 - \Pi^*(e_k, \hat{x}_k) \\ & \leq -m_0 \| [e_k, \hat{x}_k, v_{0|k}^*] \|^2 - m_1 (\|z_k\|^2 - \|w_k\|^2). \end{aligned}$$

Furthermore, since $MV^*(\hat{x}_k)$ is a feasible solution to the optimization problem at time $k+1$ [see Theorem 3.2], a lower bound on the first term on the left hand side is

$$\begin{aligned} & \| [e_{k+1}, \hat{x}_{k+1}, MV^*(\hat{x}_k)] \|_{\Omega(P, \Omega_0)}^2 = \| [e_{k+1}, \hat{x}_{k+1}] \|_{\Omega_0}^2 \\ & + \| [\hat{x}_{k+1}, MV^*(\hat{x}_k)] \|_P^2 \geq \Pi^*(e_{k+1}, \hat{x}_{k+1}). \end{aligned} \quad (21)$$

Thus, we have established the “dissipation inequality”

$$\begin{aligned} \Pi^*(e_{k+1}, \hat{x}_{k+1}) - \Pi^*(e_k, \hat{x}_k) & \leq -m_0 \| [e_k, \hat{x}_k, v_{0|k}^*] \|^2 \\ & - m_1 (\|z_k\|^2 - \|w_k\|^2), \end{aligned} \quad (22)$$

for all $k \in \mathbb{N}^+$. Summing (22) over the interval $[0, L]$, $L \in \mathbb{N}^+$, and using $\Pi^*(e_{L+1}, \hat{x}_{L+1}) \geq 0$, leads to

$$\begin{aligned} m_0 \sum_{k=0}^L \| [e_k, \hat{x}_k, v_{0|k}^*] \|^2 & \leq \Pi^*(e_0, \hat{x}_0) \\ & + m_1 \sum_{k=0}^L (\|w_k\|^2 - \|z_k\|^2). \end{aligned} \quad (23)$$

Furthermore, use of Remark 3.3 yields

$$m_0 \sum_{k=0}^L \| [e_k, \hat{x}_k, v_{0|k}^*] \|^2 \leq \Pi^*(e_0, \hat{x}_0) + m_1 \text{const}, \quad (24)$$

and since $m_0 > 0$ and the above holds for any $L \in \mathbb{N}^+$, the result follows.

Remark 3.3 *Note that we must have $m_1 > 0$ unless $B_w = 0$ and that dividing (23) by m_1 proves that the closed-loop system has an ℓ_2 -gain from $\{w_k\}$ to $\{z_k\}$ strictly bounded by one, that is, we can conclude from (23) that $\sum_{k=0}^L \|z_k\|^2 \leq \gamma_0 + \gamma \sum_{k=0}^L \|w_k\|^2$, where $L \in \mathbb{N}^+$, $\gamma < 1$ and γ_0 is a bias term due to initial conditions.*

Remark 3.4 In addition to a small-gain argument (Desoer and Vidyasagar, 1975) and other ingredients from the theory of dissipative systems such as a dissipation inequality and a storage function (Willems, 1972), in order to establish a property of the optimal solution $V^*(\hat{x}_{k+1})$, the proof of Theorem 3.4 employs the feasibility of a “candidate solution” $MV^*(\hat{x}_k)$, which is a typical argument in the MPC literature (Rawlings and Muske, 1993; Mayne, Rawlings, Rao, and Scokaert, 2000). Another related key step, which may be applicable to other prediction system formulations of MPC, is that the proposed Condition 3.1 incorporates a structural constraint on the parameterized “Lyapunov matrix” $\Omega(P, \Omega_0)$ [see (21)]. Also note from Theorem 3.2 that the set $\mathbb{E} \times \mathbb{S}$ is robustly invariant for the prediction system (11a) with input $w_k \in \mathbb{W}$ and output $q_k \in \mathbb{Q}$, where $q_k = [C_q (C_q - D_q F)]x_k^a + D_q D V_k$.

4 Choosing the Cost Function

In this section we first show that, for each candidate policy satisfying **C1**, there exists a set of cost function matrices P satisfying Condition 3.1. Thus, for each candidate policy satisfying **C1-C2**, there exists, as described in Section 3.4, an associated class of robustly stabilizing MPC policies that differ only in their choice of P . In this context, Section 4.2 proposes a technique for choosing an MPC policy within the class of robust policies, by finding P that minimizes a linear criterion subject to Condition 3.1.

4.1 Cost Function Matrices Satisfying Condition 3.1

The following theorem describes specific classes of matrices P that satisfy Condition 3.1 if **C1** holds (see Appendix A for the proof).

Theorem 4.1 Suppose that **C1** holds.

(i) For any given matrices $P_{11} \geq 0$, $P_{11} \in \mathbb{R}^{n_x \times n_x}$, and $\Psi > 0$, $\Psi \in \mathbb{R}^{n_u \times n_u}$,

$$P = \text{diag}[P_{11}, \Psi, \dots, \Psi], \quad (25)$$

satisfies Condition 3.1.

(ii) For any given matrix $P_0 \geq 0$, there exists a sufficiently large $Q_0 \in \mathbb{R}^{N_{n_u} \times N_{n_u}}$, such that

$$P = P_0 + \text{diag}[0, P_{22}], \quad (26)$$

where $M^T P_{22} M - P_{22} = -Q_0$, satisfies Condition 3.1.

Point (i) of Theorem 4.1 implies that a cost function of the type (3) employed by Kouvaritakis et al. (2000); Lee and Kouvaritakis (2001); Chisci et al. (2001) always satisfies Condition 3.1, provided that **C1** holds. Point (ii)

of Theorem 4.1 states that it is always possible to “robustify” any given $P = P_0$ so as to satisfy Condition 3.1, provided **C1** holds. Moreover, such a “robustification” of P_0 can always be obtained by modifying the associated “Hessian matrix” only (i.e., by modifying the (2,2) block entry of P_0). The following theorem shows that **C1** is necessary and sufficient for the existence of P satisfying Condition 3.1 (see Appendix A for the proof).

Theorem 4.2 There exists a P that satisfies Condition 3.1 if and only if **C1** holds.

4.2 Upper Bound on the Nominal Cost

Next we propose to determine the cost function matrix $P = P^*$, subject to Condition 3.1, as an “optimal” upper bound on the given nominal cost function matrix $P = P_{\{Q,R,F\}}$ in (15).

To find P^* , we firstly interpret $P_{\{Q,R,F\}}$ in (15) as the unique minimizer of the following optimization problem:

$$P_{\{Q,R,F\}} = \underset{P}{\text{argmin}} \{ \theta(P) \text{ s.t. } \Sigma_{\{Q,R,F\}}(P) \leq 0 \}, \quad (27)$$

where we take $\theta(P)$ to be linear and such that

$$\theta(P) \geq v \text{Trace}(P), \quad v > 0, \quad (28)$$

and where, using the matrices D, M in (11)-(12), we have introduced the function

$$\begin{aligned} \Sigma_{\{Q,R,F\}}(P) \triangleq & \begin{bmatrix} (A - BF) & BD \\ 0 & M \end{bmatrix}^T P \begin{bmatrix} (A - BF) & BD \\ 0 & M \end{bmatrix} \\ & - P + \begin{bmatrix} I & 0 \\ -F & D \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & D \end{bmatrix}. \end{aligned} \quad (29)$$

Note that $\Sigma_{\{Q,R,F\}}(P_{\{Q,R,F\}}) = 0$ and that, since $\Sigma_{\{Q,R,F\}}(P) \leq 0$ implies $P \geq P_{\{Q,R,F\}}$, the equality (27) holds for any linear criterion $\theta(P)$ satisfying (28). As a natural generalization of the latter problem we propose to consider the following semi-definite program (SDP), which incorporates the convex constraints and the associated variables of Condition 3.1:

$$\begin{aligned} & \inf_{P, \Omega_0, m_0, m_1} \{ \theta(P) \text{ s.t. } \Sigma_{\{Q,R,F\}}(P) \leq 0, \\ & (13), m_0 > 0 \text{ and } m_1 \geq 0 \}. \end{aligned} \quad (30)$$

Regarding the feasibility of this SDP we have the following result which follows directly from Theorem 4.1.

Theorem 4.3 The optimization problem (30) is feasible provided that **C1** holds.

In the sequel, we use P^* to denote a feasible and (near) optimal solution to (30). For nominal performance purposes, we would like our P^* to be a “tight” upper bound on $P_{\{Q,R,F\}}$. To clarify, we point out a special case below in which we actually have equality $P^* = P_{\{Q,R,F\}}$.

Lemma 4.4 *Suppose that **C1** holds and that F is the LQR optimal feedback gain associated with the given weighting matrices $Q \geq 0$ and $R > 0$. Then $P^* = P_{\{Q,R,F\}}$.*

Remark 4.1 *Note that Lemma 4.4 follows from the first point of Theorem 4.1 since the matrix $P_{\{Q,R,F\}}$ will be of block diagonal form (25) (Chisci et al., 2001). Also note that, since the feedback gain F of the candidate policy must be chosen so as to satisfy **C2** with a sufficiently large set $\hat{\mathbb{X}}$, it is often restrictive to assume that F is optimal with respect to the given weighting matrices Q and R . This is particularly relevant below in Section 5.1 where we choose $F = 0$ to obtain a global robust stability result.*

We next address the important issue of conservatism of Condition 3.1. Specifically, the following theorem shows that Condition 3.1 is nonconservative in the sense that $P^* \approx P_{\{Q,R,F\}}$ for “sufficiently small” model errors.

Theorem 4.5 *For any given $\epsilon > 0$, there exists a $\delta > 0$, such that, if we make the assignments $G_{11}(z) \leftarrow \delta G_{11}(z)$ and $G_{12}(z) \leftarrow \delta G_{12}(z)$, then $\text{Trace}(P^* - P_{\{Q,R,F\}}) \leq \epsilon$.*

Remark 4.2 *Replacing $G_{11}(z)$ and $G_{12}(z)$ with $\delta G_{11}(z)$ and $\delta G_{12}(z)$, respectively, amounts to “shrinking” the uncertainty by a factor δ , see Remark 2.1. Or, equivalently, scaling $\bar{C}_z$ in (14) to yield $\bar{C}_z \leftarrow \delta \bar{C}_z$.*

Theorem 4.5 holds for any linear criterion $\theta(P)$ satisfying (28), as established in Løvaas, Seron, and Goodwin (2007b). In the numerical example below, we use the simplest case $\theta(P) = \text{Trace}(P)$. A useful alternative to $\theta(P) = \text{Trace}(P)$ is described in Løvaas et al. (2007b).

5 Robust Mixed Objective MPC Design

In this section we bring together the results of the preceding sections and present a robust mixed objective MPC design. We consider two separate cases:

5.1 Design for Input-Constrained Stable Systems

Here we consider the case of input-constrained open-loop stable systems. Specifically, we consider the case when the problem data satisfies the following assumption, which incorporates our standing Assumption 2.1:

Assumption 5.1 *A is stable, $\|G_{11}(z)\|_\infty < 1$ (where $G_{11}(z) = C_z(zI - A)^{-1}B_w$), $\mathbb{Z} = \mathbb{R}^{n_z}$, $\mathbb{W} = \mathbb{R}^{n_w}$, $C_q = 0$,*

$D_q = I_{n_u}$ and the set $\mathbb{Q}$ is convex or finite² with $0 \in \mathbb{Q}$.

In this case, since $\mathbb{W} = \mathbb{R}^{n_w}$ and we cannot confine the estimation error within a bounded positively invariant set $\mathbb{E}$, we propose to choose $\mathbb{E} = \mathbb{R}^{n_x}$. Consequently, since $\mathbb{W} = \mathbb{R}^{n_w}$, $\mathbb{E} = \mathbb{R}^{n_x}$ and typically $\mathbb{Q} \subset \mathbb{R}^{n_u}$, we also make the choice $F = 0$ (basically, the only general choice for the candidate policy that satisfies **C2**) and **C2** thus holds with $\mathbb{E} = \mathbb{R}^{n_x}$ and $\hat{\mathbb{X}} = \mathbb{R}^{n_x}$, for any choice of L . Since A is stable and $G_{11}(z) = C_a(zI - A_a)^{-1}B_{a,w}$ when $F = 0$, **C1** also holds using $F = 0$ together with any estimator gain L such that $(A - LC)$ is stable. These steps lead to the following robust MPC design procedure:

Algorithm 5.1

Off-line: (i) Choose any estimator gain L such that $(A - LC)$ is stable and set $\mathbb{E} = \mathbb{R}^{n_x}$, $F = 0$ and $\hat{\mathbb{X}} = \mathbb{R}^{n_x}$. (ii) Choose any integer $N \geq 1$ to obtain $\mathbb{S}_N = \mathbb{R}^{n_x}$ in (19)-(20). (iii) Choose weighting matrices $Q \geq 0$ and $R > 0$ for the nominal cost function (15). (iv) Solve problem (30) to determine a (near) optimal and feasible $P^* \geq P_{\{Q,R,0\}}$ and set $P = P^*$. **On-line:** Implement Algorithm 3.1 using any initial state estimate $\hat{x} \in \mathbb{R}^{n_x}$.

Regarding the feasibility, stability and conservatism of Algorithm 5.1, we have the following three results, which follows from Theorems 4.3, 3.4 and 4.5, respectively.

Theorem 5.1 (Feasibility) *Under Assumption 5.1, the off-line and on-line steps of Algorithm 5.1 are always feasible.*

Theorem 5.2 (Robust Global Stability) *Under Assumption 5.1, Algorithm 5.1 robustly stabilizes the system (5) for any initial state $x_0 = x \in \mathbb{R}^{n_x}$.*

Theorem 5.3 (Convergence to Nominal Design) *For any $\epsilon > 0$, there exists a $\delta > 0$, such that if the off-line steps of Algorithm 5.1 are carried out using $G_{11}(z) \leftarrow \delta G_{11}(z)$ and $G_{12}(z) \leftarrow \delta G_{12}(z)$, then the cost function matrix satisfies $\text{Trace}(P - P_{\{Q,R,0\}}) \leq \epsilon$.*

Remark 5.1 *The constraint sets of Algorithm 5.1 (i.e., $\hat{\mathbb{Q}}_i = \mathbb{Q}$, $\mathbb{X}_F = \mathbb{R}^{n_x}$, see Remark 3.2) are the common choice for input-constrained open-loop stable systems (Rawlings and Muske, 1993; Mayne, Rawlings, Rao, and Scokaert, 2000) and, in view of Theorem 5.3, also the cost function matrix P of Algorithm 5.1 may be arbitrarily close to the common choice $P = P_{\{Q,R,0\}}$. Consequently, Algorithm 5.1 may lead to results that are arbitrarily close to those which can be obtained using a certainty equivalence implementation of the design of Rawlings and Muske (1993). In this sense, one may view*

² This allows one to tackle finite alphabet control problems (see, e.g., Quevedo, Goodwin, and De Doná (2004)).

the proposed output-feedback design as a “continuous extension” of the design of Rawlings and Muske (1993) to the case of non-zero model- and state estimation errors.

5.2 General Design Procedure

To generalize the above design for input-constrained open-loop stable systems to the general case described in Section 2, we need to change our selection of the invariant sets $\mathbb{E}$, $\hat{\mathbb{X}}$ and the feedback gains L and F . Thus we propose the following MPC design:

Algorithm 5.2

Off-line: (i) Determine the sets $\mathbb{E}$ and $\hat{\mathbb{X}}$ and the associated feedback gains L and F such that both **C1** and **C2** hold. (ii) Choose any integer $N \geq 1$ to obtain $\mathbb{S}_N \supseteq \hat{\mathbb{X}}$ in (19)-(20). (iii) Choose weighting matrices $Q \geq 0$ and $R > 0$ for the nominal cost function (15). (iv) Solve problem (30) to determine a (near) optimal and feasible $P^* \geq P_{\{Q,R,F\}}$ and set $P = P^*$. **On-line:** Implement Algorithm 3.1 using an initial state estimate $\hat{x}$ such that $\hat{x} \in \mathbb{S}_N$ and $e_0 = x - \hat{x} \in \mathbb{E}$.

Theorems 5.1-5.2 generalize as follows:

Theorem 5.4 (Feasibility) *If the first off-line (on-line) step of Algorithm 5.2 is feasible then the remaining off-line (on-line) steps are feasible.*

Theorem 5.5 (Robust Local Stability) *Algorithm 5.2 robustly stabilizes the system (5) for any initial system state $x_0 \in \mathbb{S}_N \oplus \mathbb{E} = \{\hat{x}_0 + e_0 \mid \hat{x}_0 \in \mathbb{S}_N, e_0 \in \mathbb{E}\}$.*

Remark 5.2 *Note that Theorem 5.3 generalizes straightforwardly by replacing $P_{\{Q,R,0\}}$ with $P_{\{Q,R,F\}}$. Also note from Lemma 4.4 that Algorithm 5.2 is a version of the algorithm of Lee and Kouvaritakis (2001); Chisci et al. (2001), whenever F is LQR optimal with respect to the chosen weighting matrices Q and R .*

5.2.1 Numerical Example ³

We consider a system of the form (7), where the nominal model $G_{22}(z)$ is a double integrator and where the additive term $W(z)z^{-1}\Delta(z)$ represents (high frequency) uncertainty. The system is subject to input constraints $u_k \in \mathbb{U} \triangleq \{u \mid -1 \leq u \leq 1\}$. We assume that $\|\Delta(z)\|_{\ell_\infty[\text{ind}]} \leq 1$ and choose $\mathbb{Z} = \mathbb{Q} = \mathbb{W} = \mathbb{U}$ to satisfy Assumption 2.1. The system matrices are

$$A = \begin{bmatrix} 0.06 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 1 \end{bmatrix},$$

³ The example was implemented using YALMIP (Löfberg, 2004), SeDuMi (Sturm, 1999), MPT (Kvasnica, Grieder, and Baotić, 2004) and Matlab Invariant Set Toolbox (Kerrigan).

$C = [-0.0323 \ 0.0207 \ 1 \ 0]$, $C_z = C_q = 0$ and $D_z = D_q = 1$. For our estimator design, we choose $L = AL_K$ where L_K is the Kalman gain obtained using unit weighting matrices, that is, $L_K = [-0.0068, 0.0037, 0.8217, 0.4219]$. The set $\mathbb{E}$ is computed as the maximal robustly invariant set (for the system (9) with input $w_k \in \mathbb{W}$) satisfying $\mathbb{E} \subseteq \mathbb{E}_0 \triangleq \{e \mid \|e\|_\infty \leq 1\}$. The resulting set $\mathbb{E} \subseteq \mathbb{E}_0$ is non-empty and contains the initial estimation errors used in the simulations below. To illustrate the effect of the choice of F on the size of the associated set $\hat{\mathbb{X}}$ in **C2**, we consider three alternatives: $F_1 = [0 \ 0 \ 0.009 \ 0.137]$, $F_2 = [0 \ 0 \ 0.080 \ 0.407]$ and $F_3 = [0 \ 0 \ 0.106 \ 0.471]$, derived as the LQR gains for the nominal model using: $R = 10^4$, $R = 100$ and $R = 52$, respectively, in combination with $Q = I$.⁴ It can be verified that each of these three feedback gains together with our choices of L and $\mathbb{E}$ leads to a candidate policy which satisfies both **C1** and **C2**. Hence, we have identified three possible choices for the candidate policy, $u_k = -F_i \hat{x}_k$, $i \in \{1, 2, 3\}$. Indeed, the maximal choice for the set $\hat{\mathbb{X}}$ using any of the three feedback gains is non-empty and of the form $\hat{\mathbb{X}} = \mathbb{R}^2 \times \mathbb{X}_i$, $i \in \{1, 2, 3\}$. Fig. 2 shows the sets $\mathbb{X}_i$, $i \in \{1, 2, 3\}$, corresponding to F_1 , F_2 and F_3 , respectively. We note from

Fig. 2. $\mathbb{X}_1 \supset \mathbb{X}_2 \supset \mathbb{X}_3$.

Fig. 2 that the feedback gains F_2 , F_3 , lead to significantly smaller sets $\hat{\mathbb{X}} = \mathbb{R}^2 \times \mathbb{X}_i$, $i \in \{2, 3\}$, than the “detuned” choice F_1 . Hence, to obtain a sufficiently large set of feasible initial state estimates $\mathbb{S}_N \supseteq \hat{\mathbb{X}}$, we make the choice $F = F_1$ and $\hat{\mathbb{X}} = \mathbb{R}^2 \times \mathbb{X}_1$. We next consider three different MPC designs based on this specific choice of the candidate policy and a prediction horizon set to $N = 8$. The only difference between the three designs we consider is the cost function matrix P . The first design is an instance of Algorithm 5.2 with weighting matrices $Q = C^T C$ and $R = 0.001$. The second design uses the associated nominal cost function matrix $P = P_{\{Q,R,F\}}$ (i.e., P satisfies (15) with $Q = C^T C$, $R = 0.001$ and $F = F_1$) and is referred to as NMPC. The third design uses the cost function (3) and is referred to as RMPC.

Fig. 3 shows the nominal responses obtained when $\Delta = 0$, $\hat{x}_0 = [0, 0, -100, 0] \in \mathbb{S}_N$ and $e_0 = 0 \in \mathbb{E}$. We note

⁴ The choice of an LQR design to compute F is not required by the proposed MPC design and the matrices Q and R need not be related to Q and R chosen in Algorithms 5.1-5.2.

Fig. 3. Nominal responses: Algorithm 5.2 (top), NMPC (middle) and RMPC (bottom).

from Fig. 2 that the initial state $\hat{x}_0$ is inside the set $\hat{\mathbb{X}} = \mathbb{R}^2 \times \mathbb{X}_1$ and hence from (3) and Theorem 3.2 that the RMPC design is equivalent to using the candidate policy $u_k = -F\hat{x}_k + 0$ throughout. Accordingly, the performance of the RMPC design shown in Fig. 3 is very poor. (In fact, it overshoots to a maximum of 4.2 which falls outside the scale of Fig. 3). Algorithm 5.2, on the other hand, which is based on the same de-tuned candidate policy yields a very quick response. We also note from Fig. 3 that the NMPC design, which uses $P = P_{\{Q,R,F\}}$, yields a quicker response than that of Algorithm 5.2, which uses the associated upper bound $P = P^* \geq P_{\{Q,R,F\}}$. However, unlike Algorithm 5.2 and the RMPC design, the NMPC design is not robustly stable, as illustrated next. In Fig. 4, the simulation of Fig. 3 is repeated using an initial estimation error of $e_0 = [1, 1, 1, 1] \in \mathbb{E}$ and an admissible realization of Δ given by $\Delta = \frac{0.218z - 0.013z}{z^2 + 0.390z - 0.378}$. (Accordingly, we have $\|\Delta\|_\infty = \|\Delta\|_{\ell_\infty[\text{ind}]} \approx 0.994 < 1$). In the simulation results shown in Fig. 4, the initial condition for Δ is set to an admissible value of zero, $0 \in \ell_2(\mathbb{Z})$. Hence, by Theorem 3.4 (and 5.5), it follows that both the RMPC design and Algorithm 5.2 will be closed-loop stable. In fact, their closed-loop responses are very similar to those obtained in the nominal case of Fig. 3. On the other hand, the NMPC design, which uses the nominal cost function matrix $P = P_{\{Q,R,F\}}$, fails severely when Δ is as described above. From these simulation results we con-

Fig. 4. Perturbed responses: Algorithm 5.2 (top), NMPC (middle) and RMPC (bottom).

clude that Algorithm 5.2 resulted in an appropriate ro-

bust “compromise” between the NMPC design and the RMPC design (which, for the chosen initial condition, is equivalent to the candidate policy). We note that this compromise was arrived at by choosing the cost function via the proposed optimization (30). It is also interesting that Algorithm 5.2 converges to the linear feedback law $u_k = -K\hat{x}_k$, $K \approx [0 \ 0 \ 0.146 \ 0.626]$ (from time $k = 17$ onwards) but that a robust re-design based on choosing $F = K$ is impossible, since it leads to an empty set $\hat{\mathbb{X}}$ which violates **C2**⁵. Hence, it is impossible to use $u_k = -K\hat{x}_k$ as the candidate policy and, for example, choose a cost function of the form (3) to ensure robust stability and convergence to $u_k = -K\hat{x}_k$.

6 Conclusions

We have proposed a robust mixed objective MPC design for constrained discrete-time systems with unstructured model uncertainty. The proposed design incorporates a novel closed-loop stability test and minimizes a quadratic upper bound on a nominal cost function at each time step. The parameters of the cost function are determined off-line by exploiting a novel convex parameterization of a class of cost functions that all lead to a sufficiently small closed-loop ℓ_2 -gain. We have shown, by means of both theoretical and numerical results, that this constructive approach to robust output-feedback MPC often leads to nonconservative results.

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⁵ Another instance of Algorithm 5.2 which converges to $u_k = -K\hat{x}_k$, $K \approx [0.001 \ -0.007 \ 0.363 \ 0.921]$ was designed in Løvaas et al. (2007b) using a different choice of $\theta(P)$.

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A Proofs of Theorems 4.1 and 4.2

The “only if-part” of Theorem 4.2 follows from the bounded real lemma (de Souza and Xie, 1992) and the relation $C_a(zI - A_a)^{-1}B_{a,w} = \tilde{C}_z(zI - \tilde{A})^{-1}\tilde{B}_w$, whereas the “if-part” follows directly from Theorem 4.1. Next we prove both cases of Theorem 4.1 using a scalar $\alpha > 0$ and an associated matrix $X_\alpha > 0$ as our “building blocks”. We choose our $\alpha > 0$ as any sufficiently large scalar satisfying $\|C_a(zI - A_a)^{-1}B_\alpha + D_\alpha\|_\infty < 1$, where $B_\alpha \triangleq [\frac{1}{\alpha}B_{a,u} \ B_{a,w}]$, $B_{a,u} = [0, B]$ and $D_\alpha \triangleq [\frac{1}{\alpha}D_z \ 0]$. Note that such an $\alpha > 0$ always exists whenever **C1** holds. We choose the associated matrix $X_\alpha > 0$ to be

any symmetric matrix satisfying

$$\begin{bmatrix} A_a \begin{bmatrix} B_{a,u} & B_{a,w} \end{bmatrix} \\ C_a \begin{bmatrix} D_z & 0 \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} X_\alpha & 0 \\ 0 & I_{n_z} \end{bmatrix} \begin{bmatrix} A_a \begin{bmatrix} B_{a,u} & B_{a,w} \end{bmatrix} \\ C_a \begin{bmatrix} D_z & 0 \end{bmatrix} \end{bmatrix} - \text{diag}[X_\alpha, \alpha^2 I_{n_u}, I_{n_w}] < 0. \quad (\text{A.1})$$

Note that such an $X_\alpha > 0$ exists by the bounded real lemma (de Souza and Xie, 1992).

Proof of (i): Fix the variables Ω_0, m_1 in Condition 3.1 as follows: $\Omega_0 = m_1 X_\alpha - \text{diag}[0, P_{11}]$, $m_1 = \frac{\lambda_{\min}(\Psi)}{\alpha^2}$, where $\lambda_{\min}(\Psi)$ denotes the smallest eigenvalue of Ψ . Condition 3.1 then holds with P as in (25) for some sufficiently small value for the remaining variable m_0 . This claim may be verified using (10), (11) and (A.1).

Proof of (ii): Let us introduce the notation $P = P(P_0, Q_0)$ to denote the matrix in (26). Also, as a preliminary step to proving the second statement for any given $P_0 \geq 0$, let us first consider the case when $P_0 = 0$ and prove the stronger statement that we can find a Q_0 such that Condition 3.1 holds with a strict inequality $\Phi(P(0, Q_0), \Omega_0, m_1) < 0$. This goal can be accomplished by choosing $m_0 > 0$ sufficiently small, $m_1 = 1$, $\Omega_0 = X_\alpha$ and $Q_0 = Q_\alpha$, where Q_α is any matrix such that $Q_\alpha > \alpha^2 D^T D$, leading to the desired strict inequality

$$\Phi(P(0, Q_\alpha), X_\alpha, 1) < 0, \quad (\text{A.2})$$

which can be verified using (10), (11) and (A.1). To prove the second statement for any non-zero $P_0 \geq 0$, it suffices to choose: $m_0 > 0$ sufficiently small, $m_1 > 0$ sufficiently large and

$$\Omega_0 = m_1 X_\alpha, \quad Q_0 = m_1 Q_\alpha. \quad (\text{A.3})$$

To see this, note that inequality (A.2) is strict and that, using the parameterization (A.3), we have $\Phi(P(P_0, Q_0), \Omega_0, m_1) = m_1 \Phi(P(0, Q_\alpha), X_\alpha, 1) + \Phi(P_0, 0, 0)$, where we have exploited the linearity of $\Phi(\cdot)$ and $P(\cdot)$.