On the Batchelor constant in decaying isotropic turbulence

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Direct numerical simulations of a passive scalar mixed in decaying homogeneous isotropic turbulence indicate that, when the Schmidt number is close to or exceeds unity, the Batchelor constant is about 3. This is smaller by nearly a factor of 2 than that inferred from stationary turbulence simulations. In the present case, Batchelor's spectrum approximates the present data well in the viscous-convective range while Kraichnan's spectrum provides a very good fit at large wavenumbers. © 2003 American Institute of Physics. [DOI: 10.1063/1.1577346]

A measure of mixing of a scalar by a turbulent flow is provided by the scalar spectrum. When the Prandtl number Pr ($\equiv \nu/\kappa$, ν and κ are the kinematic viscosity and thermal diffusivity, respectively) or, more generally, the Schmidt number Sc ($\equiv \nu/D$, D is the scalar diffusivity) is approximately unity, and the Reynolds number is sufficiently large, the three-dimensional scalar spectrum $G(k)$ exhibits a $k^{-5/3}$ behavior^{1,2} in the inertial-convective subrange ($L^{-1} \leq k \leq \eta^{-1}$; L is the integral length scale and $\eta \equiv \nu^{3/4}/\langle \epsilon \rangle^{1/4}$ is the Kolmogorov microscale)

$$G(k) = \beta_{ic} \langle \chi \rangle \langle \epsilon \rangle^{-1/3} k^{-5/3}, \quad (1)$$

where $\langle \epsilon \rangle$ and $\langle \chi \rangle$ are the mean energy and scalar dissipation rates, respectively, and β_{ic} is the Obukhov–Corrsin constant. $G(k)$ is defined so that $\int_0^\infty G(k) dk = \langle \theta^2 \rangle$ (θ is the scalar fluctuation; the angular brackets will denote averaging with respect to space for simulations and with respect to time for measurements). For $Sc \gg 1$, Batchelor³ argued that the rate of strain $\gamma \equiv (\langle \epsilon \rangle / \nu)^{1/2}$ should be approximately uniform over regions of space which are small compared to η and should therefore be the relevant dimensional parameter for small-scale mixing. Dimensional arguments can then be used to yield

$$G(k) = \beta_{vc} \langle \chi \rangle \gamma^{-1} k^{-1} \quad (2)$$

in the viscous-convective subrange [$\eta^{-1} < k < \eta_B^{-1}$; $\eta_B \equiv (D/\gamma)^{1/2} = \eta Sc^{-1/2}$ is the Batchelor length scale]; β_{vc} is the Batchelor constant (the symbol “ q ” has also been used). Using a Fourier model for a weakly diffusive scalar convected in isotropic turbulence, Batchelor derived an expression for $G(k)$ over the viscous-convective and viscous-diffusive subranges, viz.

$$G(k) = \beta_{vc} \langle \chi \rangle \gamma^{-1} k^{-1} \exp\left(-\beta_{vc} k^2 \frac{D}{\gamma}\right). \quad (3)$$

Gibson⁴ showed that non-Fourier mechanisms which involve γ continue to be important even when the wavelength of the smallest scalar fluctuation exceeds the scale at which uniform straining occurs. Clearly, Gibson's mixing mechanism

allows Batchelor's requirement $Sc \gg 1$ to be relaxed to the milder inequality $Sc \gtrsim 1$. Experimental data in round jets^{5,6} and DNS data for homogeneous isotropic turbulence in a periodic box (e.g., Refs. 7–10) have confirmed the importance of the compressive strain rate in forming scalar sheets, even when Sc is smaller than unity. In particular, the DNS data, e.g., Refs. 11–14, demonstrated that, for $Sc \gtrsim 1$, distributions of $G(k)$ at sufficiently large k collapse when normalized by the scales $\theta_B \equiv (\langle \chi \rangle / \gamma)^{1/2}$ and η_B . Using a dagger “ $\dagger$ ” to denote this type of normalization, (3) can be rewritten as

$$G^\dagger(k^\dagger) = \beta_{vc} k^{\dagger-1} \exp(-\beta_{vc} k^{\dagger 2}). \quad (4)$$

Kraichnan^{15,16} found that the k^{-1} behavior was unaffected by fluctuations in the rate of strain rate but the Gaussian cut-off in (4) was replaced by a linear exponential cut-off, viz.

$$G^\dagger(k^\dagger) = \beta_{vc} k^{\dagger-1} [1 + (6\beta_{vc})^{1/2} k^\dagger] \exp[-(6\beta_{vc})^{1/2} k^\dagger]. \quad (5)$$

The DNS data also support an approach to a k^{-1} behavior, even when Sc just exceeds unity. Bogucki *et al.*¹² and Brethouwer and Nieuwstadt¹³ found that $\beta_{vc} \approx 5.2$ and that (5) represented their data significantly better than (4). The latter value is appreciably larger than Batchelor's³ estimate ($\beta_{vc} \approx 2$) which was supported by the decaying grid turbulence measurements of Gibson and Schwarz¹⁷ and Clay.¹⁸

For the simulations of Bogucki *et al.*,¹² forcing was applied at low wavenumbers to both velocity and scalar fields. For those of Brethouwer and Nieuwstadt,¹³ a mean scalar gradient was imposed for almost every value of Sc except $Sc = 144$, for which simulations were run with and without the scalar gradient. The present simulations were carried out in decaying turbulence partly to provide a more meaningful comparison with available grid turbulence results but mainly to test existing theories for the similarity of the energy and scalar spectra for decaying homogeneous isotropic turbulence. The latter objective will be pursued in another paper.

FIG. 1. Product $k^\dagger G^\dagger(k^\dagger)$ for different values of Sc (— — —), $Sc=0.7$; (— — —) 1; (---) 3; (---) 7; (—) 15. For $Sc=15$, $t=60$ was used, in all other cases, $t=40$. The horizontal line has a value of 2.65. The arrow is in the direction of increasing Sc .

Here, we focus on the k^{-1} subrange with a view to throw some light on the above-mentioned discrepancy in the value of β_{vc} .

Results presented in this note were obtained with an energy conservative finite difference code and a 240^3 grid (a cubic box of size 2π was used). The simulation is second-order in space and time, with staggered velocities and with the passive scalar at the same location as that for u_3 , the velocity fluctuation in the x_3 direction. The scalar can be located at any other point in the grid without affecting the results. At $t=0$, the 3-D energy spectrum $E(k)$, defined so that $\int_0^\infty E(k)dk = \langle q^2 \rangle / 2$ ($\langle q^2 \rangle$ is the mean turbulent energy) was assumed to be given by

$$E(k,0) = \frac{\langle q^2 \rangle}{2A} \frac{1}{k_p^{\sigma+1} k^\sigma} \exp \left[-\frac{\sigma}{2} \left(\frac{k}{k_p} \right)^2 \right],$$

where k_p is the wavenumber at which $E(k,0)$ is maximum, $A = \int_0^\infty k^\sigma \exp(-\sigma k^2/2) dk$ and the parameter σ was set equal to 1.5. At $t=0$, the Taylor microscale Reynolds number $R_\lambda \equiv (\langle q^2 \rangle / 3)^{1/2} \lambda / \nu$ (the Taylor microscale λ is defined by $\lambda = (5\nu \langle q^2 \rangle / \langle \epsilon \rangle)^{1/2}$) was 850. Between $t=10$ and $t=60$, R_λ decreased from 44 to 38. For the scalar, the simulation was initiated with a random phase spectrum of the same shape as $E(k,0)$. The same scalar spectrum at $t=0$ was used for each of the five values of Sc (0.7, 1, 3, 7, 15), for which results are presented here. For $Sc=1$, the grid spacing decreased from $0.4\eta_B$ at $t=10$ to about $0.1\eta_B$ at $t=60$; for $Sc=15$, the corresponding variation was between $1.4\eta_B$ and $0.4\eta_B$. For $t > 10$, both $\langle q^2 \rangle$ and $\langle \theta^2 \rangle$ exhibited power-law decay behaviors, albeit with different power-law exponents (≈ -1.1 for $\langle q^2 \rangle$ and -1.4 for $\langle \theta^2 \rangle$). Also, the power-law range for $\langle \theta^2 \rangle$ was established at increasingly larger values of t as Sc increased.

The present distributions of $k^\dagger G^\dagger(k^\dagger)$ (Fig. 1) exhibit a modest plateau, consistent with a $k^{\dagger-1}$ behavior for $G^\dagger(k^\dagger)$. The corresponding value of β_{vc} is about 2.65 (the horizontal line in Fig. 1). For $k^\dagger \gtrsim 0.1$, all distributions are in close agreement with each other. The collapse is of comparable quality to that previously reported by Refs. 7, 12, and 13 for $Sc \gtrsim 1$ when θ_B and η_B are used to normalize the spectra. The upper end of the plateau lies below η^{-1} , which is consistent

FIG. 2. Product $k^\dagger G^\dagger(k^\dagger)$ for $Sc=7$. (— — —) DNS data ($t=40$), (---) fit using Eq. (4); (---) fit using Eq. (5).

with the observation of Bogucki *et al.*¹² and underlines the effectiveness of the strain rate when $k < \eta^{-1}$.

Another estimate of β_{vc} may be obtained by fitting Eqs. (4) and (5) to the data for all values of $k^\dagger$ greater than 0.1. This range encompasses both viscous-convective and viscous-diffusive subranges. An example is given in Fig. 2 for $Sc=7$ (similar results apply for other values of Sc). A least squares regression of Eq. (4) to the data for $k^\dagger G^\dagger(k^\dagger)$ yields $\beta_{vc} = 2.89$ whereas the regression, based on Eq. (5), yields $\beta_{vc} \approx 3.41$. The former value is close to that (≈ 2.65) inferred from Fig. 1. A measure of the quality of the fit is provided by the coefficient of determination r^2 , here equal to 0.994 for Eq. (4) and 0.990 for Eq. (5). Although the difference in r^2 is small, Eq. (4) represents the approach to the viscous-convective range ($k^\dagger \lesssim 0.2$) better than Eq. (5). When the data are plotted in the form $k^{\dagger 2} G^\dagger(k^\dagger)$ vs $k^\dagger$ (Fig. 3), Eqs. (4) and (5) (shown as symbols for clarity) provide reasonable representations over the wavenumber range covered in the figure. The magnitude of the peak in the scalar dissipation (which occurs at $k^\dagger \approx 0.4$) is overestimated by Eq. (4) and underestimated by Eq. (5). Although Eq. (4) follows the $Sc=15$ distribution, the latter is not as reliable as that for other values of Sc , because the smallest scales have not been fully resolved (the constraint that the integral under each curve is equal to 0.5 has most likely contributed to the slight distortion in the $Sc=15$ distribution) even at the largest value of t . When the data are plotted in the form $\ln G^\dagger(k^\dagger)$ vs $k^\dagger$ (Fig. 4), it is clear that the Gaussian form of Eq. (3) is inad-

FIG. 3. Batchelor normalized scalar dissipation spectra. (---) $Sc=0.7$; (— — —) 1; (---) 3; (---) 7; (—) 15. (○) Equation (4) with $\beta_{vc}=2.9$; (□) Eq. (5) with $\beta_{vc}=3.4$.

FIG. 4. Distributions of $\ln G^+(k^+)$ vs k^+ . Line types are the same as in Fig. 3; (—) Eq. (4) with $\beta_{vc}=2.9$; (---) Eq. (5) with $\beta_{vc}=4.5$.

equate beyond $k^+ \approx 1$. Equation (5), with $\beta_{vc}=4.5$, represents all the data satisfactorily up to $k^+ \approx 4$. As an aside, the exponentially linear form of Eq. (5) may provide a plausible way of extrapolating the high wavenumber of the spectrum when the smallest scales are not resolved.

The present magnitude of β_{vc} (Fig. 1) is slightly larger than that (≈ 2.2) reported in decaying grid turbulence^{17,18} for $Sc=7$. It is however significantly smaller than the value (≈ 5.3) obtained in forced simulations^{12,13} (see Ref. 2 for a tabulation of values of β_{vc} obtained by different investigators). Brethouwer and Nieuwstadt¹³ presented results at $Sc=144$ for both forced and decaying simulations. The distributions of $k^{+2}G^+(k^+)$ (Fig. 3 of their paper) exhibited significant differences below $k^+ \approx 0.2$, the rate of increase of $G^+(k^+)$ with k^+ being appreciably larger for the “decaying” than “forced” case (the “decaying” and “stationary” spectra of Wang *et al.*¹⁹ for $Sc \approx 1$ exhibit a similar relative behavior). By applying least squares regressions to the data of Ref. 13, we have found that in the forced case, Eq. (5) provides a better fit ($r^2=0.999$ and $\beta_{vc}=5.92$) to the data than Eq. (4) ($r^2=0.990$ and $\beta_{vc}=5.39$) in the range $k^+ \gtrsim 0.07$. For the decaying case, Eq. (4) ($r^2=0.974$, $\beta_{vc}=3.61$) provides a better overall fit to the data than Eq. (5) ($r^2=0.927$, $\beta_{vc} \approx 4.02$).

Although Eq. (5), which takes into account the effect of a fluctuating rate of strain, represents the high wavenumber part of the dissipation spectrum equally well for both “forced” and “decaying” simulations, one may speculate that the smaller value of β_{vc} in the decaying case reflects a more perfect alignment between the scalar gradient and the compressive strain rate than in the forced case (Batchelor’s estimate of 2 for β_{vc} implies a perfect alignment). This speculation may be worth investigating in future, possibly via the effective strain rate concept introduced by Smyth.²⁰

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