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- 1) We show that a Helmholtz resonator with symmetry and a single mode of propagation leads to extraordinary acoustic transmission in a wave guide.
- 2) We show how the solution from mode matching can be extended to complex frequencies analytically.
- 3) We show how the behaviour in the complex plane controls the solution for real frequencies and how it changes with geometry.
- 4) We show that symmetry plays a critical role in extraordinary transmission.

Extraordinary Acoustic Transmission, Symmetry, Blaschke Products and Resonators

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Abstract

The phenomenon of extraordinary acoustic transmission (EAT) in a resonator, which has recently been investigated experimentally, is studied theoretically. It is shown that the combination of a single propagating mode and a symmetry orthogonal to the direction of propagation for a resonator leads to EAT. This is accomplished by decomposing the problem using symmetry, the Blaschke product and the properties of functions of a single complex variable which have modulus one on the real axis. The conditions of a resonator requires that the solution has singularities in the analytic extension to complex frequencies (resonances) and it is precisely near these resonances that we observe EAT. The condition of a Blaschke product requires that there is a zero at the complex conjugate of the singularity and EAT occurs when the solution on the real axis passes between these complex conjugate pairs of poles and zeros. A detailed numerical study of the problem is conducted and we show that once the single mode of propagation or the symmetry is broken then EAT (at least perfect transmission) no longer holds generally.

Keywords: Waveguide; Resonance; Extraordinary Transmission; Analytic extension.

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1. Introduction

Extraordinary transmission is a phenomena in which anomalous transmission occurs, sometimes perfect transmission, for a geometry which should be strongly reflecting. It occurs only for specific frequencies and it is associated with resonances in the response. Extraordinary optical transmission (EOT) was discovered first [1] and has been the subject of significant study, e.g. [2, 3]. Recently the phenomena of extraordinary acoustic transmission (EAT) has been investigated and, unsurprisingly given the similar nature of the equations which govern electromagnetic waves and acoustic waves, many examples of EAT have been found, for example [4, 5, 6, 7, 8]. Extraordinary transmission is associated with Fabry-Pérot resonances [9, 10, 11, 12]. and Helmholtz resonators [13, 14]. These experiments are the starting point for our current investigation. It seems clear that the phenomena of EAT is closely associate with resonators. We think of a resonator as a perturbed self-adjoint operator in which, through the connection with infinity, the real eigenvalue of the self-adjoint operator has become a complex resonance, that is a singularity of the analytic extension of the resolvent, with small imaginary part. Such methods have found wide spread application in many areas of wave scattering and they are known as singularity expansion methods [15, 2].

The idea of computing the analytic extension to understand wave characteristics for real frequencies has recently been applied to investigate absorption by sub-wavelength resonances [16, 17, 18]. In this work the solution was visualized in the complex frequency plane and the position of the singularities and zeros examined. We perform here a similar analysis. In the case considered by [16, 17, 18] the analytic extension was more straightforward because a single mode approximation was made. For our problem we use bespoke computer code which allows the construction of the analytic extension of the solution to complex wavenumbers or complex frequencies with out making a single mode approximation. This analytic extension is actually one of the major challenges

of using this method of analysis and we will discuss in detail how we accomplish this for our current problem. To develop our solution we use the eigenmode matching method [19], which will restrict the geometries but is simple to code and numerically efficient. It also allow consideration of problem close to those reported by [13, 14]. The eigenmode matching method is widely used in electromagnetism and acoustics to analyze waveguides. It can be used to solve surprisingly complex problems for example it has been applied to the problem of scattering by trifurcated and penta-furcated ducts [20], to scattering by floating elastic plates [21], submerged elastic plate [22] and even to predict wave scattering in the marginal ice zone [23].

The outline of the current work is as follows. In §2 we consider the simplest case which consists of a waveguide with hard walls and a finite inner duct symmetrically (along y -axis) located within an infinite duct. We solve this problem using mode matching exploiting symmetry to decompose the solution. We explore in detail the consequences of this decomposition and show how this leads to EAT. In §3 we consider a similar problem where the cavity is not symmetrically placed in the waveguide. We show in this case the EAT exists but that the cut-off frequency is halved. Significant numerical results are given for both cases in §2 and §3 including movies which are included as supplementary material. These movies show the analytic extension of the solution for complex wavenumber which is key to our analysis. §4 consists of a more complicated problem in which there is no longer symmetry and we give brief results which show that the lack of symmetry destroys EAT. §5 is a short summary.

2. Resonator 1: Symmetry in the x and y directions

The first problem consists of a resonator with symmetry about both the x and y direction (see Figure 1). This is the simplest case of a resonator according to our definition of complex resonances with small imaginary part. It is similar to the experimental cases considered by [13, 14], although they had a narrow neck bounding the two regions and their problem was three-dimensional.

Figure 1: Schematic diagram of the waveguide for Resonator 1.

However, the symmetry properties of both problems are identical. The problem
 60 consists of a finite inner duct symmetrically located in an infinite duct with
 hard boundary conditions on all boundaries. We will show shortly that we can
 decompose the problem into a symmetric and an anti-symmetric problem. The
 solution to the original problem is then found by adding these two problems
 together using the superposition principle.

We begin with introduction of the scalar potential function $\varphi(x, y, t) = \text{Re}[\phi(x, y)e^{-i\omega t}]$, in the wave equation which governs the propagation of sound in the waveguide which gives

$$\nabla^2 \phi(x, y) + k^2 \phi(x, y) = 0, \quad (1)$$

which is the two dimensional Helmholtz equation to be solved for $\phi(x, y)$. Where the wave number $k = \omega/c$, c is the speed of sound and ω is angular frequency. We define acoustic pressure by $p = -\rho_0 \frac{\partial \varphi}{\partial t}$ and velocity vector by $\mathbf{u} = \mathbf{grad}(\varphi)(\rho_0$ indicates the density of equilibrium state). The units are non-dimensional. We will solve (1) subject to the boundary conditions

$$\phi_y = 0, \quad y = b, \quad -\infty < x < \infty, \quad (2a)$$

$$\phi_y = 0, \quad y = a, \quad -l < x < l, \quad (2b)$$

$$\phi_y = 0, \quad y = -a, \quad -l < x < l, \quad (2c)$$

$$\phi_y = 0, \quad y = -b, \quad -\infty < x < \infty, \quad (2d)$$

$$\phi_x = 0, \quad x = \pm l, \quad -b < y < -a, \quad (2e)$$

$$\phi_x = 0, \quad x = \pm l, \quad a < y < b. \quad (2f)$$

65 In addition, we have radiation conditions which insure the solution is bounded and that the scattered waves are outgoing.

2.1. Decomposition using Symmetry

The problem is symmetric about $x = 0$ and this means that we can decompose the problem into one which is symmetric and one which is anti-symmetric about $x = 0$. We will show that this decomposition has important consequences for EAT. We solve the problem using bespoke code, because we are interested in computing the solution for complex frequencies, which is not possible with most numerical codes. We use mode matching, and we only need to consider the solution for $x < 0$ (the solution for $x > 0$ following from appropriate symmetry). This problem has a further symmetry about $y = 0$ which means that if the incident mode is even about $y = 0$ (which is the case for the fundamental mode) then only even modes will be excited.

2.1.1. Region 1 $\{-b \leq y \leq b, -\infty < x < -l\}$

80 We break the solution into the symmetric and anti-symmetric solutions, which we denote by $\phi^{(s)}(x, y)$ and $\phi^{(a)}(x, y)$ respectively. The solution is written only for $x < 0$ and it is extended to $x > 0$ using the appropriate symmetry. We divide the solution into two regions, region 1 where $\{-b \leq y \leq b, -\infty < x < -l\}$

and region 2 where $\{-a \leq y \leq a, -l < x < 0\}$. In each of these regions we expand the solution using mode matching. We also impose the appropriate
 85 condition from the symmetry at $x = 0$.

For $x < -l$ the solution can be written as

$$\phi^{(s/a)}(x, y) = \sum_{n=0}^{\infty} A_n^{(s/a)} e^{-i\bar{\alpha}_{2n}(x+l)} \psi_{2n}(y) + e^{i\bar{\alpha}_0(x+l)} \psi_0(y), \quad (3)$$

which satisfies equations (1), (2a), (2d) and the radiation condition. We define eigenvalues α_n and associated eigenvalues $\bar{\alpha}_n$ by

$$\alpha_n = \frac{n\pi}{2b}, \quad n = 0, 1, 2, \dots \quad (4)$$

and

$$\bar{\alpha}_n = \sqrt{(k^2 - \alpha_n^2)}, \quad n = 0, 1, 2, \dots \quad (5)$$

respectively. Note that we retain the odd modes here because we will use them subsequently. The sign of the square root in equation (5) is chosen to be positive real or to have positive imaginary part so that the solution (3) satisfies the radiation conditions.

The eigenfunctions $\psi_n(y)$ are defined by

$$\psi_n(y) = \begin{cases} \sqrt{\frac{1}{b}} \cos \alpha_n(y - b), & n \neq 0, \\ \sqrt{\frac{1}{2b}}, & n = 0, \end{cases} \quad (6)$$

which are orthonormal, i.e.

$$\int_{-b}^b \psi_m(y) \psi_n(y) dy = \delta_{mn}. \quad (7)$$

90 2.1.2. Region 2 $\{-a \leq y \leq a, -l < x < 0\}$

The solution for $-l < x < 0$ is very similar. The vertical eigenfunctions are given by

$$\xi_n(y) = \begin{cases} \sqrt{\frac{1}{a}} \cos \beta_n(y - a), & n \neq 0, \\ \sqrt{\frac{1}{2a}}, & n = 0 \end{cases} \quad (8)$$

The eigenvalues and the associated eigenfunctions are given by

$$\beta_n = \frac{n\pi}{2a},$$

and

$$\bar{\beta}_n = \sqrt{k^2 - \beta_n^2}, \quad n = 0, 1, 2, \dots$$

They satisfy the relation

$$\int_{-a}^a \xi_n(y) \xi_m(y) dy = \delta_{mn}. \quad (9)$$

The general solution for the potential in the region 2 can be written as

$$\phi^{(s)}(x, y) = \sum_{n=0}^{\infty} B_n^{(s)} \frac{\cos \bar{\beta}_{2n} x}{\cos \bar{\beta}_{2n} l} \xi_{2n}(y) \quad (10)$$

or

$$\phi^{(a)}(x, y) = \sum_{n=0}^{\infty} B_n^{(a)} \frac{\sin \bar{\beta}_{2n} x}{-\sin \bar{\beta}_{2n} l} \xi_{2n}(y) \quad (11)$$

which satisfies equations (1), (2b) and (2c) at the appropriate condition of symmetry or antisymmetry respectively.

The solution for an incident wave traveling from the left is found by combining the solutions and is given by

$$\phi = \begin{cases} \frac{1}{2} (\phi^{(s)}(x, y) + \phi^{(a)}(x, y)), & x < 0, \\ \frac{1}{2} (\phi^{(s)}(-x, y) - \phi^{(a)}(-x, y)), & x > 0. \end{cases} \quad (12)$$

2.2. Formulation of the System of Equations

We derive the system of equations which arises from mode matching. The presentation focuses on the symmetric case which we consider first. The continuity of the pressure across $x = -l$ gives

$$\sum_{n=0}^{\infty} A_n^{(s)} \psi_{2n}(y) + \psi_0(y) = \sum_{n=0}^{\infty} B_n^{(s)} \xi_{2n}(y), \quad -a \leq y \leq a, \quad (13)$$

taking the inner product with $\xi_{2m}(y)$, integrating over $[-a, a]$ we obtain

$$\sum_{n=0}^{\infty} A_n^{(s)} \int_{-a}^a \xi_{2m}(y) \psi_{2n}(y) dy + \int_{-a}^a \xi_{2m}(y) \psi_0(y) dy = \sum_{n=0}^{\infty} B_n^{(s)} \int_{-a}^a \xi_{2n}(y) \xi_{2m}(y) dy. \quad (14)$$

Using equation (9), this can be written as

$$\sum_{n=0}^{\infty} l_{mn} A_n^{(s)} + l_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, \dots \quad (15)$$

where

$$l_{mn} = \int_{-a}^a \psi_{2n}(y) \xi_{2m}(y) dy, \quad (16)$$

$$= \begin{cases} \sqrt{\frac{a}{b}}, & m = n = 0, \\ \frac{[\sin \alpha_{2n}(a-b) + \sin \alpha_{2n}(a+b)]}{\sqrt{2ab}\alpha_{2n}}, & m = 0, n \neq 0, \\ 0, & n = 0, m \neq 0, \\ \frac{\sin \alpha_{2n}(a-b) + \sin \alpha_{2n}(3a+b) + 4a\alpha_{2n} \cos \alpha_{2n}(b-a)}{4\sqrt{ab}\alpha_{2n}}, & \alpha_{2n} = \beta_{2m}, \\ \frac{\alpha_{2n}[\sin \alpha_{2n}(a-b) + \sin \alpha_{2n}(a+b) \cos 2\alpha_{2n}a]}{\sqrt{ab}(\alpha_{2n}^2 - \beta_{2m}^2)}, & \text{other cases.} \end{cases} \quad (17)$$

The continuity of the velocity of the potential across $x = -l$ gives the following equation

$$-\sum_{n=0}^{\infty} iA_n^{(s)} \bar{\alpha}_{2n} \psi_{2n}(y) + i\bar{\alpha}_0 \psi_0(y) = \begin{cases} 0, & -b \leq y \leq -a, \\ \sum_{n=0}^{\infty} B_n^{(s)} \xi_{2n}(y) \bar{\beta}_{2n} \tan \bar{\beta}_{2n} l, & -a \leq y \leq a, \\ 0, & a \leq y \leq b. \end{cases}$$

Taking the inner product with $\psi_{2m}(y)$ and integrating over $[-b, -b]$, we obtain

$$-\sum_{n=0}^{\infty} i\bar{\alpha}_{2n} A_n^{(s)} \int_{-b}^b \psi_{2m}(y) \psi_{2n}(y) dy + i\bar{\alpha}_0 \int_{-b}^b \psi_{2m}(y) \psi_0(y) dy = \sum_{n=0}^{\infty} B_n^{(s)} \bar{\beta}_{2n} \tan \bar{\beta}_{2n} l \int_{-a}^a \xi_{2n}(y) \psi_{2m}(y) dy. \quad (18)$$

By using equation (7) and (16) the above equation becomes

$$-i\bar{\alpha}_{2m} A_m^{(s)} + i\bar{\alpha}_0 \delta_{m0} = \sum_{n=0}^{\infty} B_n^{(s)} l_{nm} \bar{\beta}_{2n} \tan \bar{\beta}_{2n} l, \quad m = 0, 1, \dots \quad (19)$$

We therefore have the following system of equations to be solved for the expansion constants

$$\sum_{n=0}^{\infty} A_n^{(s)} l_{mn} + l_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, \dots \quad (20)$$

$$-i\bar{\alpha}_{2m}A_m^{(s)} + i\bar{\alpha}_0\delta_{m0} = \sum_{n=0}^{\infty} B_n^{(s)} l_{nm} \bar{\beta}_{2n} \tan \bar{\beta}_{2n} l, \quad m = 0, 1, 2, \dots \quad (21)$$

The derivation for the antisymmetric solution is almost identical and we are lead to the following system of equations

$$\sum_{n=0}^{\infty} A_n^{(a)} l_{mn} + l_{m0} = B_m^{(a)}, \quad m = 0, 1, 2, \dots \quad (22)$$

and

$$-i\bar{\alpha}_{2m}A_m^{(a)} + i\bar{\alpha}_0\delta_{m0} = -\sum_{n=0}^{\infty} B_n^{(a)} l_{nm} \bar{\beta}_{2n} \cot \bar{\beta}_{2n} l, \quad m = 0, 1, 2, \dots \quad (23)$$

Note that the symmetric and antisymmetric solutions are almost identical so
 95 that only the smallest change to the numerical code for the symmetric problem is required to solve the antisymmetric problem.

To solve these equations numerically, we restrict ourselves to a finite number of modes. We do not need to have the same number of modes on each side and the finite system of equations is, for the symmetric case with M modes for $x < -l$ and N modes for $-l < x < 0$,

$$\sum_{n=0}^M A_n^{(s)} l_{mn} + l_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, \dots, N, \quad (24)$$

and

$$-i\bar{\alpha}_{2m}A_m^{(s)} + i\bar{\alpha}_0\delta_{m0} = \sum_{n=0}^N B_n^{(s)} l_{nm} \bar{\beta}_{2n} \tan \bar{\beta}_{2n} l, \quad m = 0, 1, 2, \dots, M. \quad (25)$$

We validate our code by ensuring that the solution satisfies the matching conditions as well as energy conservation.

2.3. Reflection and Transmission

We are interested here primarily in the far field reflection and transmission for a wave incident from the left. From equation (12) the solution to the problem of a wave of unit amplitude incident from the left propagating through the

resonator is given by averaging the symmetric and antisymmetric solutions. The far field reflection for the first mode is

$$R_0(k) = \frac{A_0^{(s)} + A_0^{(a)}}{2}, \quad (26)$$

and the far field transmission for the first mode is

$$T_0(k) = \frac{A_0^{(s)} - A_0^{(a)}}{2}. \quad (27)$$

Conservation of energy tells us that, provided there is only one propagating mode, which for our current example due to the symmetry requires that $k < \pi/b$

$$|A_0^{(s)}(k)| = |A_0^{(a)}(k)| = 1, \quad k < \frac{\pi}{b}. \quad (28)$$

100 There are many other energy balance relations which can be derived and further examples can be found in [24]

The absolute value of the reflection coefficient $|R_0|$ versus the truncation number M (with $N = M/2$) and the absolute error are show in Figure 2 for $l = 2, a = 1, b = 2$ and $k = 6$. We can see that the absolute error in the reflection coefficient becomes linear and the absolute value of the reflection becomes
105 accurate to line width after M is ten.

Figures 3 to 6 show the absolute value of the reflection $|R_0|$ and transmission $|T_0|$ versus wavenumber k . The values are $b = 2$ for all Figures and $a = 0.5$, $l = 5$, (Figure 3) $a = 0.5$, $l = 20$, (Figure 4), $a = 0.125$, $l = 5$, (Figure 5) and
110 $a = 0.125$, $l = 20$, (Figure 6). We see a pattern of EAT below $k = \pi/2$. There are regions of enhanced transmission above $k = \pi/2$ but it is nothing like the perfect transmission obtained below $k = \pi/2$.

We now offer an explanation of what is causing the EAT. We obtain perfect transmission when $A_0^{(s)} = -A_0^{(a)}$ (provided that there is only one propagating
115 mode). Since they both have modulus one this requirement is that they are in anti-phase. We now show how the presence of a resonator coupled with this symmetry gives rise to exactly this anti-phase relation.

Figure 7 shows the $\arg(A_0^{(s)}(k))$ and $\arg(A_0^{(a)}(k))$ for $b = 2$ and $l = 5$ for two values of a , $a = 0.125$ and $a = 0.5$. Note that we have plotted the argument

Figure 2: The convergence of the reflection coefficient R_0 as a function of M . Subfigure a shows the absolute value of the reflection coefficient and subfigure b shows the absolute error. The parameters are $l = 2, a = 1, b = 2$ and $k = 6$.

as a continuous function. We notice abrupt changes of phase by 2π at certain values. Exactly this phase change has been observed to be associated with EAT experimentally [13, 14]. We will see shortly that mathematically the changes in phase are associated with both resonances and perfect transmission. We also note that above a frequency of $\pi/2$ we no longer have this abrupt change of phase. Figure 8 is the same plot but with $l = 20$.

2.4. Solution for complex k

So far, while we have solved for the configuration, we have not considered the condition of a resonator. There are many ways of thinking about what creates a resonator but the key idea is that there is a mode of vibration which leaks a small amount of energy into the surrounding system. If there was no leaking

Figure 3: Reflection and Transmission against wavenumber k for $l = 5$, $a = 0.5$, $b = 2$ for frequency range $0 < k < \pi$

of energy the system would be self-adjoint and there would be an eigenvalue on the real axis. This means that the solution is not invertible at this point which appears as a singularity in the resolvent which is on the real axis. When there is leaking of energy this real eigenvalue becomes a complex resonance. In some
135 very real sense the singularity cannot be created or destroyed. It simply moves into the complex plane. If the system is close to resonant the singularity will remain close to the real axis and this is precisely what gives rise to a resonator. We will see shortly that the singularities move around the complex plane but they are never created or destroyed (although they can appear from a Riemann
140 surface).

We need to develop a method to find this complex resonance and we do this by extending the functions $|A_n^{(s)}(k)|$ and $|A_n^{(a)}(k)|$ to complex k values analytically.

Figure 4: As in Figure 3 except $l = 20$.

When we look at the system of equations we need to calculate the reflection
 145 coefficients we see that the only place where the parameter k appears is in the
 computation of the roots $\bar{\alpha}_n$ and $\bar{\beta}_n$. We therefore just need to compute the
 roots for complex k , the only difficulty being that we need to choose the roots
 consistently with the choice for real k , otherwise the analyticity will be broken.

To achieve the analyticity extension from real k to complex k we find roots
 of $\bar{\alpha}_n$ and $\bar{\beta}_n$ using a homotopy method, which is an iterative method to solve
 nonlinear systems. We define

$$\bar{\alpha}_n(\theta) = \sqrt{|k|^2 e^{2i\theta} - \alpha_n^2}, \quad n = 0, 1, 2, \dots$$

Note that this equation has roots in plus and minus pairs. In our problem for
 150 real k we only choose one from each pair and we need to insure that for complex
 k we make the same choice. We solve this equation first for $\theta = 0$ and choose
 the roots according to the rules for real k . We then slowly vary the angle, using

Figure 5: Reflection and Transmission against wavenumber for $l = 5$, $a = 0.125$, $b = 2$ for frequency range $0 < k < \pi$

the previously computed solutions as the initial guess to solve for the roots. We terminate when $\theta = \arg(k)$. This essentially allows us to track the roots as they move in the complex plane and insures we always have made the correct choice to preserve analyticity. We can validate the analyticity numerically by taking a numerical derivative in different complex directions and checking we have the same result (to numerical error).

It is not easy to visualize a complex function but we do so using the method developed by [25] and the numerical tools which accompany this book. Figure 9 is a frame from Movie 1 which is in the supplementary material. The top picture is $|R_0|$ versus real k . The inset shows the geometry with $a = 0.5$. The bottom picture is a visualization of the analytic extension of R_0 versus complex k . The colour shows the phase information and the height is the shade. Also shown are the positions of the zeros (green) and poles (black).

Figure 6: As in Figure 5 except $l = 20$.

These figures and the movie given in the supplementary material are the key to understanding EAT. We see that there is a branch cut at $k = \pi/2$. Below this cut-off the poles and zeros are complex conjugates of each other. The poles and zeros move around and but they cannot be created or destroyed, although one can be seen appearing from the Riemann surface. Note that the strange behaviour we see in the reflection coefficient at $\pi/2$, sometimes called the Wood anomaly, is explained exactly by this branch cut. The slight flickering seen in the movies we believe is real as it is not caused by numerical convergence.

2.5. Blaschke product

We know that, provided that there is only a single propagating mode (i.e. $k < \pi/2$) $|A_0^{(s)}(k)| = |A_0^{(a)}(k)| = 1$. This means that the poles and zeros of the analytic extension of these function must be at complex conjugate values. Such an expansion must take the form of a product of an exponential of an function

Figure 7: $\arg(A_0^{(s)})$ and $\arg(A_0^{(a)})$ versus wavenumber k for $b = 2$ and $l = 5$. The solid line is for $a = 0.125$, and the dashed line is for $a = 0.5$

Figure 8: As in Figure 7 except $l = 20$.

Figure 9: The top picture is $|R_0|$ versus real k . The inset shows the geometry with $a = 0.5$. The bottom picture is a visualization of the analytic extension of R_0 versus complex k . The colour shows the phase information and the height is the shade. Also shown are the positions of the zeros (green) and poles (black). The full animation can be seen in Movie 1 in the supplementary material

Figure 10: As in Figure 9 except $a = 0.125$.

and a Blaschke product. This allows us to approximate our functions by

$$A_0^{(s)}(k) \approx e^{if(k)} \prod_i \left(\frac{k - k_i}{k - \bar{k}_i} \right), \quad (29)$$

175 where the bar denotes complex conjugate. Note that there are two versions of the Blaschke product, one on the unit circle and one on the real axis, related via a Cayley transform. Here we use the one defined on the unit circle. The conjugation insures that the function has modulus one for real values of k . There is no requirement that the function $f(k)$ have special properties except
180 that it takes real values for real k . However, in practice this function is usually slowly varying and for our example well approximated by $f(k) = 0$. As the function passes between the pole and zero on the real axis there is a change of phase by 2π .

Figures 11 and 12 compare the exact solution and the approximate solution using Blaschke products for $b = 2$, and $L = 5$ with $a = 0.5$ and 0.125 respectively. The absolute value of the reflection coefficient $|R_0|$ is given as a black solid line. Note that this curve corresponds to exactly the same one which has appeared in Figures 3 and 9 (Figure 11) and Figures 5 and 10 (Figure 12). Also shown is the approximation as a sum of Blaschke products using only the poles and zeros with real part below $k = \pi/2$ (red dashed line), for which the poles and zeros are complex conjugate. We also show the approximation using all the poles and zeros (green chained line) written as

$$A_0^{(s)}(k) \approx \prod_i \left(\frac{k - k_i}{k - \kappa_i} \right), \quad (30)$$

where k_i are the zeros and κ_i are the poles of the analytic extensions of the far
185 field waves. Note that for real part greater than $k = \pi/2$ the poles and zero are no longer complex conjugates and that while the conjugacy requires energy conservation the approximation does not. Obviously perfect transmission would be lost without energy conservation. We can see that the Blaschke product with the conjugate poles and zeros very well approximates the exactly solution below
190 $\pi/2$. Including the subsequent poles and zeros actually makes the comparison worse in the case when $a = 0.5$ below $k = \pi/2$ because of the effect of the branch

Figure 11: Absolute value of the reflection coefficient $|R_0|$ as a function of k for $a = 0.5$, $b = 2$, and $L = 5$ (black solid line). Also shown is the approximation as a sum of Blaschke products using only the poles and zeros with real part below $\pi/2$ (red dashed line) and with all the poles and zeros (green chained line).

cut at $k = \pi/2$. However we get good agreement above $k = \pi/2$ especially in the case $a = 0.125$, when including all the poles and zeros.

3. Resonator 2: Symmetry in the x direction only.

195 We change the problem slightly to remove the symmetry about the line $y = 0$ but retain the symmetry about $x = 0$. This still allows the decomposition into symmetric and antisymmetric solutions and leads to perfect transmission as we will see. However, we cannot use the decomposition in even and odd modes in the y direction and this means that the cut-off frequency for perfect transmission
200 is altered. A schematic diagram of this problem is shown in Figure 13.

We solve the Helmholtz equation (1) for $\phi(x, y)$ subject to the boundary

Figure 12: As for Figure 11 except $a = 0.125$

Figure 13: Schematic diagram of the waveguide for the Resonator 2.

conditions

$$\phi_y = 0, \quad y = b, \quad -\infty < x < \infty, \quad (31a)$$

$$\phi_y = 0, \quad y = a, \quad -l < x < l, \quad (31b)$$

$$\phi_y = 0, \quad y = c, \quad -l < x < l, \quad (31c)$$

$$\phi_y = 0, \quad y = -b, \quad -\infty < x < \infty, \quad (31d)$$

$$\phi_x = 0, \quad x = \pm l, \quad -b < y < c, \quad (31e)$$

$$\phi_x = 0, \quad x = \pm l, \quad a < y < b. \quad (31f)$$

In addition, we impose the appropriate radiation conditions.

In the region 1 $\{-b \leq y \leq b, -\infty < x < -l\}$ the solution is similar to the one we found previously. We focus on the symmetric solution as the anti-symmetric solution is almost identical. The symmetric solution is given by

$$\phi^{(s)}(x, y) = \sum_{n=0}^{\infty} A_n^{(s)} e^{-i\bar{\alpha}_n(x+l)} \psi_n(y) + e^{i\bar{\alpha}_0(x+l)} \psi_0(y). \quad (32)$$

We define the vertical orthonormal eigenfunctions in region 2 $\{c \leq y \leq a, -l < x < l\}$ by

$$\hat{\xi}_n(y) = \begin{cases} \sqrt{\frac{2}{a-c}} \cos \hat{\beta}_n(y-c), & n \neq 0, \\ \sqrt{\frac{1}{a-c}}, & n = 0, \end{cases} \quad (33)$$

which satisfy the relation

$$\int_c^a \hat{\xi}_n(y) \hat{\xi}_m(y) dy = \delta_{mn}. \quad (34)$$

Here eigenvalues and the associated eigenvalues are given by

$$\hat{\beta}_n = \frac{n\pi}{a-c},$$

and

$$\bar{\beta}_n = \sqrt{k^2 - \hat{\beta}_n^2}, \quad n = 0, 1, 2, \dots$$

respectively.

As before the general solution for the symmetric potential in the region 2 can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} B_n^{(s)} \frac{\cos \bar{\beta}_n x}{\cos \bar{\beta}_n l} \hat{\xi}_n(y). \quad (35)$$

Following the same procedure as before we derive the following system of equations to be solved for expansion constants,

$$\sum_{n=0}^{\infty} A_n^{(s)} p_{mn} + p_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, \dots \quad (36)$$

$$-i\bar{\alpha}_m A_m^{(s)} + i\bar{\alpha}_0 \delta_{m0} = \sum_{n=0}^{\infty} B_n^{(s)} p_{nm} \bar{\beta}_n \tan \bar{\beta}_n l, \quad m = 0, 1, 2, \dots \quad (37)$$

where

$$p_{mn} = \int_c^a \psi_{2n}(y) \hat{\xi}_m(y) y \quad (38)$$

$$= \begin{cases} \sqrt{\frac{a-c}{2b}}, & m = n = 0, \\ \frac{[\sin \alpha_{2n}(a-b) - \sin \alpha_{2n}(c-b)]}{\sqrt{b(a-c)} \alpha_{2n}}, & m = 0, n \neq 0, \\ 0, & n = 0, m \neq 0, \\ -\frac{\sin \alpha_{2n}(c-b) + \sin \alpha_{2n}(2a-b-c) + 2\alpha_{2n}(a-c) \cos \alpha_{2n}(c-b)}{\sqrt{8b(a-c)} \alpha_{2n}}, & \alpha_{2n} = \hat{\beta}_m, \\ -\frac{\sqrt{2} \alpha_{2n} [\sin \alpha_{2n}(c-b) - \sin \alpha_{2n}(a-b) \cos \hat{\beta}_m(a-c)]}{\sqrt{b(a-c)} (\alpha_{2n}^2 - \hat{\beta}_m^2)}, & \text{other cases.} \end{cases}$$

As before the antisymmetric problem is nearly identical and is given by

$$\sum_{n=0}^{\infty} A_n^{(a)} p_{mn} + p_{m0} = B_m^{(a)}, \quad m = 0, 1, 2, \dots \quad (39)$$

$$-i\bar{\alpha}_{2m} A_m^{(a)} + i\bar{\alpha}_0 \delta_{m0} = -\sum_{n=0}^{\infty} B_n^{(a)} p_{nm} \bar{\beta}_n \cot \bar{\beta}_n l, \quad m = 0, 1, 2, \dots \quad (40)$$

3.1. Results

We can see the same perfect transmission for the resonator 2 as we have seen for the resonator 1 except the cutoff frequency is halved. Figures 14 and 15 show the absolute value of the reflection and transmission for and $b = 2$ and $L = 5$ for $a = 1.5$, $c = 1$ (Figure 14) and for $a = 1.125$, $c = 0.975$ (Figure 15). The first cutoff frequency is now at $k = \pi/4$ because we no longer have symmetry about $y = 0$.

We show the solution in the complex plane as a movie as we vary the position of the duct. Figures 16 and 17 are taken from Movie 2 in the supplementary material. For this case we set $b = 2$, and keep the distance $a - c$ to be a constant $a - c = 0.5$. This movie shows the appearance of the new branch cut at $k = \pi/4$ which appears when the symmetry is broken.

Figure 14: The reflection and transmission against wavenumber k for resonator 2 for $b = 2$ and $L = 5$ for $a = 1.5$, $c = 1$

215 4. Resonator 3: Solution without symmetry along either axis

We now consider a problem in which we have removed both symmetries. The problem is shown in Figure 18. This situation can be thought of as being formed by gluing two resonators of type 2 together. We outline the solution method rather briefly and the details of the matching equations can be derived from the previous section. In this case we cannot use the decomposition into symmetric and anti-symmetric solutions which means that the numerical solution is more difficult and we need to match at each boundary.

Helmholtz equation (1) is to be solved for $\phi(x, y)$ subject to these boundary conditions

$$\phi_y = 0, \quad y = \pm b, \quad -\infty < x < -l^-, \quad (41a)$$

Figure 15: As in Figure 14 except $a = 1.125$, $b = 2$, $c = 0.875$.

$$\phi_y = 0, \quad y = c^-, \quad -l^- < x < l, \quad (41b)$$

$$\phi_y = 0, \quad y = a^-, \quad -l^- < x < l, \quad (41c)$$

$$\phi_y = 0, \quad y = \pm b, \quad -l < x < l, \quad (41d)$$

$$\phi_y = 0, \quad y = c^+, \quad l < x < l^+, \quad (41e)$$

$$\phi_y = 0, \quad y = a^+, \quad l < x < l^+, \quad (41f)$$

$$\phi_y = 0, \quad y = \pm b, \quad l^+ < x < \infty, \quad (41g)$$

Figure 16: As in Figure 9 except for the resonator 2 with $l = 5$, $a = 1$, $b = 2$, $c = 0.5$ for frequency range $0 < k < \pi$. The full animation can be seen in Movie 2 in the supplementary material

$$\phi_x = 0, \quad x = \pm l^\pm, \quad -b < y < c^\pm, \quad (41h)$$

$$\phi_x = 0, \quad x = \pm l, \quad -b < y < c^\pm, \quad (41i)$$

$$\phi_x = 0, \quad x = \pm l^\pm, \quad a^\pm < y < b, \quad (41j)$$

In addition, we have to impose the radiation conditions.

4.1. Solution of the Problem

225 The mode matching problem is now more complicated and we need to expand the solution in five different regions and match at each boundary.

Figure 17: As in Figure 16 except $a = 1.5$, $c = 1$

4.1.1. Region 1 $\{-b \leq y \leq b, -\infty < x < -l^-\}$

In this region the solution can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} A_n e^{-i\bar{\alpha}_n(x+l^-)} \psi_n(y) + e^{i\bar{\alpha}_0(x+l^-)} \psi_0(y), \quad (42)$$

which satisfies equation (1), (41a), and radiation condition.

230 4.1.2. Region 2 $\{c^- \leq y \leq a^-, -l^- < x < -l\}$

The general solution for the potential in the region 2 can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} \left[B_n^- e^{i\bar{\beta}_n^-(x+l^-)} + C_n^- e^{-i\bar{\beta}_n^-(x+l)} \right] \xi_n^-(y), \quad (43)$$

which satisfies equations (1), (41b) and (41c) where

$$\hat{\xi}_n(y) = \begin{cases} \sqrt{\frac{2}{a^- - c^-}} \cos \hat{\beta}_n(y - c), & n \neq 0, \\ \sqrt{\frac{1}{a^- - c^-}}, & n = 0. \end{cases} \quad (44)$$

Figure 18: Schematic diagram of the waveguide for the Resonator 3.

Here eigenvalues and the associated eigenvalues are given by

$$\hat{\beta}_n^- = \frac{n\pi}{a^- - c^-},$$

and

$$\bar{\beta}_n^- = \sqrt{k^2 - \left(\hat{\beta}_n^-\right)^2}, \quad n = 0, 1, 2, \dots$$

respectively.

4.1.3. Region 3 $\{-b \leq y \leq b, -l < x < l\}$

In this region the solution can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} \left[D_n e^{i\bar{\alpha}_n(x+l)} + E_n e^{-i\bar{\alpha}_n(x-l)} \right] \psi_n(y), \quad (45)$$

which satisfies equations (1) and (41d).

235 4.1.4. Region 4 $\{c^+ \leq y \leq a^+, l < x < l^+\}$

The general solution for the potential in the region 2 can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} \left[B_n^+ e^{i\bar{\beta}_n^+(x-l)} + C_n^+ e^{-i\bar{\beta}_n^+(x-l^-)} \right] \xi_n^+(y), \quad (46)$$

which satisfies equations (1), (41e) and (41f).

$$\hat{\xi}_n^+(y) = \begin{cases} \sqrt{\frac{2}{a^+ - c^+}} \cos \hat{\beta}_n(y - c^+), & n \neq 0, \\ \sqrt{\frac{1}{a^+ - c^+}}, & n = 0, \end{cases} \quad (47)$$

Here eigenvalues and the associated eigenvalues are given by

$$\hat{\beta}_n^+ = \frac{n\pi}{a^+ - c^+},$$

and

$$\bar{\beta}_n^+ = \sqrt{k^2 - (\hat{\beta}_n^+)^2}, \quad n = 0, 1, 2, \dots$$

respectively.

4.1.5. Region 5 $\{-b \leq y \leq b, l < x < \infty\}$

In this region the solution can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} \left[F_n e^{i\bar{\alpha}_n(x-l)} \right] \psi_n(y), \quad (48)$$

which satisfies equations (1), (41g), and the radiation conditions

240 4.2. Formulation of the System of Equations

We now have four boundaries over which we need to match the potential and its normal derivative. These follow in a very similar manner to what we have previously and we give the method here rather briefly.

The continuity of the pressure across $x = -l^-$ gives

$$\sum_{n=0}^{\infty} A_n \psi_n(y) + \psi_0(y) = \sum_{n=0}^{\infty} \left[B_n^- + C_n^- e^{i\bar{\beta}_n^-(l^- - l)} \right] \xi_n(y).$$

Taking the inner product with $\eta_m(y)$ and integrating over $[c^-, a^-]$ gives

$$\sum_{n=0}^{\infty} A_n p_{mn}^- + p_{m0}^- = B_m^- + C_m^- e^{i\bar{\beta}_m^-(l^- - l)}, \quad m = 0, 1, 2, \dots \quad (49)$$

where p_{mn}^- is the just the same expression as p_{mn} except we replace a and c by a^- and c^- respectively. In a similar manner if we equate normal derivative and take the inner product with respect to ψ_m we obtain

$$-\bar{\alpha}_m A_m + \bar{\alpha}_0 \delta_{m0} = \sum_{n=0}^{\infty} \bar{\beta}_n p_{nm}^- \left(B_n^- - C_n^- e^{i\bar{\beta}_n^-(l^- - l)} \right), \quad m = 0, 1, 2, \dots \quad (50)$$

We obtain a further six equations from matching at $x = -l, l$ and l^+ which are

$$\sum_{n=0}^{\infty} p_{mn}^- \left(D_n + E_n e^{i\bar{\alpha}_n(2l)} \right) = B_m^- e^{i\bar{\beta}_m^-(l^- - l)} + C_m^-, \quad m = 0, 1, 2, \dots \quad (51)$$

$$\bar{\alpha}_m D_n - \bar{\alpha}_m E_n e^{i\bar{\alpha}_n(2l)} = \sum_{n=0}^{\infty} \bar{\beta}_n p_{nm}^- \left(B_n^- e^{i\bar{\beta}_n^-(l^- - l)} - C_m^- \right), \quad m = 0, 1, 2, \dots \quad (52)$$

$$\sum_{n=0}^{\infty} p_{mn}^+ \left(D_n e^{i\bar{\alpha}_n(2l)} + E_n \right) = B_m^+ + C_m^+ e^{i\bar{\beta}_m^+(l^+ - l)}, \quad m = 0, 1, 2, \dots$$

$$\bar{\alpha}_m D_n e^{i\bar{\alpha}_n(2l)} - \bar{\alpha}_m E_m = \sum_{n=0}^{\infty} \bar{\beta}_n p_{nm}^+ \left(B_n^+ - C_n^+ e^{i\bar{\beta}_n^+(l^+ - l)} \right), \quad m = 0, 1, 2, \dots$$

$$\sum_{n=0}^{\infty} p_{mn}^+ F_n = B_m^+ e^{i\bar{\beta}_m^+(l^+ - l)} + C_m^+, \quad m = 0, 1, 2, \dots \quad (53)$$

$$\bar{\alpha}_m F_{nm} = \sum_{n=0}^{\infty} \bar{\beta}_n p_{nm}^+ \left(B_n^+ e^{i\bar{\beta}_n^+(l^+ - l)} - C_n^+ \right), \quad m = 0, 1, 2, \dots \quad (54)$$

where p_{mn}^+ is the just the same expression as p_{mn}^- except we replace a^- and c^- by a^+ and c^+ respectively.

4.3. Results

Figure 19 plots the absolute value of reflection coefficient $|R_0|$ against k for $b = 2$, with $a_1 = 0.5$, $c_1 = -0.5$, $a_2 = 0.5$, $c_2 = -0.5$ (subfigure a) and $a_1 = 0.5$, $c_2 = -1$, $a_2 = 1$, $c_2 = -1$ (subfigure b). Figure 20 is the same as Figure 19 except $a_1 = 1.5$, $c_1 = -0.5$, $a_2 = 1.5$, $c_2 = -0.5$ (subfigure a) and $a_1 = 1.5$, $c_2 = -1$, $a_2 = 1.5$, $c_2 = -0.5$ (subfigure b). In both figures it can be seen that as soon as the symmetry about $x = 0$ is broken we no longer have perfect transmission.

5. Summary

We have shown that EAT occurs at a real frequency close to the real part of the resonance associated with the resonator, provided that the problem has a symmetry in the direction orthogonal to the direction of propagation and that there is a single propagating mode. We have shown how much the behaviour on the real axis of the reflection or transmission coefficients is determined by the properties of its analytic extension in the complex plane. The key reason for EAT

Figure 19: The absolute value of reflection coefficient $|R_0|$ against k for $b = 2$, $l_1 = L - 3 = 1$, $l_3 = 5$ with $a_1 = 0.5$, $c_1 = -0.5$, $a_2 = 0.5$, $c_2 = -0.5$ (subfigure a) and $a_1 = 0.5$, $c_2 = -1$, $a_2 = 1$, $c_2 = -1$ (subfigure b)

Figure 20: As in Figure 19 except $a_1 = 1.5$, $c_1 = -0.5$, $a_2 = 1.5$, $c_2 = -0.5$ (subfigure a) and $a_1 = 0.5$, $c_2 = -1$, $a_2 = 1$, $c_2 = -1$ (subfigure b)

is that these coefficients consist of the sum or difference of two functions of a single complex variable which are constrained to have modulus one on the real axis when there is the appropriate symmetry. This highly constrains the problem and requires that there is a matching pair of poles and zeros and that they can be written as a Blaschke product below the cutoff. We have illustrated out results with a number of simple examples for which we found the solution by mode matching and we were able to compute the analytic extension of the solution.

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