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From Vehicular Platoons to General Networked Systems: String Stability and Related Concepts*

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Abstract

Networked systems and their control are highly important and appear in a variety of applications, including vehicle platooning and formation control. Especially vehicle platoons have been intensively investigated. An interesting problem that arises in this area is string stability, which broadly speaking means that an input signal amplifies unboundedly as it travels through the vehicle string. However, various, not necessarily equivalent, definitions are commonly used. In this paper, we aim to formalise the notion of string stability and illustrate the importance of those distinctions on simulation examples. A second goal is to extend the definitions to general networked systems.

Keywords: String stability; Networked systems

1. Introduction

Networked systems and their control are studied in a variety of fields, such as vehicular platooning (Barooah and Hespanha, 2005; Cook, 2007; Herman et al., 2015a; Lestas and Vinnicombe, 2007; Levine and Athans, 1966; Martinec et al., 2016; Melzer and Kuo, 1971; Middleton and Braslavsky, 2010; Peppard, 1974; Rogge and Aeyels, 2008; Seiler et al., 2004; Swaroop and Hedrick, 1996, 1999), formation control (Fax and Murray, 2004; Yadlapalli et al., 2006; Zelazo et al., 2008), and many others. These systems consist of many agents that are performing a common task. While in early stages centralised controllers were studied (Levine and Athans, 1966; Melzer and Kuo, 1971), such controllers become infeasible if the number of agents increases. Hence, distributed and decentralised approaches are investigated, where the agents utilise local information and in some cases information transmitted by other agents (Barooah and Hespanha,

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2005; Cook, 2007; Fax and Murray, 2004; Herman et al., 2015a; Li et al., 2011; Martinec et al., 2016; Middleton and Braslavsky, 2010; Peppard, 1974; Seiler et al., 2004; Swaroop and Hedrick, 1999; Tonetti and Murray, 2010, 2011; Yadlapalli et al., 2006).

However, in some cases these distributed systems experience undesired properties such as instability, amplification of disturbances within the network, and cascading failures. It is therefore of utmost importance to understand the dynamics and limitations that are imposed on these systems with respect to the information flow as well as the underlying systems. In this work we will mostly concentrate on vehicle platoons. In that application two properties are very important, stability and so called string stability. Especially, the latter appears in several variations. In this review we will collect these variations and present them in a more unified framework. Further, we give a possible extension of this property for general networked systems. While not applicable for all systems, the property becomes critical in areas like traffic management and coordination of vehicles.

1.1. Analysis Methods

The methods used to analyse these systems range from classical control theory to spatial-temporal systems (Bamieh et al., 2002; Knorn, 2012; Knorn et al., 2014). Recent works combine control theoretic approaches with graph theory. In that context the agent's behaviour is governed by an individual dynamic system, while the information exchange among the agents is represented as a graph. The behaviour of the system is then closely linked to the Laplacian of the graph and in particular its eigenvalues, such that the study of the Laplacian becomes an integral part in the analysis of the networked system (Barooah and Hespanha, 2005; Herman et al., 2015a; Li et al., 2011; Tonetti and Murray, 2010, 2011; Yadlapalli et al., 2006; You and Xie, 2013). While some works do not consider this link to graph theory directly, their problem description and to some extent the results can be translated into the same separation of the agents individual dynamics and the graph considering the information exchange, see for example Middleton and Braslavsky (2010); Seiler et al. (2004).

An even newer approach for the analysis of networked systems utilises the so called wave approach (Herman et al., 2016, 2015b; Martinec et al., 2016). The idea is to model the state of the system as waves propagating through the graph structure. While this method is mainly used in relation to vehicle platooning, the approach can be extended for other structures (Martinec et al., 2016).

Remark 1.1. *A related field of research is that of consensus algorithms, where multiple agents aim to equalise a state variable (Moreau, 2005; Olfati-Saber et al., 2007; Zelazo et al., 2007). Normally, the dynamics of this state variable is simple, for example its derivative is set directly to a weighted average of the state variables of the neighboring agents. Nonetheless, the similarities between the two fields allow the use of analogous techniques, such as graph theory.*

1.2. Problem Specifications

The design method selected depends largely but not only on the problem specification. In this context there are four important choices to consider besides the actual controller design: 1. the control objective; 2. the individual agent's dynamic system, heterogeneous vs. homogeneous; 3. the dynamic system representation of the agents, linear vs. non-linear; and 4. the interactions between the agents and the communication structure among the agents.

1.2.1. Control Objective

The controller objective defines what the networked system should achieve as a unit. While some applications allow the control objective to be freely selected or at least relaxed, there will be systems where this is not a valid option. Also, any relaxation of the control objective will lead to a trade-off between the system properties, such as robustness and stability, versus some performance measures. Due to these reasons it is of utmost importance to investigate the properties for a given control objective, as well as investigate what certain relaxations can achieve.

For example, in the application of vehicle platooning the control objective is given as the spacing policy, which determines the inter-vehicle distance the vehicles should maintain. Two methods are prevalent in the literature, which are a constant distance and a constant time gap, respectively. The selection of this objective has a profound impact on the actual stability and performance properties of the system, but also impacts the efficiency at high speeds.

1.2.2. Individual Agent Dynamics

Independent of the nature of the dynamic system representation and model, it is important to distinguish two approaches for the dynamic systems of all considered agents:

1. homogeneous agents, i.e. the dynamic systems and their controllers of all agents are identical (Barooah and Hespanha, 2005; Cook, 2007; Herman et al., 2015a, 2016, 2015b; Martinec et al., 2016; Seiler et al., 2004; Tonetti and Murray, 2010; Yadlapalli et al., 2006);

2. heterogeneous agents, i.e. the dynamic systems and/or their controllers vary among the agents (Dunbar and Murray, 2006; Lestas and Vinnicombe, 2007; Middleton and Braslavsky, 2010; Tonetti and Murray, 2011).

The use of homogeneous agents simplifies the analysis, however idealises the systems dramatically. Hence, it is important to extend the results where possible to heterogeneous networked systems. This is also relevant in regard to model uncertainties and small model changes that are undoubtedly present. In some cases the use of heterogeneous controllers is even suggested to improve the performance of the system (Khatir and Davidson, 2004).

1.2.3. *Dynamic System Representation*

The individual dynamic system of each agent is often considered to be linear and decoupled besides the communication utilised in the controller. In that case it is represented either as a transfer function (Barooah and Hespanha, 2005; Cook, 2007; Martinec et al., 2016; Middleton and Braslavsky, 2010; Tonetti and Murray, 2010, 2011; Yadlapalli et al., 2006) or in state space (Herman et al., 2016, 2015b; Li et al., 2011; Rogge and Aeyels, 2008). Both forms of representing the dynamic system allow the usage of specific system analysis and control techniques and both are common in the literature, where in some instances both representations are used in combination to extend or facilitate the results (Cook, 2005; Fax and Murray, 2004; Herman et al., 2015a). As seen in Cook (2005) the two approaches are in effect interchangeable for the analysis of string instability in vehicle platoons. In the linear case we hence investigate individual system dynamics of similar forms to

$$Y_i(s) = \underbrace{P_i(s)K_i(s)}_{L_i(s)} U_i(s) + \Omega_i(s) \tag{1}$$

with U_i being the input, Y_i the output, and Ω_i an additive output disturbance all given as Laplace transforms. In this setting an individual controller $K_i(s)$ and individual system dynamics $P_i(s)$ for vehicles $i \in \{1, \ldots, N\}$ are used and we assume 0 initial conditions. P_i and K_i can both be matrices in the case of multi-input and multi-output systems or scalar transfer functions. While the latter is far more common, the use of transfer matrices can increase the control performance by giving the controller more freedom. The drawback is the higher dimensions and the increased complexity.

Note also that in the case of homogeneous agents, we have $P_i(s) = P(s)$ and $K_i(s) = K(s)$ for all vehicles i.

Further, the signal $Y_i(s)$ is the output of the system or some chosen performance variable. For example, in a vehicle platoon, this is often the net inter-vehicle spacing error given a certain spacing policy. Finally, the signals $U_i(s)$ are the inputs to the controller and include the information from other agents, possibly communicated among them, as well as a given references. Hence, with this set-up the individual agents interact through these inputs $U_i(s)$ with each other, but are otherwise independent. This interaction among the agents will be discussed in detail in Section 1.2.4.

Remark 1.2. *It is possible that there exist other interactions among the individual systems that are not captured by $U_i(s)$, see Stankovic et al. (2000). To a certain degree these interactions can be incorporated as restrictions in both the controller $K_i(s)$ and constraints on $U_i(s)$. The main advantage of decoupling the systems is that it separates the interaction among agents from the single agent's dynamics. In this setting the interlink is decided by the communication structure, which might contain some restrictions due to the present interactions among the agents. This is discussed in Section 1.2.4.*

As is well known, if $L_i(s) = K_i(s)P_i(s)$ is a proper transfer function a state space representation can be found, which is not unique. In this context we look at a linear representation of the form

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) \tag{2a}$$

$$y_i(t) = C_i x_i(t) + D_i u_i(t) + \omega_i(t), \tag{2b}$$

where $x_i(t)$ is the state of the i-th subsystem and $u_i(t), y_i(t)$, and $\omega_i(t)$ are the time signals corresponding to the Laplace transforms $U_i(s)$, $Y_i(s)$, and $\Omega_i(s)$. In the following, we will consider the special case $D_i = 0$, which is the most commonly studied. Very often one investigates either the reaction to a change in the reference input $y_{ref}(t)$ (see Section 1.2.4) or the reaction to the output disturbance $\omega_i(t)$.

Sometimes the use of non-linear models and controllers is preferred. The reason thereof is partly the fact that in many networked systems the dynamics are inherently non-linear and linear models make use of linearisation around an operating point. While for a broad range of applications the linearisation works well within the regulated bounds, once the operation deviates considerably from the operating point the linear dynamics may no longer represent the system accurately enough. Secondly, it seems possible to avoid certain undesirable effects by the use of non-linear control approaches (Yanakiev and Kanellakopoulos, 1998). While, we mainly focus on linear dynamics we will include some comments regarding non-linear approaches where appropriate.

1.2.4. Interaction Among Agents

The other main design choice relates to the interaction among the agents and hence the graph structure that is used. These graphs are either un-directed (Li et al., 2011; Yadlapalli et al., 2006) leading to symmetric, bi-directional communication structures, or directed (Herman et al., 2015a; Martinec et al., 2016; Rogge and Aeyels, 2008; Tonetti and Murray, 2010, 2011) allowing for asymmetric control structures that can improve the performance of the system. Further, some works include weighted edges (Herman et al., 2015a,b), which allows for more complex control strategies. The graph is commonly described using its Laplacian, which we denote $\mathbb{L}$ in the following.

Using the Laplacian $\mathbb{L}$, we can write the control input as

$$u(t) = \mathbb{L}(y_{ref}(t) - y(t)), \tag{3}$$

where $u(t)$ and $y(t)$ are the vectors containing the inputs and measured outputs of each agent and $y_{ref}(t)$ is a reference signal that reflects the objective of the controller. For example, in vehicle platooning $y(t)$ could be the inter-vehicle distance or the position of the vehicles and $y_{ref}(t)$ is given by the selected spacing policy. $Y_{ref}(s)$, $Y(s)$, and $U(s)$ denote their Laplace transforms.

Then, using this control input and the individual system dynamics of each agent as in (1), the networked system can be described by

$$Y(s) = (I + \operatorname{diag}(P_i(s)) \operatorname{diag}(K_i(s))\mathbb{L})^{-1} \operatorname{diag}(P_i(s)) \operatorname{diag}(K_i(s))\mathbb{L}Y_{ref}(s) \\ + (I + \operatorname{diag}(P_i(s)) \operatorname{diag}(K_i(s))\mathbb{L})^{-1} \Omega(s). \quad (4)$$

Equivalently, the networked system can be expressed in state space form as

$$\dot{x}(t) = (\operatorname{diag}(A_i) - \operatorname{diag}(B_i)\mathbb{L} \operatorname{diag}(C_i))\, x(t) \\ + \operatorname{diag}(B_i)\mathbb{L}y_{ref}(t) - \operatorname{diag}(B_i)\mathbb{L}\omega(t) \quad (5a)$$

$$y(t) = \operatorname{diag}(C_i)x(t) + \omega(t). \quad (5b)$$

Therefore, the Laplacian $\mathbb{L}$ is very important for the analysis of such networked systems and many works investigate the eigenvalues of $\mathbb{L}$ and their influence on the whole system (Barooah and Hespanha, 2005; Herman et al., 2015b; You and Xie, 2013).

Remark 1.3. *As mentioned earlier, it is possible to model additional interactions among the agents as well as control decisions by imposing conditions on the Laplacian $\mathbb{L}$. In many situations it then makes sense to split the Laplacian in two parts. The first part of $\mathbb{L}$ is typically fixed by the system set-up and the second part can be chosen more freely according to the desired control configuration. Such a case also occurs when the reference/performance variable links the individual agents, while the measured output is given individually. Then, one part of $\mathbb{L}$ describes the map between the measured output and the reference, while the second part describes the control configuration used. For example, in the case of vehicle platoons, the measured output is usually the position of each vehicle, the reference is given as the inter-vehicle distance depending on the spacing policy, and the control configuration selects for each vehicle the information available from other vehicles. Such a set-up is for example followed in Middleton and Braslavsky (2010). Naturally, it is possible to reformulate the system to eliminate the split of the Laplacian.*

Remark 1.4. *In some works the use of multiple Laplacians for different inputs to the agents is considered, see for example Herman et al. (2016, 2015b). This case can be alternatively covered by proper modeling of the transfer matrices $P_i(s)$ and $K_i(s)$, see for example Fax and Murray (2004). The works in Herman et al. (2016, 2015b) take the approach to split the Laplacian $\mathbb{L}$ into multiple Laplacians where each Laplacian contains the links among the vehicles considering one single input, such as the velocity or the position of the vehicles. By splitting $\mathbb{L}$ in this way, they obtained results on necessary configurations depending on the input, i.e. they found that for the position input a symmetric communication structure has to be deployed to achieve string stability.*

1.2.5. Extensions and Other Considerations

A possible extension of these models is to consider time dependent information exchanges which would result in time dependent Laplacians. While this is investigated for consensus systems (Moreau, 2005), to the best of our knowledge it has not been investigated directly for networked systems as considered here.

While a lot of work is focused on the effects of using an idealised communication set-up, there are some works that investigate properties when communication constraints occur, such as delays or losses, (Guo and Yue, 2011; Öncü et al., 2012; Peters et al., 2014; Rivas, 2015; Seiler and Sengupta, 2001). These additional constraints make the control more difficult and can degrade the performance of the controller considerably. These issues become more prominent when the communication occurs over a shared network, where there are additionally bandwidth limitations and packet drop outs. These issues worsen the conditions and impair both state estimation and control of such systems. The related field of networked control, see Hespanha et al. (2007); Yang (2006); You and Xie (2013) and references therein for a summary, deals solely with these problems. In this work, we concentrate instead on limitations of the performance caused by the communication structure and the individual systems. Both approaches are of importance for the future utilisation of networked systems.

Other important practical issues present that deserve special consideration include sensor and actuator limitations (Guo and Yue, 2012), sensor and actuator failures (Guo and Yue, 2014), as well as security considerations (Amoozadeh et al., 2015; Dadras et al., 2015; DeBruhl et al., 2015; Gerdes et al., 2013).

In this paper, we will focus mostly on vehicle platooning in an ideal setting, which is a special case of networked systems that have widely been studied. In this regard, a property denoted string stability is commonly mentioned as a performance criterion for vehicle platoons. String stability is not directly linked to stability in the usual sense. In fact, it is better understood as a scaling measure. Note that there is some connection to stability, which we will briefly elaborate on later in the paper. Another, property, which we denote eventual instability, is much closer related to instability in the usual sense. In Section 2, we will introduce both properties. Further, we will formalise different variations of string stability commonly used which, while related, contain subtle differences. Additionally, we will discuss the results found for vehicle platoons in regard to stability as well as the property of string instability. In Section 3 results for more general systems are revised that illustrate how the property of string stability can be extended to such more general systems. Finally, in Section 4 we will conclude our findings and pinpoint to some open problems in this field.

2. Vehicle Platoons

One of the applications that is very well studied is vehicle platooning, which is often mentioned in conjunction with Automated Highway Systems (AHS). In this application multiple vehicles driving in a string are controlled to keep a pre-specified distance to each other, which defines the control objective. This pre-specified distance is known as the spacing policy and is an important design parameter of the system. The two most commonly used policies are constant distance, i.e. an absolute distance between the vehicles, and constant time gap, a velocity dependent distance between the vehicles.

The inter-vehicle spacing in both cases can be either measured from front bumper to front bumper of two vehicles or from front bumper to rear bumper of the preceding vehicle. In particular, when considering the velocity dependent policy the first notion is known as time headway, while the latter as time gap.

In a qualitative aspect both notions lead to the same behaviour. In fact proper adaptation of the target inter-vehicle spacing allows to convert one situation into the other. Hence, in this paper we will solely refer to inter-vehicle spacing without further distinguishing the exact notion and to time gap in case of a velocity dependent spacing policy to avoid unnecessary confusion. Note that we use this terminology when reviewing existing work, even if the referred paper actually considers the time headway.

In fact vehicle platoons following such policies (mostly a constant time gap) have been successfully used on public roads, in projects such as PATH, the SARTRE project, or the Truck platooning challenge (European Truck Platooning, 2017; Volvo Trucks, 2012). While these tests were applied to relatively small scale platoons, in more realistic scenarios these platoons can increase to tremendous size given the current traffic on our roads. Hence, it is important to investigate the potential benefits and risks. These risks are linked to undesired effects in the distributed system, such as the amplification of disturbances, and can cause safety issues.

In Section 2.1, we introduce the usual structural properties that are present in this application. Then, we introduce in Sections 2.2 and 2.3 the properties of stability and string stability in this context. Finally, in Section 2.4 we revisit the main contributions in this field.

2.1. Structural Properties

The individual open loop dynamics of vehicle platoon systems, consisting of the vehicle model and the controller, contain usually two integrators, which allows them to follow given ramp inputs. Also, these are mostly modeled as single input, single output systems, where the input is commonly a positional error and the output is the position of the vehicle. Hence, in the linear case we investigate open loop dynamics of single agents of the form

$$L_i(s) = \frac{N_i(s)}{s^2 D_i(s)} \tag{6}$$

with

$$N(s) = n_0 + n_1 s + n_2 s^2 + \dots \tag{7a}$$
$$D(s) = d_0 + d_1 s + d_2 s^2 + \dots \tag{7b}$$

if considering transfer functions. Note that, when considering a state space realisation of (6), there are certain structural limitations in A_i due to the double integrator. For example, we can find A_i in the controller canonical form as

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \\ \vdots & \vdots & 0 & \ddots & \\ 0 & 0 & -d_0 & \cdots & -d_{m-3} \end{bmatrix}, \tag{8}$$

where the two zeros in the last row arise from the double integrator present in the system.

Further, these systems lead to a special structure of the graph that represents the information exchange. There are two main categories; the first is a string connection where the first vehicle is independent or following a virtual leader (Cook, 2007; Martinec et al., 2016; Middleton and Braslavsky, 2010; Rivas, 2015; Seiler et al., 2004), the second is a cyclic interconnection where the first vehicle uses the last vehicle as its predecessor (Herman et al., 2015b; Rivas, 2015; Rogge and Aeyels, 2008).

The first category leads to graphs that form chains, such that the associated adjacency matrix is a band diagonal matrix. In some cases additional communication links are present that disturb this band diagonal matrix structure. For example Seiler et al. (2004) broadcasts the leader position additionally to local information to avoid disturbance amplification.

The second category leads to cyclic graphs where the adjacency matrix is a circulant matrix. Systems that can be represented in such a form are for example light rail or subway circuits. Further, such circular structures can be useful to establish results for the non-cyclic case, since the analysis is simplified. Such an approach is followed in Cantos et al. (2016a); Herman et al. (2016, 2015b).

2.2. Stability

As is common in control theory the closed loop stability of the vehicle platoon is important. Here, unlike in normal conditions, the stability is often investigated not just for a given platoon length but for increasing number of vehicles, i.e. a system can become unstable for large enough N (Barooah and Hespanha, 2005; Rivas, 2015; Rogge and Aeyels, 2008). We will use the term eventual instability to denote situations where instability occurs for long enough vehicle platoons. Otherwise stability is understood in the usual sense of asymptotic stability or bounded input bounded output (BIBO) stability in the case of input-output models. It is important to remark that in vehicle platoons eventual instability is mostly observed for bi-directional or cyclic interconnections.

2.3. String Stability

Once the stability of the vehicle platoon is assured, i.e. it is not eventually unstable, string stability is an important characteristic of vehicle platoons. In a broad sense, this property implies that a disturbance will not amplify when propagating through the platoon (Peppard, 1974). Besides disturbances the effect of reference changes or other inputs to the system can be considered. String instability is a very different effect from eventual instability. While eventual instability means that the interconnected system loses stability for some length N, this does not occur for a string unstable system. However, the magnitude of the response, i.e. the amplification of an input signal, will increase as the string length increases. Fig. 1 shows a typical response to an impulse disturbance acting on the first vehicle for a string unstable system that is not eventual unstable. The response with the smallest peaks (shown in dark green) is that of the last vehicle when the string consists of 5 vehicles whilst the response with the largest peaks (in light yellow) corresponds to the response of the last vehicle when the string consists of 100 vehicles.

In this sense string stability is better understood as a scalability measure rather than as a stability property in the common sense. This means that for any fixed length, the vehicle platoon exhibits a stable response. On the other hand scaling the system by increasing the platoon length, leads to an unwanted behaviour. The link to stability becomes clear when looking at another possible representation of the system, namely as a spatial-temporal system. In this form the system contains two independent variables: 1. time, which is continuous; and 2. location within the string, which is discrete. In this special form, string stability can be interpreted as stability in the second variable, location within the string.

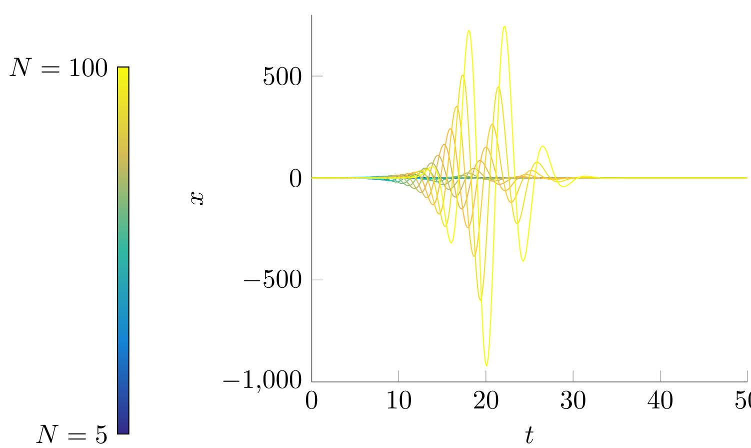


Figure 1: Example for a string unstable system that uses homogeneous agents with open loop transfer function $L(s) = \frac{6s+5}{s^2}$ and a uni-directional communication structure. The plot shows the response of the last vehicle in the string to an impulse disturbance acting on the first vehicle.

There are different definitions of string stability, for example based on the H_∞ norm (Eyre et al., 1998; Herman, 2016; Middleton and Braslavsky, 2010; Peppard, 1974) or the $\mathcal{L}_\infty$ signal norm (Rogge and Aeyels, 2008; Swaroop and Hedrick, 1996, 1999; de Wit and Brogliato, 1999), and related stability issues such as flock stability (Cantos et al., 2016a; Herman et al., 2016), harmonic stability (Herman et al., 2015a), and eventual string stability (Khatir and Davidson, 2004) in the literature. The work of Swaroop and Hedrick (1996) gives a more formal definition of string instability based on the $\mathcal{L}_\infty$ signal norm with respect to initial conditions. The work of Rogge and Aeyels (2008) extends these definitions, leading to a more strict form of string stability that includes avoidance of the so called slinky effect, i.e. amplification from one vehicle to the next. Besides this extension they give a more formal definition of string stability with respect to input signals. Cook (2005) gives formal definitions of so called practical string stability in terms of more general signal norms. Note that depending on the actual definition of string stability some systems achieve one form of string stability while they remain "unstable" in another sense. This fact makes the comparison of different results more challenging due to the various nuances present.

We now formalise the definitions of the most commonly used forms of string stability, including the ones mentioned above. Note that all the definitions have in common that the amplification of a signal does not approach infinity with increasing platoon lengths. The main distinction among the various versions occurs in relation to which signals are considered and the characteristics of the bound on the amplification. In the following, we will look at these distinctions and their importance. To this end, we first introduce formally various versions of input-output string stability, where the main difference lies in the considered inputs and outputs. Afterwards, we will introduce the very important notion of internal string stability that is commonly used. Finally, we will look at other notions of string stability that allow a more precise classification.

To clarify the definitions, we will use the following notation for the rest of this section. With $z_i(t)$, we denote an input to a single vehicle. This could be either the reference input, i.e. the entries of $y_{ref}(t)$ concerning the i-th vehicle, or the disturbance $\omega_i(t)$. While very often either one or the other input is investigated, it is important in a practical setting to assess the system behaviour both in terms of reference as well as disturbance changes. The reason thereof is that in any practical vehicle platoon there will both be disturbances present as well as reference changes. The latter for example can occur if the road conditions suddenly change, such as in case of a weather change or changing traffic conditions. Further, we use $z(t)$ for an input to all vehicles. The input is again either the reference $y_{ref}(t)$ or the collection of the disturbances acting on the vehicles, i.e. the vector containing $\omega_i(t)$. Similarly, we denote by $y_i(t)$ and $y(t)$ the output of a single vehicle and all the vehicles, respectively. The number of vehicles in the platoon is denoted by N and can be any natural number.

All the definitions we consider in this paper are notions of $\mathcal{L}_{p,q}$ string stability or $\mathcal{L}_p$ string stability, if $p = q$. This means that as a measure, we use $\mathcal{L}_p$ and $\mathcal{L}_q$ signal norms. Other norms could possibly be considered such as power signal norms. However, to the best of our knowledge the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms are by far the most considered ones, so that usually $p = q = 2$ or $p = q = \infty$. Especially, the relation between the $\mathcal{L}_2$ signal norm and the $\mathcal{H}_\infty$ system norm is commonly used for the investigation of string stability in linear systems. This link gives necessary conditions for $\mathcal{L}_2$ string stability that are commonly used in some works even as a definition for string stability itself.

Before introducing string stability formally, we will illustrate the importance of the signal norm considered. Therefore, we reproduce an example given in Klinge (2008), which shows that $\mathcal{L}_2$ and $\mathcal{L}_\infty$ string stability are not equivalent. To this end a non-homogeneous string of vehicles is interconnected as in Fig. 2. The parameters are chosen such that $T_i > T_{i+1}$ holds for all i, which is achieved by using $T_i = \left(\frac{1}{2}\right)^{i-1}$. Setting the initial states 0, except for the first vehicle which is set to 1, produces the initial condition response plotted in Fig. 3, where the dark blue shows the result for a string of length 2 and the bright yellow for a string of length 5. We see that while the $\mathcal{L}_\infty$ norm of the response is not bounded with increasing string length, since the initial peak increases with increasing number of vehicles in the string, the $\mathcal{L}_2$ norm remains bounded, since the decay speed increases fast enough with increasing number of vehicles in the string. Hence, the system is $\mathcal{L}_2$ string stable but not $\mathcal{L}_\infty$ string stable (see Definition 2.1 for a formal definition of string stability). We refer the reader to Klinge (2008) for a detailed mathematical analysis of the example.

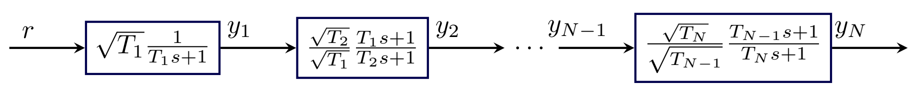


Figure 2: Example for $\mathcal{L}_2$ vs $\mathcal{L}_\infty$ string stability, see also Fig. 1.1 in Klinge (2008)

Not only is the choice of norm of utmost importance in a mathematical sense, but it also has a profound influence on the practicality of the string stability notion. There are two reasons for that. Firstly, in all variations of $\mathcal{L}_{p,q}$ string stability, the input and/or the initial state of one or all vehicles are assumed to be bounded in the $\mathcal{L}_p$ signal norm and l_p vector norm, respectively. This means the use of different norms has a strong implication on the considered signals and initial states. Secondly, the output of a string stable system is bounded by the $\mathcal{L}_q$ norm. Again this has a specific outcome in mind.

Due to these two reasons $\mathcal{L}_2$ string stability is less practical than the $\mathcal{L}_\infty$ string stability, even though $\mathcal{L}_2$ string stability is more commonly considered due to its link to the $\mathcal{H}_\infty$ system norm. The impact on the output becomes evident in the example given above. While the system is $\mathcal{L}_2$ string stable, the initial peak in the output increases as the string length grows. In a practical setting such an increase is dangerous and can cause collision among the vehicles. The use of the $\mathcal{L}_\infty$ norm does render any system containing such an increasing peak response $\mathcal{L}_\infty$ string unstable. In that sense for an $\mathcal{L}_\infty$ string stable system the

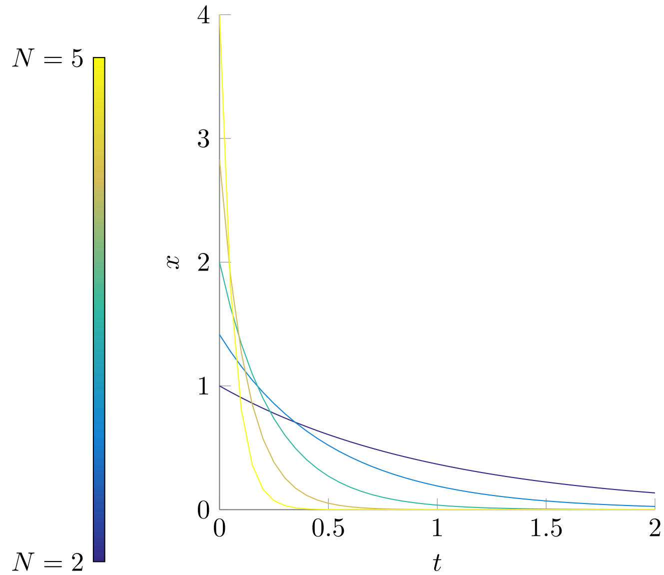


Figure 3: Example for $\mathcal{L}_2$ vs $\mathcal{L}_\infty$ string stability, see also Fig. 1.2 in Klinge (2008).

safety is assured.

Also, when looking at the input, the $\mathcal{L}_2$ norm is not usable in a practical setting. Especially, in the case of the general $\mathcal{L}_{p,q}$ string stability for the response to a disturbance, as defined formally in Definition 2.1, where the complete vector of all disturbances to each vehicle is assumed to be bounded, i.e.

$$||\omega||_p = \left|\left|\begin{bmatrix}\omega_1 & \omega_2 & \dots & \omega_N\end{bmatrix}\right|\right|_p < \infty. \tag{11}$$

For example, when considering the $\mathcal{L}_2$ norm and assuming that a similar disturbance acts on each vehicle, the bound on $||\omega||_2$ means that as the length of the vehicle platoon increases the absolute peak disturbance acting on each vehicle is lowered. However, in a practical sense it is not logical that by adding additional vehicles to the platoon the disturbances acting on the vehicles already in the platoon will change. In this sense the $\mathcal{L}_2$ norm is too restrictive and may have limited practical value. On the other hand the $\mathcal{L}_\infty$ norm does not suffer from that flaw.

2.3.1. Input-Output Versions of String Stability

Considering input-output stability, there exist several variations depending on which input and output signals are considered. For example, string stability usually considers a disturbance or reference change as input and the absolute inter-vehicle distance as output. In other cases the velocity of the vehicles is used as the output variable (Cantos et al., 2016a; Herman et al., 2016). The considered input and output variables are important and for practical reasons any combination should be investigated for string stability. Another distinction can be made where the input acts and the output is measured. For example, the input can act on the first vehicle or it can affect multiple vehicles simultaneously. We will give the definitions concerning this second kind using a general input, $z(t)$, and a general output, $y(t)$.

A brief overview over these variations is given in Table 1 with shortened definitions using the same notation as in the text and some references that utilise string stability in such a form.

The first notion that we introduce is general $\mathcal{L}_{p,q}$ string stability, defined below.

Definition 2.1 (General $\mathcal{L}_{p,q}$ string stability)**.** *A vehicle platoon is $\mathcal{L}_{p,q}$ string stable, if for every input signal $z(t)$ with*

$$||z||_p < \infty \tag{12}$$

there exists $R < \infty$ for all $N \in \mathbb{N}$ such that

$$||y||_q < R.$$

In terms of practical vehicle platoons, it is the most versatile input-output string stability, since it includes any possible combination of inputs. Especially, disturbances will be present at all times and for each vehicle. In this context this variation has the most practical value. However, the investigation of this form of string stability is more complex.

The main distinction between the other variations lies in the choice of the input considered. One very common variation is *single with input α final $\mathcal{L}_{p,q}$ string stability.* Here, one selects a single vehicle indexed by α, often the first vehicle in the platoon, and investigates the input from this vehicle to the output of the last vehicle. We formally define this notion below.

Definition 2.2 (Single with input α final $\mathcal{L}_{p,q}$ string stability)**.** *A vehicle platoon is $\mathcal{L}_{p,q}$ single with input α final string stable, if for every input signal $z(t)$ such that for $i \neq \alpha$ $z_i(t) = 0$ and*

$$||z_\alpha||_p < \infty \tag{13}$$

there exists R for all $N \in \mathbb{N}$ such that for the output of the last vehicle $y_N(t)$

$$||y_N||_q < R \tag{14}$$

holds.

This variation considers only the input of one selected vehicle and its influence on the output of the last vehicle, in contrast with the general form where any input and output to the vehicles is considered. This also means that the practicality of this definition is limited. In a system that exhibits single with input α final string stability, there are several occasions where collisions among vehicles can occur.

For example, if a disturbance acts on a different vehicle, two disturbances act on two different vehicles simultaneously, or the output of intermediate vehicles increases unbounded. On the other hand the complexity of the problem is reduced significantly, which allows to find some general conditions for string instability.

The remaining variations are different relaxations of the more specific definition of single with input α final string stability towards the general definition. In this sense their practical implications lie between those of the general and of the more specific definition of single with input α final string stability. Whilst relaxing the input signal resolves the issue that in a practical setting disturbances can occur at any vehicle and simultaneously, the relaxation of the output deals with the issue that some intermediate vehicles could collide. Due to the reduced complexity of these restricted variations they can be useful to obtain general limitations for string instability.

The first relaxation we consider is that we do no longer select the specific vehicle α as an input, but demand that any input that only acts on one single vehicle leads to a bounded response of the last vehicle. We call this notion *single final $\mathcal{L}_{p,q}$ string stability* and formally define it below.

Definition 2.3 (Single final $\mathcal{L}_{p,q}$ string stability)**.** *A vehicle platoon is $\mathcal{L}_{p,q}$ single final string stable, if for every $i \in \{1, 2, \ldots, N\}$ and every input signal $z(t)$ such that for $j \neq i$ $z_j(t) = 0$ and*

$$||z_i||_p < \infty \tag{15}$$

there exists R for all $N \in \mathbb{N}$ such that for the output of the last vehicle $y_N(t)$

$$||y_N||_q < R \tag{16}$$

holds.

Further, relaxing the input signal leads to the so called *final $\mathcal{L}_{p,q}$ string stability* notion. Here, as in the general form every possible input signal is considered. This includes the case where the input affects multiple vehicles simultaneously. We formally define this notion below.

Definition 2.4 (Final $\mathcal{L}_{p,q}$ string stability)**.** *A vehicle platoon is $\mathcal{L}_{p,q}$ final string stable, if for every input signal $z(t)$ with*

$$||z||_p < \infty \tag{17}$$

there exists R for all $N \in \mathbb{N}$ such that for the output of the last vehicle $y_N(t)$

$$||y_N||_q < R \tag{18}$$

holds.

So far, we relaxed the input signal considered. It is possible to similarly relax the considered output. So instead of investigating the influence on the output of the last vehicle in the string $y_N(t)$, we consider the output of all vehicles $y(t)$. We can combine this relaxation with the relaxation for the input signals above, which yields two additional notions, denoted *single with input* α and *single* $\mathcal{L}_{p,q}$ *string stability*. These are formally defined below.

Definition 2.5 (Single with input α $\mathcal{L}_{p,q}$ string stability)**.** *A vehicle platoon is* $\mathcal{L}_{p,q}$ *single with input* α *final string stable, if for every input signal* $z(t)$ *such that for* $i \neq \alpha$ $z_i(t) = 0$ *and*

$$||z_\alpha||_p < \infty \tag{19}$$

there exists R *for all* $N \in \mathbb{N}$ *such that for the output*

$$||y||_q < R \tag{20}$$

holds.

Definition 2.6 (Single $\mathcal{L}_{p,q}$ string stability)**.** *A vehicle platoon is* $\mathcal{L}_{p,q}$ *single final string stable, if for every* $i \in \{1, 2, \dots, N\}$ *and every input signal* $z(t)$ *such that for* $j \neq i$ $z_j(t) = 0$ *and*

$$||z_i||_p < \infty \tag{21}$$

there exists R *for all* $N \in \mathbb{N}$ *such that for the output*

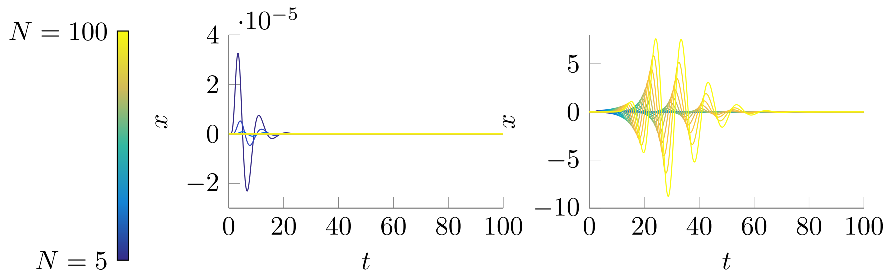
$$||y||_q < R \tag{22}$$

holds.

To illustrate that it is important to consider these different variations, we show an example of a system that exhibits single with input 1 final string stability, however is clearly general string unstable. Fig. 4 shows the responses of a vehicle platoon with increasing N to a disturbance input. First, we assume that the disturbance acts on the first vehicle alone and we are interested in the last vehicle's response, which is shown in Fig. 4a. From this, we see that the system is single with input 1 final string stable, since with increasing N the peak does not grow. However, this observation alone does not guarantee that the system is general string stable or even single string stable. This becomes clear when we assume that the disturbance acts on the last vehicle instead and we are interested in the effect on the first vehicle, which is shown in Fig. 4b. Here, the peak in the position clearly increases with increasing N which means the system is general string unstable as well as single string unstable.

These different variations are linked to a certain degree. Specifically, using in all cases the same norms $\mathcal{L}_p$ and $\mathcal{L}_q$ we find the relations below, see also Fig. 5 for a visual overview. Note that these relations follow directly from the definitions. For completeness, the proofs can be found in Appendix A.

1. General $\mathcal{L}_{p,q}$ string stability implies final $\mathcal{L}_{p,q}$ string stability.
2. General $\mathcal{L}_{p,q}$ string stability implies single $\mathcal{L}_{p,q}$ string stability.
3. General $\mathcal{L}_{p,q}$ string stability implies single final $\mathcal{L}_{p,q}$ string stability.



(a) Last vehicle response to a disturbance acting on the first vehicle.

(b) First vehicle response to a disturbance acting on the last vehicle.

Figure 4: Example for a system that experiences single final string stability, but not general string stability. The system uses homogeneous agents with open loop transfer function $L(s) = \frac{6s+5}{s^2}$ and a bi-directional communication structure.

4. Single $\mathcal{L}_{p,q}$ string stability implies single final $\mathcal{L}_{p,q}$ string stability.
5. Final $\mathcal{L}_{p,q}$ string stability implies single final $\mathcal{L}_{p,q}$ string stability.
6. Single $\mathcal{L}_{p,q}$ string stability implies single with input α $\mathcal{L}_{p,q}$ string stability.
7. Single final $\mathcal{L}_{p,q}$ string stability implies single with input α final $\mathcal{L}_{p,q}$ string stability.

As mentioned previously, in the special case of $\mathcal{L}_2$ string stability, we can find the following necessary condition that uses the transfer matrix of the vehicle platoon. In this relation we use $X(s)$ to represent the transfer matrix from the input $z(t)$ to the output $y(t)$. As previously mentioned, the input can be either the reference input $y_{ref}(t)$ or the output disturbance $\omega(t)$.

Fact 2.1. *The vehicle platoon is general $\mathcal{L}_2$ string unstable if the $\mathcal{H}_\infty$ norm of the MIMO transfer matrix $X(s)$ is not bounded independently of the string length N, where $X(s)$ is the transfer matrix from the input $z(t)$ to the output $y(t)$.*

In other words, $\mathcal{L}_2$ string stability is present only if

$$\sup_{\omega}(\sigma_{max}(X(\mathbf{j}\omega))) < \infty \tag{23}$$

for all $N \in \mathbb{N}$.

Similar conditions can be obtained for the other variations such as the ones given below. Here, we use subscripts to specify certain entries of the transfer matrix $X(s)$. So we use $X_{i,j}(s)$ to refer to the entry in column j and row i. Further, we use : to indicate all entries, i.e. if we refer to a complete row or column.

Fact 2.2. *The vehicle platoon is single with input 1 $\mathcal{L}_2$ string stable only if the $\mathcal{H}_\infty$ norm of the MIMO transfer matrix $X_{:,1}(\boldsymbol{j}\omega)$ is bounded independently of*

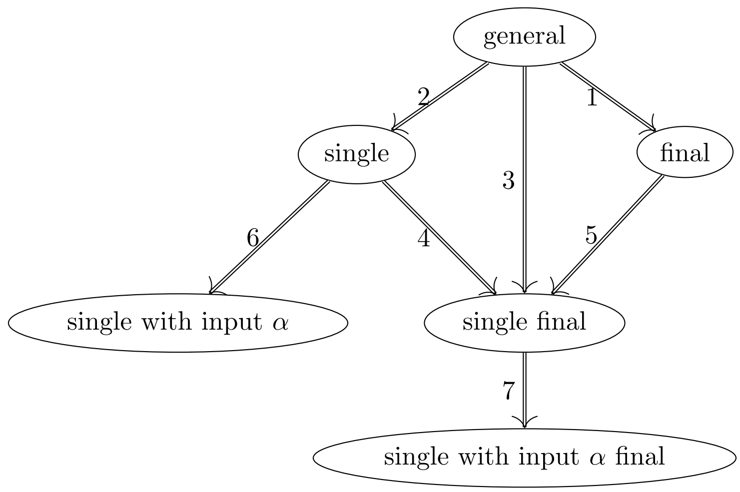


Figure 5: Overview over relations among different variations of $\mathcal{L}_p$ string stability.

the string length N, where $X(\boldsymbol{j}\omega)$ is the transfer matrix from the input $z(t)$ to the output $y(t)$.

Fact 2.3. *The vehicle platoon is single with input 1 final $\mathcal{L}_2$ string stable only if the $\mathcal{H}_\infty$ norm of the MIMO transfer matrix $X_{1,n}(\boldsymbol{j}\omega)$ is bounded independently of the string length N, where $X(\boldsymbol{j}\omega)$ is the transfer matrix from the input $z(t)$ to the output $y(t)$.*

The reasoning behind the above conditions follows directly from the definitions of the $\mathcal{L}_2$ signal norm and the $\mathcal{H}_\infty$ system norm. Note that similar conditions can be obtained for $\mathcal{L}_{2,\infty}$ string stability in terms of the $\mathcal{H}_2$ system norm or for $\mathcal{L}_\infty$ string stability in terms of the $\mathcal{L}_1$ norm.

2.3.2. Internal String Stability Notions

An equally important notion of string stability is internal string stability (Cook, 2005; Klinge, 2008; Swaroop and Hedrick, 1996; de Wit and Brogliato, 1999). For this notion, instead of considering an input-output relation one considers autonomous systems with initial conditions and the relation to the internal states of the system. As for input-output string stability, it is important to note what norm is used. Hence, we define *internal $\mathcal{L}_{p,q}$ string stability* as follows.

Definition 2.7 (Internal $\mathcal{L}_{p,q}$ string stability)**.** *A networked system is internal $\mathcal{L}_{p,q}$ string stable, if there is a point x^* such that for any initial conditions $x(0)$ with $||x(0) - x^*||_p < \infty$, there exists R for all $N \in \mathbb{N}$ such that*

$$||x(t) - x^*||_q \leq R. \tag{24}$$

In general internal string stability is a very different notion from input-output string stability.

Remark 2.1. *As for input-output $\mathcal{L}_{p,q}$ string stability, different variations can be defined depending on the set-up of the initial conditions and output signal considered, see for example Klinge (2008). This means, that we can use similar prefixes as before, such as single, single final, final, single with initial condition α and single with initial condition α final. For the sake of shortness, we will not give definitions for each of these variations.*

Since internal string stability and input-output notions of string stability are not equivalent, these two notions can be combined. For example, we can require that the response to the initial state must decay and the output must remain bounded for all inputs. Such an approach is for example taken in Knorn et al. (2014) in terms of $\mathcal{L}_{2,\infty}$ string stability and in Ploeg et al. (2014b) in terms of $\mathcal{L}_p$ string stability. In particular, the latter proposes a very general definition of (strict) string stability in terms of the $\mathcal{L}_p$ norm. This general definition combines internal $\mathcal{L}_p$ stability with single $\mathcal{L}_p$ string stability using as investigated input the reference to the first vehicle. Here, we will call this notion *input-to-state string stability.*

Definition 2.8 (Input-to-state string stability)**.** *A networked system is input-to-state $\mathcal{L}_{p,q}$ string stable, if for any initial condition $x(0)$ with $||x(0) - x^*||_q < \infty$ and input $z(t)$ with $||z||_p < \infty$, there exists R such that*

$$||x(t) - x^*||_q < R. \tag{25}$$

Remark 2.2. *For a linear networked system input-to-state string stability and internal string stability are equivalent, which is a direct consequence of the system linearity.*

Remark 2.3. *As for the internal and input-output string stability different variations can be defined depending on the initial conditions, the inputs and outputs considered, e.g. Ploeg et al. (2014b). It is possible to use again the same prefixes to denote the several variations. However, for input-to-state string stability it is possible to vary both the input and the initial condition considered. This means when checking any form of "single" string stability it has to be specified which vehicle is considered for the input and which for the initial condition. We suggest that for the sake of simplicity, the same vehicle is chosen for both. The case where different vehicles are selected is covered by the general notion.*

2.3.3. Characteristics of the Bound

Finally, the last main distinction among string stability notions is due to the characteristics of the amplification bound, which includes the rate of increase and an enforced decrease. While the above variations described different versions, these notions allow for a more detailed description of the string instability that occurs.

We give an overview of the main notions in Table 2 focusing on input-output general $\mathcal{L}_{p,q}$ string stability (that is, no specific choice for the form or location of the considered input/output signals is taken). Analogous definitions can be obtained for all other notions and versions defined in this paper. To emphasise that we talk about the input-output general $\mathcal{L}_{p,q}$ string stability as defined in Definition 2.1, we refer to this as *normal* $\mathcal{L}_{p,q}$ *string stability.*

The first notion is strict string stability and is very widely used (though not always under this name) (Herman, 2016; Peppard, 1974; Ploeg et al., 2014b; Rogge and Aeyels, 2008). It is only defined for vehicle strings, which clearly determine a leader follower structure, such as present in uni-directional communication structures. These vehicle strings can be represented using a directed path graph. Hence, when considering strict string stability, we assume that a single leader and a well defined order of followers can be identified. Then, we can formally define *strict* $\mathcal{L}_{p,q}$ *string stability* as follows.

Definition 2.9 (Strict string stability from vehicle i)**.** *Assume a vehicle platoon with clearly defined single leader and followers. Then, the platoon is strict* $\mathcal{L}_{p,q}$ *string stable from vehicle* i *if it is* $\mathcal{L}_{p,q}$ *string stable and for any input signal* $z_i(t)$ *with* $||z_i||_p < \infty$ *and vehicles* j *and* k*, where* j *is any vehicle following* i *and* k *is a direct follower of* j*,*

$$||y_k||_q \leq ||y_j||_q \tag{26}$$

for all $j \geq i, k = j + 1$*.*

We say that the system is *strict* $\mathcal{L}_{p,q}$ *string stable* if it is string string stable from vehicle 1. In this case a condition in terms of transfer functions can be given such that a vehicle platoon is strict $\mathcal{L}_2$ string stable in the sense of single final if the transfer function from a vehicle k to its follower $k+1$ is smaller than 1 for any vehicle k, i.e.

$$\left|\frac{Y_{k+1}(\mathbf{j}\omega)}{Y_k(\mathbf{j}\omega)}\right| < 1. \tag{27}$$

This form of string stability is practically very important since it enforces the disturbance effect to decrease for large strings rather than just being bounded. This means that a disturbance will ultimately die out rather than being passed down the complete string. Some works refer to such a behaviour as the slinky effect not being present or as strong string stability.

On the other hand a weaker form of string stability is *harmonic string stability* formally defined below.

Definition 2.10 (Harmonic string stability)**.** *For every input signal* $z(t)$ *with* $||z||_p < \infty$ *there exists* R *independent of* N *such that*

$$||y||_q < R^N \qquad \forall N. \tag{28}$$

In fact this weaker form allows the bound to increase with the length of the string, but the increase is sub-exponential. Hence, a harmonic string stable system is not necessarily normal string stable. In a practical sense this definition is useful to deal with systems that are unavoidably normal string unstable.

When considering the velocity as the output variable, it is sometimes referred to as *flock stability* (Cantos et al., 2016a; Herman et al., 2016).

Remark 2.4. *It is possible to define another variation that allows for a sub linear increase. However, to the best of our knowledge this has not yet been considered.*

The following relations follow directly from the definition of the various forms. For the sake of completeness the proofs are given in Appendix B.

1. Normal string stability implies harmonic string stability.
2. Strict string stability from $\bar{N}$ implies normal string stability.
3. Strict string stability from $\bar{N}$ implies harmonic string stability.

Finally, we define *weak string stability* as follows.

Definition 2.11 (Weak string stability)**.** *For any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ (independent of N) such that for any input $||z||_p < \delta(\epsilon)$ it follows that*

$$||y||_q < \epsilon \quad \forall N.$$

The concept of weak string stability (Knorn et al., 2014; Rogge and Aeyels, 2008), includes a concept of locality, which is not present in the other notions, i.e. the input can be selected small enough to guarantee an arbitrarily small bound on the output.

Remark 2.5 (Eventual string stability)**.** *In Khatir and Davidson (2004) another form of string stability is introduced denoted eventual string stability. Their definition is that a vehicle platoon is eventual string stable if there exists a vehicle $\bar{N}$ by which the resultant system is string stable for all following vehicles $i \geq \bar{N}$. This implies, as for the notion of strict string stability, that the considered vehicle platoon has a defined leader and following ordering. We believe that this form of stability is covered indirectly by the definitions given in Tables 1 and 2.*

Remark 2.6. *Another important characteristic of a vehicle platoon, however not a safety issue, is coherence or rigidity (Bamieh et al., 2012), which measures the notion of how well the platoon resembles a solid object. In vehicle platoons this phenomenon appears as slow spatial wave modes viewed over the whole string that propagate unregulated, even though the local spacing between the vehicles is well regulated. This phenomenon is not directly related to string stability and looks at a macroscopic view of the system. In fact it is possible for a platoon to be string stable without being coherent.*

2.4. Selected Contributions

It has been shown that for linear-time invariant systems under certain conditions string instability can occur for certain strings (Seiler et al., 2004). While in some cases, e.g. a priori known short string length, eventual instability and string instability are not crucial, in many applications these are important indications for the viability of the vehicle platoon system. There are hence many results analysing stability and string stability as well as finding methods to design controllers that ensure these properties. Methods that are suggested to avoid string instability include non-linear control structures (Yanakiev and Kanellakopoulos, 1998), non-homogeneous control structures (Barooah et al., 2009; Khatir and Davidson, 2004; Lestas and Vinnicombe, 2007), relaxing the formation rigidity (Swaroop and Hedrick, 1996), increasing the communication among the vehicles (Cook, 2007; Middleton and Braslavsky, 2010; Seiler et al., 2004; Swaroop and Hedrick, 1999), and appropriate selection of the spacing policy (Chien and Ioannou, 1992; Cook, 2007; Herman, 2016; Klinge and Middleton, 2009; Middleton and Braslavsky, 2010).

Due to these reasons it is important to obtain results regarding stability and string stability of the platoon in relation to the information exchange, control objective, and controller selection, as well as find limitations that exist independent of the actual controller and the communication involved. In the remainder of this section, we will summarise the main results related to stability and string stability of the various notions of vehicle platoons both for non-cyclic, in a first instance, and cyclic structures. In Section 3 we then discuss briefly how the concepts of eventual instability and string instability can be extended to more general networked systems and mention available results for these systems.

2.4.1. Non-cyclic Interconnections

Most studies concentrate on cases with linear dynamics, and where the spacing policy is to maintain a constant inter-vehicle distance. We will here first consider works that investigate vehicle platoons under this control objective. In a second step, we look at the possibility of relaxing the control objective by changing the spacing policy.

Often there is solely relative position information available from the immediate predecessor and possibly the follower. If only the predecessors information is used the control is termed uni-directional (Cook, 2007; Middleton and Braslavsky, 2010; Rivas, 2015; Seiler et al., 2004), while if the follower is included in the control we speak of bi-directional control (Barooah and Hespanha, 2005; Cook, 2007; Herman et al., 2015a,b; Lestas and Vinnicombe, 2007; Martinec et al., 2016; Middleton and Braslavsky, 2010; Rivas, 2015). The use of local information alone eliminates the need for communication devices on board, since the information can be obtained using sensors alone. The communication pattern is reflected in the structure of the resulting graph Laplacian. For example, for bi-directional communication, we find a Laplacian of the form (Barooah and

Hespanha, 2005)

$$\begin{bmatrix} 1 & -0.5 & & & \\ -0.5 & 1 & -0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.5 & 1 & -0.5 \\ & & & -0.5 & 0.5 \end{bmatrix}.$$

Note, that the first and last vehicle are special since they do not possess a leader and follower, respectively.

Remark 2.7. *The term directional here does not relate to the previous use in terms of the associated graph being un-directed or directed. In fact a symmetric bi-directional control leads to un-directed graph Laplacians, while directed graph Laplacians are mostly linked to bi-directional but asymmetric control.*

The motivation for more general controllers and communication structures stems from the fact that string instability is unavoidable for uni-directional configurations in linear vehicle platoons irrespective of the individual controller used as long as the objective is to maintain a constant spacing (shown in Seiler et al. (2004)). Hence, as an alternative to expanding the communication among the agents, relaxing the spacing policy, or resorting to non-linear control structures, bi-directional controllers are of high interest.

In Barooah and Hespanha (2005) it is shown for a bi-directional controller that if $L(s)$ contains more than 2 integrators the system is eventual unstable, i.e. it becomes unstable for large N. In the case of two integrators they obtain two additional conditions to maintain closed loop stability independent of N. The first condition limits the steady state magnitude of the open loop transfer function $L(s)$ disregarding the integrators, i.e. $\frac{N(0)}{D(0)} > 0$. The second condition is developed linking the stability of the networked system to the stability of N transfer functions defined as

$$G_i(s) = \frac{1}{1 + \lambda_i L(s)}, \tag{29}$$

where λ_i are the eigenvalues of the Laplacian $\mathbb{L}$. This second condition holds as well for other communication structures, including cyclic structures (e.g. Stüdli et al. (2017)).

In regard to string stability it has been shown that bi-directional control can significantly improve the issue (Lestas and Vinnicombe, 2007; Middleton and Braslavsky, 2010). In fact, by proper selection of the control gains in $K(s)$ these systems are $\mathcal{L}_2$ string stable (de Wit and Brogliato, 1999). A similar result is obtained in Cook (2007), where it is shown that bi-directional control can avoid the need for a velocity dependent spacing policy, even though the inter vehicle spacing will still be dependent on the length of the vehicle string, under appropriate selection of the controller gains, when the jerk of the vehicles should be bounded for comfort reasons. However, with this approach long transients are caused if the string length grows (Cook, 2007; Herman, 2016; Middleton and Braslavsky, 2010; Rivas, 2015).

One method to deal with these long transients is the use of asymmetric control, i.e. the control law uses the information from leading and following vehicles differently (Barooah et al., 2009; Cook, 2007; Herman, 2016; Herman et al., 2015a; Martinec et al., 2016). In Barooah et al. (2009) the authors show that so called mistuning, i.e. the selection of different controller gains for predecessor and follower, can reduce the sensitivity to disturbances in the sense of $\mathcal{L}_2$ string stability, but does not necessarily achieve string stability. However, their approach assumes that the vehicles know the desired velocity and make use of this information in their controller. Similarly, this allows the author in Cook (2007) to avoid the introduced limitation on the spacing policy completely in the case where the jerk for the vehicles is limited.

Herman et al. (2015a) extends the results on asymmetric control strategies where the asymmetry is modeled as a different weighting factor in the Laplacian graph. They conclude that a vehicle platoon with eigenvalues of the Laplacian uniformly bounded away from zero is harmonically string unstable independent of the linear controller utilised without further knowledge, such as desired velocity as is used in Barooah et al. (2009).

This motivated a condition of positional symmetry, derived in Cantos et al. (2016b); Martinec et al. (2016) using the wave approach, which if not satisfied leads to string instability for homogeneous agents. This condition does not hold for other forms of asymmetry, for example in velocity feedback, where asymmetric graph structures can improve the performance of the vehicle platoon in terms of the transient behaviour. Their results are related to previous work of the authors in Herman et al. (2016) where a cyclic interconnection of vehicles with three integrators is investigated to find results for the non-cyclic interconnection of such vehicles. As in Martinec et al. (2016) positional symmetry has to hold for asymptotic stability and flock stability of the non-cyclic interconnection.

Besides introduction of asymmetry and bi-directional control, the increase in information exchange is suggested to alleviate string instability. For example, Barooah et al. (2009) assumes the knowledge of a desired velocity in combination with bi-directional asymmetric control, whereas Rivas (2015); Seiler et al. (2004) additionally use the position or velocity of the leader in the controller. While the first approach does not necessarily require additional communication among the vehicles in an ideal setting, the latter means that the communication will increase with the platoon length. Note however, that in a real system the desired velocity is still given by the first (leader) vehicle and might change, for example due to changed road conditions. In view of these changes also the first approach requires an ever increasing communication range. This additional communication may cause broadcasting delays that need to be considered, which is done in Rivas (2015). There, the broadcast of the velocity of the leader vehicle is investigated in detail. In particular it is shown that leader velocity broadcast provides string stability if the controller gains are selected properly, without the need to know the number of vehicles in the platoon nor the individual inter-vehicle spacing of other vehicles. Further, for velocity broadcast string stability is not affected by communication delays that are linearly proportional to the position in the platoon.

Alternatively, it is possible to increase the communication range locally rather than including leader position or velocity. This approach is investigated in Cook (2007) for unidirectional control and Herbrych et al. (2017) for next nearest neighbour communication. While the use of a wider communication range does not avoid string stability issues, they can be lowered considerably. The same holds for bi-directional vehicle strings as shown in Middleton and Braslavsky (2010), which proves that an increased but limited communication range does not eliminate single final $\mathcal{L}_2$ string instability, even though the amplification can be reduced. This fact has been used in Stüdli et al. (2017) to achieve string stability in cyclic interconnections. Note that the benefits of increased local communication are not as clear for relaxed spacing policies. Such a case has been investigated in Ploeg et al. (2014a), where it is shown that a locally increased communication range, when using a constant time gap policy, is only beneficial in the presence of large time delays in the communication path.

Remark 2.8. *There is a subtle difference between the two approaches of increasing the communication range locally while keeping it bounded, and the broadcast of the leader velocity or position. In the latter the communication range is in fact not bounded, since the total communication length increases with growing string length. This means that for a local communication approach to achieve string stability the communication range may need to increase with a growing vehicle string.*

Finally, a commonly accepted way to achieve string stability is the relaxation of the spacing policy. The works reviewed above all aimed for a constant spacing between the vehicles, which is ideal in terms of efficiency. However, the use of a velocity dependent spacing by introduction of a constant time gap can achieve string stability (Chien and Ioannou, 1992; Cook, 2007; Herman, 2016; Klinge and Middleton, 2009; Middleton and Braslavsky, 2010). One main condition in this case for string stability is the size of the time gap, i.e. a large enough time gap achieves stability. The drawback of this technique is that it increases the actual inter-vehicle spacing for high speeds and hence lowers the efficiency.

The time gap can be incorporated into the model by changing the reference input to include, in addition to a constant reference, a velocity dependent part, i.e. the reference itself becomes state dependent. This can be incorporated by a change in the Laplacian $\mathbb{L}$ instead. The inclusion of this term becomes straightforward when using the position of the vehicles as measured output and the inter-vehicle distance as reference. In that case, as remarked previously, the graph Laplacian can be split in two parts. Then, $\mathbb{L}$ is the product of those two parts, the one that maps the output to the reference, M, and the part that maps the reference to the control inputs, Γ. In this set-up the constant time gap can be modeled by adding a diagonal matrix H containing the selected time gap for each vehicle to M, such that we use $M + sH$ instead. This means that $\mathbb{L}$ is no longer constant, which requires more detailed analysis methods.

Middleton and Braslavsky (2010) concludes that while a time gap for a uni-directional string does not change the string instability qualitatively, a large enough time gap can lead to single $\mathcal{L}_2$ and $\mathcal{L}_\infty$ string stability of the vehicle platoon (Klinge, 2008). Also, in Cook (2007) the constraint of the jerk introduces a lower bound on the time gap. The bound on the time gap can be lowered by the increase of the communication range, which also lowers the absolute inter-vehicle spacing.

In Yanakiev and Kanellakopoulos (1998) the approach of velocity dependent spacing is extended and nonlinear spacing policies are suggested to achieve string stability while decreasing the actual spacing between vehicles.

While most of the above mentioned results consider the homogeneous case, it is important to discuss the implications of non-homogeneous dynamics (Middleton and Braslavsky, 2010). In some instances it is the approach taken to achieve string stability by using heterogeneous controllers. For example, in Khatir and Davidson (2004) controllers with linear increasing gains are chosen to achieve eventual string stability. However, in Middleton and Braslavsky (2010) it is shown that heterogeneous control in general does not overcome string instability unless the control bandwidths are allowed to diverge with increasing string lengths.

As we discussed in this section non-cyclic vehicle platoons suffer from string instability, which is present for uni-directional as well as bi-directional communication structures with limited communication range for constant distance spacing policies or a too short time gap. In the case of bi-directional communication it is important to distinguish between symmetric and asymmetric approaches. The former can actually achieve $\mathcal{L}_2$ string stability with proper controller selection, however introduces long transients. The latter, while reducing the issue of transients, does no longer achieve string stability. Hence, the notion of positional symmetry has been introduced in Herman et al. (2016). In this approach, the control based on position is symmetric, while other control inputs, such as velocity, are treated asymmetrically. This seems to achieve a good trade off between the two control strategies. The two main approaches that successfully achieve string stability, are an unbounded increase of the communication range, for example by broadcasting the velocity of the leader, and the introduction of other spacing policies, most commonly a velocity dependent policy by maintaining a constant or variable time gap.

2.4.2. Cyclic Interconnections

In parallel to work reported on the non-cyclic structures research has also focused on cyclic structures. In some cases cyclic system properties are used to obtain results for non-cyclic interconnections (Herman et al., 2016, 2015b). The reason to start with cyclic interconnections is that the analysis simplifies considerably, since the Laplacian is a circulant matrix. For instance Cantos et al. (2016a); Herbrych et al. (2017); Herman et al. (2015b) use the conjecture that an unstable cyclic system implies that the system using the associated path graph as communication structure is asymptotically or flock unstable. This conjecture is then used in Herman et al. (2016) to obtain the results for non-cyclic systems.

While for non-cyclic systems stability is normally not critical, cyclic systems do not exhibit generally a stable behaviour. Hence, the investigation of cyclic systems is focused on eventual instability. Only for eventually stable systems the notion of string stability becomes important.

As for the non-cyclic case the communication structure can both be uni-directional (Peters et al., 2016) or bi-directional, and can include the use of information from the immediate predecessor and successor (Herman et al., 2015b; Rivas, 2015) or extended communication ranges (see for example Herbrych et al. (2017) where 2-nd order systems and an extended communication range of 2 vehicles is investigated). In the uni-directional case it is shown in Peters et al. (2016); Rivas (2015) that eventual instability is unavoidable. This aligns with the results in Seiler et al. (2004), where string instability of a uni-directional string of vehicles is unavoidable. Note that this again holds only for the case where a constant inter-vehicle distance spacing policy is selected.

Hence, similar techniques as for non-cyclic structures can be used, including bi-directional control, both symmetric and asymmetric, increase of the communication range, use of other spacing policies, as well as the use of heterogeneous controllers and non-linear controllers. In most cases the results found for cyclic interconnections reflect the ones found for non-cyclic ones discussed above.

For example, the relation between a time gap, i.e. a velocity dependent inter-vehicle distance spacing policy, and the friction present in vehicles is investigated in Rogge and Aeyels (2008). They find conditions that guarantee stability and string stability, respectively. These conditions rely on the friction present, the time gap and the controller gain. Their model is a simple point mass model including friction. Using these conditions they are able to conclude, in agreement with the other results, see Peters et al. (2016); Rivas (2015), that a frictionless system becomes unstable for large enough N if the time gap is 0. These reflect results in Klinge and Middleton (2009).

Similarly, in Herman et al. (2015b) the authors show that a positional symmetry has to hold for stability. Further, they extended the result to show that if the vehicles open loop systems contain more than two integrators instability is unavoidable for growing string lengths independent on the symmetry or asymmetry present. This is consistent with the result in Herman et al. (2016); Martinec et al. (2016).

3. General Networked Systems

Similar stability and performance results are sought for general networked systems, for example in areas of formation control. Especially, limitations on the performance are very important for the design of controllers. To this end it is important to generalise the notion of string stability to general networked systems, which is for example done in Li et al. (2011); Yadlapalli et al. (2006).

In regard to stability the identical principle of eventual instability can be used for any network. Ideally, similar string stability definitions can also be utilised. In fact, we can reuse the definitions for general, single, and internal $\mathcal{L}_{p,q}$ string stability, as well as harmonic string stability, without any change, for example Yadlapalli et al. (2006) uses both single and general $\mathcal{L}_\infty$ string stability considering as input either the velocity or disturbances. However, other versions and notions cannot be directly applied. For example, the version of final string stability does no longer make sense when there is no final vehicle. Under these considerations, in Section 3.1 we will propose generalisations of the string stability notions given in Section 2.3 for networked systems. Afterwards, we will pinpoint to some selected contributions in that direction in Section 3.2.

3.1. Networked Stability

One remarkable difference between a vehicle platoon and a more general networked system is that the communication structure as well as the performance measure are more complex. So in the case of a vehicle platoon if the string is increased by one vehicle, this vehicle appends to the last vehicle, and its performance measure is usually well defined in terms of the distance between some of its predecessors and maybe virtual followers. However, in the case of general networks there are usually multiple locations where the additional agent can be inserted, as well as other performance measures that could be considered. Hence, to be able to generalise the notion of string stability to a form of networked stability, the networked system has to grow in a structured and well defined way. Here, we assume this structure as given such that if the notion of networked stability makes reference to all $N \in \mathbb{N}$, it is inherently clear how the network will expand with increasing N. Not in all networked systems such an increase in size will necessarily occur in practice. However, especially in transportation systems and distribution systems, the actual system size can grow significantly. In these areas it is where a generalised form of string stability becomes important.

Remark 3.1. *In the future, it might prove useful to include certain structural expansion within the notion of networked stability.*

We will base our definitions of networked stability on our notions of string stability in Section 2.3. In that context we define input-output $\mathcal{L}_{p,q}$ networked stability with different variations. Note that the variations differ in the considered input and outputs and are related with general, single, and final $\mathcal{L}_{p,q}$ string stability. An overview is given in Table 3. We use the same notation as in Section 2.3.

General $\mathcal{L}_{p,q}$ string stability can be generalised as follows.

Definition 3.1 (General $\mathcal{L}_{p,q}$ networked stability)**.** *A networked system is general $\mathcal{L}_{p,q}$ networked stable, if for every input signal $z(t)$ with $||z||_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that*

$$||y||_q < R.$$

A more restrictive form of networked stability can be based on single final string stability. Since in a network no final agent exists, this notion has been adjusted such that it investigates the output of a single agent rather than the final vehicle. We call this variation *single input-output networked stability* and define it as follows.

Definition 3.2 (Single input-output $\mathcal{L}_{p,q}$ networked stability)**.** *A networked system is single input-output $\mathcal{L}_{p,q}$ networked stable, if for every $i \in \{1, \ldots, N\}$ and input signal $z(t)$ with $z_j(t) = 0$ for $j \neq i$ and $||z_i||_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that*

$$||y_\ell||_q < R \qquad \forall \ell \in \{1, \ldots, N\}.$$

Then, as for string stability, the remaining variations relax the restriction on the input and output. These variations are given in the two formal definitions below.

Definition 3.3 (Single input $\mathcal{L}_{p,q}$ networked stability)**.** *A networked system is single input $\mathcal{L}_{p,q}$ networked stable, if for every $i \in \{1, \ldots, N\}$ and input signal $z(t)$ with $z_j(t) = 0$ for $j \neq i$ and $||z_i||_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that*

$$||y||_q < R.$$

Definition 3.4 (Single output $\mathcal{L}_{p,q}$ networked stability)**.** *A networked system is single output $\mathcal{L}_{p,q}$ networked stable, if for every input signal $z(t)$ with $||z||_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that*

$$||y_\ell||_q < R \qquad \forall \ell \in \{1, \ldots, N\}.$$

Further, the notion of internal and input-to-state string stability can be translated directly to general networked systems without any changes. The same holds for the notions of harmonic and weak string stability.

The notion of strict string stability has to be more carefully treated. To this extend we limit the notion of strict networked stability to networks with a single leader and tree structure. Further, we use the term agent in level k as an agent where there are $k - 1$ agents between itself and the leader. So for example an agent i following directly the leader is on level 1, an agent directly following agent i is on level 2, and so forth. Then, we can define a form of strict networked stability as follows.

Definition 3.5 (Strict $\mathcal{L}_{p,q}$ networked stability). *A networked system with a single leader and a well defined tree structure, is strict $\mathcal{L}_{p,q}$ networked stable from level $\bar{N}$, if it is $\mathcal{L}_{p,q}$ networked stable and in addition for any input signal $z_i(t)$ with $||z_i||_p < \infty$ and agents j and k, where j is any agent following i on a level higher or equal than $\bar{N}$ and k is a direct follower of j,*

$$||y_k||_q \leq ||y_j||_q \tag{30}$$

for all j, k.

3.2. Selected Contributions

Li et al. (2011) uses a notion of $\mathcal{L}_2$ and $\mathcal{L}_{2,\infty}$ networked stability. Their goal is to design a controller such that the total system is asymptotically stable and the H_∞ or H_2 norm of the transfer function from a disturbance to an output is smaller than a design parameter γ. For this purpose they define performance regions that are based on a parameter σ. Then, the performance region spans the values of this parameter that lead to a stable and well performing system in regard to the design parameter γ. The parameter σ gives a direct bound on the eigenvalues of the Laplacian. They also find using this approach a limit for the performance of any controller. Their investigations use an un-directed, un-weighted graph as well as state feedback, but holds for arbitrary graph structures.

An approach based on $\mathcal{L}_\infty$ string stability is used in Yadlapalli et al. (2006). Here, they consider a homogeneous control structure with linear dynamics and un-directed information exchange. Their performance variable is the spacing error of the vehicles to a reference vehicle, as well as the relative velocity to this vehicle. Note that even though they use this as a performance indication they do not assume that all vehicles have access to that information. In that regard they show that the overall system is unstable if the open loop systems of the individual vehicles have more than two integrators. This result generalises results found for vehicle strings (Barooah and Hespanha, 2005; Fax and Murray, 2004). Additionally, they find that at least one vehicle needs to communicate with a considerable part of the complete formation for stability and string stability in their sense. This links well with the use of the leader position to avoid string instability in vehicle platoons (Seiler et al., 2004).

In Tonetti and Murray (2010, 2011) classic control principles combined with Mason's Direct rule are used to obtain performance limitations similar in nature to Bode's integral formula. They show that control merely distributes disturbance rejection at low frequencies between agents. This means by improving one agent's disturbance rejection another agent's performance becomes worse. While the cyclic interconnection influences the low frequency disturbance rejection properties, the Laplacian spectrum influences the peak in the sensitivity function. Hence, they conclude that for disturbance rejection a two-degree of freedom controller should be used. Their analysis is based on arbitrary networks and includes results both for homogeneous and heterogeneous systems. However, they do not consider weighting of the topology.

4. Concluding Remarks

The field of networked systems has been studied in detail, with emphasis on vehicle strings. In this context the notion of string stability becomes important, since it captures the critical property that the errors will increase unbounded with increasing numbers of participating agents, even though the system can be stable in the usual sense. This property is mostly studied for vehicle strings, where it has been shown that without any counter measure, such as increasing communication range, velocity dependent spacing policies, etc., string instability occurs. While the case of vehicle platoons is very particular, similar effects can occur in more general networked systems, as discussed briefly. Hence, the investigations of why and when these occur become extremely important for the design of controllers.

In this paper we reviewed the main results obtained for vehicle platoons. Especially, we formalised different variations of string stability that are commonly investigated and illustrated the importance in the distinction between those with some examples. We hope that this will help in the future to clarify and unify the way string stability is presented. Further, we summarised briefly how these definitions could be generalised for any networked system.

Even though the field of string stability and networked systems led to a wide range of results, there are several open problems. These are for example:

- Constant distance spacing policy and infinite communication length: While it has been shown that increased, but limited communication range does not avoid string instability in non-cyclic vehicle strings (Middleton and Braslavsky, 2010), the use of the leader velocity or position achieves string stability (Seiler et al., 2004). This latter set-up achieves a form of unbounded communication. It is yet to be investigated whether a local unbounded communication structure results equally in string stability. For cyclic interconnection it has been shown that a linearly increasing local communication range achieves string stability (Stüdli et al., 2017).

- Incorporation of weighting: Most research assumes equal weights when treating the information from various sources. However, it has been shown that in vehicle strings the use of asymmetry can improve the response (Barooah et al., 2009). In a similar way asymmetric weights in multi-hop vehicle strings might have a benefit in regard to performance.

- Limitations in control: As mentioned before, for some configurations string instability is unavoidable regardless of the linear controller used. It is important to obtain performance limitations for networked systems to understand what is achievable. These limitations are particularly insightful if they are independent of the controller that is used. Naturally, these might depend on other design parameters, such as the control objective, communication structures, and system characteristics.

- Other performance measures: As we mentioned previously, it is important to distinguish between the different measures that are used, e.g. difference between $\mathcal{L}_\infty$ and $\mathcal{L}_2$ string stability. While $\mathcal{L}_2$ is the most often discussed string stability notion it is important to investigate and verify the results for other measures, such as $\mathcal{L}_\infty$, $\mathcal{L}_{2,\infty}$ or power signal norm. Further, it is essential to establish relations between the different measures for obtaining an overall understanding. Especially, the more practical notion of $\mathcal{L}_\infty$ string stability can benefit of such results.

- Communication structure: The communication structure can be of great importance in determining eventual stability or string stability. The conjecture in Herman et al. (2015b) links eventual stability for vehicle platoons with cyclic communication to string stability in platoons with non-cyclic communication, for single final string stability. Similarly, Cantos et al. (2016a) uses this property for short communication ranges. It is desirable to expand such relations between various communication structures in a formal manner including rigorous proofs and the various types of string stability.
- Uncertainties: The robustness analysis mostly concentrates on disturbance amplification. Another, important aspect however is the presence of uncertainties in the model. At present, the approaches to this topic are fragmented, and primarily rely on either restrictive assumptions or are based on small perturbations.
- Faults and security: Due to the reliance on sensors, actuators, and partly communication, it is also important to venture in the area of fault or irregularity detection and correction. Initially, these irregularities or faults that can be considered are actually a piece of hardware that is defective or not operating normally. On the other hand the analysis should be extended to include security aspects towards malicious acts, for example individuals that aim for an advantage in the case of vehicle platoons. Recent works that concentrate on this aspect are Amoozadeh et al. (2015); Dadras et al. (2015); DeBruhl et al. (2015); Gerdes et al. (2013).
- Generalisation of the concept of string stability: We gave in Section 3.1 some examples of how the concept of string stability could be generalised for generic networked systems. In doing so we assumed that a structured way is used to expand the network, i.e. if N grows. It is important to investigate the structural needs to retain networked stability and consider cases that include complex cyclic or bi-directional structures, this is the case especially for the concept of strict networked stability.

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A. Proofs of the relations among the various variations of $\mathcal{L}_{p,q}$ string stability.

In this section, we give the proofs of the relations identified in Section 2.3.1 among the various variations of $\mathcal{L}_{p,q}$ string stability.

1. General $\mathcal{L}_{p,q}$ stability implies that for any bounded input z the output y is bounded, i.e. (i.e.) using the $\mathcal{L}_q$ signal norm we find that

$$||y||_q \triangleq \left(\int_{-\infty}^{\infty} \sum_i |y_i|^q \, \mathrm{d}t \right)^{1/q} \tag{A.1}$$

is bounded. Then, (A.1) is an upper bound for

$$\left(\int_{-\infty}^{\infty} |y_N|^q \, \mathrm{d}t \right)^{1/q} = ||y_N||_q \tag{A.2}$$

and final $\mathcal{L}_{p,q}$ stability is shown.

2. We select as input signal all $z(t)$ where all besides one entry is zero, wlg. we say this input is i. Note that for such a signal $||z||_p = ||z_i||_p$. Then, general $\mathcal{L}_{p,q}$ string stability guarantees the existence of R independent of N such that the output $||y||_q$ is bounded for any input signal such that $||z||_p < \infty$, which shows single $\mathcal{L}_{p,q}$ string stability since the input considered is a particular case.

3. We select as input signal all $z(t)$ where all besides one entry is zero, wlg. we say this input is i. Note that for such a signal $||z||_p = ||z_i||_p$. Then, general $\mathcal{L}_{p,q}$ string stability guarantees the existence of R independent of N such that $||y||_q < R$. Since $||y||_q$ is an upper bound for $||y_N||_q$, this implies single final $\mathcal{L}_{p,q}$ string stability.

4. According to the definition of single $\mathcal{L}_{p,q}$ string stability, any bounded input that acts on a single vehicle produces a bounded output, i.e.

$$||y||_q \triangleq \left(\int_{-\infty}^{\infty} \sum_i |y_i|^q \, \mathrm{d}t \right)^{1/q} < \infty. \tag{A.3}$$

   Again this is an upper bound for

$$\left(\int_{-\infty}^{\infty} |y_N|^q \, \mathrm{d}t \right)^{1/q} \triangleq ||y_N||_q \, , \tag{A.4}$$

   which is the condition for single final $\mathcal{L}_{p,q}$ string stability.

5. The definition of final $\mathcal{L}_{p,q}$ string stability implies that for any input $z(t)$, the output y_N remains bounded. We select the input such that the input to all vehicles besides one remain equal to zero. For any of these inputs the output y_N remains bounded which shows single final string stability.

6. We select the input signal such that $||z_\alpha||_p < \infty$ and $z_i(t) = 0$ for all $i \neq \alpha$. Since the system is single string stable this implies that there exists R such that $||y||_q < R$, which shows single with input α string stability.

7. We select the input signal such that $||z_\alpha||_p < \infty$ and $z_i(t) = 0$ for all $i \neq \alpha$. Since the system is single final string stable there exists R such that $||y_n||_q < R$, which shows single with input α final $\mathcal{L}_{p,q}$ string stability.

B. Proofs of the relation between normal, strict, and harmonic string stability

In this section, we give the proofs of the relations identified in Section 2.3.3 between normal, strict, and harmonic string stability.

1. Note that harmonic string stability in fact allows increasing amplification with increasing string length, but bounds the rate to be sub-exponential. Hence, if a system is string stable, i.e. the amplification is bounded independent of the string length it is also harmonic string stable. More formally, given R_0 such that $||y||_q < R_0$ for all $N \in \mathbb{N}$ it follows that we can find $R_1 > 1$ such that $||y||_q < R_1 \leq R_1^N$. The existence of R_0 is given due to normal string stability.
2. This follows directly from the definition of strict string stability.
3. This is a direct consequence of items 1 and 2.

	Definition	used in
general	If for every input signal $z(t)$ with $\|\|z\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y\|\|_q < R.$$	Barooah et al. (2009); Seiler et al. (2004)
single with input α	If for every input signal $z(t)$ with $z_i(t) = 0$ for $i \neq \alpha$ and $\|\|z_\alpha\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y\|\|_q < R.$$	Barooah and Hespanha (2005)
single	If for every i and input signal $z(t)$ with $z_j(t) = 0$ for $j \neq i$ and $\|\|z_i\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y\|\|_q < R.$$	
final	If for every input signal $z(t)$ with $\|\|z\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y_N\|\|_q < R.$$	Rivas (2015)
single with input α final	If for every input signal $z(t)$ with $z_i(t) = 0$ for $i \neq \alpha$ and $\|\|z_\alpha\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y_N\|\|_q < R.$$	Khatir and Davidson (2004); Middleton and Braslavsky (2010)
single final	If for every i and input signal $z(t)$ with $z_j(t) = 0$ for $j \neq i$ and $\|\|z_i\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y_N\|\|_q < R.$$	Khatir and Davidson (2004); Middleton and Braslavsky (2010)

Table 1: $\mathcal{L}_{p,q}$ string stability or $\mathcal{L}_p$ string stability if $p = q$. Most commonly $p = q = 2$ or $p = q = \infty$ is used. Several variations are presented that are differentiated by the inputs and outputs considered. The definitions are shortened to yield a better overview and use the notation introduced in Section 2.3.

	Definition	used in
harmonic	For every input signal $z(t)$ with $\|\|z\|\|_p < \infty$ there exists R such that $$\|\|y\|\|_q < R^N \quad \forall N. \qquad (9)$$	Cantos et al. (2016a); Herman et al. (2015a)
strict from vehicle i	Assume a vehicle platoon with a clear defined single leader and followers. Then, the platoon is strict $\mathcal{L}_{p,q}$ string stable from vehicle i if it is $\mathcal{L}_{p,q}$ string stable and for any input signal $z_i(t)$ with $\|\|z_i\|\|_p < \infty$ and vehicles j and k, where j is any vehicle following i and k is a direct follower of j, $$\|\|y_k\|\|_q \leq \|\|y_j\|\|_q \qquad (10)$$ for all $j \geq i, k = j + 1$.	Herman (2016); Peppard (1974); Ploeg et al. (2014b); Rogge and Aeyels (2008)
weak	For any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ (independent of N) such that for any input $\|\|z\|\|_p < \delta(\epsilon)$ it follows that $$\|\|y\|\|_q < \epsilon \quad \forall N.$$	Knorn et al. (2014); Rogge and Aeyels (2008); Swaroop and Hedrick (1996)

Table 2: Different notions of string stability in terms of general $\mathcal{L}_{p,q}$ string stability.

	Definition
general	If for every input signal $z(t)$ with $\|\|z\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y\|\|_q < R.$$
single input	If for every $i \in \{1, \ldots, N\}$ and input signal $z(t)$ with $z_j(t) = 0$ for $j \neq i$ and $\|\|z_i\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y\|\|_q < R.$$
single output	If for every input signal $z(t)$ with $\|\|z\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y_\ell\|\|_q < R \quad \forall \ell \in \{1, \ldots, N\}$$
single input-output	If for every $i \in \{1, \ldots, N\}$ and input signal $z(t)$ with $z_j(t) = 0$ for $j \neq i$ and $\|\|z_i\|\|_p < \infty$ there exists R for all $N \in \mathbb{N}$ such that $$\|\|y_\ell\|\|_q < R \quad \forall \ell \in \{1, \ldots, N\}$$

Table 3: $\mathcal{L}_{p,q}$ networked stability or $\mathcal{L}_p$ networked stability if $p = q$. Various definitions are collected differentiated by the inputs and outputs considered.