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Ultimate bound minimisation by state feedback in discrete-time switched linear systems under arbitrary switching

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Abstract

We present a novel state feedback design method for perturbed discrete-time switched linear systems. The method aims at achieving (a) closed-loop stability under arbitrary switching and (b) minimisation of ultimate bounds for specific state components. Objective (a) is achieved by computing state feedback matrices so that the closed-loop subsystem evolution matrices generate a solvable Lie algebra (namely, they are all upper triangular in a common coordinate basis). Previous results derived an iterative algorithm that computes the required feedback matrices, and established conditions under which this procedure is possible. Based on these conditions, objective (b) is achieved by exploiting available degrees of freedom in the iterative algorithm.

Keywords: Control of switched systems, Eigenstructure assignment, Ultimate bounds, Bounded disturbances.

1. Introduction

In the last decade there has been increasing research activities in the areas of stability and stabilisability of switched systems; see, for example, [1, 2, 3]. An example of switched systems is a system with time-varying dynamics, which switches within a

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known set of modes, or subsystems, indexed by a switching signal. A problem of interest is that of stability under arbitrary switching, which consists in obtaining conditions that guarantee stability of the switched system for every switching signal. Finding these conditions in general is not a simple task except for special cases, such as when the subsystems are pairwise commutative, symmetric or normal [1, Chapter 2]. A well-known necessary and sufficient condition for exponential stability under arbitrary switching is the existence of a common Lyapunov function (CLF) for all subsystems [4]. As a CLF might be rather complex and difficult to find, most developed work focuses on the existence of a common quadratic Lyapunov function (CQLF).

While most efforts on stability and stabilisation of switched systems deal with asymptotic stability of the origin (as equilibrium point of the system), it might not be possible to achieve asymptotic stability in some situations, such as when the switched system is subject to non-vanishing disturbances. In these situations, one may seek practical stability of the system, in the sense that the system trajectories are required to ultimately lie inside a bounded region around the origin. The *ultimate bounds* of the states of the system characterise such region, which should be sufficiently small for good asymptotic disturbance attenuation performance.

Estimates of the state ultimate bounds can be obtained through the use of level sets of suitable Lyapunov functions (see for example [5, Section 9.2]). This approach is applicable to a general class of nonlinear systems, but may produce overly conservative bounds in linear systems, since the structure of the system is generally lost in the Lyapunov function [6]. Tighter estimates of ultimate bounds in linear systems can be obtained through a *componentwise* analysis technique proposed in [6]–[7], which preserves system structure and dispenses with Lyapunov functions.

The problem of reducing the effect of disturbances by feedback has been long studied and a number of robust control methods exist for the minimisation of ultimate bounds in linear systems (e.g., [8, 9, 10]), although typically relying on Lyapunov analysis. The minimisation of componentwise ultimate bounds by feedback design has been studied in [11] for linear time-invariant (LTI) systems. The authors in [11] have shown that arbitrarily small ultimate bounds can be guaranteed in continuous-time systems by assigning closed-loop eigenvalues with arbitrarily large negative real part when disturbances are “matched” to the control input (that is, disturbances in the span of the system’s control input matrix). For discrete-time systems, however, there is a fundamental limitation in rendering these ultimate bounds arbitrarily small, depending on the way the disturbance affects the state equations. In this regard, the problem of ultimate bound minimisation for discrete-time LTI systems has recently been studied in [12], where conditions were derived so that the ultimate bound on one (or more) state components can be minimised to its least possible value via eigenvalue-eigenvector assignment.

Ultimate boundedness of switched systems has been the focus of attention recently. In [13], uniform ultimate boundedness of a class of switched linear systems is addressed

through designing a switching control sequence. A control design based on the existence of a CLF is proposed in [14] and [15] to ensure uniform ultimate boundedness for switched linear systems under arbitrary switching. Furthermore, the stability and componentwise state ultimate bounds of autonomous switched systems under arbitrary switching have been analysed in [16, 17, 18], where an iterative algorithm that derives a CQLF is proposed.

The present paper examines the problem of feedback design for practical stabilisation with ultimate bound minimisation and proposes an iterative algorithm to obtain the smallest possible bounds for specific state components in discrete-time switched linear systems under arbitrary switching. To this purpose, we exploit an algorithm from [19], which iteratively seeks a set of stabilising state feedback gains that render the closed-loop subsystem matrices simultaneously upper-triangular after a change of coordinates common to all subsystems. This is a desirable property, since then the stable closed-loop subsystem matrices will generate a solvable Lie-algebra, which guarantees the existence of a CQLF [20]. We modify the algorithm from [19] to impose additional structure to the closed-loop subsystems to achieve disturbance attenuation through the minimisation of ultimate bounds by feedback design.

The first contribution of this paper is to derive conditions in terms of eigenstructure of the perturbed switched system in order for the trajectories of one or more components of the state to lie within the smallest possible bound in at most one time step. Next, we extend the results of [19] by exploiting the available degrees of freedom in the iterative triangularisation algorithm by imposing a set of conditions on the common eigenvectors at each iteration of the algorithm. The main contribution of the paper is an eigenstructure assignment procedure embedded in the extended version of the aforementioned algorithm such that the resulting stabilising feedback laws achieve the minimum possible ultimate bound for one or more states of the switched system under arbitrary switching. The results in this paper build upon preliminary work communicated in the conference paper [21].

The layout of the remainder of the paper is as follows. In Section 2, structural conditions on the system matrices for one or more ultimate bound components to be the minimum possible are presented. In Section 3, ultimate bound minimisation is addressed through iterative eigenstructure assignment, which is performed via modifications in the iterative algorithm of [19]. In Section 4, a numerical example shows the effectiveness of the proposed algorithm and finally, Section 5 concludes the paper.

Notation. The index set $\{1, 2, \dots, N\}$ is denoted by $\underline{N}$. The kernel (null space) of a matrix or linear map $A : \mathcal{X} \rightarrow \mathcal{Y}$ is denoted by $\ker A$ and its image (range), $\text{img } A$. For $x \in \mathbb{C}^{n \times m}$, its j -th row is denoted by $x_{(j,:)}$, its transpose by x' , its conjugate transpose by x^* and its Moore-Penrose generalised inverse by $x^\dagger$. Given a set of indices $\mathcal{J} \subset \{1, \dots, n\}$, $x_{(\mathcal{J},:)}$ denotes the matrix formed by the rows of x with indices in $\mathcal{J}$ and $x_{(\mathcal{J},k)}$ denotes the column vector constructed by the elements of the k -th column of x with indices in

107 $\mathcal{J}$. Also, given another set of indices $\mathcal{Z} \subset \{1, \dots, n\}$, $x_{(\mathcal{J}, \mathcal{Z})}$ is a matrix formed by the
 108 elements of x with indices (j, z) such that $j \in \mathcal{J}$ and $z \in \mathcal{Z}$. The cardinality of a set $\mathcal{J}$
 109 is denoted by $\#\mathcal{J}$. If $\mathcal{S}$ is a vector space, then $\dim(\mathcal{S})$ denotes the dimension of $\mathcal{S}$. For a
 110 column vector $x \in \mathbb{C}^n$, x_j denotes its j -th component, and $x_{\mathcal{J}}$ denotes its elements with
 111 their indices in $\mathcal{J}$. An eigenvalue $\lambda \in \mathbb{C}$ is *stable* if $|\lambda| < 1$. By extension, a matrix is
 112 stable if all its eigenvalues are stable. Two pairs of numbers (a, b) and (c, d) are distinct
 113 if $a \neq c$ and $b \neq d$.

114 2. Problem statement: Tightest ultimate bounds by feedback

115 The problem of ultimate bound minimisation for discrete-time switched systems is
 116 studied in this section. Consider a perturbed discrete-time switched linear system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u_{\sigma(k)}(k) + H_{\sigma(k)}d(k) \quad (1)$$

117 where the switching function $\sigma(\cdot)$ takes values in a finite index set $\underline{N}$, $x \in \mathbb{R}^n$, and for
 118 all $i \in \underline{N}$, $u_i \in \mathbb{R}^{m_i}$, the matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m_i}$ and $H_i \in \mathbb{R}^{n \times z}$ are known and
 119 B_i have full column rank. The disturbance variable $d \in \mathbb{R}^z$ is componentwise bounded
 120 with a known bound $|d(\cdot)| \leq \mathbf{d}$, where $\mathbf{d} \in \mathbb{R}^z$ is a nonnegative vector.

121 The state of an asymptotically stable system driven by a bounded non-vanishing
 122 disturbance will not converge to the origin in general, but rather to a bounded neigh-
 123 bourhood of the origin. The state ultimate bounds, defined as the set $\{x \in \mathbb{R}^n : x \leq$
 124 $\sup_{\sigma} \limsup_{k \rightarrow \infty} |x(k)|\}$, provide a componentwise characterisation of the smallest neigh-
 125 bourhood of the origin that will ultimately contain the perturbed state trajectories. This
 126 paper aims to minimise state ultimate bounds by feedback design.

127 *Problem statement.* Assuming $\sigma(k)$ is known at time k , we are interested in designing a
 128 set of state feedback matrices $\{K_i\}_{i \in \underline{N}}$ such that the system (1) with

$$u_{\sigma(k)}(k) = K_{\sigma(k)}x(k) \quad (2)$$

129 is uniformly exponentially stable for every switching sequence $\{\sigma(k)\}$ (when $d = 0$) and
 130 minimises the ultimate bound of state components when $d \neq 0$. That is, the resulting
 131 perturbed closed-loop system

$$x(k+1) = A_{\sigma(k)}^{cl}x(k) + H_{\sigma(k)}d(k), \quad (3)$$

132 with $A_i^{cl} = A_i + B_iK_i$, admits a CLF and achieves the minimum possible ultimate bound
 133 for one or more state components.

134 For discrete-time switched systems under arbitrary switching, there exists an inher-
 135 ent limitation to reducing the ultimate bound for any state component, which can never
 136 be smaller than the effect of the perturbation on that component.

137 **Lemma 1.** (Lowest ultimate bound [12]) For the switched system (3), the ultimate
 138 bound on the state vector's j -th component is bounded below as

$$\sup_{\sigma} \limsup_{k \rightarrow \infty} |x_j(k)| \geq \max_{i \in \underline{N}} [\max_{|d| \leq \mathbf{d}} |[H_i]_{(j,:)} d|] \doteq \mathbf{b}_j^{\min}. \quad (4)$$

139 **PROOF.** From (3), since $x_j(k+1) = [A_{\sigma(k)}^{cl}]_{(j,:)} x(k) + [H_{\sigma(k)}]_{(j,:)} d(k)$, then the ultimate
 140 bound on the j -th state component can never be smaller than that corresponding to the
 141 case when the j -th row of A_i^{cl} is zero for every $i \in \underline{N}$. Then, the result follows from
 142 direct analysis of (3). ■

143 Note that $\mathbf{b}_j^{\min}$ in (4) is independent of the closed-loop matrices A_i^{cl} , $i \in \underline{N}$, because
 144 it corresponds to the case when the j -th row of every $A_i^{cl} = A_i + B_i K_i$ is zero. We next
 145 single out a case where the smallest possible ultimate bound is not attainable.

146 **Lemma 2.** (Unfeasibility of ultimate bound minimisation) For the switched system (3)
 147 with (A_i, B_i) controllable for all $i \in \underline{N}$, suppose that for an arbitrary $j \in \{1, \dots, n\}$, the
 148 j -th row of at least one of the B_i matrices is identically zero. Then, the j -th ultimate
 149 bound of the system cannot be minimised to its lowest value $\mathbf{b}_j^{\min}$ as in (4).

150 **PROOF.** To minimise the j -th ultimate bound of the switched system (2), we need to
 151 achieve

$$[A_i^{cl}]_{(j,:)} = [A_i + B_i K_i]_{(j,:)} = [A_i]_{(j,:)} + [B_i]_{(j,:)} K_i = 0 \quad (5)$$

152 for all $i \in \underline{N}$. If for the k -th subsystem $[B_k]_{(j,:)} = 0$, then for any choice of K_k we have
 153 $[A_k^{cl}]_{(j,:)} = [A_k]_{(j,:)} \neq 0$ by controllability and hence, the j -th ultimate bound cannot be
 154 minimised to $\mathbf{b}_j^{\min}$ as in (4). ■

155 Lemma 2 establishes that to achieve the minimum ultimate bound for a state in
 156 the arbitrarily switched system (1) with controllable subsystems requires that the corre-
 157 sponding row in all the input matrices B_i be not identically zero.

158 Having the case of Lemma 2 ruled out, to minimise the ultimate bounds of the
 159 switched system, we give conditions in Lemma 3 below on the structure of a common
 160 transformation matrix V and resulting transformed matrices M_i , such that

$$A_i^{cl} \doteq A_i + B_i K_i = V M_i V^{-1} \quad (6)$$

161 has its j -th row equal to zero for all i and, hence, the j -th ultimate bound of (3) is
 162 minimised.

163 **Lemma 3.** (Closed-loop structure with lowest ultimate bound in a single state) Con-
 164 sider the discrete-time switched system (3) under arbitrary switching. For an arbitrary

165 $j \in \{1, \dots, n\}$, suppose that there exist feedback matrices K_i for all $i \in \underline{N}$ and an invert-
 166 ible transformation V such that $M_i = V^{-1}(A_i + B_i K_i)V$ are stable, upper triangular, and
 167 have the form¹

$$M_i = \begin{bmatrix} \Delta_{k \times k}^{i,1} & \delta_{k \times 1}^i & \Delta_{k \times (n-k-1)}^{i,2} \\ 0_{1 \times k} & 0 & 0_{1 \times (n-k-1)} \\ 0_{(n-k-1) \times k} & 0_{(n-k-1) \times 1} & \Delta_{(n-k-1) \times (n-k-1)}^{i,3} \end{bmatrix}, \quad (7)$$

168 where $\Delta^{i,1}$ and $\Delta^{i,3}$ are upper-triangular matrices, $\Delta^{i,2}$ is an arbitrary matrix, δ^i is an
 169 arbitrary vector, and the transformation matrix V is such that its j -th row has a nonzero
 170 entry at the $(k+1)$ -th column and is zero everywhere else, that is,

$$V_{(j,:)} = \begin{bmatrix} 0_{1 \times k} & V_{j,k+1} & 0_{1 \times (n-k-1)} \end{bmatrix}, \quad V_{j,k+1} \neq 0. \quad (8)$$

171 Then the j -th ultimate bound of the switched system is equal to its minimum possible
 172 value $\mathbf{b}_j^{\min}$ defined in (4)

173 **PROOF.** Using (7) and (8), the j -th row of the closed-loop matrix of each subsystems is

$$\begin{aligned} [A_i^{cl}]_{(j,:)} &= [A_i + B_i K_i]_{(j,:)} \\ &= [VM_i V^{-1}]_{(j,:)} = V_{(j,:)} M_i V^{-1} \\ &= \begin{bmatrix} 0 & V_{j,k+1} & 0 \end{bmatrix} \begin{bmatrix} \Delta^{i,1} & \delta^i & \Delta^{i,2} \\ 0 & 0 & 0 \\ 0 & 0 & \Delta^{i,3} \end{bmatrix} V^{-1} = 0_{1 \times n} \end{aligned}$$

174 and hence, the ultimate bound on the j -th state component is minimised to $\mathbf{b}_j^{\min}$ as
 175 in (4). ■

176 In the next lemma, the result in Lemma 3 is extended to the minimisation of more
 177 than one ultimate bound.

178 **Lemma 4.** (Closed-loop structure with lowest ultimate bounds multiple states) Let $\mathcal{J} \subset$
 179 $\{1, \dots, n\}$ with cardinality $\#\mathcal{J}$ contain indices of a set of desired states. Suppose that
 180 the matrices M_i and V in (6) satisfy

- 181 • M_i are upper triangular and stable and have $\#\mathcal{J}$ zero rows with indices in an
 182 arbitrary set $\mathcal{Z} \subset \{1, \dots, n\}$ with $\#\mathcal{Z} = \#\mathcal{J}$.
- 183 • V is invertible and such that $V_{(\mathcal{J}, \mathcal{Z})} \in \mathbb{C}^{\#\mathcal{J} \times \#\mathcal{J}}$ is invertible (for example, any
 184 permutation of the identity matrix) and $V_{(\mathcal{J}, \{1, \dots, n\} \setminus \mathcal{Z})} = 0$.

¹For clarity, the dimensions of some of the matrices are indicated as subscripts.

185 Then the ultimate bounds of the switched system with their indices specified in the set
 186 $\mathcal{J}$ can be minimised to their minimum possible values $\mathbf{b}_{\mathcal{J}}^{\min}$ as in (4). $\circ$

187 In [19] an algorithm that iteratively seeks feedback matrices K_i and the transforma-
 188 tion V so that $M_i = V^{-1}(A_i + B_i K_i)V$ are stable and upper triangular was developed. In
 189 the next section, we modify this algorithm in order to achieve the additional conditions
 190 of Lemmas 3 and 4 and hence yield closed-loop matrices $A_i^{cl} = A_i + B_i K_i$ with the desired
 191 zero rows.

192 3. Stabilisation and ultimate bound minimisation by feedback

193 In this section, eigenstructure assignment guaranteeing stability and ultimate bound-
 194 edness of a switched linear system under arbitrary switching is considered. We extend
 195 the iterative algorithm proposed in [19], which exploits state-feedback control and a
 196 common transformation to obtain subsystems with upper-triangular matrices.

197 3.1. The ITBF Algorithm

198 In [19], conditions were given on the number of states n , the number of subsystems
 199 N , and the number of inputs of each subsystem m_i , $i \in \underline{N}$, so that the stabilising feedback
 200 matrices K_i and the simultaneous triangularisation transformation V will exist for almost
 201 every set of system parameters, i.e. for almost all possible entries of the matrices A_i and
 202 B_i , for all $i \in \underline{N}$. When these conditions are satisfied, [19] also shows that, in addition,
 203 the closed-loop eigenvalues for every subsystem can be arbitrarily selected. In this
 204 section, we modify the feedback design algorithm of [19] so that all available degrees
 205 of freedom are exploited to achieve minimum ultimate bounds through the selection
 206 of some closed-loop eigenvalues and the construction of a unitary matrix with specific
 207 properties.

208 Consider the discrete-time switched linear system (1) with state-feedback law (2),
 209 yielding the closed-loop system (3). The proposed modified algorithm is shown below
 210 as Algorithm ITBF in Figure 1. This algorithm seeks feedback matrices K_i so that

- 211 1. the closed-loop matrices $A_i^{cl} = A_i + B_i K_i$ are stable and simultaneously triangular-
 212 isable, and
- 213 2. for a selected set of state components with indices in $\mathcal{J} \subset \{1, 2, \dots, n\}$, whose car-
 214 dinality will be specified later, their corresponding ultimate bounds are minimised
 215 to their smallest values.

216 The Algorithm ITBF (shown in Figure 1) is an extension of the algorithm in [19],
 217 where the main modifications are: (a) the set of state components to be minimised,
 218 namely $\mathcal{J} \subset \{1, \dots, n\}$, has to be supplied as input data, (b) the common eigenvector
 219 assignment (CEA) procedure of [19] is replaced by the *structured* common eigenvector

Algorithm ITBF: Iterative triangularisation and ultimate bound minimisation by feedback

Data: $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m_i}$ for $i \in \underline{N}$, and $\mathcal{J}$

Output: $V \in \mathbb{C}^{n \times n}$, $K_i \in \mathbb{R}^{n \times n}$ for $i \in \underline{N}$

Initialisation: $A_i^1 \doteq A_i$, $B_i^1 \doteq B_i$, $K_i^0 \doteq 0$, $U_1 \doteq I$, $\ell \leftarrow 0$, $\mathcal{J}^1 \doteq \mathcal{J}$, $m_i^1 = m_i$ for $i \in \underline{N}$

repeat

$$\ell \leftarrow \ell + 1, \quad n_\ell \leftarrow n - \ell + 1, \quad (9)$$

if $p_\ell = n_\ell + \sum_{i=1}^N m_i^\ell - N n_\ell < n_\ell$ **then**

$$[\xi^\ell, \{F_i^\ell\}_{i=1}^N] \leftarrow \text{SCEA}(\{A_i^\ell\}_{i=1}^N, \{B_i^\ell\}_{i=1}^N, \mathcal{J}^\ell), \quad (10)$$

$$A_i^{\ell, \text{cl}} \doteq A_i^\ell + B_i^\ell F_i^\ell, \quad (11)$$

$$K_i^\ell \leftarrow K_i^{\ell-1} + F_i^\ell \left(\prod_{r=1}^\ell U_r^* \right), \quad (12)$$

$$V_{(:, \ell)} \leftarrow \left(\prod_{r=1}^\ell U_r \right) \xi^\ell. \quad (13)$$

Construct a unitary matrix (15) with arbitrary set of distinct elements

$$\mathcal{J}^{\ell+1} \subset \{1, 2, \dots, n_\ell - 1\}, \quad \#\mathcal{J}^{\ell+1} = \#\mathcal{J} \quad (14)$$

where for each distinct pair $(j^\ell, j^{\ell+1})$, with $j^\ell \in \mathcal{J}^\ell$, $j^{\ell+1} \in \mathcal{J}^{\ell+1}$

$$[\xi^\ell | v_1^\ell | \dots | v_{n_\ell-1}^\ell] \in \mathbb{C}^{n_\ell \times n_\ell}, \quad (15)$$

$$\begin{cases} v_{j^\ell, i}^\ell = 0 : i = 1, \dots, n_\ell - 1, i \neq j^{\ell+1} \\ v_{j^\ell, j^{\ell+1}}^\ell \neq 0 \end{cases} \quad (16)$$

Assign (17)–(19):

$$U_{\ell+1} \leftarrow [v_1^\ell | \dots | v_{n_\ell-1}^\ell], \quad (17)$$

$$A_i^{\ell+1} \leftarrow U_{\ell+1}^* A_i^{\ell, \text{cl}} U_{\ell+1}, \quad (18)$$

$$B_i^{\ell+1} \leftarrow U_{\ell+1}^* B_i^\ell. \quad (19)$$

end if

until $p_\ell = n_\ell$;

Let $V^\ell = I_{n_\ell}$;

Select $\{\lambda_i^\ell, \dots, \lambda_i^n\}$ stable with $\lambda_i^{\ell-1+j} = 0, \forall j \in \mathcal{J}^\ell$;

Construct $\Delta_i \doteq \text{diag}\{\lambda_i^\ell, \dots, \lambda_i^n\}$;

Compute

$$F_i^n \leftarrow (B_i^\ell)^{-1} (\Delta_i I_{n_\ell} - A_i^\ell);$$

$$K_i \leftarrow K_i^{\ell-1} + F_i^n \left(\prod_{r=1}^\ell U_r^* \right);$$

$$V_{(:, \ell:n)} \leftarrow \left(\prod_{r=1}^\ell U_r \right) V^\ell.$$

Figure 1: Algorithm for iterative triangularisation and ultimate bound minimisation by feedback (ITBF)

assignment (SCEA) procedure in (10), and (c) the unitary matrix construction (15) has to satisfy the additional constraints (16). As in [19], the proposed algorithm seeks feedback matrices K_i so that the closed-loop matrices A_i^{cl} in (3) are stable and simultaneously triangularisable, but with the additional requirement that the conditions in Lemma 4 are fulfilled.

A brief description of the algorithm is as follows. After initialisation, the algorithm iterates the following steps: common eigenvector computation for the internal subsystems identified by A_i^ℓ, B_i^ℓ (performed by procedure SCEA in (10)), state feedback and transformation update (performed at (12) and (13)), and internal matrices' update for the next iteration (at (15) to (19)). The internal subsystem matrices change dimensions during the execution of the algorithm because one state dimension is eliminated at each iteration. At start, the internal matrices for iteration $\ell = 1$ are set to coincide with the subsystem matrices: $A_i^1 = A_i, B_i^1 = B_i$. Then the SCEA procedure in (10) seeks a unit vector ξ^ℓ having specific structure (which will be explained later), and corresponding (internal) feedback matrices F_i^ℓ , so that ξ^ℓ is a feedback-assignable eigenvector common to all internal subsystems, with corresponding stable eigenvalues. That is, if Procedure SCEA is successful, then ξ^ℓ satisfies $\|\xi^\ell\| = 1$ and $(A_i^\ell + B_i^\ell F_i^\ell)\xi^\ell = \lambda_i^\ell \xi^\ell$ for some scalars λ_i^ℓ satisfying $|\lambda_i^\ell| < 1$ for all $i \in \underline{N}$.

3.2. Structural condition for simultaneous triangularisation by feedback

As noted above, successful satisfaction of the algorithm at each iteration requires the existence of the vector ξ^ℓ . Existence of such ξ^ℓ (without the additional requirement for ultimate bound minimisation) is ensured by the structural condition of [19], as we next explain. Define $m_i^\ell \doteq \text{rank}(B_i^\ell) = \text{d}(\text{img } B_i^\ell)$, and factor $B_i^\ell = b_i^\ell r_i^\ell$, where $r_i^\ell : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i^\ell}$ has full row rank and $b_i^\ell : \mathbb{R}^{m_i^\ell} \rightarrow \mathbb{R}^{n_i}$ has full column rank. Note that $\text{img } B_i^\ell = \text{img } b_i^\ell$. Let Λ^ℓ be a vector with components $\lambda_i^\ell, i \in \underline{N}$, i.e.

$$\Lambda^\ell \doteq [\lambda_1^\ell \ \lambda_2^\ell \ \dots \ \lambda_N^\ell]', \quad (20)$$

and build the matrix

$$Q_\ell(\Lambda^\ell) \doteq [R_\ell(\Lambda^\ell) - B_\ell], \text{ where} \quad (21)$$

$$R_\ell(\Lambda^\ell) \doteq \begin{bmatrix} \lambda_1^\ell I - A_1^\ell \\ \vdots \\ \lambda_N^\ell I - A_N^\ell \end{bmatrix}, \quad B_\ell \doteq \text{blkdiag}[b_1^\ell, \dots, b_N^\ell],$$

and where blkdiag denotes block diagonal concatenation. The following result from [19] presents conditions for the existence of a common feedback assignable eigenvector based on properties of the matrix (21), and provides a mechanism to obtain the common eigenvector and the associated feedback matrices when the problem is solvable.

250 **Lemma 5.** (Structural condition) ([19]) Let

$$p_\ell \doteq n_\ell + \sum_{i=1}^N m_i^\ell - Nn_\ell. \quad (22)$$

251 Then,

252 (a) $\dim(\ker Q_\ell(\Lambda^\ell)) \geq p_\ell$ for every choice of Λ^ℓ as in (20).

253 (b) A vector that can be assigned by feedback as a common eigenvector with corre-
254 sponding eigenvalues λ_i^ℓ for $i \in \underline{N}$ exists if and only if $\dim(\ker Q_\ell(\Lambda^\ell)) > 0$. Conse-
255 quently, if

$$p_\ell > 0, \quad (23)$$

256 then a feedback-assignable common eigenvector exists for every choice of corre-
257 sponding eigenvalues.

258 (c) If $Q_\ell(\Lambda^\ell)w^\ell = 0$ with $w^\ell \neq 0$ partitioned as

$$w^\ell \doteq [v' \ u_1' \ \dots \ u_N']', \quad \text{then } v \neq 0, \text{ and} \quad (24)$$

$$(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda_i^\ell v, \quad \text{for } i \in \underline{N}, \quad (25)$$

259 for every F_i^ℓ satisfying $r_i^\ell F_i^\ell v = u_i$. For each $i \in \underline{N}$ one such F_i^ℓ is

$$F_i^\ell = (r_i^\ell)^\dagger u_i v^\dagger, \quad (26)$$

260 where † denotes the Moore-Penrose generalised inverse. ◦

261 If the structural condition (23) holds, the nullspace of $Q_\ell(\Lambda^\ell)$ is not empty and, thus,
262 we can find $w^\ell \in \ker Q_\ell(\Lambda^\ell)$. Suppose $\dim(\ker Q_\ell(\Lambda^\ell)) = \psi_\ell \geq p_\ell$, define $d_\ell \doteq n_\ell + \sum_{i=1}^N m_i^\ell$
263 and let $W^\ell \in \mathbb{C}^{d_\ell \times \psi_\ell}$ be a basis for $\ker Q_\ell(\Lambda^\ell)$. Then, from Lemma 5(c), the vector $w^\ell \neq 0$
264 has the form

$$w^\ell = W^\ell \alpha^\ell \quad (27)$$

265 where $\alpha^\ell \in \mathbb{C}^{\psi_\ell}$ is an arbitrary vector. Once (27) is obtained, a feedback-assignable
266 common eigenvector ξ^ℓ provided by procedure SCEA at iteration ℓ of algorithm ITBF
267 can be computed by selecting the first n_ℓ components of w^ℓ to construct v [cf. (24)] and
268 then letting

$$\xi^\ell = v/\|v\|. \quad (28)$$

269 The quantity p_1 given by (22) for the first iteration $\ell = 1$ is central to the solvability
270 of the proposed algorithm. It also determines an upper bound on the number of ultimate

bound components that can be minimised by appropriate use of the available degrees of freedom. These facts will be discussed in the following sections.

In the next subsection we provide a technical result that will be required to analyse the solvability of the ITBF algorithm at all iterations. Then, in Subsection 3.4 we analyse in detail the mechanism for ultimate bound minimisation embedded in the algorithm.

Procedure SCEA (Structured Common Eigenvector Assignment)

Input: $A_i^\ell \in \mathbb{R}^{n_\ell \times n_\ell}$, $B_i^\ell \in \mathbb{R}^{n_\ell \times m_i}$, for $i \in \underline{N}$ and $\mathcal{J}^\ell$

Output: ξ^ℓ with $\xi_{\mathcal{J}^\ell}^\ell = 0$, F_i^ℓ for $i \in \underline{N}$

Factor $B_i^\ell = b_i^\ell r_i^\ell$ with $b_i^\ell \in \mathbb{R}^{n_\ell \times m_i^\ell}$ and $m_i^\ell = \text{rank}(B_i^\ell)$;

Select $\lambda_i^\ell \in \mathbb{R}$ stable and construct Λ^ℓ as in (20);

Construct $Q_\ell(\Lambda^\ell)$ as in (21)

Compute W^ℓ as in (27);

Select α^ℓ as in Lemma 9 satisfying Theorem 7;

Compute w^ℓ as in (27) and partition it as in (24);

Compute F_i^ℓ as in (26)

Determine $\xi^\ell = v/\|v\|$;

Figure 2: Procedure SCEA when the structural condition is satisfied.

Remark 1. Note that the iterations in Algorithm ITBF (see Figure 1) are divided into two categories. The first category iteratively runs with the structural condition $p_\ell < n_\ell$ while the second one deals with the case when $p_\ell = n_\ell$ for which the algorithm terminates in just one step. $\circ$

3.3. Solvability of Algorithm ITBF

Solvability of the Algorithm ITBF builds on the solvability of the iterative algorithm proposed in [19]. In this section we first revisit some results from [19] that analyse the structural condition (23). Then we present a new result on how to exploit the available degrees of freedom in the iterative algorithm proposed in [19] to guarantee satisfaction of (23) at each iteration.

The structural condition (23) depends on m_i^ℓ , the rank of B_i^ℓ . At the first iteration $\ell = 1$, $m_i^\ell = m_i$ and thus, $p_1 = (1 - N)n + \sum_{i=1}^N m_i$. At subsequent iterations, $m_i^{\ell+1}$ depends on the vector ξ^ℓ returned by Procedure SCEA as follows:

$$m_i^{\ell+1} = \begin{cases} m_i^\ell & \text{if } \xi^\ell \notin \text{img } B_i^\ell, \\ m_i^\ell - 1 & \text{if } \xi^\ell \in \text{img } B_i^\ell. \end{cases} \quad (29)$$

From (29), then $m_i^{\ell+1} = m_i^\ell - 1$ when $m_i^\ell = n_\ell$, because $\xi^\ell \in \mathbb{R}^{n_\ell} = \text{img } B_i^\ell$. The next key result from [19] follows from (22) and (29).

292 **Lemma 6 ([19]).** Consider Algorithm ITBF at iteration ℓ and p_ℓ as in (22), with $m_i^\ell =$
 293 $\text{rank}(B_i^\ell)$. Then, $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if

$$\xi^\ell \in \mathcal{B}^\ell, \quad \text{with } \mathcal{B}^\ell \doteq \bigcap_{i \in \underline{N}} \mathcal{B}_i^\ell \quad \text{and } \mathcal{B}_i^\ell \doteq \text{img } B_i^\ell. \quad (30)$$

294

○

295 According to Lemma 6, if at iteration ℓ , $p_\ell = 1$ and $\xi^\ell \in \bigcap_{i \in \underline{N}} \mathcal{B}_i^\ell$, then $p_{\ell+1} =$
 296 $p_\ell - 1 = 0$ and it is possible that no stable common eigenvector can be found; thus, the
 297 ITBF algorithm terminates unsuccessfully. We are thus interested in finding conditions
 298 to ensure that $p_\ell > 0$ will hold at every iteration. Following [19], we derive these
 299 conditions by considering certain subspaces associated with the internal subsystems
 300 (A_i^ℓ, B_i^ℓ) .

301 Let $\mathcal{S}_i^\ell$ denote the set of vectors $v \in \mathcal{B}_i^\ell = \text{img } B_i^\ell$ for which there exist a matrix F_i^ℓ
 302 and a stable scalar λ so that

$$(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda v. \quad (31)$$

303 By definition, $\mathcal{S}_i^\ell$ is the set of feedback-assignable eigenvectors for the internal subsys-
 304 tem (A_i^ℓ, B_i^ℓ) , with associated stable eigenvalues, which are contained in $\mathcal{B}_i^\ell$.

305 Suppose $\{\mathcal{S}_i^\ell : i \in \underline{N}\}$ are transverse subspaces (a generic property of randomly
 306 selected subspaces, see, [19, Lemma 3]). Define the following quantities:

$$\rho_i^\ell \doteq \text{d}(\mathcal{S}_i^\ell), \quad q_\ell \doteq n_\ell + \sum_{i \in \underline{N}} \rho_i^\ell - N n_\ell, \quad (32)$$

$$\mathcal{S}^\ell \doteq \bigcap_{i \in \underline{N}} \mathcal{S}_i^\ell, \quad \rho^\ell \doteq \text{d}(\mathcal{S}^\ell). \quad (33)$$

307 From (33) and Lemma 6, if the common eigenvector ξ^ℓ lies within $\mathcal{S}^\ell$, then $\xi^\ell \in$
 308 $\mathcal{B}^\ell$ as in (30), and the structural condition $p_{\ell+1} = p_\ell - 1 > 0$ might not be satisfied.
 309 To avoid this situation and guarantee that the structural condition (23) continues to be
 310 satisfied at every iteration of the algorithm, we exploit the available degrees of freedom
 311 to choose the common eigenvector such that $\xi^\ell \notin \mathcal{S}^\ell$ and hence, $p_{\ell+1} \geq p_\ell$. This result,
 312 presented in Theorem 7 below, will come into use in the next section where we lay out
 313 the procedure for ultimate bound minimisation.

314 **Theorem 7.** (Conditions for solvability of Algorithm ITBF) Let $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be trans-
 315 verse, $q_1 \geq 0$ and (A_i, B_i) be controllable for all $i \in \underline{N}$. Then, it is always possible to
 316 select α^ℓ in (27) such that $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if $p_\ell = n_\ell$. ○

317 Before we proceed to the proof of the theorem, note that under the assumptions of
 318 Theorem 7, it was shown in [19, Theorem 2] that

- $p_\ell > 0$ for $\ell = 1, \dots, n$.
- There exist feedback gains K_i such that the set $\mathcal{Z} = \{(A_i + B_i K_i, B_i) : i \in \underline{N}\}$ consists of stable matrices and generates a solvable Lie-algebra. Hence, such that the closed-loop system admits a CQLF.

Namely, these are sufficient conditions to guarantee stabilisability of the switched system. Theorem 7 above shows that without any further restrictions, it is possible to select feedback gains that, in addition to providing closed-loop stability, ensure that p_ℓ is non-decreasing for all iterations (until $p_\ell = n_\ell$ and remains equal to n_ℓ afterwards). This means that starting with ‘enough freedom’ (i.e., p_1 ‘large enough’) to select the common eigenvector, this freedom is maintained for all iterations and hence lays the groundwork to tackle the ultimate bound minimisation problem, as we will show in Section 3.4.

As a preliminary result required to prove Theorem 7, the next lemma relates the quantities p_ℓ defined in (22) and ρ^ℓ defined in (33), and is central to the solvability of the proposed algorithm (SCEA Procedure) at all iterations.

Lemma 8. *Consider p_ℓ defined in (22), q_ℓ defined in (32), let $p_\ell > 0$, $\{S_i^\ell : i \in \underline{N}\}$ be transverse and (A_i^ℓ, B_i^ℓ) be controllable. Then $p_\ell \geq \rho^\ell = \max\{0, q_\ell\}$, with $p_\ell = \rho^\ell$ if and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$.*

PROOF. Recalling properties of transverse subspaces, the dimension of the intersection of transverse subspaces $\mathcal{S}^\ell$ satisfies

$$\rho^\ell = \max\{0, q_\ell\}. \quad (34)$$

Since $\mathcal{S}_i^\ell \subset \mathcal{B}_i^\ell$, then $\rho_i^\ell \leq m_i^\ell$ and $q_\ell \leq p_\ell$ which together with (34) and $p_\ell > 0$ yields $\rho^\ell \leq p_\ell$. Next, we prove that $\rho^\ell = p_\ell$ if and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$.

($\rightarrow$) If $\rho^\ell = p_\ell > 0$, then from (34) we have $\rho^\ell = q_\ell = p_\ell > 0$ which from (32) means $\rho_i^\ell = m_i^\ell$ for $i \in \underline{N}$. Controllability of (A_i^ℓ, B_i^ℓ) and the fact that ρ_i^ℓ is the number of controllability indices of (A_i, B_i) equal to 1 (see [19, Lemma 5]) then yield $m_i^\ell = n_\ell$, for $i \in \underline{N}$.

($\leftarrow$) If $m_i^\ell = n_\ell$ for all $i \in \underline{N}$, then $\rho_i^\ell = m_i^\ell$ for $i \in \underline{N}$ and thus, from (32) $p_\ell = q_\ell$. From (34) and the assumption $p_\ell > 0$, we have $\rho^\ell = q_\ell = p_\ell > 0$. ■

Now we are ready to proceed to the proof of Theorem 7.

PROOF OF THEOREM 7. Here we show that by proper selection of α^ℓ in (27), p_ℓ is non-decreasing for all iterations until $p_\ell = n_\ell$ and remains equal to n_ℓ afterwards.

From Lemma 5(a), we have $\dim(\ker Q_\ell(\Lambda^\ell)) = \psi_\ell \geq p_\ell$. Thus, if the latter holds with equality, a basis for the nullspace of $Q_\ell(\Lambda^\ell)$ has the form (see (27))

$$W^\ell = [w_1^\ell \dots w_{\psi_\ell}^\ell] = \begin{bmatrix} v_1 & \dots & v_{\psi_\ell} \\ u_{11} & \dots & u_{1\psi_\ell} \\ \vdots & & \vdots \\ u_{N1} & \dots & u_{N\psi_\ell} \end{bmatrix}, \quad (35)$$

$$\text{rank}(W^\ell) = \psi_\ell \geq p_\ell,$$

where the partition of each vector follows from (24). From (24) and (27), the common eigenvector is determined as

$$v = [v_1 \dots v_{\psi_\ell}] \alpha^\ell. \quad (36)$$

First, we show that the subspace generated by the v_r vectors is also of rank ψ_ℓ , i.e.

$$\text{rank}([v_1 \dots v_k \dots v_{\psi_\ell}]) = \psi_\ell. \quad (37)$$

This is shown by contradiction. Suppose that v_k , for some $k \in \{1, \dots, \psi_\ell\}$, is a linear combination of other columns, that is

$$v_k = \sum_{r=1, r \neq k}^{\psi_\ell} v_r \gamma_r. \quad (38)$$

where at least one coefficient γ_r is nonzero. Then, since W^ℓ in (35) is in the nullspace of $Q_\ell(\Lambda^\ell)$ we have

$$Q_\ell(\Lambda^\ell) w_r^\ell = 0, \quad r = 1, \dots, \psi_\ell \quad (39)$$

and by replacing (21) and (35) in (39), for $i \in \underline{N}$ we obtain

$$(\lambda_i^\ell I - A_i^\ell) v_r - b_i^\ell u_{ir} = 0, \quad r = 1, \dots, \psi_\ell. \quad (40)$$

For $r = k$, using (38) in (40) we obtain

$$\begin{aligned} & (\lambda_i^\ell I - A_i^\ell) v_k - b_i^\ell u_{ik} \\ &= \sum_{r=1, r \neq k}^{\psi_\ell} (\lambda_i^\ell I - A_i^\ell) v_r \gamma_r - b_i^\ell u_{ik} = b_i^\ell \sum_{r=1, r \neq k}^{\psi_\ell} u_{ir} \gamma_r - b_i^\ell u_{ik} \\ &= b_i^\ell \left(\sum_{r=1, r \neq k}^{\psi_\ell} u_{ir} \gamma_r - u_{ik} \right) = 0. \end{aligned} \quad (41)$$

Since the b_i^ℓ matrices have full column rank, (41) implies

$$\sum_{r=1, r \neq k}^{\psi_\ell} u_{ir} \gamma_r - u_{ik} = 0. \quad (42)$$

362 This means that $u_{ik} = \sum_{r=1, r \neq k}^{\psi_\ell} u_{ir} \gamma_r$, for $i \in \underline{N}$, which together with (38) yields

$$w_k^\ell = \sum_{r=1, r \neq k}^{\psi_\ell} w_r^\ell \gamma_r, \quad (43)$$

363 i.e. $\text{rank}(W^\ell) < \psi_\ell$ which contradicts our assumption in (35). Hence, (37) holds. There-
364 fore, the common eigenvector v in (24) can be chosen in a space of rank ψ_ℓ .

365 In [19, Theorem 2] it is proved that when $\{S_i^1 : i \in \underline{N}\}$ is transverse, $q_1 \geq 0$ and
366 (A_i, B_i) is controllable, then for $\ell = 1, \dots, n$, $\{S_i^\ell : i \in \underline{N}\}$ is transverse and (A_i^ℓ, B_i^ℓ) is
367 controllable by induction. Hence, by Lemma 8, we know that $p_\ell \geq \rho^\ell$ with equality if
368 and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$. We consider two cases, $p_\ell = \rho^\ell$ and $p_\ell > \rho^\ell$, separately as
369 follows.

370 When $p_\ell = \rho^\ell$, then $m_i^\ell = n_\ell$ for $i \in \underline{N}$ and all input matrices are invertible. We
371 then have $p_\ell = \rho^\ell = n_\ell$ and hence, the dimension of $\mathcal{S}^\ell$ is n_ℓ (see (33)) and any common
372 eigenvector $\xi^\ell \in \mathbb{C}^{n_\ell}$ is also in $\mathcal{S}^\ell$. Then, from Lemma 6, $p_{\ell+1} = p_\ell - 1 = n_\ell - 1$. On
373 the other hand, the reduction of subsystems dimension in the next step (see (9)) yields
374 $n_{\ell+1} = n_\ell - 1$ which results in $p_{\ell+1} = n_{\ell+1}$.

375 When $p_\ell > \rho^\ell$, from (37), it is always possible to select α^ℓ in (36) such that the
376 resulting vector v is not in $\mathcal{S}^\ell$. From (28) we then have $\xi^\ell \notin \mathcal{S}^\ell$, hence $\xi^\ell \notin \mathcal{B}^\ell$ and (30)
377 does not hold. Thus, from Lemma 6 we have $p_{\ell+1} > p_\ell - 1$. ■

378 3.4. Iterative ultimate bound minimisation

379 In [12], the problem of ultimate bound minimisation for non-switched discrete-time
380 systems required a separate analysis depending on the number of control inputs. For
381 single-input systems, eigenvalue assignment based on the roots of certain polynomials
382 associated with the system matrices characterises the possibility of having the minimum
383 ultimate bound. For multiple-input systems, on the other hand, the smallest ultimate
384 bound on a state is achievable by eigenstructure assignment under certain conditions.
385 The above analysis can be deployed in switched discrete-time systems as explained
386 below.

387 For switched systems of the form (1), if the structural condition (23) holds at each
388 iteration of Algorithm ITBF, the nullspace of the matrix $Q_\ell(\Lambda^\ell)$ defined in (21) is non-
389 empty and thus, there exists a common eigenvector for all subsystems. At each iteration,
390 $\psi_\ell(\geq p_\ell)$ represents the degrees of freedom to choose the common eigenvector such that
391 a desirable property is satisfied. Indeed, as seen from (27), the vector α^ℓ needs to be
392 selected to shape the common eigenvector ξ^ℓ given in (28) in a specific way. This is
393 shown in the following result.

394 **Lemma 9.** *At iteration ℓ of Algorithm ITBF, consider $w^\ell = W^\ell \alpha^\ell$, $w^\ell \in \mathbb{C}^{d_\ell}$, where $W^\ell \in$
395 $\mathbb{C}^{d_\ell \times \psi_\ell}$ is a basis for $\ker Q_\ell(\Lambda^\ell)$, with the matrix $Q_\ell(\Lambda^\ell)$ defined in (21) and $\alpha^\ell \in \mathbb{C}^{\psi_\ell}$ an
396 arbitrary vector. Let $\mathcal{J}^\ell \subset \{1, 2, \dots, n_\ell\}$ be a subset of desired indices with cardinality*

397 $\#\mathcal{J}^\ell$. If $p_\ell > \#\mathcal{J}^\ell$, then the elements of the common eigenvector ξ^ℓ given in (28), with
 398 their indices specified in $\mathcal{J}^\ell$, can be made zero, that is $\xi_{\mathcal{J}^\ell}^\ell = W_{(\mathcal{J}^\ell, :)}^\ell \alpha^\ell / \|\mathbf{v}\| = 0$.

399 **PROOF.** Since $d(\ker W_{(\mathcal{J}^\ell, :)}^\ell) \geq \psi_\ell - \#\mathcal{J}^\ell \geq p_\ell - \#\mathcal{J}^\ell \geq 1$, then there exists a nonzero
 400 nullspace in which α^ℓ can be determined. ■

401 The above lemma states that if $p_\ell > \#\mathcal{J}^\ell$ at each iteration of Algorithm ITBF, then
 402 by proper selection of $\alpha^\ell \neq 0$ such that

$$\alpha^\ell \in \ker W_{(\mathcal{J}^\ell, :)}^\ell, \quad (44)$$

403 it is possible to assign common eigenvector so that the desired elements of the matrix V
 404 in (6) are zero. Otherwise, if $p_\ell \leq \psi_\ell \leq \#\mathcal{J}^\ell$, then, noting that $W^\ell \triangleq W^\ell(\Lambda^\ell)$ depends
 405 on the eigenvalue vector Λ^ℓ of the form (20), the desired zero elements could still be
 406 achieved by eigenvalue assignment provided the solution of the equation $W_{(\mathcal{J}^\ell, :)}^\ell(\Lambda^\ell) =$
 407 0 has elements with magnitude smaller than one. This will generically not hold so
 408 we concentrate on the case $p_\ell > \#\mathcal{J}^\ell$. Since we consider the cases where p_ℓ is non-
 409 decreasing (see Theorem 7), hence, we make the following assumption.

410 **Assumption 10.** For the system (3), (6), the set $\mathcal{J} \subset \{1, \dots, n\}$ that contains the indices
 411 of the desired state components for ultimate bound minimisation has cardinality $\#\mathcal{J} \leq$
 412 $p_1 - 1$, with p_1 defined in (22) for the first iteration.

413 **Theorem 11.** (Minimisation of ultimate bounds by feedback) Consider the perturbed
 414 switched discrete-time system (1), and let $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be transverse, $q_1 \geq 0$, $p_1 >$
 415 1 and (A_i, B_i) be controllable for all $i \in \underline{N}$. Let $\mathcal{J} \subset \{1, 2, \dots, n\}$, with cardinality
 416 $\#\mathcal{J}$ satisfying Assumption 10, contain the indices of the desired state components for
 417 ultimate bound minimisation. Suppose that at every iteration of Algorithm ITBF, there
 418 exists a vector $\alpha^\ell \in \mathbb{C}^{n_\ell}$ satisfying (44) which yields a $p_{\ell+1} \geq p_\ell - 1$ with equality if
 419 and only if $p_\ell = n_\ell$. Then, the ultimate bounds of system (1) with indices in $\mathcal{J}$ can be
 420 minimised to their minimum possible values $\mathbf{b}_{\mathcal{J}}^{\min}$ defined in (4) by executing Algorithm
 421 ITBF in Figure 1.

422 **PROOF.** The assumption $q_1 \geq 0$ together with $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ being transverse and (A_i, B_i)
 423 being controllable for all $i \in \underline{N}$, satisfy the conditions in Theorem 7. From the as-
 424 sumption $p_1 > 1$ and Theorem 7, with the appropriate selection of α^ℓ , p_ℓ can be made
 425 non-decreasing at all iterations while $p_\ell < n_\ell$, thereby $p_\ell > 1$ for all those iterations.
 426 (When $p_\ell = n_\ell$, whether $n_\ell = 1$ or $n_\ell > 1$, no more iterations are necessary since the
 427 algorithm can be terminated in one step as explained later in the proof.) If the selection
 428 of α^ℓ as described in Theorem 7 is also compatible with condition (44), then ultimate
 429 bound minimisation can be achieved. In the remainder of the proof, the iterative ultimate
 430 bound minimisation is explained.

Let $\mathcal{J}^1 = \mathcal{J}$ with cardinality $\#\mathcal{J}$ contain the indices of the desired state components for ultimate bound minimisation. The aim is to iteratively apply Lemma 3 by constructing the blocks of the M_i matrices for $i \in \underline{N}$ through eigenvalue assignment and the columns of the matrix V through (13)–(17) to achieve the final matrices M_i and V , where M_i has the form (7) and the matrix V has rows of the form (8). With regard to the matrix M_i , the algorithm accomplishes the required upper triangular structure through (10)–(12), provided its diagonal entries are set as the desired eigenvalues. However, for the desired matrix V , since its columns are the result of a product of matrices (c.f. (13)), the idea is to propagate the location of zero and nonzero elements in relevant rows of these matrices so that the end result is the j -th row of V , $j \in \mathcal{J}^1$, having all zero elements except for one nonzero element different for each $j \in \mathcal{J}^1$.

The proof is divided into two parts. First, we analyse the ultimate bound minimisation problem at iterations with $p_\ell < n_\ell$. At these iterations, for arbitrary eigenvalues and by common eigenvector assignment, we determine successive columns of the common triangularising transformation matrix V with zero elements at places specified in $\mathcal{J}^1$. Next, we show that if at some iteration κ we reach $p_\kappa = n_\kappa$, then Algorithm ITBF can be terminated in just one more step.

At the first iteration, from (22) and considering that the B_i matrices are of full column rank ($m_i \leq n$), we have $p_1 \leq n$ with equality if and only if $m_i = n$ for all $i \in \underline{N}$. If $p_1 = n$, the algorithm can be terminated in one step as explained below. If $p_1 < n$, then, $p_1 > p_1 - 1 \geq \#\mathcal{J}$ admits the existence of enough degrees of freedom to execute the common eigenvector assignment as in (44). Since $U_1 = I_n$, to have $V_{(\mathcal{J}^1, 1)} = 0$, Procedure SCEA needs to select the common eigenvector such that $\xi_{\mathcal{J}^1}^1 = 0$ (see (13)). Then, we construct a unitary matrix as in (15) with ξ^1 as its first column, such that for all $j^1 \in \mathcal{J}^1$, the j^1 -th row of the unitary matrix will have $n - 1$ zero entries and one nonzero element. To this end, choose an arbitrary set of distinct elements $\mathcal{J}^2 \subset \{1, 2, \dots, n - 1\}$, $\#\mathcal{J}^2 = \#\mathcal{J}$. Pair each index $j^1 \in \mathcal{J}^1$ with an index $j^2 \in \mathcal{J}^2$ (where no two pairs have the same first element or the same second element), and let the j^1 -th row of the unitary matrix be zero except for a nonzero entry at the $(j^2 + 1)$ -th place. Thus, the matrix U_2 as in (17) will be such that for all $j^1 \in \mathcal{J}^1$, its j^1 -th row has $n_2 - 1 = (n - 1) - 1$ zeros and a nonzero entry at component j^2 . Then, if $(j^1, j^2), (i^1, i^2)$ are two of those pairs, with $i^1, j^1 \in \mathcal{J}^1$ and $i^2, j^2 \in \mathcal{J}^2$ we have

$$U_1 U_2 = I_n \begin{bmatrix} * & \dots & * & \dots & * & \dots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & v_{j^1, j^2}^1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & v_{i^1, i^2}^1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \dots & * & \dots & * & \dots & * \end{bmatrix}_{n \times n_2}$$

where v_{j^1, j^2}^1 and v_{i^1, i^2}^1 are nonzero and $*$ is a non-specified entry. Thus, the matrix $U_1 U_2$ has $\#\mathcal{J}$ rows, with their indices in $\mathcal{J}^1$ (only two rows, with indices j^1 and i^1 , are illustrated above), with just one nonzero entry at the place indexed by the second element of the pair, and otherwise zero.

For the second iteration of Algorithm ITBF (assuming that $p_2 < n_2$), since by assumption α^ℓ can be chosen to satisfy (44) and the condition in Theorem 7, we obtain $p_2 \geq p_1 > \#\mathcal{J}$ and hence, the common eigenvector assignment can be performed. For the second column of the matrix V to have zero elements at places specified in $\mathcal{J}^1$, i.e. $V_{(\mathcal{J}^1, 2)} = 0$, from (13) and considering the nonzero entries of $U_1 U_2$ in the intersections of rows with indices in $\mathcal{J}^1$ and columns with indices in $\mathcal{J}^2$, we need to have $\xi_{\mathcal{J}^2}^2 = 0$ for the common eigenvector ξ^2 . Accordingly, the unitary matrix U_3 as in (17) constructed using this ξ^2 , will have $\#\mathcal{J}$ rows with indices $j^2 \in \mathcal{J}^2$ having $n_3 - 1$ zeros and one nonzero entry at their j^3 -th places, for $j^3 \in \mathcal{J}^3$, for an arbitrary set of distinct elements $\mathcal{J}^3 \subset \{1, 2, \dots, n_2 - 1\}$, $\#\mathcal{J}^3 = \#\mathcal{J}$ (that is, each index $j^2 \in \mathcal{J}^2$ is paired with an index $j^3 \in \mathcal{J}^3$, as explained above for the first iteration). For $i^1, j^1 \in \mathcal{J}^1$, $i^2, j^2 \in \mathcal{J}^2$ and $i^3, j^3 \in \mathcal{J}^3$ (correspondingly paired) we have

$$\begin{aligned} \prod_{r=1}^3 U_r &= (U_1)_{n \times n} (U_2)_{n \times n_2} (U_3)_{n_2 \times n_3} \\ &= I_n \begin{bmatrix} * & \dots & * & \dots & * & \dots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & v_{j^1, j^2}^1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & v_{i^1, i^2}^1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \dots & * & \dots & * & \dots & * \end{bmatrix} \begin{bmatrix} * & \dots & * & \dots & * & \dots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & v_{j^2, j^3}^2 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & v_{i^2, i^3}^2 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \dots & * & \dots & * & \dots & * \end{bmatrix} \\ &= \begin{bmatrix} * & \dots & * & \dots & * & \dots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \pi_{j^1, j^3}^2 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & \pi_{i^1, i^3}^2 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \dots & * & \dots & * & \dots & * \end{bmatrix}_{n \times n_3}, \pi_{j^1, j^3}^2 \doteq v_{j^1, j^2}^1 v_{j^2, j^3}^2. \end{aligned}$$

Following the same procedure for each subsequent iteration ℓ with $p_\ell < n_\ell$ and satisfying the conditions on α^ℓ , we have $p_\ell > \#\mathcal{J}$. At this iteration, the matrix $\prod_{r=1}^\ell U_r$ is of size $n \times n_\ell$ and has its rows with indices in $\mathcal{J}^1$ equal to zero except for their j^ℓ -th

482 entries, $j^\ell \in \mathcal{J}^\ell$, $\mathcal{J}^\ell \subset \{1, 2, \dots, n_\ell\}$, $\#\mathcal{J}^\ell = \#\mathcal{J}$, that is, for $i^1, j^1 \in \mathcal{J}^1$ and $i^\ell, j^\ell \in \mathcal{J}^\ell$

$$\prod_{r=1}^{\ell} U_r = \begin{bmatrix} * & \dots & * & \dots & * & \dots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \pi_{j^1, j^\ell}^{\ell-1} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & \pi_{i^1, i^\ell}^{\ell-1} & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \dots & * & \dots & * & \dots & * \end{bmatrix}_{n \times n_\ell}, \pi_{j^1, j^\ell}^{\ell-1} = \prod_{i=1}^{\ell-1} v_{j^i, j^{i+1}}^i. \quad (45)$$

483 Thus, to have $V_{(\mathcal{J}^1, \ell)} = 0$, the common eigenvector assignment should satisfy $\xi_{\mathcal{J}^\ell}^\ell = 0$.

484 At iteration κ , with $p_\kappa = n_\kappa$, the same procedure yields a matrix $\prod_{r=1}^{\kappa} U_r$ of the form
 485 (45) with dimension $n \times n_\kappa$. Then since $p_\kappa = n_\kappa$ and from Lemma 8, all the B_i^κ matrices
 486 for $i \in \underline{N}$ have rank n_κ . That is, the control input matrices are invertible, and we can
 487 assign an arbitrary eigenvector matrix and select a diagonal matrix $\Delta_i^3 = \text{diag}\{\lambda_i^\kappa, \dots, \lambda_i^n\}$
 488 to complete the M_i matrices construction as follows

$$M_i = \begin{bmatrix} \Delta_i^1 & \Delta_i^2 \\ 0 & \Delta_i^3 \end{bmatrix}$$

489 where $\Delta_i^1 \in \mathbb{C}^{(\kappa-1) \times (\kappa-1)}$ is an upper triangular matrix with diagonal elements $\{\lambda_i^1, \dots, \lambda_i^{\kappa-1}\}$
 490 computed in the previous iterations and $\Delta_i^2 \in \mathbb{C}^{(\kappa-1) \times (n-\kappa+1)}$ is an arbitrary matrix.

491 We assign a common eigenvector matrix I_{n_κ} , that is, all remaining iterations from κ
 492 to n of Algorithm ITBF can be subsumed in one step by taking

$$V^\ell = I_{n_\kappa}.$$

493 Since the matrix $(\prod_{r=1}^{\kappa} U_r) \in \mathbb{C}^{n \times n_\kappa}$, as mentioned before, has $\#\mathcal{J}$ rows (with indices
 494 in $\mathcal{J}^1$) with one nonzero entry and otherwise zero, by multiplying this matrix with the
 495 common eigenvector matrix I_{n_κ} , the last $n - \kappa + 1$ columns of the matrix V in (13) will
 496 be the matrix $\prod_{r=1}^{\kappa} U_r$ and thus,

$$V_{(:, \kappa:n)} = \left(\prod_{r=1}^{\kappa} U_r \right) V^\ell = \begin{bmatrix} * & \dots & * & \dots & * & \dots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \pi_{j^1, j^\kappa}^{\kappa-1} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & \pi_{i^1, i^\kappa}^{\kappa-1} & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \dots & * & \dots & * & \dots & * \end{bmatrix},$$

$$V = [V_{(:, 1:\kappa-1)} \quad V_{(:, \kappa:n)}],$$

$$V_{(\mathcal{J}^1, :)} = \begin{bmatrix} 0 & \dots & 0 & \dots & \pi_{j^1, j^\kappa}^{\kappa-1} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & \pi_{i^1, i^\kappa}^{\kappa-1} & \dots & 0 \end{bmatrix}.$$

Then, the condition in Lemma 3 and Lemma 4 for M_i , can be satisfied by assigning to zero the eigenvalues associated with the nonzero entries of the matrix $V_{(\mathcal{J}^1, :)}$, that is, the eigenvalues in Δ_i^3 satisfy $\lambda_i^{\kappa-1+j^\kappa} = 0, \forall j^\kappa \in \mathcal{J}^\kappa$. Note that in the matrix M_i , the row corresponding to the zero eigenvalue $\lambda_i^{\kappa-1+j^\kappa}$, $j^\kappa \in \mathcal{J}^\kappa$ is zero and hence, together with its corresponding nonzero entry in $V_{(j^1, :)}$, $j^1 \in \mathcal{J}^1$, the conditions of Lemma 3 are satisfied and the j^1 -th ultimate bound is minimised to its lowest possible value. ■

3.5. Successful Termination of Algorithm ITBF

As mentioned in Theorem 11, Algorithm ITBF is successful if at each iteration, α^ℓ that satisfies (44) also renders $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if $p_\ell = n_\ell$ (see Theorem 7). The latter property is achieved if the vector α^ℓ satisfying (44) is such that the resulting common eigenvector ξ^ℓ as in (28) does not lie in the intersection subspace $\mathcal{S}^\ell$ unless $p_\ell = n_\ell = \rho^\ell$, in which case the minimisation problem is trivially solved in one step (See Remark 1) due to invertibility of all input matrices.

We next give sufficient conditions for the ℓ -th iteration of Algorithm ITBF ensuring the common eigenvector $\xi^\ell \in \mathbb{C}^{n_\ell}$ satisfying (44) is not inside the intersection subspace $\mathcal{S}^\ell$ with $\rho^\ell < n_\ell$.

Lemma 12. *At the ℓ -th iteration of Algorithm ITBF, consider p_ℓ defined in (22) and let $\mathcal{J}^\ell \subset \{1, 2, \dots, n_\ell\}$, $\#\mathcal{J}^\ell \leq p_\ell - 1$ be the target zero elements of the common eigenvector ξ^ℓ . Let $S \in \mathbb{C}^{n_\ell \times \rho^\ell}$, $\rho^\ell < n_\ell$, be a basis of the intersection subspace $\mathcal{S}^\ell$, and from (35), consider the matrix $W_{(1:n_\ell, :)}^\ell \in \mathbb{C}^{n_\ell \times \psi_\ell}$ where W^ℓ is a basis for $\ker Q^\ell$, with the matrix Q^ℓ defined in (21). Suppose that $d(\ker S_{(\mathcal{J}^\ell, :)}) < d(\ker W_{(\mathcal{J}^\ell, :)}^\ell)$, then, a common eigenvector ξ^ℓ with $\xi_{\mathcal{J}^\ell}^\ell = 0$ can be found such that $\xi^\ell \notin \mathcal{S}^\ell$, namely, the intersection subspace can be avoided.*

PROOF. From (24), (27), (36) and the matrix $W_{(1:n_\ell, :)}^\ell$, we have $v = W_{(1:n_\ell, :)}^\ell \alpha^\ell$ and thus, the common eigenvector (28) is

$$\xi^\ell = W_{(1:n_\ell, :)}^\ell \alpha^\ell / \|v\|. \quad (46)$$

For the condition $\xi^\ell \in \mathcal{S}^\ell$ to hold, there should exist a nonzero vector $\beta \in \mathbb{C}^{\rho^\ell}$ such that

$$S\beta = \xi^\ell = W_{(1:n_\ell, :)}^\ell \alpha^\ell / \|v\|. \quad (47)$$

For simplicity, reorder the rows of the matrices S and $W_{(1:n_\ell, :)}^\ell$, respectively, to form $\bar{S} = [S'_1 \ S'_2]'$ and $\bar{W} = [W'_1 \ W'_2]'$, such that $S_1 = S_{(\mathcal{J}^\ell, :)}$ and $W_1 = W_{(\mathcal{J}^\ell, :)}^\ell$. According to these reordered matrices, reorder β and α^ℓ (for simplicity we keep the same symbols). Then, combining (47) with (44) yields

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \beta = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \alpha^\ell / \|v\|, \quad S_1 \beta = W_1 \alpha^\ell = 0, \quad (48)$$

We divide the proof into two parts to obtain a sufficient condition to avoid (48).

528 (i) If $d(\ker S_1) = 0$, then there exists no $\beta \neq 0$ such that $S_1\beta = 0$ and hence, any α^ℓ in
 529 the nullspace of W_1 avoids the intersection space.

530 (ii) If $d(\ker S_1) > 0$, then $S_1\beta = W_1\alpha^\ell = 0$ is feasible and in order to avoid (48) we
 531 need to derive conditions preventing the occurrence of the equality $S_2\beta = W_2\alpha^\ell$.

532 Let $\Phi_{S_1} \in \mathbb{C}^{\rho^\ell \times d(\ker S_1)}$ be a basis of $\ker S_1$. Then, we can choose $\beta \in \mathbb{C}^{\rho^\ell}$ in $\text{img } \Phi_{S_1}$
 533 as $\beta = \Phi_{S_1}\eta_\beta$ with $\eta_\beta \in \mathbb{C}^{d(\ker S_1)}$ arbitrary. Also, let $\Phi_{W_1} \in \mathbb{C}^{p_\ell \times d(\ker W_1)}$ be a matrix
 534 of a set of basis vectors of $\ker W_1$. The vector $\alpha^\ell \in \mathbb{C}^{p_\ell}$ can then be selected in
 535 $\text{img } \Phi_{W_1}$ as $\alpha^\ell = \Phi_{W_1}\eta_\alpha$ with $\eta_\alpha \in \mathbb{C}^{d(\ker W_1)}$ arbitrary.

536 The equality $S_2\beta = W_2\alpha^\ell$ then takes the form

$$S_2\Phi_{S_1}\eta_\beta = W_2\Phi_{W_1}\eta_\alpha. \quad (49)$$

537 Note that $W_2\Phi_{W_1}$ is not rank deficient. If otherwise, it means that $\text{img } \Phi_{W_1} \cap \ker W_2 \neq$
 538 $\emptyset$, that is, $\ker W_1 \cap \ker W_2 \neq \emptyset$ which is in contradiction with $\bar{W}$ being full column
 539 rank (cf. (37)).

540 Hence, $\text{rank}(W_2\Phi_{W_1}) = d(\ker W_1)$. Then, if $d(\ker S_1) < d(\ker W_1)$, proper selection
 541 of η_α ensures that no η_β in $\ker S_1$ can be found to satisfy (49).

542 ■

543 **Remark 2.** In Lemma 2, we singled out a case for which the ultimate bound minimi-
 544 sation is not feasible. When the j -th row of at least one of the B_i matrices is zero, then
 545 the corresponding ultimate bound cannot be minimised to its minimum value $\mathbf{b}_j^{\min}$ as in
 546 (4). In the Appendix A we show how this case causes Algorithm ITBF to break down
 547 at some iteration for which the conditions in Lemma 9 cease to hold. ◦

548 4. Numerical example

549 Consider a switched system formed by two subsystems with matrices

$$\begin{aligned}
A_1 &= \begin{bmatrix} -1.1680 & 4.0080 & 0.4535 & -1.5970 & 2.0732 & -4.5139 \\ 2.1211 & 2.5822 & 1.6484 & -4.9292 & 0.0862 & 0.1569 \\ 3.5424 & 0.8391 & 3.1124 & -4.2657 & -0.4706 & -1.5780 \\ 1.9897 & -0.9946 & 3.0450 & -0.8680 & -3.1999 & -0.7216 \\ 4.5828 & -4.8787 & 0.8283 & -2.3990 & -3.7715 & 3.4032 \\ -4.7401 & -2.1672 & -2.7062 & 0.4180 & -0.2747 & -1.1410 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 1.5627 & -1.5909 & -1.9712 & -2.9420 & 0.3049 & 1.4167 \\ 0.1561 & 4.6490 & 4.4860 & 1.6537 & -4.4574 & -1.6594 \\ 0.8228 & -4.1364 & -3.9620 & -3.1066 & -2.4372 & -1.6533 \\ 0.4060 & 4.6741 & 4.3958 & 1.9146 & -0.1129 & 2.3545 \\ 3.9448 & 0.5444 & -1.1935 & -2.2255 & -2.2387 & 4.5450 \\ 0.6211 & -0.1799 & -0.2757 & 2.5900 & -4.6122 & -2.9693 \end{bmatrix} \\
B_1 &= \begin{bmatrix} -1.2420 & 2.3311 & -4.6076 & 0.9842 & -0.1453 \\ -4.2966 & 0.2976 & -1.0773 & -4.4021 & -0.1310 \\ -1.1432 & -2.2856 & -0.9735 & -4.7206 & 3.1869 \\ -4.5608 & 0.6943 & -4.0939 & 3.5796 & -0.0540 \\ -3.6490 & 4.1370 & -1.0558 & 4.3792 & 1.7705 \\ 3.8813 & 4.6013 & 4.3709 & 0.1758 & -0.1383 \end{bmatrix}, H_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
B_2 &= \begin{bmatrix} -4.6122 & -0.6965 & -1.4818 & -0.0505 \\ -1.9460 & 1.3177 & -4.5455 & -1.1453 \\ 0.9533 & 3.4591 & -2.9129 & -2.6257 \\ 4.8218 & -0.2982 & 4.8561 & -2.7867 \\ -3.4101 & 1.0458 & 4.1421 & 2.4541 \\ 3.7593 & 3.7462 & 4.1393 & -3.8830 \end{bmatrix}, H_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d} = 1
\end{aligned}$$

From (22) we can compute $p_1 = 3$. Hence, as in Assumption 10, we aim at minimising up to $p_1 - 1 = 2$ ultimate bounds of the switched system under arbitrary switching. For this 6-th order system, we choose the 5-th and 6-th states for ultimate bound minimisation and thus, $\mathcal{J} = \{5, 6\}$, $\#\mathcal{J} = 2$.

At the first iteration, ($q_1 = 0$ and) $p_1 = 3 > \#\mathcal{J}$ and thus, for an arbitrary set of eigenvalues $\Lambda^1 = [\lambda_1^1 \ \lambda_2^1]' = [0.0551 \ 0.3846]'$ Procedure SCEA successfully results in $V_{(:,1)} = \xi^1 = [-0.6042 \ 0.6026 \ -0.4930 \ -0.1696 \ 0 \ 0]^T$ with $\alpha^1 = [0.2242 \ -0.9283 \ 0.2968]^T$. The matrix U_2 as in (17) with $\mathcal{J}^2 = \{4, 5\}$ is

$$U_2 = \begin{bmatrix} 0.7968 & 0 & 0 & 0 & 0 \\ 0.4569 & 0.6543 & 0 & 0 & 0 \\ -0.3738 & 0.7151 & -0.3254 & 0 & 0 \\ -0.1286 & 0.2461 & 0.9456 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

At the next iteration, ($q_2 = 3$ and) $p_2 = 4 > \#\mathcal{J}$ and thus, for an arbitrary set of eigenvalues $\Lambda^2 = [\lambda_1^2 \ \lambda_2^2]' = [0.4242 \ 0.5993]'$, Procedure SCEA for $\mathcal{J}^2$ gives ξ^2 and α^2 as shown below $\xi^2 = [0.4273 \ 0.3156 \ 0.8472 \ 0 \ 0]^T$ with $\alpha^2 = [0.0723 \ 0.5665 \ 0.9188 \ 0.9108]^T$. The second column of the matrix V in (13) is thus $V_{(:,2)} = I_6 U_2 \xi^2 = [0.3405 \ 0.4018 \ -0.2097 \ 0.8238 \ 0 \ 0]^T$. Also the matrix U_3 as in (17) with $\mathcal{J}^3 = \{3, 4\}$ is

$$U_3 = \begin{bmatrix} -0.9041 & 0 & 0 & 0 \\ 0.1492 & -0.9371 & 0 & 0 \\ 0.4005 & 0.3491 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

At the next iteration, $n_3 = 4$ and $m_1^3 = m_2^3 = 4$ and hence, the input matrices are invertible. In this case, the common eigenvector matrix associated with arbitrary eigenvalues can be taken to be the identity matrix. Since at this iteration with subsystems of order 4 we have $\mathcal{J}^3 = \{3, 4\}$, we need the last two eigenvalues of both subsystems to be zero. Therefore, assigning the remaining eigenvalues at

$$\begin{aligned}\Delta_1^3 &= \text{diag}\{\lambda_1^3, \lambda_1^4, \lambda_1^5, \lambda_1^6\} = \text{diag}\{-0.2854, -0.5910, 0, 0\} \\ \Delta_2^3 &= \text{diag}\{\lambda_2^3, \lambda_2^4, \lambda_2^5, \lambda_2^6\} = \text{diag}\{-0.9381, 0.3268, 0, 0\}\end{aligned}$$

568 and computing the eigenvectors as in Part 2 of the proof of Theorem 11, $V^3 = I_4$, yields

$$V_{(:,3:6)} = I_6 U_2 U_3 V^3 = \begin{bmatrix} -0.7204 & 0 & 0 & 0 \\ -0.3155 & -0.6131 & 0 & 0 \\ 0.3144 & -0.7837 & 0 & 0 \\ 0.5317 & 0.0995 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

569 The resulting feedback gains, triangularising transformation matrix and upper trian-
570 gular closed-loop matrices are

$$\begin{aligned}K_1 &= \begin{bmatrix} -0.1709 & -1.2207 & 0.9183 & -0.8941 & -1.2457 & 0.8258 \\ 0.2483 & -0.2276 & 0.3587 & 0.5143 & -0.2647 & 0.2418 \\ 0.9193 & 1.7895 & -0.5789 & 0.1856 & 1.4308 & -0.7072 \\ -0.3254 & 0.7433 & 0.1064 & -0.6640 & 0.4182 & -0.4880 \\ -2.1678 & 0 & -0.0218 & 0.0633 & 0 & 0 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} 0.2890 & -0.1019 & -0.2887 & 0.1020 & -0.3859 & -0.4916 \\ -0.8009 & -0.0091 & 0.2280 & 0.8341 & 1.1647 & -0.9145 \\ -0.1930 & -0.0747 & 0.0430 & -0.3186 & 0.1153 & -0.0082 \\ -0.5388 & -0.2334 & -0.0847 & 1.2309 & -0.3149 & -2.1316 \end{bmatrix}, \\ V &= \begin{bmatrix} -0.6042 & 0.3405 & -0.7204 & 0 & 0 & 0 \\ 0.6026 & 0.4018 & -0.3155 & -0.6131 & 0 & 0 \\ -0.4930 & -0.2097 & 0.3144 & -0.7837 & 0 & 0 \\ -0.1696 & 0.8238 & 0.5317 & 0.0995 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\ M_1 &= \begin{bmatrix} 0.0551 & 6.6301 & -6.4030 & 0.6186 & 4.3531 & 1.6601 \\ 0 & 0.4242 & 1.5813 & -1.3306 & -1.6077 & -3.5081 \\ 0 & 0 & -0.2854 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5910 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ M_2 &= \begin{bmatrix} 0.3846 & 4.9649 & 3.7414 & -6.1443 & -2.7784 & -3.5492 \\ 0 & 0.5993 & -3.1425 & -8.1900 & -1.6449 & 6.7434 \\ 0 & 0 & -0.9381 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3268 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

571 The matrices V and M_i satisfy the conditions of Lemma 1 and Lemma 4. For a particular
572 realisation of the disturbance signal shown in Figure 3, and two random switching sig-
573 nals (Figures 4 and 6), the trajectories of the perturbed switched system (Figures 5 and 7)
574 and zoomed trajectories of the perturbed switched system depicted in Figure 8, corre-
575 sponding to the initial condition $x(0) = [0.6146, 1.1240, 1.7603, 2.1086, 1.8297, 3.5015]'$,
576 show that the 5-th and 6-th states are ultimately bounded within their smallest possible
577 regions. As expected, the 5-th and 6-th state trajectories become equal after one time
578 step since they correspond to $\max_i H_i(5)\mathbf{d} = \max_i H_i(6)\mathbf{d} = \mathbf{d}$, that is, only the distur-
579 bance.

Figure 3: Random bounded disturbance

Figure 4: First random switching signal

Figure 5: Switched system trajectories under first random switching signal

Figure 6: Second random switching signal

Figure 7: Switched system trajectories under second random switching signal

Figure 8: Zoomed in trajectories under both random switching signals

5. Conclusion

This paper has considered discrete-time switched linear systems in the presence of non-vanishing perturbations and derived sufficient conditions to achieve the minimum ultimate bounds by state feedback for one or more components of the state of the closed-loop switched system under arbitrary switching. These conditions are expressed as design constraints on the eigenstructure of the subsystems. A constructive procedure to satisfy these constraints has been presented in the form of an iterative algorithm that simultaneously triangularises all closed-loop subsystem matrices and achieves the lowest ultimate bounds for the target states provided they are feasible.

The reliance of the proposed method on the construction of stable, simultaneously triangularisable closed-loop feedback matrices inherits the conservatism of this technique, which is only sufficient (and with a CQLF) for closed-loop stability under arbitrary switching. However, there is not much work available on control of switched systems under arbitrary switching, and the little available focuses on stabilisation. The present paper goes beyond stabilisation by integrating in the feedback design disturbance attenuation through the minimisation of state componentwise ultimate bounds. A distinctive feature of the proposed approach is that it is entirely analytic, which enables us to algebraically characterise when the problem can and cannot be solved in terms of the system structure.

Appendix A. Unfeasibility of ultimate bound minimisation

Let the matrix B_1 of size $n \times m_1$ be of full column rank and its j -th row be zero, that is

$$B_1 = \begin{bmatrix} * & \dots & * \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ * & \dots & * \end{bmatrix}. \quad (\text{A.1})$$

From Lemma 2 we know that the j -th ultimate bound cannot be minimised by any feedback control. Here we want to show where the ITBF algorithm breaks down.

At the first iteration, the common eigenvector assignment yields ξ^1 with $\xi_j^1 = 0$. Then we should check whether $\xi^1 \in \text{img } B_1$ or not. In the worst case, if $m_1 = n - 1$, then the part of the matrix B_1 associated with the elements of ξ^1 excluding its j -th element is square and thus, $\xi^1 \in \text{img } B_1$. Then, depending on whether or not $\xi^1 \in \text{img } B_i$ for all $i = 2, \dots, N$, we obtain $\xi^1 \in S^1$ or $\xi^1 \notin S^1$.

Assume that we have $m_1 < n - 1$ and $\xi^1 \notin S^1$. Then, from (14)–(17), for $j^1 = j$ and an arbitrary j^2 , the matrix U_2 is of the form

$$U_2 = \begin{bmatrix} * & \dots & 0 & \dots & * \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & v_{j^1, j^2}^1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ * & \dots & 0 & \dots & * \end{bmatrix}$$

611 which from (19) yields the input matrix B_1^2 for the second iteration with a zero j^2 -th row

$$B_i^2 = U_2^* B_i^1 = \begin{bmatrix} * & \dots & 0 & \dots & * \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & v_{j^1, j^2}^1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ * & \dots & 0 & \dots & * \end{bmatrix}^* \begin{bmatrix} * & \dots & * \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ * & \dots & * \end{bmatrix} = \begin{bmatrix} * & \dots & * \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ * & \dots & * \end{bmatrix}.$$

612 As it can be seen, the zero row of the matrix B_1 has been shifted to the input matrix of
613 the second iteration such that the j^2 -th row of the matrix B_1^2 is zero.

614 At this iteration α^2 can be selected from (44) so that $\xi_{j^2}^2 = 0$. Again we need to
615 check whether $\xi^2 \in \mathcal{S}^2$ which includes checking $\xi^2 \in \text{img } B_1^2$. Similar to the previous
616 iteration, if $m_2 = n_2 - 1$, then it is certain that $\xi^2 \in \text{img } B_1^2$.

617 We suppose that up to the iteration ℓ , where $m_\ell = n_\ell - 1$, we have $\xi^\ell \notin \text{img } B_1^\ell$ and
618 thus, $\xi^\ell \notin \mathcal{S}^\ell$. Computing the $U_{\ell+1}$ matrices as in (14)–(17) and determining the matrices
619 B_i^ℓ from (19), similar to the above we see that the matrix B_1^ℓ has its j^ℓ -th row equal to
620 zero. At this iteration we certainly have $\xi^\ell \in \text{img } B_1^\ell$ due to $m_\ell = n_\ell - 1$. However,
621 having $\xi^\ell \in \mathcal{S}^\ell$ still needs checking whether $\xi^\ell \in \text{img } B_i^\ell$ for all $i = 2, \dots, N$. Suppose
622 that $\xi^\ell \notin \text{img } B_i^\ell$ until the B_i matrices are invertible for all $i = 2, \dots, N$.

623 For the iteration $\ell + 1$, since $\xi^\ell \in \text{img } B_1^\ell$, we have $m_{\ell+1} = m_\ell - 1$ which together
624 with $n_{\ell+1} = n_\ell - 1$ results in $m_{\ell+1} = n_{\ell+1} - 1$. Also, considering the shift of the zero row
625 to the $B_1^{\ell+1}$ matrix, similar to the previous iteration, we will have $\xi^{\ell+1} \in \text{img } B_1^{\ell+1}$. The
626 same situation happens at the subsequent iterations.

627 Continuing with Algorithm ITBF and considering the above arguments and assump-
628 tions, at some iteration l we obtain invertible B_i^l matrices for all $i = 2, \dots, N$ and B_1^l of
629 size $n_l \times (n_l - 1)$. From the previous discussion we know that $\xi^l \in \text{img } B_1^l$. Together with
630 invertibility of the other B_i matrices, the common eigenvector lies inside the intersection
631 subspace $\mathcal{S}^l$, i.e. $\xi^l \in \mathcal{S}^l$ which results in $p_{l+1} = p_l - 1$. Thus the condition in Theorem 7
632 does not hold. However, we might still have $p_{l+1} > \#\mathcal{J}$ and the vector α^{l+1} could be
633 selected as in (44) so that the algorithm continues without any problem. In this case,
634 the next iterations of Algorithm ITBF will follow the same pattern as the l -th iteration
635 and reduce the quantity p_l until we reach $p_l \not> \#\mathcal{J}$, at which point the condition in
636 Lemm 9 does not hold and the required α^l cannot be found. Therefore, Algorithm ITBF
637 is unsuccessful.

References

- [1] D. Liberzon, *Switching in systems and control*, Birkhäuser, 2003.
- [2] R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King, Stability criteria for switched and hybrid systems, *SIAM Review* 49 (4) (2007) 545–592.
- [3] H. Lin, P. J. Antsaklis, Stability and stabilisability of switched linear systems: a survey of recent results, *IEEE Trans. on Aut. Control* 54 (2) (2009) 308–322.
- [4] A. P. Molchanov, Y. S. Pyatnitskiy, Criteria of asymptotic stability of differential and difference inclusions encountered in control theory, *Systems and Control Letters* 13 (1) (1989) 59–64.
- [5] H. Khalil, *Nonlinear Systems*, Prentice-Hall, New Jersey, 2002.
- [6] E. Kofman, Non conservative ultimate bound estimation in LTI perturbed systems, *Automatica* 41 (10) (2005) 1835–1838.
- [7] H. Haimovich, Quantisation issues in feedback control, Ph.D. thesis, School of Electrical Engineering and Computer Science, The University of Newcastle, Australia (Mar. 2006).
URL <http://usuarios.fceia.unr.edu.ar/~haimo>
- [8] B. Barmish, I. Petersen, A. Feuer, Linear ultimate boundedness control of uncertain dynamical systems, *Automatica* 19 (5) (1983) 523–532.
- [9] Y. Cao, Y. Sun, Robust stabilization of uncertain systems with time-varying multistate delay, *IEEE Trans. on Aut. Control* 43 (10) (1998) 1484–1488.
- [10] S. Oucheriah, Robust tracking and model following of uncertain dynamic delay systems by memoryless linear controllers, *IEEE Trans. on Aut. Control* 44 (7) (1999) 1473–1477.
- [11] E. Kofman, M. M. Seron, H. Haimovich, Control design with guaranteed ultimate bound for perturbed systems, *Automatica* 44 (7) (2008) 1815–1821.
- [12] R. Heidari, M. Seron, J. Braslavsky, On eigenvalue-eigenvector assignment for componentwise ultimate bound minimisation in MIMO LTI discrete-time systems, in: *Australian Control Conf.*, Perth, Australia, 2013.
- [13] X. Zhang, J. Zhao, An algorithm of uniform ultimate boundedness for a class of switched linear systems, *International Journal of Control* 75 (16/17) (2010) 1399–1405.

- 669 [14] L. Zhang, Y. Chen, Z. Sun, N. Mastorakis, A. Alexsandrov, Uniformly ultimate
670 boundedness control for switched linear systems with parameter uncertainties,
671 Proc. American Control Conference, Seattle, WA, USA (2008) 3971–3974.
- 672 [15] L. Zhang, Y. Lu, Y. Chen, N. Mastorakis, Robust uniformly ultimate boundedness
673 control for uncertain switched linear systems, Computers and Mathematics with
674 Applications (2008) 1709–1714.
- 675 [16] H. Haimovich, M. Seron, Componentwise ultimate bound and invariant set com-
676 putation for switched linear systems, Automatica 46 (11) (2010) 1897–1901.
- 677 [17] H. Haimovich, M. Seron, Bounds and invariant sets for a class of switching sys-
678 tems with delayed-state-dependent perturbations, Automatica 49 (2013) 748–754.
- 679 [18] H. Haimovich, M. Seron, Bounds and invariant sets for a class of discrete-time
680 switching systems with perturbations, International Journal of Control 87 (2014)
681 371–383.
- 682 [19] H. Haimovich, J. Braslavsky, Sufficient conditions for generic feedback stabilis-
683 ability of switching systems via Lie-algebraic solvability, IEEE Trans. on Aut.
684 Control 58 (3) (2013) 814–820.
- 685 [20] D. Liberzon, J. Hespanha, A. Morse, Stability of switched systems: a Lie-algebraic
686 condition, Systems and Control Letters 37 (1999) 117–122.
- 687 [21] R. Heidari, M. Seron, J. Braslavsky, H. Haimovich, On eigenstructure assignment
688 for componentwise ultimate bound minimisation in discrete-time switched linear
689 systems, in: Proc. of the IFAC World Congress, Cape Town, South Africa, 2014,
690 pp. 4085–4090.