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Nonlinear consolidation of vertical drains with coupled radial-vertical flow considering well resistance

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Abstract: The consolidation behavior of ground with vertical drains is known to be greatly affected by the finite permeability of the sand drains, also called the effect of well resistance. However, up to now, no analytical methods have been reported for evaluating this effect on the nonlinear consolidation behavior of vertical drains. In this paper, by considering the nonlinear compressibility and permeability of soil during consolidation, the effect of well resistance was incorporated into the derivation of the equations that govern the nonlinear consolidation of a vertical drain with coupled radial-vertical flow. In addition, the smear effect was considered by assuming three decay patterns for the radial permeability coefficients of the soil toward the sand drain in the smeared zone. After obtaining the governing equations, a simplified analytical solution is derived for a general time-variable surcharge loading. Based on the general solution obtained, detailed solutions are provided for three special types of loading schemes: constant loading, single-stage loading, and multi-stage loading. The validity of the solution is verified by reducing it to several special cases and comparing these to existing solutions. Finally, the effect of the well resistance, the ratios of the compression index to the radial and vertical permeability indices, various loading schemes, and various variation patterns of the radial permeability coefficient of the soil in the smeared soil zone are investigated using parametric analysis.

Keywords: consolidation, vertical drain, nonlinearity, well resistance, coupled radial-vertical flow, instantaneous loading, single-stage loading, multi-stage loading

1. Introduction

Vertical drains, such as prefabricated vertical drains (PVDs) and sand drains, are used worldwide to accelerate the consolidation of soft soils. The first analytical study on the consolidation of vertical drains was performed by Barron (1948), where the water was assumed to flow radially into the sand drain and the vertical flow in the soil was ignored. After this study, many works, including both linear

1 and nonlinear consolidation theories for vertical drains, were performed to extend this theory to reflect
2 more realistic engineering problems.

3 The linear consolidation theory for vertical drains, which commonly assumes a constant permeability
4 and compressibility during consolidation, has experienced great progress in the past few decades. In
5 Barron's theory (1948), the surcharge loading was assumed to be fully applied instantaneously on the
6 ground. Obviously, this instantaneous loading cannot be achieved in practice because the construction
7 of superstructures requires time to be accomplished. In fact, the loading rate and the loading pattern
8 have been proven to greatly influence the consolidation behavior of soils with vertical drains (Tang
9 and Onitsuka, 2000). For this reason, many studies (Zhu and Yin, 2004; Leo, 2004; Conte and
10 Troncone, 2009; Lu et al., 2011; Geng et al., 2012) were performed to account for some commonly
11 used loadings, such as ramp loading, multi-stage loading, time- and depth-dependent loading, and
12 vacuum-surcharge loading. Apart from the loading scheme, the smear effect is also known to influence
13 the consolidation behavior of vertical drains. Walker and Indraratna (2006, 2007) studied the smear
14 effect by considering a parabolic distribution of the permeability coefficient of the soil and the
15 overlapping smear zone due to the installation of sand-drains. Their studies provided a possible
16 explanation for why a reduction in the size of the influence zone has a limited effect on accelerating
17 the consolidation rate. In addition, Walker et al. (2009) numerically analyzed the consolidation of
18 multi-layered vertical drains using a spectral method. However, this method is difficult for engineers
19 to use due to its complex programming requirements. By introducing the double porosity model into
20 the analysis of consolidation with vertical drains, Wang and Jiao (2004) provided a simplified
21 approach for considering the smear effect of vertical drains, which has since been used by Lu et al.
22 (2011, 2014). To develop an appropriate equation to convert the rectangular shape of a PVD to an
23 equivalent circular shape, Abuel-Naga et al. (2012) assessed the validity of several existing
24 approaches numerically and identified the most appropriate one for use in design. Incorporating the
25 elastic visco-plastic constitutive equation of Yin and Graham (1989, 1994, 1996), Hu et al. (2014)
26 suggested that including the large-strain condition and the creep effects was essential in the analysis of

the consolidation of very soft clay layers with vertical drains. Compared to the aforementioned linear consolidation theories, fewer developments have been made for the nonlinear consolidation of vertical drains due to the complexity of the problem. For this reason, Lekha et al. (1998) proposed an approximate analytical method to investigate the nonlinear radial consolidation of vertical drains under ramp loading. Likewise, using the same nonlinear relation of a soil's compression and permeability with effective stress, Indraratna et al. (2005) solved the consolidation of vertical drains under instantaneous loading analytically and compared their results with data measured in the laboratory with good agreement found between the two sets of results. For this reason, Lu et al. (2014) used the same method to derive a simplified analytical solution for the nonlinear consolidation of vertical drains that are subjected to a general time-variable loading. In a different approach, Walker et al. (2012) analyzed the nonlinear consolidation of vertical drains numerically where a non-Darcian flow law was incorporated. However, this numerical method is difficult for engineers to use in an actual design.

Thus, although some progress has been made on the consolidation behavior of vertical drains, particularly for linear consolidation theory, no analytical solutions for the nonlinear consolidation of vertical drains allowing for the effect of well resistance are available. As Hansbo et al. (1981) noted, the effect of well resistance should be taken into account, especially when the drains are deeply installed. Well resistance in the field can be caused by several factors, such as the reduction in cross-sectional area of the PVDs, the deformation of the PVDs, and fine soil particles infiltrating into the PVD core and clogging the drainage channels (Deng et al., 2013). In this context, this paper addresses the problem by incorporating a finite permeability coefficient for a sand drain into the analysis of the consolidation of a vertical drain with coupled radial and vertical flow. In addition, to account for the unfavorable effects of installation on the adjacent soil, the decay of the radial permeability of the smeared soil towards a sand drain is also considered in the analysis.

2. Basic assumptions, objectives, equations, and solutions

Fig. 1. Schematic diagram of the vertical drain showing the variation of the radial coefficient of soil permeability

The main objective of this paper is to develop an easy-to-use analytical solution for the nonlinear consolidation of a vertical drain with coupled radial and vertical flows by considering the finite permeability of the vertical drain (well resistance), the variation of the radial permeability of the soil in the smeared soil zone (the smear effect), and time-varying surcharge loading. The unit cell for analysis can be idealized as shown in Fig. 1, and the assumptions made in the analysis are listed below.

① The following relations are adopted for the nonlinear response of the soil permeability and compressibility during consolidation:

$$e = e_0 - C_c \log \frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \quad (1.a)$$

$$e = e_0 + C_{kh} \log \frac{k_h}{k_{h0}} \quad (1.b)$$

$$e = e_0 + C_{kv} \log \frac{k_v}{k_{v0}} \quad (1.c)$$

where C_c , C_{kh} , and C_{kv} are the compression index, the radial permeability index, and the vertical permeability index, respectively; k_h and k_{h0} are the radial permeability coefficients of the unsmeared soil at any time and at the initial time, respectively; k_v and k_{v0} are the vertical permeability coefficients of the unsmeared soil at any time and at the initial time; e and e_0 are the void ratios of the soil at any time and at the initial time; and $\bar{\sigma}'_s$ and $\bar{\sigma}'_{s0}$ are the average effective stresses in the soil at any time and at the initial time.

From Eq. (1.a–c), the following basic relations are obtained:

$$\frac{k_v}{k_{v0}} = \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{\frac{C_c}{C_{kv}}} \quad (2.a)$$

$$\frac{k_v}{k_h} = \frac{k_{v0}}{k_{h0}} \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{\frac{C_c}{C_{kh}} - \frac{C_c}{C_{kv}}} \quad (2.b)$$

$$\frac{a_v}{a_{v0}} = \frac{(\partial \bar{e} / \partial \bar{\sigma}'_s)}{(\partial \bar{e} / \partial \bar{\sigma}'_s)_{t=0}} = \frac{\bar{\sigma}'_{s0}}{\bar{\sigma}'_s} = \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{-1} \quad (2.c)$$

where a_v and a_{v0} are the compression coefficients of the soil at any time and at the initial time, respectively.

② During the consolidation process, the soil permeability reduces because the void ratio of soil diminishes as the effective stress on the soil skeleton increases. The sand fill in the vertical-drain is much denser than the surrounding soil, which may lead to little change in the permeability of the sand drains during the entire process. Therefore, the coefficient of permeability of the vertical drain is assumed to be finite and constant. Moreover, the finite permeability of the sand drain (well resistance) is considered by assuming that the water flowing into the vertical drain from its circumferential

direction is equal to that flowing out of it in the vertical direction:

$$\left[2\pi r \frac{k_r(r)}{\gamma_w} \frac{\partial u_s}{\partial r} \right]_{r=r_w} = -\pi r_w^2 \frac{k_w}{\gamma_w} \frac{\partial^2 u_w}{\partial z^2} \quad (3)$$

where u_w is the average excess pore water pressure in the vertical drain at any depth; u_s is the excess pore water pressure in the surrounding soil at any point; r_w , r_s , and r_e are the radii of the vertical drain, smeared zone, and influence zone, respectively; $k_r(r)$ is the radial permeability coefficient of the surrounding soil; k_w is the permeability coefficient of the vertical drain, which is held as a constant during consolidation; and γ_w is the unit weight of water.

③ To reflect the smear effect, the radial permeability coefficient of the soil is assumed to vary with the radial distance away from the vertical drain:

$$k_r(r) = k_h f(r) \quad (4)$$

where k_h varies during consolidation as shown in Eq. (1.b) and simultaneously increases from k_s to k_h in the smeared soil zone with respect to radial distance, as shown in Fig. 1. The expressions of $f(r)$ for various distribution patterns of horizontal permeability of soil can be found in the study of Xie et al. (2009).

④ The surcharge loading is assumed to be gradually applied to the ground surface:

$$p(t) = p_u g(t) \quad (5)$$

where p_u is the maximum loading value and $g(t)$ is a function that governs the loading variation with time.

The other assumptions are identical to Barron's theory, such as the validity of Darcy's law and the supposition that the soil particles and water are incompressible. As demonstrated by Xie (1987) and Tang and Onitsuka (2000), the governing equations of a vertical drain under equal-strain conditions follow the law of conservation of mass as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{k_r(r)}{\gamma_w} r \frac{\partial u_s}{\partial r} \right) + \frac{k_v}{\gamma_w} \frac{\partial^2 \bar{u}_s}{\partial z^2} = - \frac{\partial \varepsilon_v}{\partial t} \quad (6)$$

where $\bar{u}_s$ is the average excess pore water pressure in the surrounding soil at any depth and ε_v is the vertical or volumetric strain of the soil and the vertical drain. The boundary conditions in the radial direction are:

$$\begin{cases} r = r_e : & \frac{\partial u_s}{\partial r} = 0 \\ r = r_w : & u_s = u_w \end{cases} \quad (7)$$

Integrating Eq. (6) twice and using the radial boundary conditions given in Eq. (7) leads to:

$$u_s = u_w + \frac{\gamma_w}{2k_h} \left(\frac{\partial \varepsilon_v}{\partial t} + \frac{k_v}{\gamma_w} \frac{\partial^2 \bar{u}_s}{\partial z^2} \right) [r_e^2 \bar{A}(r) - \bar{B}(r)] \quad (8)$$

$$\text{where } \bar{A}(r) = \int_{r_w}^r \frac{d\xi}{\xi f(\xi)} \text{ and } \bar{B}(r) = \int_{r_w}^r \frac{\xi d\xi}{f(\xi)}.$$

Let us define the average excess pore water pressure in the soil at any depth combine with Eq.

$$(8) \text{ yields: } \bar{u}_s = \frac{1}{\pi(r_e^2 - r_w^2)} \int_{r_w}^{r_e} 2\pi r u_s(r, z, t) dr = u_w + \frac{\gamma_w r_e^2 F_c}{2k_h} \left(\frac{\partial \varepsilon_v}{\partial t} + \frac{k_v}{\gamma_w} \frac{\partial^2 \bar{u}_s}{\partial z^2} \right) \quad (9)$$

11

$$\text{where } F_c = \frac{2(\bar{\bar{A}}r_e^2 - \bar{\bar{B}})}{r_e^2 r_w^2 (n^2 - 1)}, \bar{\bar{A}} = \int_{r_w}^{r_e} r \bar{A}(r) dr, \bar{\bar{B}} = \int_{r_w}^{r_e} r \bar{B}(r) dr, \text{ and } n \text{ is the radius ratio of the}$$

influence zone to the sand drain, $n = r_e / r_w$.

F_c is a comprehensive parameter that reflects the three variation patterns of the radial permeability coefficient in the smeared soil zone, as shown in Fig. 1. In fact, the effect of the variations of the radial permeability coefficient of soil is only related to this parameter. A detailed calculation of F_c can be found in the study of Xie et al. (2009).

Combining Eq. (3) with Eq. (8) yields:

$$1 \quad \frac{k_w}{\gamma_w} \frac{\partial^2 u_w}{\partial z^2} = -\left(n^2 - 1\right) \left(\frac{\partial \varepsilon_v}{\partial t} + \frac{k_v}{\gamma_w} \frac{\partial^2 \bar{u}_s}{\partial z^2} \right) \quad (10)$$

2 and inserting Eq. (9) into Eq. (10) gives:

$$3 \quad \bar{u}_s = u_w - \frac{r_e^2 F_c}{2(n^2 - 1) k_h} \frac{k_w}{\gamma_w} \frac{\partial^2 u_w}{\partial z^2} \quad (11)$$

4 Substituting Eq. (11) into Eq. (9) yields:

$$5 \quad \frac{r_e^2 F_c}{2} \frac{k_v}{k_h} \frac{\partial^4 u_w}{\partial z^4} - \frac{(n^2 - 1) \gamma_w}{k_w} \frac{\partial \varepsilon_v}{\partial t} - \left((n^2 - 1) \frac{k_v}{k_w} + 1 \right) \frac{\partial^2 u_w}{\partial z^2} = 0 \quad (12)$$

6 Furthermore, in combination with Eq. (2.c), the volumetric strain rate can be obtained by replacing the

7 real time variable t by the time factor T_h in the radial direction:

$$8 \quad \frac{\partial \varepsilon_v}{\partial t} = \frac{1}{1 + e_0} \frac{\partial e}{\partial t} = \frac{1}{1 + e_0} \frac{\partial e}{\partial \bar{\sigma}'_s} \frac{\partial \bar{\sigma}'_s}{\partial T_h} \frac{\partial T_h}{\partial t} = \frac{k_{h0}}{4r_e^2 \gamma_w} \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{-1} \left(p_u \frac{dg(T_h)}{dT_h} - \frac{\partial \bar{u}_s}{\partial T_h} \right) \quad (13)$$

9 where T_h is the radial time factor of the soil, which has the form

$$10 \quad T_h = \frac{c_{h0} t}{4r_e^2} = \frac{k_{h0} (1 + e_0) t}{4r_e^2 \gamma_w a_{v0}} = \frac{k_{h0} (1 + e_0) t}{4r_e^2 \gamma_w (\partial \bar{e} / \partial \bar{\sigma}')_{t=0}} \quad \text{and } c_{h0} \text{ is the radial consolidation coefficient at}$$

11 the initial time.

12 Substituting Eq. (13) into Eq. (12) and combining with Eqs. (2.a), (2.b), and (11) gives:

$$13 \quad A' \frac{\partial^4 u_w}{\partial z^4} + B' \frac{\partial \bar{u}_s}{\partial T_h} + C' \frac{\partial^2 u_w}{\partial z^2} = B' p_u \frac{dg(T_h)}{dT_h} \quad (14)$$

$$14 \quad \text{where } A' = \frac{r_e^2 F_c}{2} \frac{k_{v0}}{k_{h0}} \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{1 + \frac{C_c}{C_{kh}} - \frac{C_c}{C_{kv}}} , \quad B' = \frac{(n^2 - 1) k_{h0}}{4r_e^2} \frac{k_w}{\gamma_w} \quad \text{and}$$

$$15 \quad C' = - \left[(n^2 - 1) \frac{k_{v0}}{k_w} \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{1 - \frac{C_c}{C_{kv}}} + \frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right] .$$

16 To remove the variable $\bar{u}_s$ in the above equation, we must rewrite Eq. (10) as:

$$\bar{u}_s = u_w - \frac{r_e^2 F_c}{2(n^2 - 1)} \frac{k_w}{k_{h0}} \left(\frac{\bar{\sigma}'_s}{\bar{\sigma}'_{s0}} \right)^{\frac{C_c}{C_{kh}}} \frac{\partial^2 u_w}{\partial z^2} = u_w + B \frac{\partial^2 u_w}{\partial z^2} \quad (15)$$

where $B = -\frac{r_e^2 F_c}{2(n^2 - 1)} \frac{k_w}{k_{h0}} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right)^{\frac{C_c}{C_{kh}}}$; $\bar{\sigma}'_s$ is the average effective stress at any depth in the surrounding soil, which is expressed as $\bar{\sigma}'_s = \bar{\sigma}'_{s0} + p(t) - \bar{u}_s$; and $\bar{\sigma}'_{s0}$ is the initial stress. The stress $\bar{\sigma}'_s$ increases from its initial value $\bar{\sigma}'_{s0}$ to its final value at the end of the consolidation $\bar{\sigma}'_{s0} + p_u$. To simplify the above equation, the approach of Lekha et al. (1998) and Indraratna et al. (2005) was adopted by replacing $\bar{\sigma}'_s / \bar{\sigma}'_{s0}$ with its average value $[\bar{\sigma}'_{s0} + (\bar{\sigma}'_{s0} + p_u)] / (2\bar{\sigma}'_{s0})$ over the consolidation period.

With the substitution of Eq. (15), Eq. (14) can be expressed as a nonhomogeneous partial differential equation in u_w :

$$A \frac{\partial^4 u_w}{\partial z^4} + B \frac{\partial^3 u_w}{\partial T_h \partial z^2} + C \frac{\partial^2 u_w}{\partial z^2} + \frac{\partial u_w}{\partial T_h} = p_u \frac{dg(T_h)}{dT_h} \quad (16)$$

where

$$A = \frac{A'}{B'} = \frac{2r_e^4 F_c}{(n^2 - 1)} \frac{k_{v0}}{k_{h0}} \frac{k_w}{k_{h0}} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right)^{1 + \frac{C_c}{C_{kh}} - \frac{C_c}{C_{kv}}}$$

$$C = \frac{C'}{B'} = -4r_e^2 \frac{k_w}{k_{h0}} \left[\frac{k_{v0}}{k_w} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right)^{1 - \frac{C_c}{C_{kv}}} + \frac{1}{(n^2 - 1)} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right) \right]$$

Eqs. (15) and (16) need to be solved in combination with the vertical boundary conditions and the initial conditions. As indicated above, the coefficients A' and C' contain the effective stress of $\bar{\sigma}'_s$ that varies with time. Obviously, solving Eq. (14) analytically is difficult with varying coefficients. Therefore, by replacing $\bar{\sigma}'_s / \bar{\sigma}'_{s0}$ with the approximation $[\bar{\sigma}'_{s0} + (\bar{\sigma}'_{s0} + p_u)] / (2\bar{\sigma}'_{s0})$, $\bar{\sigma}'_s$ is removed and the variable coefficients of the nonlinear partial differential equations are changed to

1 constant coefficients, allowing Eqs. (15) and (16) to be solved analytically. This simplifies the
 2 problem substantially and can provide an explicit analytical solution. Despite the simplification, the
 3 predicted results from the analytical solution are consistent with data measured in laboratory tests
 4 (Indraratna et al., 2005).

5 The appropriate boundary conditions in the vertical direction depend on the drainage path at the top
 6 and bottom of the vertical drains. As shown in Fig. 1, the top surface is assumed to be pervious and the
 7 bottom surface is assumed to be impervious. Thus, the boundary conditions are shown as follows:

$$8 \quad \begin{cases} z = 0: & \bar{u}_s(z, T_h) = 0, & u_w(z, T_h) = 0 \\ z = H: & \frac{\partial \bar{u}_s(z, T_h)}{\partial z} = 0, & \frac{\partial u_w(z, T_h)}{\partial z} = 0 \end{cases} \quad (17)$$

9 The consolidation is caused only by the newly applied surcharge loading, so that the initial condition
 10 can be written as:

$$11 \quad T_h = 0: \quad \bar{u}_s(z, T_h) = p_u g(T_h = 0) = p_0 \quad (18)$$

12 where p_0 is the initial value of the surcharge loading.

13 Eqs. (15) and (16) need to be solved to satisfy the boundary conditions in Eq. (17) and the initial
 14 condition in Eq. (18). Referring to the study of Lu et al. (2010), the solutions of the inhomogeneous
 15 partial differential equations (15) and (16) can be obtained as:

$$16 \quad u_w = \sum_{m=1}^{\infty} \frac{2}{M \left[1 - B \left(\frac{M}{H} \right)^2 \right]} \left(p_0 e^{-\beta_m T_h} + p_u e^{-\beta_m T_h} \int_0^{T_h} \frac{dg(\tau)}{d\tau} e^{\beta_m \tau} d\tau \right) \sin \left(\frac{M}{H} z \right) \quad (19)$$

$$17 \quad \bar{u}_s = \sum_{m=1}^{\infty} \frac{2}{M} \left(p_0 e^{-\beta_m T_h} + p_u e^{-\beta_m T_h} \int_0^{T_h} \frac{dg(\tau)}{d\tau} e^{\beta_m \tau} d\tau \right) \sin \left(\frac{M}{H} z \right) \quad (20)$$

$$18 \quad \text{where } \beta_m = \frac{A \left(\frac{M}{H} \right)^4 - C \left(\frac{M}{H} \right)^2}{1 - B \left(\frac{M}{H} \right)^2} \text{ and } Q_m = \frac{2p_u}{M \left[1 - B \left(\frac{M}{H} \right)^2 \right]}; \text{ and the expressions for } A, B$$

and C can be seen in the text following Eqs. (15) and (16).

According to the work of Lu et al. (2011), the total average degree of consolidation, which is defined in terms of stress, can be derived in combination with Eq. (24) as follows:

$$U(T_h) = \frac{\int_0^H [p(T_h) - \bar{u}_s] dz}{\int_0^H p_u dz} = g(T_h) - \sum_{m=1}^{\infty} \frac{2}{M^2} \left(\frac{p_0}{p_u} e^{-\beta_m T_h} + e^{-\beta_m T_h} \int_0^{T_h} \frac{dg(\tau)}{d\tau} e^{\beta_m \tau} d\tau \right) \quad (21)$$

At this stage, the general solutions for the excess pore water pressure and degree of consolidation have been obtained.

3. Solutions for several special cases

To check its validity, the present solution can be simplified to several special cases that have been studied by others.

If we let $k_w \rightarrow \infty$, β_m converges to:

$$\beta_m = \left(\frac{M}{H} \right)^2 \frac{4r_e^2 k_{v0}}{k_{h0}} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right)^{1-\frac{C_c}{C_{kv}}} + \frac{8}{F_c} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right)^{1-\frac{C_c}{C_{kh}}} \quad (22)$$

This expression is the nonlinear solution for the coupled radial and vertical consolidation of a vertical drain without considering the effect of well resistance. The solutions for the nonlinear consolidation case have the same unified form as the corresponding linear case, which was provided by Tang and Onitsuka (2000), i.e., the coupled radial and vertical solution can be written as the summation of two terms in β_m that correspond to radial consolidation and vertical consolidation.

If we let $k_w \rightarrow \infty$, $C_c/C_{kv} = 1$ and $C_c/C_{kh} = 1$, β_m can be reduced to:

$$\beta_m = \frac{4r_e^2 k_{v0}}{k_{h0}} \left(\frac{M}{H} \right)^2 + \frac{8}{F_c} \quad (23)$$

This expression is the linear solution for the consolidation of a vertical drain without well resistance, which can be obtained from Tang and Onitsuka (2000), where the compression and permeability of the soil are assumed to remain constant during the consolidation process.

If we let $k_w \rightarrow \infty$ and $k_v \rightarrow 0$, the expression for β_m becomes:

$$\beta_m = \frac{8}{F_c} \left(\frac{2\bar{\sigma}'_{s0} + p_u}{2\bar{\sigma}'_{s0}} \right)^{1 - \frac{C_c}{C_{kh}}} \quad (24)$$

This relation is the solution for the radial nonlinear consolidation of a vertical drain, which was provided by Lu et al. (2014), where only radial flow is considered and the well resistance is ignored.

Based on the above solutions for a general time-variable loading, detailed solutions are derived for several special cases: instantaneous loading, single-stage loading, and multi-stage loading, as shown in

Fig. 2.

Fig. 2. Surcharge loading schemes

(1) Instantaneous loading

As shown in Fig. 2(a), when the load is fully applied suddenly and then held constant, we have:

$$g(T_h) = 1, \quad p_0 = p_u \quad (25)$$

Substituting Eq. (25) into Eqs. (20) and (21) yields:

$$\bar{u}_s = \sum_{m=1}^{\infty} \frac{2p_u}{M} e^{-\beta_m T_h} \sin\left(\frac{M}{H} z\right) \quad (26)$$

$$U_p = \frac{\int_0^H (p(t) - \bar{u}_s) dz}{\int_0^H p_u dz} = 1 - \sum_{m=1}^{\infty} \frac{2}{M^2} e^{-\beta_m T_h} \quad (27)$$

2 (2) Single-stage loading

As shown in Fig. 2(b), if the surcharge load is gradually increased to its final value and then held constant we have what is commonly termed single-stage loading (or ramp loading). In this case, the following relations apply:

$$p_0 = 0; \quad g(T_h) = \begin{cases} \frac{T_h}{T_{h1}}, & T_h \leq T_{h1} \\ 1, & T_h > T_{h1} \end{cases} \quad (28)$$

Substituting Eq. (28) into Eqs. (20) and (21) then yields:

$$\bar{u}_s = \begin{cases} \sum_{m=1}^{\infty} \frac{2p_u (1 - e^{-\beta_m T_h})}{M \beta_m T_{h1}} \sin\left(\frac{M}{H} z\right), & T_h \leq T_{h1} \\ \sum_{m=1}^{\infty} \frac{2p_u (e^{-\beta_m (T_h - T_{h1})} - e^{-\beta_m T_h})}{M \beta_m T_{h1}} \sin\left(\frac{M}{H} z\right), & T_h > T_{h1} \end{cases} \quad (29)$$

$$U_p = \begin{cases} \frac{T_h}{T_{h1}} - \sum_{m=1}^{\infty} \frac{2}{M^2 \beta_m T_{h1}} (1 - e^{-\beta_m T_h}), & T_h \leq T_{h1} \\ 1 - \sum_{m=1}^{\infty} \frac{2}{M^2 \beta_m T_{h1}} (e^{-\beta_m (T_h - T_{h1})} - e^{-\beta_m T_h}), & T_h > T_{h1} \end{cases} \quad (30)$$

10 (3) Multi-stage loading

When the soil is too soft to support a large surcharge load, the load can be gradually applied stage-by-stage, as shown in Fig. 2(c). This loading scheme consolidates the soil in stages, thus increasing its strength in a controlled manner. This loading scheme can be expressed as:

$$p_0 = 0; \quad g(T_h) = \begin{cases} a_{i-1} + R_i (T_h - T_{h(2i-2)}), & T_{h(2i-2)} < T_h \leq T_{h(2i-1)} \\ a_i, & T_{h(2i-1)} < T_h \leq T_{h(2i)} \end{cases} \quad (31)$$

1 where $a_i = \frac{p_i}{p_u}$ and $R_i = \frac{a_i - a_{i-1}}{T_{h(2i-1)} - T_{h(2i-2)}}$; i denotes the i^{th} stage of loading and has the values of

2 1, 2, 3, ...; and p_i is the final value of the i^{th} stage of loading.

3 The average excess pore water pressure can be obtained in a unified form in the ranges of both

4 $T_{h(2i-2)} < T_h \leq T_{h(2i-1)}$ and $T_{h(2i-1)} < T_h \leq T_{h(2i)}$ by substituting Eq. (31) into Eq. (20):

$$5 \quad \bar{u}_s = \sum_{m=1}^{\infty} \frac{2p_u}{M\beta_m} \sin\left(\frac{M}{H}z\right) \sum_{j=1}^i R_j \left(e^{-\beta_m(T_h - T_s)} - e^{-\beta_m(T_h - T_{h(2j-2)})} \right) \quad (32)$$

6 where $T_s = \min(T_h, T_{h(2j-1)})$ and $j = 1, 2, \dots, i$.

7 In addition, the average degree of consolidation due to the applied stress can be obtained by
8 substituting Eq. (31) into Eq. (21):

$$9 \quad U_p = g(T_h) - \sum_{m=1}^{\infty} \frac{2}{M^2\beta_m} \sum_{j=1}^i R_j \left(e^{-\beta_m(T_h - T_s)} - e^{-\beta_m(T_h - T_{h(2j-2)})} \right) \quad (33)$$

10 4. Analysis of the consolidation behavior

11 In this section, the nonlinear consolidation behavior of a vertical drain is investigated using parametric
12 analysis. The first contribution of this paper is to introduce the well resistance into the analysis of the
13 nonlinear consolidation of vertical drains. As indicated by Fig. 3, an increase in the permeability
14 coefficient of a vertical drain significantly enhances the average degree of consolidation. This
15 influence becomes increasingly small with an increase in the value of k_w/k_{h0} , especially when
16 k_w/k_{h0} exceeds 1000. Therefore, from a practical standpoint, the effect of well resistance can be
17 ignored when the value of k_w/k_{h0} exceeds 1000.

Fig. 3. Effect of the permeability of sand drain on the average degree of consolidation (Pattern I , two-stage loading)

Fig. 4. Effect of C_c/C_{kh} on the average degree of consolidation (Pattern I , two-stage loading)

In this paper, the radial and vertical flows in the soil are coupled and included in the derivation of the nonlinear consolidation solutions. Consequently, the nonlinear consolidation behavior of a vertical drain is affected by the ratio of the compression index to the permeability indices in both the radial and vertical directions. Fig. 4 shows the influence of the ratio of the compression index to the radial permeability index C_c/C_{kh} on the average degree of consolidation. The average degree of

consolidation is greatly affected by the value of C_c/C_{kh} , and the maximum difference between the average degree of consolidation with various values of C_c/C_{kh} may reach 10%. Moreover, a smaller value of C_c/C_{kh} always gives larger average degrees of consolidation because a low value of C_c/C_{kh} implies either a smaller value of C_c or a larger value of C_{kh} . As shown in Eq. (1.a), a smaller value of C_c indicates a smaller decrease in the compression coefficient ($a_v = -de/d\sigma$) or a larger increase in the compression modulus ($E_s = (1+e_0)/a_v$), i.e., a smaller value of C_c indicates stiffer soil with a more rapid consolidation rate. On the other hand, with the same change in the void ratio, a larger C_{kh} indicates a smaller decrease in the soil permeability coefficient as indicated by Eq. (1.b). This, in turn, will also accelerate the dissipation of excess pore water pressure.

Fig. 5. Effect of C_c/C_{kv} on the average degree of consolidation (Pattern I, two-stage loading)

The effect of the ratio of the compression index to the vertical permeability index C_c/C_{kv} is shown in Fig. 5. A similar conclusion can be drawn as with Fig. 4: a smaller value of C_c/C_{kv} always results in a larger average degree of consolidation. However, unlike the significant influence of C_c/C_{kh} on the degree of consolidation, the effect of C_c/C_{kv} is limited and the difference between the average

degrees of consolidation is so small that it can be ignored in practice. The reason for this is that the vertical flow makes a much smaller contribution than the radial flow because the excess water pressure within the surrounding soil is mainly dissipated along the radial direction due to the much shorter drainage path, which results in the radial flow having a dominant role in the consolidation process.

Fig. 6. Comparison of the average degree of consolidation with various variation patterns of the radial permeability of soil in the smeared soil zone (two-stage loading)

Fig. 6 compares the average degrees of consolidation for the three variation patterns of the radial permeability coefficient of the smeared soil as shown in Fig. 1. A large difference between pattern I and pattern III can be observed; pattern III has the most rapid consolidation rate, while pattern I has the slowest rate. In comparison, the difference between the average degrees of consolidation for patterns II and III are very small, but the expression for the parameter F_c , which reflects the effect of the various patterns, is much more complex for pattern III than for pattern II. Thus, pattern II is recommended for practical use when considering the smear effect.

Fig. 7. Effect of various loading schemes on the dissipation of excess pore water pressure (Pattern I)

Fig. 7 shows the dissipation of excess pore water pressure under three different types of loading: instantaneous loading, single-stage loading and two-stage loading. As expected, for the same final value of surcharge loading, the maximum induced excess pore water pressure can be reduced greatly by gradually applying the load in stages. For example, if the total surcharge loading is applied in two stages, the maximum excess pore pressure is much smaller than that when the full surcharge loading is imposed suddenly. However, although the maximum excess pore pressure is decreased, a large excess pore water pressure in the final stage of consolidation is also observed, which means the consolidation process is delayed. Therefore, when a large excess pore water pressure is undesirable, a multi-stage loading scheme can be used. This is often the case when a soft soil layer with a small shear strength is present because a large excess pore water pressure may diminish the effective stress in the soil (and hence the shear strength) and trigger collapse. For stronger soils, however, where the shear strength is not a concern, a rapid consolidation rate can be obtained by applying a larger surcharge loading.

Fig. 8. Comparisons between this study and previous studies (instantaneous loading, Pattern I)

Finally, the present solution is compared to several existing solutions in Fig. 8, including the solution provided by Tang and Onitsuka (2000) for the linear consolidation of vertical drains with coupled radial and vertical flow and the solution provided by Indraratna et al. (2005) for the nonlinear consolidation of vertical drains with only radial flow. First, the only difference between the present study and that of Tang and Onitsuka (2000) is that the former considers the nonlinear compressibility and permeability of soil, whereas the latter ignored these factors. In this context, when $C_c/C_{kh} = 0.5 < 1$ and $C_c/C_{kv} = 0.8 < 1$, the nonlinear consolidation rate is more rapid than the linear consolidation rate. In contrast, the reverse is true when $C_c/C_{kh} = 1.5$ and $C_c/C_{kv} = 2 > 1$; i.e., the nonlinear consolidation rate is slower than the linear consolidation rate. Second, the difference between the present solution and the solution of Indraratna et al. (2005) is that the former considers the well resistance and couples the radial and vertical flows, whereas the latter ignores the well resistance and considers only radial flow. As previously mentioned, the contribution of the vertical flow to the consolidation rate is limited if radial flow is considered simultaneously. Therefore, although vertical flow may promote the consolidation rate, the well resistance may have a more evident and opposite effect in reducing the consolidation rate. For this reason, although the vertical

flow is incorporated into the present solution, the consolidation rate is still slower than that predicted by the Indraratna et al. (2005) solution because the well resistance was ignored in the latter. This highlights the importance of the effect of well resistance on the nonlinear consolidation behavior of a vertical drain.

5. Conclusions

This paper derived a simplified analytical solution for the nonlinear consolidation of a vertical drain with coupled radial-vertical flow by considering the well resistance, the variation of soil compressibility and permeability with effective stress, three variation patterns of the radial permeability coefficient of soil in the smeared soil zone, and a general time-variable loading. Then, detailed solutions were provided for instantaneous, single-stage, and multi-stage loading. The present solution was verified by considering a number of simple cases. Finally, the nonlinear consolidation behavior of a vertical drain was investigated using parametric analysis, and the following conclusions can be drawn:

① The well resistance significantly affects the consolidation rate of sand drains. With an increase in the permeability coefficient of a sand drain, the average degree of consolidation is greatly enhanced. However, this influence may be small and can be ignored in practice when the values of k_w/k_{h0} exceed 1000.

② The ratios of the compression index to the horizontal permeability indices (C_c/C_{kh}) significantly affect the dissipation of excess pore water pressure. With an increase in the value of C_c/C_{kh} , the consolidation rate is decreased by a maximum of 10%. Compared with C_c/C_{kh} , the influence of C_c/C_{kv} is so small that it can be ignored when predicting the average degree of consolidation. In addition, when $C_c/C_{kv}, C_c/C_{kh} < 1$, the nonlinear consolidation rate is larger than the linear one, and when $C_c/C_{kv}, C_c/C_{kh} > 1$, the reverse is true.

③ For the same final value of surcharge loading, the maximum value of the induced excess pore water pressure can be reduced by applying the load in stages. However, the time to attain the same

average degree of consolidation will also be significantly extended.

④ A linear decay of the radial permeability coefficient of soil towards a sand drain in the smeared soil zone is sufficient to reflect the smear effect caused by the installation of the drain.

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Figure Captions

Fig. 1. Schematic diagram of the vertical drain showing the variation of the radial coefficient of soil permeability

Fig. 2. Surcharge loading schemes. (a) Instantaneous loading; (b) Single-stage loading; (c) Multi-stage

1 loading.

2 Fig. 3. Effect of the permeability of sand drain on the average degree of consolidation (Pattern I ,

3 two-stage loading)

4 Fig. 4. Effect of C_c/C_{kh} on the average degree of consolidation (Pattern I , two-stage loading)

5 Fig. 5. Effect of C_c/C_{kv} on the average degree of consolidation (Pattern I , two-stage loading)

6 Fig. 6. Comparison of the average degree of consolidation with various variation patterns of the radial

7 permeability of soil in the smeared soil zone (two-stage loading)

8 Fig. 7. Effect of various loading schemes on the dissipation of excess pore water pressure (Pattern I)

9 Fig. 8. Comparisons between this study and the previous studies (instantaneous loading, Pattern I)

Notations

14 a_v, a_{v0} compression coefficients of soil at any time and at the initial time, respectively;

15 A, B, C constant parameters in Eqs. (16) and (17);

16 $\bar{A}(r), \bar{B}(r)$ functions of radial distance that are relevant to the variation in the radial permeability

17 coefficients of the surrounding soil, shown in Eq. (8);

18 $\bar{\bar{A}}, \bar{\bar{B}}$ constant parameters that are relevant to the variation in the radial permeability

19 coefficients of the surrounding soil, shown in Eq. (10);

20 C_c, C_{kh}, C_{kv} compressive index, radial permeability index, and vertical permeability index,

21 respectively;

22 c_{h0} initial coefficient of consolidation in the radial direction;

23 e, e_0 void ratios of the soil at any time and at the initial time, respectively;

1	$f(r)$	a function of radius to depict the variation pattern of the horizontal permeability
2		coefficient of soil along the radial direction;
3	F_c	a group parameter to reflect the effect of the geometry and property of the vertical drain
4		system;
5	$g(t)$	a function of time to depict the loading schemes;
6	k_w	permeability coefficient of the vertical drain, which is maintained as a constant during
7		consolidation;
8	$k_r(r)$	radial permeability coefficients of the surrounding soil, which vary with the radial
9		distance, $k_r(r) = k_h f(r)$;
10	k_h, k_{h0}	radial permeability coefficients of the unsmeared soil at any time and at the initial time,
11		respectively;
12	k_v, k_{v0}	vertical permeability coefficients of the unsmeared soil at any time and at the initial
13		time, respectively;
14	m	integers, $m = 1, 2, 3 \dots$;
15	M	parameters in Eq. (20), $M = \frac{2m-1}{2} \pi$, $m = 1, 2, 3 \dots$;
16	n	radius ratio of the effect zone to the vertical drain, $n = r_e / r_w$;
17	$p(t)$	general expression for the loading;
18	p_0, p_u	initial and final loading values, respectively;
19	Q_m	parameter in Eq. (23);
20	r_w, r_s, r_e	radii of the vertical drain, smeared zone, and influence zone, respectively;
21	s	radius ratio of the smeared soil zone to the vertical drain, $s = r_s / r_w$;
22	T_h	time factor of the soil in the radial direction;