LMI approach to suboptimal guaranteed cost control for uncertain time-delay systems

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Abstract: Results on the design of robust memoryless state feedback controllers for uncertain time-delay systems with norm bounded uncertainty are presented. It is proved that the feasibility of a linear matrix inequality (LMI) problem is necessary and sufficient for the quadratic stabilisation of an uncertain time-delay system. A robust state feedback controller can be constructed using the corresponding feasible solution of the LMI problem. A procedure is given to select a suitable state feedback controller that is also suboptimal in the sense of minimising a bound on a quadratic performance index.

1 Introduction

During recent years, the problem of quadratic stabilisation of uncertain linear systems with norm-bounded uncertainty has received considerable attention in the control community (e.g. see [1–3]). The stabilisation approach presented in [3] is based on the use of a fixed quadratic Lyapunov function for uncertain systems which leads to a parameter-dependent Riccati equation. Using any positive definite solution of this parameter-dependent Riccati equation, a robust controller is constructed that quadratically stabilises the linear uncertain system. In [4], the problem of designing a quadratic stabilising controller, which also yields an adequate level of performance for a closed loop system, is proposed. The performance is measured with the standard quadratic cost function, and an upper bound for this cost function is defined. Finally, a corresponding robust controller that minimises this bound is found by a numerical search over the stabilising solutions of the parameter-dependent Riccati equation.

The problem of designing robust controllers for uncertain time-delay systems has attracted the attention of a number of researchers (e.g. see [5–7]). Again, the Lyapunov approach is used to design a robust controller. However, the problem of designing a controller

which would guarantee an adequate level of performance in terms of a quadratic cost function is still under consideration, especially for the case when no matching assumptions are made. In [8], this problem is addressed for a particular class of uncertain time-delay systems.

In this paper, we propose a linear matrix inequality (LMI) approach to the stabilisation problem for uncertain time-delay systems. It should be noted that the idea of employing LMIs to stabilise linear time-delay systems was first proposed in [15]. The results of [15] are a special case of the results of this paper. Moreover, our approach also guarantees an upper bound on a quadratic cost function.

2 Quadratic stability of uncertain time-delay systems

We consider the uncertain time-delay system described by the state equation of the form:

$$\dot{x}(t) = (A + D\Delta(t)E_1)x(t) + (F + D\Delta(t)E_3)x(t-\tau) \quad (1)$$

where $x(t) \in R^n$ is the state and $\Delta(t) \in R^{p \times q}$ is a matrix of uncertain parameters satisfying the bound $\Delta'(t)\Delta(t) \leq I$. The rest of the matrices are real constant matrices with compatible dimensions and τ is a scalar representing the delay in the system. The initial condition is specified as $(x(0), x(s)) = (x_0, \phi(s))$ where $\phi(\cdot) \in L_2[-\tau, 0]$.

For the system in eqn. 1, we define the Lyapunov functional:

$$V(x, t) = x'(t)Px(t) + \int_0^\tau x(t-s)'Nx(t-s)ds$$

where P and N are symmetric positive definite matrices. It is straightforward to show that

$$\begin{aligned} \dot{V}(x, t) &= [x(t)' \quad x(t-\tau)'] \\ &\times \begin{bmatrix} (A+D\Delta E_1)'P + P(A+D\Delta E_1) + N & P(F+D\Delta E_3) \\ (F+D\Delta E_3)'P & -N \end{bmatrix} \\ &\times \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \end{aligned} \quad (2)$$

If $\dot{V} < 0$, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$. This will guarantee the asymptotic stability of the uncertain time-delay system (eqn. 1).

Definition 1: The uncertain time-delay system (eqn. 1) is quadratically stable if there exist $P > 0$ and $N > 0$ such that:

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$$\begin{bmatrix} (A+D\Delta E_1)'P + P(A+D\Delta E_1) + N & P(F+D\Delta E_3) \\ (F+D\Delta E_3)'P & -N \end{bmatrix} < 0 \quad (3)$$

for all Δ satisfying $\Delta'\Delta \leq I$.

If an uncertain time-delay system is quadratically stable, it is straightforward to show that, for any admissible uncertainty $\Delta(t)$, the resulting realisation of the uncertain time-delay system will be asymptotically stable.

Remark 1: Definition 1 generalises the notion of quadratic stability for uncertain time-delay systems given in [8]. The corresponding definition in [8] is a special case of this definition whenever the positive definite symmetric matrix N is restricted to be of the form $N = \delta I$. This enlarges the possible solution set in the quadratic stability and stabilisation problems, and also leads to a less conservative value for the guaranteed cost bound. Furthermore, in definition 1, the uncertainty is also allowed to enter the delay matrix of the system. This helps us to consider a broader range of uncertain time-delay systems.

Remark 2: Definition 1 can be extended to the case of uncertain systems with multiple time-delays as described below:

$$\dot{x}(t) = (A + D\Delta(t)E_1)x(t) + \sum_{i=1}^L (F_i + D\Delta(t)E_{i+2})x(t - \tau_i)$$

where L is the number of time-delays. This system is said to be quadratically stable if there exist matrices $P > 0$ and $N_i > 0$, $i = 1, \dots, L$ such that

$$\begin{bmatrix} (A+D\Delta E_1)'P + P(A+D\Delta E_1) + \sum_{i=1}^L N_i & P(F_1+D\Delta E_3) & \dots & P(F_L+D\Delta E_{2+L}) \\ (F_1+D\Delta E_3)'P & -N_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (F_L+D\Delta E_{2+L})'P & 0 & \dots & -N_L \end{bmatrix} < 0$$

for all Δ satisfying $\Delta'\Delta \leq I$.

The next theorem reduces the question of quadratic stability for the system in eqn. 1 to the solvability of a quadratic matrix inequality.

Theorem 1: The uncertain time-delay system (eqn. 1) is quadratically stable if and only if there exist $P > 0$, $N > 0$ and $\epsilon > 0$ such that

$$\begin{bmatrix} A'P + PA + \epsilon PDD'P + \frac{1}{\epsilon}E_1'E_1 + N & PF + \frac{1}{\epsilon}E_1'E_3 \\ F'P + \frac{1}{\epsilon}E_3'E_1 & -N + \frac{1}{\epsilon}E_3'E_3 \end{bmatrix} < 0 \quad (4)$$

Proof (Sufficiency): Assume that there exist $P > 0$ and $N > 0$ that satisfy eqn. 4 for some value of $\epsilon > 0$. Subtracting the left-hand side of eqn. 4 from eqn. 3, we have

$$\begin{bmatrix} \epsilon PDD'P + \frac{1}{\epsilon}E_1'E_1 - PDD\Delta E_1 - E_1'\Delta'D'P & \frac{1}{\epsilon}E_1'E_3 - PDD\Delta E_3 \\ \frac{1}{\epsilon}E_3'E_1 - E_3'\Delta'D'P & \frac{1}{\epsilon}E_3'E_3 \end{bmatrix} \quad (5)$$

Thus, to establish eqn. 3 using eqn. 4, it is sufficient to prove that this matrix is positive semidefinite. For all

nonzero $x, x_\tau \in R^n$, we have

$$\begin{aligned} & \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} PDD\Delta E_1 + E_1'\Delta'D'P & PDD\Delta E_3 \\ E_3'\Delta'D'P & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \\ &= 2x'PD\Delta E_1x + 2x'PD\Delta E_3x_\tau \\ &= 2 \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} PD \\ 0 \end{bmatrix} \Delta \begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \end{aligned}$$

Now, claim 1 of [3] implies that, given any $\epsilon > 0$, for all nonzero $x, x_\tau \in R^n$ and for any Δ such that $\Delta'\Delta \leq I$, we have

$$\begin{aligned} & 2 \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} PD \\ 0 \end{bmatrix} \Delta \begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \\ & \leq \epsilon \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} PDD'P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \\ & \quad + \frac{1}{\epsilon} \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} E_1'E_1 & E_1'E_3 \\ E_3'E_1 & E_3'E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \end{aligned}$$

From this, it follows that the matrix in eqn. 5 is positive semi-definite. This completes the proof of sufficiency.

(Necessity) eqn. 3 implies that

$$\begin{aligned} & x' \{ (A + D\Delta E_1)'P + P(A + D\Delta E_1) + N \} x \\ & + 2x'P(F + D\Delta E_3)x_\tau - x_\tau'Nx_\tau \\ &= x'(A'P + PA + N)x + 2x'PFx_\tau - x_\tau'Nx_\tau \\ & \quad + 2x'PD\Delta(E_1x + E_3x_\tau) \\ & < 0 \end{aligned}$$

for all $x \neq 0, x_\tau \neq 0$ and Δ satisfying $\Delta'\Delta \leq I$. Now, as in lemma 3.1 and observation 3.1 of [3], we have

$$\begin{aligned} & \max \{ x'PD\Delta(E_1x + E_3x_\tau) : \Delta'\Delta \leq I \} \\ &= \max \left\{ \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} PD \\ 0 \end{bmatrix} \Delta \begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} : \Delta'\Delta \leq I \right\} \\ &= \left\| \begin{bmatrix} D'P & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \right\| \left\| \begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \right\| \end{aligned}$$

where $\|\cdot\|$ denotes the standard 2-norm. Hence, we obtain

$$\begin{aligned} & x'(A'P + PA + N)x + 2x'PFx_\tau - x_\tau'Nx_\tau \\ & < -2 \left\| \begin{bmatrix} D'P & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \right\| \left\| \begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \right\| \end{aligned}$$

This implies

$$\begin{bmatrix} A'P + PA + N & PF \\ F'P & -N \end{bmatrix} < 0$$

and

$$\begin{aligned} & \left\{ \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} A'P + PA + N & PF \\ F'P & -N \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \right\}^2 \\ & > 4 \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} PDD'P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \\ & \quad \times \begin{bmatrix} x' & x_\tau' \end{bmatrix} \begin{bmatrix} E_1'E_1 & E_1'E_3 \\ E_3'E_1 & E_3'E_3 \end{bmatrix} \begin{bmatrix} x \\ x_\tau \end{bmatrix} \end{aligned}$$

for all $x \neq 0, x_\tau \neq 0$. Now, theorem 4.7 of [2] implies that there exists a parameter $\epsilon > 0$ such that

$$\begin{bmatrix} A'P + PA + \epsilon PDD'P + \frac{1}{\epsilon}E_1'E_1 + N & PF + \frac{1}{\epsilon}E_1'E_3 \\ F'P + \frac{1}{\epsilon}E_3'E_1 & -N + \frac{1}{\epsilon}E_3'E_3 \end{bmatrix} < 0$$

This completes the proof of the theorem.

Now, suppose P and N are positive definite solutions of the quadratic matrix inequality (eqn. 4) for some value of $\varepsilon > 0$. Hence $\bar{P} = \varepsilon P$ and $\bar{N} = \varepsilon N$ satisfy

$$\begin{bmatrix} A'\bar{P} + \bar{P}A + \bar{P}DD'\bar{P} + E_1'E_1 + \bar{N} & \bar{P}F + E_1'E_3 \\ F'\bar{P} + E_3'E_1 & -\bar{N} + E_3'E_3 \end{bmatrix} < 0 \quad (6)$$

Also, from this it follows that $\bar{N} - E_3'E_3 > 0$. Conversely, if the condition in eqn. 6 is satisfied, then it follows that eqn. 4 is satisfied with $\varepsilon = 1$. Thus, to check the quadratic stability of an uncertain time-delay system (eqn. 1), we need to find positive definite matrices $\bar{P}$ and $\bar{N}$ such that eqn. 6 is satisfied. Using Schur complements, the quadratic matrix inequality (eqn. 6) will be equivalent to the inequality

$$\begin{bmatrix} A'\bar{P} + \bar{P}A + E_1'E_1 + \bar{N} & \bar{P}D & \bar{P}F + E_1'E_3 \\ D'\bar{P} & -I & 0 \\ F'\bar{P} + E_3'E_1 & 0 & -\bar{N} + E_3'E_3 \end{bmatrix} < 0$$

which is an LMI in the matrix variables $\bar{P}$ and $\bar{N}$. Therefore, the problem of determining the quadratic stability of the uncertain time-delay system (eqn. 1) can be considered as an LMI feasibility problem. This problem is convex and can be effectively solved by corresponding LMI software; (e.g. see [14]). If this problem is infeasible, it necessarily implies that the system in eqn. 1 is not quadratically stable.

Remark 3: Using the definition of quadratic stability for uncertain systems with multiple time-delays given in remark 2, it follows that an uncertain system with multiple time-delays is quadratically stable if and only if the LMI

$$\begin{bmatrix} A'\bar{P} + \bar{P}A + E_1'E_1 + \sum_{i=1}^L \bar{N}_i & \bar{P}D & \bar{P}F_1 + E_1'E_3 \\ D'\bar{P} & -I & 0 \\ F_1'\bar{P} + E_3'E_1 & 0 & -\bar{N}_1 + E_3'E_3 \\ \vdots & \vdots & \vdots \\ F_L'\bar{P} + E_{L+2}'E_1 & 0 & E_{L+2}'E_3 \\ & \dots & \bar{P}F_L + E_1'E_{L+2} \\ & \dots & 0 \\ & \dots & E_3'E_{L+2} \\ & \ddots & \vdots \\ & \dots & -\bar{N}_L + E_{L+2}'E_{L+2} \end{bmatrix} < 0$$

has a feasible solution $\bar{P} > 0$ and $\bar{N}_i > 0, i = 1, \dots, L$.

Considering the strict bounded real lemma (theorem 2.1 in [9]), it is possible to state the quadratic stability condition (eqn. 6) in terms of a Riccati equation or an H_∞ norm condition.

Theorem 2: The following statements are equivalent:

- (i) The uncertain time-delay system (eqn. 1) is quadratically stable.
- (ii) The LMI

$$\begin{bmatrix} A'P + PA + E_1'E_1 + N & PD & PF + E_1'E_3 \\ D'P & -I & 0 \\ F'P + E_3'E_1 & 0 & -N + E_3'E_3 \end{bmatrix} < 0$$

has a positive definite solution P and N .

- (iii) There exists a matrix $N > E_3'E_3$ such that the Riccati equation:

$$\begin{aligned} & (A + F(N - E_3'E_3)^{-1}E_3'E_1)'P \\ & + P(A + F(N - E_3'E_3)^{-1}E_3'E_1) \\ & + P(DD' + F(N - E_3'E_3)^{-1}F')P \\ & + E_1'\{I + E_3(N - E_3'E_3)^{-1}E_3'\}E_1 + N = 0 \end{aligned}$$

has a positive definite stabilising solution P .

- (iv) There exists a matrix $N > E_3'E_3$ such that $A + F(N - E_3'E_3)^{-1}E_3'E_1$ is stable and

$$\begin{aligned} & \left\| \begin{bmatrix} E_1 \\ N^{1/2} \\ (N - E_3'E_3)^{-1/2}E_3'E_1 \end{bmatrix} \right\|_\infty \\ & \times (sI - A - F(N - E_3'E_3)^{-1}E_3'E_1)^{-1} \\ & \times [D \quad F(N - E_3'E_3)^{-1/2}] \left\|_\infty < 1 \end{aligned}$$

Proof: The proof is established by applying Schur complements to the matrix inequality (eqn. 6) and then using the strict bounded real lemma.

Observation 1: It follows from the above theorem that the three conditions:

- (i) A is a stability matrix;
- (ii) $\|E_1(sI - A)^{-1}D\|_\infty < 1$;
- (iii) $\|N^{1/2}(sI - A)^{-1}FN^{-1/2}\|_\infty < 1, N > 0$,

are necessary for quadratic stability of the system in eqn. 1. In the third condition, it is assumed that $E_1 = E_3 = 0$.

Observation 2: Consider the case that $E_1 = E_3 = 0$. In this case, eqn. 1 represents a linear time-delay system without uncertainty. By using the result of theorem 2, this time-delay system is quadratically stable if, and only if, A is stable and

$$\|N^{1/2}(sI - A)^{-1}FN^{-1/2}\|_\infty < 1$$

for some $N > 0$. Therefore, quadratic stability of a delay system can be considered as a scaled H_∞ analysis problem with scaling matrix $N > 0$.

Now assume that the system in eqn. 1 is quadratically stable; the next step is to consider a cost function:

$$J = \int_0^\infty x'(t)R_1x(t)dt \quad (7)$$

with $R_1 > 0$ as a performance measure for this system. We are interested in finding the least upper bound for this cost function.

Theorem 3: Consider the system in eqn. 1 and cost function in eqn. 7 and suppose that there exist $P > 0$ and $N > 0$ such that

$$\begin{bmatrix} (A + D\Delta E_1)'P + P(A + D\Delta E_1) + N + R_1 & P(F + D\Delta E_3) \\ (F + D\Delta E_3)'P & -N \end{bmatrix} < 0 \quad (8)$$

for all Δ such that $\Delta'\Delta \leq I$. Then the system in eqn. 1 is quadratically stable and the cost function in eqn. 7 satisfies the bound:

$$J \leq x_0'Px_0 + \int_{-\tau}^0 x'(s)Nx(s)ds \quad (9)$$

for all $\Delta(t)$ satisfying $\Delta(t)\Delta'(t) \leq I$.

Conversely, if the system in eqn. 1 is quadratically stable, then given any $R_1 > 0$, there will exist $\hat{P} > 0$ and $\hat{N} > 0$ such that eqn. 8 is satisfied.

Proof: Suppose $P > 0$ and $N > 0$ satisfy eqn. 8. We have from eqn. 2 that

$$\begin{aligned} \dot{V}(x, t) &= [x(t)' \quad x(t - \tau)'] \\ &\times \begin{bmatrix} (A + D\Delta E_1)'P + P(A + D\Delta E_1) + N & P(F + D\Delta E_3) \\ (F + D\Delta E_3)'P & -N \end{bmatrix} \\ &\times \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} \\ &< -x(t)R_1x(t) \end{aligned}$$

Therefore, the system in eqn. 1 is quadratically stable. Integrating both sides of the above inequality from 0 to ∞ , we obtain

$$V(x(\infty), \infty) - V(x(0), 0) \leq - \int_0^\infty x'(t)R_1x(t)dt$$

Since the quadratic stability of the system follows from theorem 1, we can conclude that $V(x(t), t) \rightarrow 0$ as $t \rightarrow \infty$. Hence,

$$J \leq x_0'Px_0 + \int_{-\tau}^0 x'(s)Nx(s)ds$$

To prove the second part of the theorem, suppose that the system in eqn. 1 is quadratically stable. This implies that there exist $P > 0$ and $N > 0$ such that the matrix inequality (eqn. 2) is satisfied for all admissible Δ . From the inequality (eqn. 2), it is easily verified that we can find $\mu > 0$ such that

$$\begin{bmatrix} (A + D\Delta E_1)'P + P(A + D\Delta E_1) + N + \mu R_1 & P(F + D\Delta E_3) \\ (F + D\Delta E_3)'P & -N \end{bmatrix} < 0$$

is satisfied. This implies that there exist matrices $\hat{P} = P/\mu$ and $\hat{N} = N/\mu$ such that

$$\begin{bmatrix} (A + D\Delta E_1)'\hat{P} + \hat{P}(A + D\Delta E_1) + \hat{N} + R_1 & \hat{P}(F + D\Delta E_3) \\ (F + D\Delta E_3)'\hat{P} & -\hat{N} \end{bmatrix} < 0$$

Now, if we consider the result of theorem 1, it follows that the matrix inequality (eqn. 8) is equivalent to the existence of positive definite matrices P and N such that

$$\begin{bmatrix} A'P + PA + \frac{1}{\epsilon}E_1'E_1 + N + R_1 & PD & PF + \frac{1}{\epsilon}E_1'E_3 \\ D'P & -\frac{1}{\epsilon}I & 0 \\ F'P + \frac{1}{\epsilon}E_3'E_1 & 0 & -N + \frac{1}{\epsilon}E_3'E_3 \end{bmatrix} < 0 \quad (10)$$

for some value of $\epsilon > 0$. In this case, theorem 3 suggests a corresponding upper bound for the cost function (eqn. 7). This bound is a linear and convex function in matrix variables P and N . Hence, the problem of finding the least upper bound for cost function J amongst all the possible choices of P , N and ϵ which satisfy eqn. 10 will be an LMI eigenvalue problem (see [15]). This problem is a convex optimisation problem and is stated as follows:

$$\text{minimise } x_0'Px_0 + \int_{-\tau}^0 x'(s)Nx(s)ds$$

subject to eqn. 10 and $P > 0$, $N > 0$, $\epsilon > 0$

The solution to this problem gives the global minimum for the upper bound (eqn. 9).

Remark 4: Suppose that the system in eqn. 1 is quadratically stable and that the cost function (eqn. 7) is defined with some weighting matrix R_1 . Then the minimum of the upper bound (eqn. 9) is achieved for some value of the parameters P , N and ϵ ; e.g. P_1 , N_1 and ϵ_1 . Using Schur complements, we can relate the Riccati inequality

$$\begin{aligned} &(A + F(\epsilon N - E_3'E_3)^{-1}E_3'E_1)'P \\ &+ P(A + F(\epsilon N - E_3'E_3)^{-1}E_3'E_1) \\ &+ \epsilon P(DD' + F(\epsilon N - E_3'E_3)^{-1}F')P \\ &+ \frac{1}{\epsilon}E_1'\{I + E_3(\epsilon N - E_3'E_3)^{-1}E_3'\}E_1 + N + R_1 < 0 \end{aligned} \quad (11)$$

to the LMI (eqn. 10). Since we have assumed that the uncertain time-delay system (eqn. 1) is quadratically stable, there will exist some $P_1 > 0$ which satisfy eqns. 10 and 11 for corresponding values of N_1 and ϵ_1 . Now if we apply the strict bounded real lemma of [9], it follows that the Riccati equation corresponding to eqn. 11 will have a stabilising solution $\hat{P}_1 < P_j$. In this case, if N_j and R_j are positive definite, then P_1 is also positive definite. Hence, the minimum in the LMI eigenvalue problem defined above occurs at P_1 which is equal to $\hat{P}_1$.

3 Quadratic stabilisation of uncertain time-delay systems

In this Section, we consider the problem of designing state feedback quadratically stabilising controllers for uncertain time-delay systems. The system under consideration is described by the state equation:

$$\begin{aligned} \dot{x}(t) &= (A + D\Delta(t)E_1)x(t) + (F + D\Delta(t)E_3)x(t - \tau) \\ &\quad + (B + D\Delta(t)E_2)u(t) \end{aligned} \quad (12)$$

For this system, $u(t) = Kx(t)$ is a quadratically stabilising controller if the closed loop uncertain time-delay system

$$\begin{aligned} \dot{x}(t) &= [A + BK + D\Delta(t)(E_1 + E_2K)]x(t) \\ &\quad + (F + D\Delta(t)E_3)x(t - \tau) \end{aligned} \quad (13)$$

is quadratically stable. Based on definition 1, this requires that there exist $P > 0$ and $N > 0$ such that

$$\begin{bmatrix} \begin{pmatrix} (A+BK)'P + P(A+BK) + (D\Delta(E_1+E_2K))'P \\ + P(D\Delta(E_1+E_2K)) + N \end{pmatrix} & P(F + D\Delta E_3) \\ (F + D\Delta E_3)'P & -N \end{bmatrix} < 0$$

for all Δ satisfying $\Delta'\Delta \leq I$.

Theorem 2 implies that $Kx(t)$ is quadratically stabilising if, and only if, there exist $P > 0$ and $N > 0$ such that quadratic matrix inequality

$$\begin{bmatrix} \begin{pmatrix} (A+BK)'P + P(A+BK) + PDD'P + N \\ + (E_1+E_2K)'(E_1+E_2K) \end{pmatrix} & PF + (E_1+E_2K)'E_3 \\ F'P + E_3'(E_1+E_2K) & -N + E_3'E_3 \end{bmatrix} < 0 \quad (14)$$

is satisfied. This problem is not jointly convex in matrix variables P , N and K . However, using the method of changing the variables as in [15] we can obtain an equivalent condition which is an LMI in its matrix variables.

First, we multiply every block of the above matrix inequality on the left and on the right by P^{-1} . Then, we define new matrix variables $Q = P^{-1}$, $Y = KP^{-1}$ and

$M = P^{-1}NP^{-1}$. Hence we obtain the condition

$$\begin{bmatrix} \left(\begin{array}{c} QA' + AQ + BY + Y'B' + DD' + M + \\ (E_1Q + E_2Y)'(E_1Q + E_2Y) \end{array} \right) & FQ + (E_1Q + E_2Y)'E_3Q \\ QF' + QE_3'(E_1Q + E_2Y) & -M + QE_3'E_3Q \end{bmatrix} < 0$$

Next, by applying Schur complements and a standard matrix inversion formula [17], it follows that the above matrix inequality is equivalent to the inequality

$$\begin{bmatrix} QA' + AQ + BY + Y'B' + DD' + M & FQ & (E_1Q + E_2Y)' \\ QF' & -M & QE_3' \\ (E_1Q + E_2Y) & E_3Q & -I \end{bmatrix} < 0 \quad (15)$$

Since this matrix inequality is an LMI in matrix variables $Q > 0$, $M > 0$ and Y , the quadratic stabilisation of an uncertain time-delay system (eqn. 12) can be considered as an LMI feasibility problem. Whenever this problem is infeasible, the uncertain time-delay system (eqn. 12) is not quadratically stabilisable by memoryless state feedback.

Remark 5: It follows from the above result that the uncertain system with multiple time-delays given in remark 2 is quadratically stabilisable if, and only if, the LMI

$$\begin{bmatrix} \{QA' + AQ + BY + Y'B' + DD' + \sum_{i=1}^L M_i\} & F_1Q & \dots & F_LQ & (E_1Q + E_2Y)' \\ QF_1' & -M_1 & \dots & 0 & QE_3' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ QF_L' & 0 & \dots & -M_L & QE_L' \\ (E_1Q + E_2Y) & E_3Q & \dots & E_LQ & -I \end{bmatrix} < 0$$

Observation 3: We have proved that the uncertain time-delay system (eqn. 12) is quadratically stabilisable if, and only if, the matrix inequality (eqn. 14) has a solution. We now consider some necessary conditions for the existence of such a solution. It is required that the pair (A, B) is stabilisable. Now assume that $E_3 = 0$ and E_2 is full rank. Hence, we have an uncertain time-delay system as follows:

$$\begin{aligned} \dot{x}(t) = & (A + D\Delta(t)E_1)x(t) + Fx(t - \tau) \\ & + (B + D\Delta(t)E_2)u(t) \end{aligned} \quad (16)$$

For this system, the matrix inequality (eqn. 14) is equivalent to the inequality

$$(A + BK)'P + P(A + BK) + PDD'P + PFN^{-1}F'P + (E_1 + E_2K)'(E_1 + E_2K) + N < 0$$

If we use the method of completing the square, we obtain the Riccati inequality:

$$\begin{aligned} & (A - B(E_2'E_2)^{-1}E_2'E_1)'P + P(A - B(E_2'E_2)^{-1}E_2'E_1) \\ & + PDD'P + PFN^{-1}F'P - PB(E_2'E_2)^{-1}B'P \\ & + E_1'\{I - E_2(E_2'E_2)^{-1}E_2'\}E_1 + N < 0 \end{aligned} \quad (17)$$

which is required to be satisfied for some $N > 0$. Now assume that, for some $N > 0$, this Riccati inequality has a positive definite solution P . Hence, the corresponding stabilising controller will be

$$K = -(E_2'E_2)^{-1}[B'P + E_2'E_1]$$

Thus, eqn. 16 is quadratically stabilisable if, and only if, eqn. 17 has a positive definite solution P for some value of $N > 0$. This condition is also a necessary condition for quadratic stabilisation of eqn. 12. We can also relax the full rank assumption on E_2 . This leads to a more complicated expression in place of the inequality in eqn. 17.

Now, assume that eqn. 17 is satisfied for some value of $P > 0$ and $N > 0$. We note that $I - E_2(E_2'E_2)^{-1}E_2'$ is positive semidefinite. Therefore, using some results on the solution of the algebraic Riccati equation and Riccati inequality [13], it follows that the algebraic Riccati equation

$$\begin{aligned} & (A - B(E_2'E_2)^{-1}E_2'E_1)'P + P(A - B(E_2'E_2)^{-1}E_2'E_1) \\ & + PDD'P + PFN^{-1}F'P - PB(E_2'E_2)^{-1}B'P \\ & + E_1'\{I - E_2(E_2'E_2)^{-1}E_2'\}E_1 + N = 0 \end{aligned}$$

will have a positive definite stabilising solution. However, we need to search over $N > 0$ to find a suitable solution to this algebraic Riccati equation. In a particular case, if we assume that the matrix N is of the form $N = \delta I$, this search is reduced to a search over real variable δ . This case is similar to the result of [8]. In our approach we do not require any assumption on the form of N . Moreover, we carry out the required numerical search by solving a corresponding LMI problem.

We now consider the problem of finding a suitable feedback controller K which not only quadratically stabilises the system in eqn. 12, but also guarantees some level of performance for a closed loop system. Similar to our results in Section 2, associated with the system in eqn. 12, we consider the cost function

$$J = \int_0^\infty \{x'(t)R_1x(t) + u'(t)R_2u(t)\} dt \quad (18)$$

where R_1 and R_2 are positive definite matrices.

Theorem 4: Consider the system in eqn. 12 and the cost function in eqn. 18 and suppose that there exist matrices $Q > 0$, $M > 0$ and Y such that

$$\begin{bmatrix} \left(\begin{array}{c} Q(A + D\Delta E_1)' + (A + D\Delta E_1)Q \\ + (B + D\Delta E_2)Y + Y'(B + D\Delta E_2)' \\ + M + Q R_1 Q + Y' R_2 Y \end{array} \right) & (F + D\Delta E_3)Q \\ Q(F + D\Delta E_3)' & -M \end{bmatrix} < 0 \quad (19)$$

for all Δ such that $\Delta' \Delta \leq I$. Then the control law $u(t) = Kx(t)$, $K = YQ^{-1}$, quadratically stabilises the uncertain time-delay system (eqn. 12). Furthermore, the corresponding value of the cost (eqn. 18) satisfies the bound

$$J \leq x_0'Q^{-1}x_0 + \int_{-\tau}^0 x'(s)Q^{-1}MQ^{-1}x(s)ds \quad (20)$$

for all admissible $\Delta(t)$ satisfying $\Delta'(t)\Delta(t) \leq I$.

Conversely, suppose $u(t) = Kx(t)$ is a control law such that the resulting closed loop system (eqn. 13) is quadratically stable and let R_1 and R_2 be given weighting matrices in the cost function (eqn. 18). Then, there exist positive definite matrices $\hat{Q}$ and $\hat{M}$ that satisfy eqn. 19 with $\hat{Y} = K\hat{Q}$.

Proof: The proof is similar to the proof of theorem 3 and therefore is omitted.

The matrix inequality (eqn. 19) is equivalent to the LMI

$$\begin{bmatrix} (QA' + AQ + BY + Y'B' + \epsilon DD' + M) & FQ & (E_1Q + E_2Y)' & QR_1^{1/2} & Y'R_2^{1/2} \\ QF' & -M & QE_3' & 0 & 0 \\ (E_1Q + E_2Y) & E_3Q & -\epsilon I & 0 & 0 \\ R_1^{1/2}Q & 0 & 0 & -I & 0 \\ R_2^{1/2}Y & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (21)$$

In contrast to our result in Section 2, here the corresponding upper bound for the cost function (eqn. 20) is apparently not a convex function in Q and M . Hence, finding the minimum of this upper bound cannot be considered as an LMI eigenvalue problem. Instead, we propose a method to find a suboptimal value for this bound.

First, we consider the LMI eigenvalue problem:

minimise $x_0'Q^{-1}x_0$

subject to eqn. 21 and $Q > 0$, $M > 0$, $\epsilon > 0$, $\text{trace}(M) = \gamma$.

Here we add an additional constraint for the trace of M . Then we vary the parameter γ in a certain domain. For each value of γ , we find the solution of the above LMI eigenvalue problem. Using this solution, we calculate the upper bound (eqn. 20). This bound can be considered as a function of γ and we select its minimum as a suboptimal value for the corresponding upper bound of the cost function (eqn. 18).

4 Illustrative examples

In this Section we present a number of examples to illustrate our stabilisation approach developed in Sections 2 and 3.

Example 1: Consider the uncertain time-delay system

$$\begin{aligned} \dot{x}(t) = & \begin{bmatrix} -1 + r_1(t) & 1 \\ 3 & 2 + r_2(t) \end{bmatrix} x(t) \\ & + \begin{bmatrix} s(t) & s(t) \\ -1 & -0.5 \end{bmatrix} x(t - \tau) \\ & + \begin{bmatrix} 1 + q(t) \\ 3 \end{bmatrix} u(t) \end{aligned}$$

where $|r_1(t)| \leq 1$, $|r_2(t)| \leq 2$, $|s(t)| \leq 1$, $|q(t)| \leq 0.1$ and $\tau = 1$. We assume that the information of initial condition is not available. The type of uncertainty in this example is called 'Rank-1' uncertainty. For this case, the uncertainty description in matrices A , F and B can be written in the following form:

$$\begin{aligned} \Delta A &= \begin{bmatrix} r_1(t) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & r_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ 0] r_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1 \ 0] r_2(t) \\ \Delta F &= \begin{bmatrix} s(t) & s(t) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ 1] s(t) \\ \Delta B &= \begin{bmatrix} q(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} q(t) \end{aligned}$$

This type of uncertainty is not of the same type as we have considered previously in this paper. However, if we consider the notion of overbounding; [3], our result

for quadratic stabilisation can be considered as a sufficient condition for this type of uncertainty. We define:

$$\begin{aligned} D &= \begin{bmatrix} 1 & 0 & 1 & 0.1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}' \\ E_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}' \quad E_2 = [0 \ 0 \ 0 \ 1]' \\ \Delta(t) &= \text{diag}\{\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t)\}, \\ |\delta_i(t)| &\leq 1, \quad i = 1, \dots, 4 \end{aligned}$$

We wish to know whether this system is quadratically stabilisable and if it is stabilisable, select a suitable state feedback controller. We consider $R_1 = I_{2 \times 2}$ and $R_2 = 1$ in the cost function (eqn. 18). Now we need to consider the feasibility problem for the LMI (eqn. 21). One solution to this problem is:

$$\begin{aligned} Q &= \begin{bmatrix} 0.045 & -0.030 \\ -0.030 & 0.029 \end{bmatrix} \quad M = \begin{bmatrix} 0.030 & 0.028 \\ 0.028 & 0.097 \end{bmatrix} \\ Y &= [-0.046 \quad -0.137] \end{aligned}$$

$\epsilon = 0.048$ and the corresponding value of K will be $[-10.42 \quad -15.561]$.

Example 2: Consider the uncertain time-delay system:

$$\begin{aligned} \dot{x}(t) = & \begin{bmatrix} 1 & 0 \\ 0 & -2 + \Delta(t) \end{bmatrix} x(t) \\ & + \begin{bmatrix} 0 & 0 \\ 0.5 + \Delta(t) & \Delta(t) \end{bmatrix} x(t - \tau) \\ & + \begin{bmatrix} -1 \\ 2 + 0.5\Delta(t) \end{bmatrix} u(t) \end{aligned}$$

where $\tau = 1$, $x_1(t) = e^{t+1}$ and $x_2(t) = 0$ for $t \in [-1, 0]$. We wish to construct a suitable quadratically stabilising controller for this system such that a corresponding upper bound for the cost function:

$$J = \int_0^\infty \{x_1(t)^2 + x_2(t)^2 + u(t)^2\} dt$$

is obtained. Thus, we apply our approach to find the state feedback controller that minimises the cost bound (eqn. 20). We have:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ D &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad E_1 = [0 \ 1] \quad E_2 = 0.5 \quad E_3 = [1 \ 1] \end{aligned}$$

First, we assume that $E_2 = E_3 = 0$. In this case, our example will be the same as the example in [8]. Using the approach detailed in Section 3, we obtain a plot of the upper bound (eqn. 20) with respect to $\text{trace}(M)$ as shown in Fig. 1. The minimum occurs at $\text{trace}(M) = 0.26$ and the corresponding value for upper bound (3.9) is 49.196. The other variables are as follows:

$$\begin{aligned} Q &= \begin{bmatrix} 0.2442 & -0.4795 \\ -0.4795 & 2.7160 \end{bmatrix} \\ M &= \begin{bmatrix} 0.0535 & -0.1051 \\ -0.1051 & 0.2065 \end{bmatrix} \end{aligned}$$

$$Y = [1.0000 \quad -2.0000] \quad K = [4.0558 \quad -0.0203] \quad \text{and } \epsilon = 1.3639$$

It should be noted that, in this case, the stabilising feedback and the corresponding value of the upper bound are very close to those obtained in [8]. The rea-

son for this is that, for this class of uncertain time-delay systems ($E_3 = 0$), the definition of quadratic stability in [8] can be considered as a special case of our definition. In example 3, we give an example in which our LMI approach will lead to a less conservative value for cost bound.

Fig. 1 Upper bound for closed loop cost against trace(M)

Fig. 2 Upper bound for closed loop cost against trace(M)

Now, we consider the case that E_2 and E_3 are not zero. This case cannot be handled using the results of [8]. Fig. 2 shows a plot of the cost bound against trace(M) for this case. The minimum occurs at trace(M) = 3.66 and the corresponding value of the upper bound is 169.605. The other variables are

$$Q = \begin{bmatrix} 0.199 & -0.622 \\ -0.622 & 2.543 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.149 & -0.719 \\ -0.719 & 3.511 \end{bmatrix}$$

$$Y = [1.110 \quad -2.812] \quad K = [9.027 \quad 1.101]$$

$$\text{and } \epsilon = 1.750$$

Example 3: In this example, we show that the LMI approach developed in Sections 2 and 3 leads to a sub-

optimal guaranteed cost controller and results in a less conservative value for the cost bound than the controller obtained using the Riccati equation approach in [8].

Consider the uncertain time-delay system

$$\dot{x}(t) = \begin{bmatrix} 1 + 0.33\Delta(t) & 0.42\Delta(t) \\ 0.53\Delta(t) & -2 + 0.67\Delta(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0.32 \\ 0.5 & -0.5 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 0.47 \\ 0.75 \end{bmatrix} u(t)$$

where $\tau = 1$, $x_1(t) = 3e^{t+1} - 1$ and $x_2(t) = 0$ for $t \in [-1, 0]$. It is assumed that $R_1 = I_{2 \times 2}$ and $R_2 = 1$ in the quadratic cost function (eqn. 18). We have:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad F = \begin{bmatrix} 0.5 & 0.32 \\ 0 & -0.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.47 \\ 0.75 \end{bmatrix} \quad D = \begin{bmatrix} 0.47 \\ 0.75 \end{bmatrix}$$

$$E_1 = [0.71 \quad 0.89] \quad E_2 = 0 \quad E_3 = 0$$

We wish to construct a suitable guaranteed cost controller for this system, such that the corresponding cost bound is minimised.

We first apply the Riccati equation approach of [8]. In this case, we need to search over the stabilising solutions of the Riccati equation

$$\begin{aligned} & (A - B(\epsilon R_2 + E'_2 E_2)^{-1} E'_2 E_1)' P \\ & + P(A - B(\epsilon R_2 + E'_2 E_2)^{-1} E'_2 E_1) \\ & + \epsilon P D D' P + P F N^{-1} F' P \\ & - \epsilon P B (\epsilon R_2 + E'_2 E_2)^{-1} B' P \\ & + \frac{1}{\epsilon} E'_1 \{ I - E_2 (\epsilon R_2 + E'_2 E_2)^{-1} E'_2 \} E_1 \\ & + N + R_1 = 0 \end{aligned} \quad (22)$$

and minimise the cost bound

$$J \leq x'_0 P x_0 + \int_{-\tau}^0 x(s)' N x(s) ds$$

In this Riccati equation, the matrix N is required to be of the form $N = \delta I$. The result of this search is plotted in Fig. 3. The minimum of the cost bound occurs at $\epsilon = 0.20$ and $\delta = 12.25$. The minimum value for the cost bound is 2.6×10^3 . With these values, the stabilising solution to the Riccati equation is

$$P = \begin{bmatrix} 47.8438 & -9.5550 \\ -9.5550 & 5.4005 \end{bmatrix}$$

This solution leads to the guaranteed cost controller $K = [-15.4312 \quad 0.4787]$

We now apply the LMI approach to find our suboptimal guaranteed cost controller. In this case, the numerical search leads to the plot in Fig. 4.

In this plot, the minimum occurs at trace(M) = 0.75 and the corresponding suboptimal value for the cost bound (eqn. 20) is 1.35×10^3 . This value for the cost bound is significantly less than the cost bound obtained using the Riccati equation approach of [8]. The other variables in this case are as follows:

$$Q = \begin{bmatrix} 0.0513 & -0.0010 \\ -0.0010 & 0.2217 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.0462 & 0.1753 \\ 0.1753 & 0.7038 \end{bmatrix}$$

$$Y = [-0.4714 \quad -0.7454]$$

$$K = [-9.2536 \quad -3.4020] \quad \text{and } \epsilon = 1.335$$

Fig.3 Upper bound for closed loop cost against ϵ and δ

Fig.4 Upper bound for closed loop cost against trace(M)

To complete this example, we compute the variables $P = Q^{-1}$ and $N = Q^{-1}MQ^{-1}$:

$$P = \begin{bmatrix} 19.4975 & 0.0838 \\ 0.0838 & 4.5110 \end{bmatrix}$$

$$N = \begin{bmatrix} 18.1542 & 15.7588 \\ 15.7588 & 14.4537 \end{bmatrix}$$

With these values for P and N , the Riccati equation (eqn. 22) is satisfied. However, the matrix N is not of the form $N = \delta I$ as is the case in the Riccati equation approach of [8]. This is why our approach leads to a less conservative value for the cost bound.

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