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ONE-DIMENSIONAL TEST PROBLEMS FOR DYNAMIC CONSOLIDATION

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ABSTRACT: Closed-form solutions are presented for some one-dimensional problems involving the dynamic response of saturated porous media. These solutions are useful for validating finite element codes for dynamic consolidation of soil. While they consider only elasticity and small strains, they do allow a check on the concurrent wave transmission and consolidation processes.

INTRODUCTION

In this note we consider the basic equations governing the dynamics of a saturated porous medium. They were first derived by De Josselin de Jong (1956) and Biot (1956) and a clear exposition of them, together with some useful solutions, may be found in Chapter 5 of the book by Verruijt (2010).

In particular, the basic equations will be presented for the one-dimensional case of propagation of plane waves and the associated coupled consolidation. The solution for the problem of step loading applied to a layer of saturated soil with a linear elastic skeleton and a compressible pore fluid is presented. This solution may be useful in the validation of finite element codes developed for the solution of dynamic consolidation problems.

BASIC DIFFERENTIAL EQUATIONS

As indicated by Verruijt (2010) the governing differential equations for the one-dimensional case of plane wave propagation in a soft soil, saturated with a compressible pore fluid, are as follows.

1. Mass conservation of the fluid and solid particles, i.e., total mass conservation:

$$\alpha \frac{\partial w}{\partial x} + S_p \frac{\partial p}{\partial t} = - \frac{\partial [n(v-w)]}{\partial x} \quad (1)$$

2. Stress-strain relationship of the solid soil skeleton:

$$m_v \frac{\partial \sigma'}{\partial t} = - \frac{\partial w}{\partial x} \quad (2)$$

3. Conservation of total momentum:

$$n\rho_f \frac{\partial v}{\partial t} + (1-n)\rho_s \frac{\partial w}{\partial t} = -\frac{\partial \sigma'}{\partial x} - \alpha \frac{\partial p}{\partial x} \quad (3)$$

4. Conservation of momentum of the pore fluid, i.e., the generalization of Darcy's law to the dynamic case:

$$n\rho_f \frac{\partial v}{\partial t} + \tau n\rho_s \frac{\partial (v-w)}{\partial t} = -n \frac{\partial p}{\partial x} - \frac{n^2 \mu}{\kappa} (v-w) \quad (4)$$

These are four governing equations in the four basic field quantities, defined as follows:

- v = the velocity of the pore fluid,
- w = the velocity of the solid particles,
- σ' = the isotropic effective stress, and
- p = the pore water pressure.

The symbols x and t represent the one-dimensional spatial coordinate and time, respectively.

The other symbols appearing in these equations represent the material properties, as follows:

- n = the porosity of the soil,
- α = Biot's coefficient for a saturated soil,
- m_v = the one-dimensional compressibility of the porous medium under fully drained conditions,
- S_p = the storativity of the pore space,
- ρ_f = the mass density of the pore fluid,
- ρ_s = the mass density of the solid particles,
- μ = the viscosity of the pore fluid,
- κ = the permeability of the porous medium, and
- τ = a tortuosity factor, describing the added mass due to the tortuosity of the fluid flow path.

In developing these equations it was assumed that the total stress, σ , can be decomposed into the isotropic effective stress, σ' , and the pore pressure, p , as follows:

$$\sigma = \sigma' + \alpha p \quad (5)$$

It can also be shown that the storativity of the pore space can be written as:

$$S_p = nC_f + (\alpha - n)C_s \quad (6)$$

and Biot's coefficient can be written as:

$$\alpha = 1 - \frac{C_s}{C_m} \quad (7)$$

where C_f , C_s and C_m are the compressibility of the pore fluid, the solid particle material and the porous medium, respectively.

If the soil skeleton can be represented by an ideal isotropic linear elastic material, then the one-dimensional compressibility, m_v , can be expressed in terms of elasticity coefficients as:

$$m_v = \frac{1}{K + \frac{4}{3}G} \quad (8)$$

where K and G represent the elastic bulk modulus and shear modulus, respectively.

SPECIAL CASE

Consider the special case where $\tau = 0$ and $\alpha = 1$. This corresponds to a soil where the tortuosity is insignificant and the compressibility of the solid particles is much less than that of the saturated soil overall. These are reasonable approximations of many cases of soils encountered in engineering practice. It is also reasonable to assume that variations in the porosity of the soil are of second order importance, so that the porosity n may be assumed as approximately constant.

With these assumptions, the governing equations for this special case simplify to the following:

$$\frac{\partial w}{\partial x} + S_p \frac{\partial p}{\partial t} = -n \left(\frac{\partial v}{\partial x} - \frac{\partial w}{\partial x} \right) \quad (9)$$

$$m_v \frac{\partial \sigma'}{\partial t} = -\frac{\partial w}{\partial x} \quad (10)$$

$$n\rho_f \frac{\partial v}{\partial t} + (1-n)\rho_s \frac{\partial w}{\partial t} = -\frac{\partial \sigma'}{\partial x} - \frac{\partial p}{\partial x} \quad (11)$$

$$n\rho_f \frac{\partial v}{\partial t} = -n \frac{\partial p}{\partial x} - \frac{n^2 \gamma_w}{k} (v - w) \quad (12)$$

In equation (12) account has also been taken of the following relationship:

$$\frac{\mu}{\kappa} = \frac{\rho_f g}{k} = \frac{\gamma_w}{k} \quad (13)$$

where g is the acceleration due to gravity and k is the hydraulic conductivity of the soil that is familiar from Darcy's law.

SOLUTION OF THE GOVERNING EQUATIONS

Obtaining solutions of the governing equations, (9) – (12), can be obtained by a variety of means. For example, analytical solutions can be pursued using the method described by Verruijt (2010), in which the fundamental solutions for harmonic variations in the field quantities applied at the boundaries can be combined appropriately as Fourier series to represent the required boundary conditions.

Alternatively, closed form solutions may also be obtained using the technique that involves taking Laplace transforms of the governing equations, solving these equations in Laplace transform space, and then inverting the solution for the Laplace transforms, numerically if necessary, to recover the original field quantities.

A third option is to apply the numerical technique of finite differences to solve equations (9) to (12) directly, subject to the appropriate boundary conditions.

In this note we will use the Laplace transform method and also check the solutions so obtained using an independent finite difference approach.

STEP LOADING APPLIED TO A SOIL LAYER

Consider first the problem of a layer of saturated porous soil subjected to a sudden increase in pore water pressure applied at the soil surface.

The problem of an infinitely deep layer was considered previously by Verruitt (2010) who solved it both numerically and using the Fourier series technique. As indicated, we will proceed using the method of taking Laplace transforms.

Taking Laplace transforms of the governing equations (9) to (12) provides:

$$(n-1)\frac{\partial \bar{w}}{\partial x} = n\frac{\partial \bar{v}}{\partial x} + S_p \bar{p}s \quad (14)$$

$$m_v \bar{\sigma}' s = -\frac{\partial \bar{w}}{\partial x} \quad (15)$$

$$n\rho_f \bar{v}s + (1-n)\rho_s \bar{w}s = -\frac{\partial \bar{\sigma}'}{\partial x} - \frac{\partial \bar{p}}{\partial x} \quad (16)$$

$$n\rho_f \bar{v}s = -n\frac{\partial \bar{p}}{\partial x} - n^2\left(\frac{\gamma_w}{k}\right)(\bar{v} - \bar{w}) \quad (17)$$

where the superior bar indicates a Laplace transform quantity and s is the Laplace transform variable.

If equations (16) and (17) are both differentiated with respect to the coordinate x , and appropriate substitutions are made, making use of equations (14) and (15), these four governing equations expressed in terms of Laplace transforms can be reduced to the following two equations in terms of the transforms of the effective stress and pore pressure, i.e.,

$$A\bar{p} + B\bar{\sigma}' + \frac{\partial^2 \bar{\sigma}'}{\partial x^2} + \frac{\partial^2 \bar{p}}{\partial x^2} = 0 \quad (18)$$

$$C\bar{p} + D\bar{\sigma}' + \frac{\partial^2 \bar{p}}{\partial x^2} = 0 \quad (19)$$

where

$$A = -\rho_f S_p s^2 \quad (20)$$

$$B = (\rho_f - \rho_s)(1-n)m_v s^2 \quad (21)$$

$$C = -\left[\rho_f s + n \left(\frac{\gamma_w}{k} \right) \right] \left(\frac{S_p s}{n} \right) \quad (22)$$

$$D = \left\{ \left[\rho_f s + n \left(\frac{\gamma_w}{k} \right) \right] \left(\frac{1-n}{n} \right) + n \left(\frac{\gamma_w}{k} \right) \right\} m_v s \quad (23)$$

Further simplification provides the following governing equation for the transform of the pore water pressure:

$$\frac{\partial^4 \bar{p}}{\partial x^4} + X \frac{\partial^2 \bar{p}}{\partial x^2} + Y \bar{p} = 0 \quad (24)$$

where

$$X = B + C - D \quad (25)$$

and

$$Y = BC - AD \quad (26)$$

The solution of (24) is well known and in general it takes the form:

$$\bar{p} = E_1 e^{-\alpha_1 x} + E_2 e^{\alpha_1 x} + F_1 e^{-\alpha_2 x} + F_2 e^{\alpha_2 x} \quad (27)$$

where the coefficients E_1, E_2, F_1, F_2 must be determined from the boundary conditions of the problem. It can also be shown that the terms α_1 and α_2 are given by:

$$\alpha_1 = \sqrt{\left(\frac{\sqrt{X^2 - 4Y}}{2} - \frac{X}{2} \right)} \quad (28)$$

$$\alpha_2 = \sqrt{\left(-\frac{\sqrt{X^2 - 4Y}}{2} - \frac{X}{2} \right)} \quad (29)$$

Infinitely Deep Layer

Let us first consider the case of an infinitely deep layer. In such cases the solution must remain bounded as x approaches ∞ , which means that $E_2 = F_2 = 0$. The boundary condition at $x = 0$ corresponds to a step loading in the pore pressure p , which in turn implies the following:

$$\bar{p} = \frac{p_o}{s} \quad (30)$$

where p_o is the magnitude of the step rise in pore pressure. Also at $x = 0$ the effective stress boundary condition is expressed as:

$$\bar{\sigma}' = 0 \quad (31)$$

Applying these boundary conditions provides the following solutions for the non-zero constants E_1 and F_1 :

$$E_1 = -\left(\frac{p_o}{s}\right)\left(\frac{\alpha_2^2 + C}{\alpha_1^2 - \alpha_2^2}\right) \quad (32)$$

$$F_1 = \left(\frac{p_o}{s}\right)\left(\frac{\alpha_1^2 + C}{\alpha_1^2 - \alpha_2^2}\right) \quad (33)$$

The solution for the Laplace transform of the pore pressure is given by the combination of equations (27), (32) and (33). It then remains to invert this transform to recover values of the pore pressure p . For this problem analytical inversion of the transform is difficult, if not impossible, so that numerical inversion is required. In evaluating the solutions to this problem, the transforms have been inverted numerically using the algorithm suggested by Talbot (1979).

Solution Evaluation

Solutions have been evaluated for the case of an infinitely deep layer of saturated soil to which a step loading in pore water pressure (and total stress) of magnitude p_o is applied at the surface $x = 0$. These solutions are presented in Figures 1 and 2 and correspond to the material properties listed in Table 1. They show the pore water pressure as a function of time at a location given by $x = 0.2$ m.

Table 1. Soil properties

Symbol	Property	Value
n	Porosity of the soil (-)	0.4
α	Biot's coefficient for a saturated soil (-)	1
τ	Tortuosity (-)	0
ρ_f	Density of the pore fluid (kg/m ³)	1000
ρ_s	Density of the solid particles (kg/m ³)	2650
k	Hydraulic conductivity of soil (m/s)	0.001 and 0.0005
g	Gravitational constant (m/s ²)	10
m_v	Compressibility of soil (m ² /N)	2×10^{-10}
C_f	Compressibility of pore fluid (m ² /N)	5×10^{-10}
C_s	Compressibility of solid particles (m ² /N)	0

The solutions plotted in Figure 1 correspond to the case of a soil with $k = 0.001 \text{ m/s}$.¹

In Figure 1(a), corresponding to small values of time, it can be clearly seen that two waves of dynamic pore pressure are developed and pass through the given location. As indicated by Verruijt (2010), the first arrives at a time of approximately 0.00009 s (moving with a velocity of 2242 m/s) and is what is known as an “undrained wave” because the soil skeleton and the pore fluid move in phase with each other, i.e., for this type of wave the velocities of the solid particles (w) and the pore fluid (v) are the same. The second wave observed in Figure 1(a) corresponds to the case where the velocities of the solid particles and the fluid are equal in magnitude but opposite in direction. For this type of wave the velocity is much slower, i.e., at approximately 1180 m/s it is about one-half of the velocity of the undrained wave.

Figure 1(b) shows the solution for the same case as depicted in Figure 1(a), but for larger values of time. It can be observed that with the passage of time, after the initial shock due to the arrival of the dynamic waves at $x = 0.2$, the pore pressure gradually increases and approaches the value p_o applied at the boundary $x = 0$. The mechanism causing this increase is consolidation, as the pore fluid flows through the solid skeleton of the soil. Evidence for pseudo static consolidation can be found in the predicted consolidation curve, but this is best illustrated by considering a layer of finite thickness rather than an infinitely thick layer. We will turn to the latter problem in due course.

Meanwhile, as observed by Verruijt (2010), the second type of wave observed in this problem attenuates reasonably quickly. This attenuation or damping arises principally because the water must flow through the solid skeleton (i.e., v and w are different) and in doing so it meets resistance. The undrained wave is not attenuated in the same way because the soil and water move together.

As also noted by Verruijt, this attenuation is a function of the hydraulic conductivity of the soil; the lower the value of hydraulic conductivity the more quickly the wave is damped. An example of this effect may be seen in Figure 2, which shows results plotted for the case where $k = 0.0005 \text{ m/s}$, i.e., a soil only one-half as permeable as that shown in Figure 1. Comparison of Figure 1(a) and Figure 2 reveals that by the time the second wave of pore pressure arrives at $x = 0.2$ it already has a smaller amplitude, while the first wave type appears to have the same magnitude in each case. Note that the overall pressure immediately after the arrival of

¹ The minor oscillations in the plotted solution are simply an artifice of the numerical algorithm used to invert the Laplace transforms and are not physically real.

the second wave is slightly more than $0.8p_o$ in Figure 1(a), while it is lower at approximately $0.7p_o$ in Figure 2.

Finite Layer

Consider now the case of a layer of finite thickness, H . Conceptually, this case is no more challenging to solve than the infinitely deep layer, involving only slightly different yet significant boundary conditions. As we shall see revealed in the evaluated solution, the presence of a rigid impermeable boundary at the bottom of the layer produces some very interesting effects.

For this case we have two additional boundary conditions that must be applied at $x = H$. These are:

$$\frac{\partial \bar{p}}{\partial x} = \frac{\partial \bar{\sigma}'}{\partial x} = 0, \quad x = H \quad (34)$$

Application of these conditions at $x = H$, together with those already considered at $x = 0$, provides four equations allowing solutions to be obtained for the coefficients E_1 , E_2 , F_1 and F_2 in the general solution expressed as equation (27). These equations can be written as:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_1^2 + C & \alpha_1^2 + C & \alpha_2^2 + C & \alpha_2^2 + C \\ -\alpha_1 e^{-\alpha_1 H} & \alpha_1 e^{\alpha_1 H} & -\alpha_2 e^{-\alpha_2 H} & \alpha_2 e^{\alpha_2 H} \\ -\alpha_1 (\alpha_1^2 + C) e^{-\alpha_1 H} & \alpha_1 (\alpha_1^2 + C) e^{\alpha_1 H} & -\alpha_2 (\alpha_2^2 + C) e^{-\alpha_2 H} & \alpha_2 (\alpha_2^2 + C) e^{\alpha_2 H} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{p_o}{s} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (35)$$

Values of these coefficients are required in the Laplace transform solution of the problem of a finite layer. Otherwise, inversion of the transforms proceeds as for the infinitely deep layer.

Solution Evaluation

Solutions have been evaluated² for the case of a 1 m deep layer of saturated soil to which a step loading in pore water pressure (and total stress) of magnitude p_o is applied at the surface $x = 0$. These solutions are presented in Figures 3 and 4, and they correspond to the material properties listed in Table 1, with the smaller hydraulic conductivity, $k = 0.0005$ m/s, being

² For convenience the results in Figures 3 and 4 were computed using the finite difference approach, rather than Laplace transforms. This explains the slight numerical "overshoot" when a wave arrives. It should also be noted that the Talbot method of inversion of the Laplace transforms proved to be problematic for times greater than about 0.0008 s at $x = 0.2$ m, presumably due to singularities in the transform of pore water pressure. It is curious that this occurred at about the time the first reflected wave arrived at $x = 0.2$ m. This issue requires further investigation but is beyond the scope of the present note.

adopted. Figure 3 shows the variation of the pore pressure at $x = 0.2$ m, while Figure 4 shows the pore pressure at the bottom of the layer, $x = 1$ m.

There are several interesting features depicted in the plots shown in Figures 3 and 4, which are now described.

First, a comparison of Figures 3(a) and 4 further illustrates the point about the damping of the second type of wave. Closer to the source of the disturbance, at $x = 0.2$ m, two distinct types of wave can be seen arriving at different times, as previously discussed. However, further from the source, at $x = 1$ m, the undrained wave type clearly arrives at a time of approximately 0.00045 s, corresponding to a velocity of 2242 m/s. Only a very weak second pulse can be observed in the time trace shown in Figure 4, i.e., at a time of about 0.00085 s, corresponding to the speed of a wave of the second type of 1180 m/s. So although the second type of wave can just be observed it has almost completely attenuated by the time it reaches the bottom of the 1 m deep layer. Thereafter, it should play no significant part in the on-going pore pressure history of the finite layer of saturated soil.

Figure 3(b) shows the variation of the pore pressure at $x = 0.2$ m for a longer period than depicted in Figure 3(a). The series of square pulses traced out in this plot correspond to a sequence of undrained waves reflected from the boundaries of the layer, both top and bottom. The first reflection arrives at a total elapsed time of approximately 0.0008 s, which is consistent with a wave travelling at 2242 m/s generated at the surface at $t = 0$ and travelling 1 m to the bottom of the layer and being reflected to arrive back at the location $x = 0.2$ m after having travelled a total distance of 1.8 m in about 0.0008 s. Note that this first reflected wave causes an increase in the pore water pressure. In other words, the reflection from the fixed boundary has caused the reflected wave pulse to have the same sign as the incoming wave.

The reflected wave then continues to travel back towards the surface where it again is reflected, in this case from the free boundary. Travelling at a speed of 2242 m/s it arrives back at $x = 0.2$ m after a further period of approximately 0.0002 s, corresponding to the time required to traverse a distance of 2×0.2 m = 0.4 m. It passes through $x = 0.2$ m again at a total elapsed time of approximately 0.001 s. On this occasion it causes a reduction in the pore water pressure, having been reflected from a free surface. In other words, reflection from the free surface has caused a change in sign of the reflected pulse.

This process of sequential reflection from the fixed and free surfaces of the layer continues, as is evidenced by the series of regular spiked pulses in the pore pressure history.

Meanwhile, the mean pore pressure, ignoring the pulsing, rises consistently with time, driven by the underlying consolidation process taking place in the saturated soil. It is worth noting that in Terzaghi's theory of consolidation for static loading applied to a finite layer with one-way (surface) drainage, the non-dimensional time for about 90% consolidation is approximately 1. For the example studied here it can be shown that coefficient of consolidation of the soil, $c_v = k/(m_v \gamma_w) = 250 \text{ m}^2/\text{s}$, which implies a real time for about 90% consolidation in a 1 m thick layer of approximately 0.004 s. The curve in Figure 3(b) indicates that a mean pore water pressure of about 90% of the applied pressure occurs around $t = 0.004 \text{ s}$, supporting the contention that the rise in mean pore pressure is driven by consolidation.

VALIDATION

The example solutions plotted in Figures 3 and 4 might be useful for validating finite element codes for dynamic consolidation. While they consider only elasticity and small strains, they do allow a check on the concurrent wave transmission and consolidation processes.

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(a)

(b)

Figure 1 Pore pressure at $x = 0.2$ m in infinitely deep layer with $k = 0.001$ m/s

Figure 2 Pore pressure at $x = 0.2$ m in infinitely deep layer with $k = 0.0005$ m/s

(a)

(b)

Figure 3 Pore pressure at $x = 0.2$ m in finite (1 m thick) layer with $k = 0.0005$ m/s

Figure 4 Pore pressure at $x = 1\text{ m}$ in finite (1 m thick) layer with $k = 0.0005\text{ m/s}$