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On Eigenvalue-Eigenvector Assignment for Componentwise Ultimate Bound Minimisation in MIMO LTI Discrete-Time Systems

Rahmat Heidari¹, Maria M. Seron¹, Julio H. Braslavsky²

Abstract—We consider eigenvalue-eigenvector assignment in order to minimise ultimate bounds on the states of a linear time-invariant (LTI) discrete-time system in the presence of non-vanishing bounded disturbances. As opposed to continuous-time systems, for which eigenstructure assignment with large magnitude stable eigenvalues can yield arbitrarily small ultimate bounds for “matched” perturbations, for discrete-time systems, ultimate bounds cannot be smaller than certain values depending on the disturbance bounds. Moreover, these smallest bounds are not achievable, in general, by assigning the closed-loop eigenvalues to zero (an intuitive conjecture that parallels the continuous-time case). The first contribution of the paper, for single-input systems, are conditions on the zeros of the transfer function between the control input and a state to minimise the ultimate bound corresponding to that state. These conditions generalise a result recently presented by the authors. The second, and main, contribution of the current paper is to characterise, for multiple-input systems, the eigenstructure of the closed-loop system so that some ultimate bounds are minimised to their minimum values. The number of ultimate bound components that can be minimised is constrained by the number of control inputs. For m -input system, the minimisation problem of $m - 1$ ultimate bound components can be solved without restrictions, while in order to minimise an additional bound, an additional restrictive condition should be satisfied.

I. INTRODUCTION

Asymptotic stability is in general not achievable in dynamic systems subject to persistent disturbances. One can instead investigate practical stability, as the state trajectories become ultimately bounded within a region of the state space. It is well-known that ultimate bounds that characterise such regions can be estimated using level sets of suitable Lyapunov functions (e.g., [1, §9.2]). Although the Lyapunov approach has the strength that it can be applied to a general class of nonlinear systems, it may produce overly conservative bounds in linear systems, as state and disturbance geometric structure is generally lost [2].

A new method to obtain *componentwise* ultimate bounds of linear system states, potentially tighter than with Lyapunov function level sets, has been proposed in [2]–[3]. This new method dispenses with the need to compute Lyapunov functions by directly analysing the system in modal coordinates to derive state componentwise ultimate bounds.

Componentwise ultimate bound computation has been developed for both continuous and discrete-time systems [4].

For continuous-time systems the authors in [5] showed that ultimate bounds can be made arbitrarily small by assigning closed-loop eigenvalues with sufficiently large magnitudes when disturbances are “matched” with the control inputs; namely, when they perturb the system within the span of the system control inputs. However, the parallel result does not generally hold for discrete-time systems. Indeed, as shown in [6], assigning the system closed-loop eigenvalues to the origin (“high gain” control in discrete-time systems) does not necessarily render the lowest achievable ultimate bounds. It turns out for single-input systems that the ultimate bounds of a state component to a disturbance input can be minimised to its least possible value if the transfer function between the control input and that state is minimum-phase. If such transfer function has relative degree one, the optimal closed-loop pole placement strategy then is to cancel all zeros and place the remaining pole at the origin.

The main contribution of the current paper is the extension of the results in [6] to multi-input systems. We first provide an extension of the aforementioned result for single-input systems by relaxing the requirement in [6] that the minimum-phase transfer function between the control input and a state whose ultimate bound is to be minimised could not have zeros coinciding with eigenvalues of the matrix A and a zero at the origin.

The second and main contribution of the paper is to extend the single-input system results to n -th order systems with m -inputs. First, we tackle the minimisation problem for $m - 1$ ultimate bounds. This problem is solved by placing (at least) $m - 1$ eigenvalues at zero and executing eigenvector assignment for the remaining eigenvalues, which need to be stable but otherwise arbitrary. If for each eigenvalue at zero, the j -th component (where j is the index of each of the desired $m - 1$ elements) of the associated eigenvector could not be assigned to be nonzero, an additional eigenvalue should be set to zero with an associated generalised eigenvector having nonzero j -th component. Then, for m -input systems, we establish conditions to attain m minimum ultimate bounds. These conditions become more restrictive as the number of inputs m increases.

The results presented in the current paper complete the analysis of ultimate bound minimisation by eigenvalue-eigenvector assignment for discrete-time linear systems. These results have implications in the study of system fundamental structural and dynamic limitations that may preclude the achievement of the lowest possible ultimate bounds. The analysis of these limitations is the subject of current and future research.

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Notation: $\mathbb{R}$, $\mathbb{R}_{+0}$ and $\mathbb{C}$ denote the sets of real, nonnegative real and complex numbers, respectively. $|M|$ denotes the *elementwise* magnitude of a matrix or vector M . The i -th row of a matrix M is denoted by $M_{(i,:)}$ and the i -th element of a vector v is denoted by v_i . Given an $n \times n$ matrix M and a set of indices $\mathcal{M} \subset \{1, \dots, n\}$, $M_{(\mathcal{M},:)}$ denotes the matrix formed by selecting the rows of M with indices in $\mathcal{M}$. The cardinality of a set $\mathcal{M}$ is denoted by $\#\mathcal{M}$. $\mathcal{N}(M)$ and $\mathcal{R}(M)$ denote the null-space and the range-space of a matrix M . A generalised inverse of a matrix M , denoted by M^g , satisfies $MM^gM = M$. If $x(t)$ is a vector-valued function, then $\limsup_{t \rightarrow \infty} x(t)$ denotes the vector obtained by taking $\limsup_{t \rightarrow \infty}$ of each component of $x(t)$. If $x, y \in \mathbb{R}^n$, ' $x \preceq y$ ' denotes the set of componentwise inequalities $x_i \leq y_i$, $i = 1, \dots, n$. The notation $(\text{block})\text{diag}(M_1, \dots, M_k)$ represents a (block) diagonal matrix with the (matrix/vector) elements $M_1, \dots, M_k$ as the (block) diagonal entries. I_n is the $n \times n$ identity matrix.

II. PRELIMINARIES

In this section we present the componentwise ultimate bound computation formula of [4] for discrete-time systems, which constitutes the basis for our analysis in the forthcoming sections. We discuss the problem of eigenvalue-eigenvector assignment for these systems and characterise the lowest achievable ultimate bound for a component of the state vector.

Consider the LTI discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + Hw(k), \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are a controllable pair of evolution and control input matrices, respectively, with rank $B = m$, and $H \in \mathbb{R}^{n \times p}$ is the disturbance input matrix. The disturbance w is bounded as $|w(\cdot)| \leq \mathbf{w}$, where $\mathbf{w} \in \mathbb{R}_{+0}^p$ is a componentwise nonnegative vector.

The assignment by feedback of an eigenvalue-eigenvector pair (λ, v) for the system (1) requires the existence of a feedback gain K such that $(A+BK)v = \lambda v$. The conditions for eigenvalue-eigenvector (eigenstructure) assignability are laid out in the following result from [7].

Lemma 2.1: [7, Lemma 3.1.1] Given a state-space system $x(k+1) = Ax(k) + Bu(k)$, and an eigenvalue-eigenvector pair (λ, v) , where $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$, the following conditions are equivalent.

- (i) There exists a feedback gain $K \in \mathbb{R}^{m \times n}$ such that (λ, v) is an eigenvalue-eigenvector pair of the closed-loop matrix $A+BK$, namely, $(A+BK)v = \lambda v$.
- (ii) The vector $v \in \mathbb{C}^n$ belongs to the null space of the matrix $\phi = (BB^\dagger - I_n)(\lambda I_n - A)$,

$$v \in \mathcal{N}(\phi). \quad (2)$$

- (iii) The vector $(\lambda I_n - A)v$ belongs to the range of the control input matrix B ,

$$(\lambda I_n - A)v \in \mathcal{R}(B). \quad (3)$$

○

We consider eigenvalue-eigenvector assignment by state feedback $u = Kx$ so that the closed-loop trajectories of (1),

$$x(k+1) = A^{cl}x(k) + Hw(k), \quad (4)$$

with $A^{cl} = A + BK$, have the smallest achievable ultimate bound $\limsup_{k \rightarrow \infty} |x(k)|$.

A formula for the computation of ultimate bounds for the system (4) is presented in Theorem 2.2 below.

Theorem 2.2 ([4][3]): Consider LTI system (4) where the disturbance variable w is bounded as $|w(\cdot)| \leq \mathbf{w}$, for some nonnegative vector $\mathbf{w} \in \mathbb{R}_{+0}^p$. Suppose the matrix A^{cl} is stable (i.e., all its eigenvalues have magnitude less than one) and let $\Lambda = V^{-1}A^{cl}V$ be its Jordan canonical form. Then, the states of the system are ultimately bounded as

$$\limsup_{k \rightarrow \infty} |x(k)| \preceq |V| (I - |\Lambda|)^{-1} |V^{-1}H| \mathbf{w} \triangleq \mathbf{b}. \quad (5)$$

○

It was shown in [5] that for continuous-time systems with “matched” perturbations, ultimate bounds may be arbitrarily reduced by high-gain control. In contrast, for discrete-time systems the ultimate bound on a state vector component can never be smaller than the effect of the perturbation on that component in one time step.

Lemma 2.3: For system (4) where A^{cl} is stable and the disturbance variable w is bounded as $|w(\cdot)| \leq \mathbf{w}$, with $\mathbf{w} \in \mathbb{R}_{+0}^p$, the lowest possible ultimate bound on the state vector's j -th component is

$$\mathbf{b}_j^{\min} \triangleq |H_{(j,:)}| \mathbf{w}. \quad (6)$$

Proof: Follows from direct analysis of (4). ■

III. ULTIMATE BOUND MINIMISATION

In this section we study eigenstructure assignment to achieve the lowest possible ultimate bound (6) for one or more state components. We base our analysis on the formula (5) and treat the single- and multiple-input cases separately.

Lemma 3.1: The j -th component $\mathbf{b}_j$ of the ultimate bound vector $\mathbf{b}$ in (5) can be minimised to its minimum possible value $\mathbf{b}_j^{\min} = |H_{(j,:)}| \mathbf{w}$ if the matrix Λ is diagonal with one zero eigenvalue, say at column $k \in \{1, \dots, n\}$,

$$\Lambda = \text{diag}(\lambda_1, \dots, \underbrace{0}_{k\text{-th component}}, \dots, \lambda_n), \quad (7)$$

and otherwise arbitrary eigenvalues, and the eigenvector matrix V is invertible and such that its j -th row has a nonzero element at column k and is zero otherwise, that is,

$$V_{(j,:)} = [0 \quad \dots \quad v_{j,k} \quad \dots \quad 0], \quad v_{j,k} \neq 0. \quad (8)$$

Proof: Define $Z \triangleq |V^{-1}H| \mathbf{w}$ and let $\{u_{k,i} : i = 1, \dots, n\}$ be the k -th row elements of $U \triangleq V^{-1}$. Using these definitions and (7)-(8) in (5), yields for the j -th component of the ultimate bound vector

$$\mathbf{b}_j = [0 \dots |v_{j,k}| \dots 0] \begin{bmatrix} \frac{1}{1-|\lambda_1|} & \dots & 0 \\ & \ddots & \\ \vdots & & 1 \\ & & \ddots & \frac{1}{1-|\lambda_n|} \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_n \end{bmatrix}$$

$$\begin{aligned}
&= |v_{j,k}| Z_k = |v_{j,k}| \begin{bmatrix} u_{k,1} & \dots & u_{k,j} & \dots & u_{k,n} \end{bmatrix} H \mathbf{w} \\
&= \begin{bmatrix} |v_{j,k}| u_{k,1} & \dots & |v_{j,k}| u_{k,j} & \dots & |v_{j,k}| u_{k,n} \end{bmatrix} H \mathbf{w} \\
&= \begin{bmatrix} 0 & \dots & \pm 1 & \dots & 0 \end{bmatrix} H \mathbf{w} = \mathbf{b}_j^{\min} \quad (9)
\end{aligned}$$

where the second last equation follows from $V_{(j,:)} V^{-1} = I_{n(j,:)}$. The result then follows. ■

According to Lemma 3.1, the minimum ultimate bound of one state component can be achieved by performing an eigenvalue-eigenvector assignment so that (7) and (8) hold. We proceed by applying Lemma 2.1, first to single-input systems, and then to multi-input systems.

A. Single-Input Systems

For single input systems, since the dimension of $\mathcal{N}(\phi)$ and $\mathcal{R}(B)$ in (2), (3) is one, there is no freedom to select the eigenvector direction associated with a particular eigenvalue. In this case, enforcing the conditions of Lemma 3.1 can be approached by searching for eigenvalues λ_i whose associated eigenvectors have the j -th component satisfying $v_{j,i} = 0$.

From (2) and (3), the eigenvector associated with the eigenvalue λ_i is

$$v_i = (I_n - \phi_i^g \phi_i) \gamma_i, \quad \gamma_i \in \mathbb{C}^n \text{ arbitrary} \quad (10a)$$

$$v_i = (\lambda_i I_n - A)^{-1} B \delta_i, \quad \delta_i \in \mathbb{C} \text{ arbitrary} \quad (10b)$$

where $\phi_i = (BB^\dagger - I_n)(\lambda_i I_n - A)$. Note that if λ_i is an eigenvalue of A , (10b) is not valid and the eigenvector should be computed from (10a). The j -th component of v_i is

$$v_{j,i} = [(I_n - \phi_i^g \phi_i)]_{(j,:)} \gamma_i \quad (11a)$$

$$v_{j,i} = [(\lambda_i I_n - A)^{-1}]_{(j,:)} B. \quad (11b)$$

To have $v_{j,i} = 0$, we should search for an eigenvalue that makes the numerator of the right hand side of (11) zero. Since $d(\mathcal{N}(\phi_i)) = d(\mathcal{R}(B)) = 1$, the zeros of the right hand sides of (11a) and (11b) are the same and thus, we can consider the numerator of (11b) without loss of generality.

Suppose the j -th component of B is nonzero and let

$$P_{n-1}^j(\lambda) = [\text{adj}(\lambda I_n - A)]_{(j,:)} B \quad (12)$$

be the $(n-1)$ -th order polynomial¹ associated with the j -th component of an eigenvector. The roots of this polynomial can determine whether or not its corresponding ultimate bound component can be minimised to the smallest possible value. This is analysed in the following result.

Theorem 3.2: Given a single-input system of the form (4) where A^{cl} is stable and B is a vector such that its j -th component is nonzero, if zero is not an eigenvalue of A , then its j -th ultimate bound component can be minimised to the minimum possible value if all roots of $P_{n-1}^j(\lambda)$ defined in (12) are distinct and stable.

Proof: If all $n-1$ roots of the polynomial $P_{n-1}^j(\lambda)$ are distinct and stable, the placement of eigenvalues at these roots results in having $n-1$ eigenvectors with $v_{j,i} = 0$. Thus, the conditions of Lemma 3.1 are met if we place the remaining eigenvalue at zero and V is invertible.

¹It can easily be shown that the assumption that the j -th element of B is nonzero implies that (12) has order $n-1$, see [6].

An important observation is that if zero is one of the distinct stable roots of the polynomial (12), then placing the eigenvalues at the roots of the polynomial and the last eigenvalue at zero results in two zero eigenvalues and hence, a generalised eigenvector needs to be considered for a single-input system to achieve an invertible eigenvector matrix V .

Consider first the case that $P_{n-1}^j(\lambda)$ has $n-1$ non-zero distinct stable roots. Let $\{\lambda_i : |\lambda_i| < 1, \lambda_i \neq 0, i = 1, \dots, n-1\}$ be the distinct roots of $P_{n-1}^j(\lambda)$. Eigenvalue assignment as described above at the beginning of the proof leads to the minimum ultimate bound following Lemma 3.1. Note that since zero is not a root of $P_{n-1}^j(\lambda)$, then assignment of $\lambda_n = 0$ implies that $v_{j,n} \neq 0$.

Now we turn to the case where $P_{n-1}^j(\lambda)$ has $n-1$ distinct stable roots, one of which is zero. Let $\{\lambda_i : |\lambda_i| < 1, i = 1, \dots, n-2, \lambda_{n-1} = 0\}$ be the distinct roots of $P_{n-1}^j(\lambda)$. Place closed-loop eigenvalues at $[\lambda_1, \dots, \lambda_{n-2}, 0, 0]$. Then, due to the repeated eigenvalue at zero, the eigenvalue and eigenvector matrices take the form

$$\Lambda = \text{blockdiag}(\text{diag}(\lambda_1, \dots, \lambda_{n-2}), \Lambda_0) \text{ with } \Lambda_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$V = [v_1 \dots v_{n-2} v_0], \text{ with } v_0 = [-A^{-1}B \ A^{-2}B] \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \end{bmatrix},$$

$\alpha_1 \in \mathbb{R}$ is arbitrary, and $\{v_i : i = 1, \dots, n-2\}$ are obtained from (11) [from (11a) if λ_i is an eigenvalue of A].

Due to the placement of the closed-loop eigenvalues at the $n-1$ stable roots of $P_{n-1}^j(\lambda)$, the j -th row of V satisfies

$$V_{(j,:)} = [0 \dots 0 \ v_{j,n}], \quad (13)$$

where $v_{j,n} \neq 0$ if the j -th element of $A^{-2}B$ is nonzero. Note that by controllability of (A, B) and invertibility of A , there will be a power $k^* \in \{1, \dots, n\}$ such that the j -th element of $A^{-k^*}B$ is nonzero.²

Similarly to (9), the j -th component of the ultimate bound is shown to be minimised to its lowest possible value as follows

$$\begin{aligned}
\mathbf{b}_j &= |V_{(j,:)}| \begin{bmatrix} \frac{1}{1-|\lambda_1|} & \dots & 0 \\ & \ddots & \\ 0 & & 1 & \frac{1}{1} \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_{n-1} \\ Z_n \end{bmatrix} \\
&= [0 \dots 0 \ |v_{j,n}|] \begin{bmatrix} \frac{Z_1}{1-|\lambda_1|} \\ \vdots \\ Z_{n-1} + Z_n \\ Z_n \end{bmatrix} = |H_{(j,:)}| \mathbf{w} = \mathbf{b}_j^{\min}.
\end{aligned}$$

■

B. Multi-Input Systems

The number of ultimate bound components that can be minimised to their lowest possible value using eigenstructure assignment depends on the number of available inputs. In the sequel, we first consider for an n -th order, m -input system, the minimisation of $m-1$ ultimate bound components, which requires the assignment of $n-m+1$ eigenvectors and $m-$

² Indeed, if we factor $C \triangleq [B \ AB \ \dots \ A^{n-1}B]$ as $C = A^n [A^{-n}B \ A^{-n+1}B \ \dots \ A^{-1}B]$, the latter bracket is nonsingular since $\text{rank}(C) = \text{rank}(A^n) = n$. Then if $v_{j,n} = 0$ in (13), one can assign more eigenvalues at zero, whose associated generalised eigenvectors are given in terms of higher powers of A^{-1} , until the power k^* is reached.

1 eigenvalue-eigenvector pairs. Then, we derive conditions for the minimisation of m ultimate bound components by assigning n eigenvalue-eigenvector pairs.

Throughout this section we assume that the eigenvalues to be assigned are not eigenvalues of A . Note that this condition is not restrictive for the minimisation of up to $m-1$ ultimate bound components since, as we will see, the choice of eigenvalues does not affect the minimisation.

1) *Minimising $m-1$ ultimate bounds:* From Lemma 3.1, an ultimate bound can be minimised to its lowest possible value if the eigenstructure assignment satisfies (7) and (8). From (2) and (3), it follows that if the ranks of $\mathcal{N}(\phi)$ and $\mathcal{R}(B)$ are greater than one, we have freedom to choose the eigenvector so that $v_{j,i} = 0$.

Lemma 3.3: (Eigenvector assignment) For system (4), the eigenvector associated with an arbitrary eigenvalue can be assigned to have $m-1$ zero elements.

Proof: Recalling Lemma 2.1, an eigenvector v_i associated with $\lambda = \lambda_i$ satisfies (3). The column space of B can be represented in matrix form as $\mathcal{R}(B) = B\alpha_i$ where $\alpha_i \in \mathbb{C}^m$ is an arbitrary vector to be assigned. Then v_i satisfies

$$v_i = (\lambda_i I_n - A)^{-1} B \alpha_i. \quad (14)$$

Let $\mathcal{M} \subset \{1, 2, \dots, n\}$ be such that $\#\mathcal{M} = m-1$. We will show that we can select the vector $\alpha_i \in \mathbb{C}^m$ in (14) such that the components of v_i with indices in $\mathcal{M}$ are zero.

Consider the matrix

$$G_i = [(\lambda_i I_n - A)^{-1} B]_{(\mathcal{M},:)} \in \mathbb{C}^{(m-1) \times m}. \quad (15)$$

Then, since G_i has $m-1$ rows and m columns, there exists

$$\alpha_i \in \mathcal{N}(G_i) \quad \text{where} \quad \alpha_i \in \mathbb{C}^m \quad (16)$$

that assigns $m-1$ components of v_i to zero, that is,

$$v_{i(\mathcal{M},:)}^T = G_i \alpha_i = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}_{m-1}^T. \quad (17)$$

Theorem 3.4: Given an n -th order m -input system ($1 < m < n$), if zero is not an eigenvalue of A , then $m-1$ components of the ultimate bound vector can be minimised to their smallest possible values.

Proof: Let $\mathcal{M} \subset \{1, 2, \dots, n\}$, $\#\mathcal{M} = m-1$, contain the indices of the desired state components for ultimate bound minimisation. Recalling Lemma 3.1, for each bound to be minimised, the corresponding row of the eigenvector matrix should have $n-1$ zeros associated with arbitrary eigenvalues and a nonzero element associated with a zero eigenvalue. Thus, in order to minimise $m-1$ ultimate bounds, $m-1$ eigenvalues must be placed at zero.

For each $\lambda_i = 0$, eigenvector assignment is successful if for each $j \in \mathcal{M}$ a vector α_i can be selected such that

$$v_{j,i} = [-A^{-1}]_{(j,:)} B \alpha_i \neq 0. \quad (18)$$

From (18), it can be seen that if $[-A^{-1}]_{(j,:)} B = 0_{1 \times m}$, regardless of the choice of α_i we have $v_{j,i} = 0$. This happens

when zero is a common root of the polynomials

$$[P_1^j(\lambda), \dots, P_m^j(\lambda)] \triangleq [\text{adj}(\lambda I - A)]_{(j,:)} [b_1, \dots, b_m]. \quad (19)$$

In this case, the eigenvector assignment is unsuccessful to make $v_{j,i} \neq 0$ by any selection of α_i . The proof of Theorem 3.4 is then divided into two sections: (a) when zero is not a common root of the polynomials (19) for all $j \in \mathcal{M}$ and (b) when zero is a common root of the polynomials (19) for some $j \in \mathcal{M}$.

a) *Zero is not a common root of P_k^j , $k = 1, \dots, m$ in Equation (19) for all $j \in \mathcal{M}$:* For each $j \in \mathcal{M}$, we perform the eigenvalue-eigenvector assignment

$$\lambda_j = 0, \quad v_j = (\lambda_j I_n - A)^{-1} B \alpha_j, \quad (20)$$

where α_j satisfies

$$[(\lambda_j I_n - A)^{-1} B]_{(j,:)} \alpha_j \neq 0 \quad \text{and} \quad \alpha_j \in \mathcal{N}(G_j),$$

with

$$G_j = [(\lambda_j I_n - A)^{-1} B]_{(\mathcal{M}_j,:)}, \quad \mathcal{M}_j \triangleq \{k \in \mathcal{M} : k \neq j\}. \quad (21)$$

For $k \in \mathcal{M}$, the above assignment results in

$$v_{k,j} \begin{cases} = 0 & \text{for } k \neq j \\ \neq 0 & \text{for } k = j. \end{cases} \quad (22)$$

The remaining $n-m+1$ eigenvalues are arbitrary, and so are the remaining components of the eigenvectors $v_j, j \in \mathcal{M}$ (as long as the full set of eigenvectors remains linearly independent). The resulting eigenstructure has the form (assuming, for simplicity of notation, that $\mathcal{M}$ contains the first $m-1$ components),

$$\Lambda = \text{diag} \left(\underbrace{0, \dots, 0}_{m-1}, \underbrace{\lambda_1, \dots, \lambda_{n-m+1}}_{n-m+1} \right), \quad (23)$$

$$V = \begin{bmatrix} v_{1,1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & v_{m-1,m-1} & 0 & \dots & 0 \\ v_{m,1} & \dots & v_{m,m-1} & v_{m,m} & \dots & v_{m,n} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & \dots & v_{n,m-1} & v_{n,m} & \dots & v_{n,n} \end{bmatrix}, \quad (24)$$

where $v_{j,j} \neq 0$ for $j \in \mathcal{M}$ and the entries not specifically set to zero are arbitrary provided V is invertible.

The ultimate bounds in the equation below are minimised to their smallest possible values:

$$\begin{aligned} \mathbf{b}_{(1:m-1)} &= |V_{(1:m-1,:)}| \begin{bmatrix} 1 & \dots & 0 \\ & \ddots & \\ & & 1 \\ \vdots & & & \frac{1}{1-|\lambda_m|} & \vdots \\ 0 & \dots & & & \frac{1}{1-|\lambda_n|} \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_{m-1} \\ Z_m \\ \vdots \\ Z_n \end{bmatrix} \\ &= \begin{bmatrix} v_{1,1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & v_{m-1,m-1} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_{m-1} \\ \frac{Z_m}{1-|\lambda_m|} \\ \vdots \\ \frac{Z_n}{1-|\lambda_n|} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^{\min} \\ \vdots \\ \mathbf{b}_{m-1}^{\min} \end{bmatrix}. \end{aligned}$$

b) Zero is a common root of $P_k^j, k = 1, \dots, m$ in Equation (19) for some $j \in \mathcal{M}$. In this case, $v_{j,i} = 0$ for any choice of α_i . Thus, we need to assign a generalised eigenvector so that its j -th element is not zero. We proceed as follows. First, the eigenvector $v_i = (-A)^{-1}B\alpha_i$ associated with a zero eigenvalue is assigned so that $\{v_{k,i} = 0 : k \in \mathcal{M}\}$. The generalised eigenvector $\bar{v}_i$ associated with v_i is

$$\bar{v}_i = (-A)^{-1}B\alpha_i - (-A)^{-2}B\beta_i, \quad \beta_i \in \mathbb{C}^m \text{ arbitrary.}$$

We want to assign the above generalised eigenvector so that $\{\bar{v}_{k,i} = 0 : k \in \mathcal{M}, k \neq j\}$ and $\bar{v}_{j,i} \neq 0$.³ Define the matrices

$$G_i \triangleq [(-A)^{-1}B]_{(\mathcal{M},j,:)}, \quad \bar{G}_i \triangleq [(-A)^{-2}B]_{(\mathcal{M},j,:)}, \quad (25)$$

with $\mathcal{M}_j$ defined in (21). Compute α_i and β_i satisfying

$$\alpha_i \in \mathcal{N}(G_i), \quad \beta_i \in \mathcal{N}(\bar{G}_i), \quad [(-A)^{-2}B]_{(j,:)}\beta_i \neq 0, \quad (26)$$

and for $k \in \mathcal{M}$, the generalised eigenvector satisfies

$$\bar{v}_{k,i} = [G_i]_{(k,:)}\alpha_i + [\bar{G}_i]_{(k,:)}\beta_i \begin{cases} = 0 & \text{for } k \neq j \\ \neq 0 & \text{for } k = j \end{cases}$$

The worst case is when $(-A)^{-1}B$ has the largest possible number of zero rows. For a system of order n with m inputs, $(-A)^{-1}B$ can have at most $n - m$ zero rows. (To see this, note that for $(-A)^{-1}B$ to have zero rows, B has to be in the null space of the corresponding row of A^{-1} and this can happen for at most $n - m$ rows of A^{-1} . Otherwise $\text{rank}(A^{-1}) < n$).

If $m - 1 \leq n - m$, then $(m - 1)$ pairs of eigenvector-generalised eigenvectors associated with $2(m - 1)$ zero eigenvalues need to be assigned. The remaining $n - 2(m - 1)$ eigenvalues are arbitrarily assigned with eigenvectors computed to have zeros in the entries with indices in $\mathcal{M}$. Note that $n - 2(m - 1) \geq n - (m - 1) - (n - m) = 1$, so at least one arbitrary eigenvalue needs to be assigned apart from the $2(m - 1)$ zero eigenvalues.

If $m - 1 > n - m$, then we need to assign $n - m$ pairs of eigenvector-generalised eigenvector associated with $2(n - m)$ zero eigenvalues to generate $n - m$ nonzero elements. Moreover, we should have another $(m - 1) - (n - m)$ nonzero elements of eigenvectors associated with zero eigenvalues. The remaining $n - 2(n - m) - (m - 1) + (n - m) = 1$ eigenvalue is arbitrarily assigned with its eigenvector computed to have zeros in the entries with indices in $\mathcal{M}$.

Suppose that $A^{-1}B$ has n_z zero rows. The eigenvalues assigned in this case are

$$\begin{bmatrix} \underbrace{0 \ 0 \ \dots \ 0 \ 0}_{2n_z} & \underbrace{0 \ \dots \ 0}_{(m-1)-n_z} & \lambda_1 \ \dots \ \lambda_r \end{bmatrix} \quad (27)$$

where $r \triangleq n - (m - 1) - n_z$. By eigenvector assignment as in (20)-(21) and generalised-eigenvector assignment as in (25)-(26), the eigenvalue and eigenvector matrices take the

³If $[(-A)^{-2}B]_{(j,:)} = 0$ we keep assigning eigenvalues at zero until a power $k^* \in \{1, \dots, n\}$ is reached such that the j -th row of $A^{-k^*}B$ is nonzero, see footnote 2.

form (assuming, as before, that $\mathcal{M}$ contains the first $m - 1$ components),

$$\Lambda = \text{blockdiag} \left(\Lambda_1, \dots, \Lambda_{n_z}, \text{diag} \left(\underbrace{0, \dots, 0}_{(m-1)-n_z}, \lambda_1, \dots, \lambda_r \right) \right) \quad (28)$$

$$V = \begin{bmatrix} V_1 \dots V_{n_z} & \bar{V}_1 \dots \bar{V}_{(m-1)-n_z} & \hat{V}_1 \dots \hat{V}_r \end{bmatrix} \quad (29)$$

where

$$\Lambda_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (V_i)_{(1:m-1,:)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & \bar{v}_{i,i} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

$$(\bar{V}_i)_{(1:m-1)} = \begin{bmatrix} 0 \\ \vdots \\ v_{n_z+i, 2n_z+i} \\ \vdots \\ 0 \end{bmatrix}, \quad (\hat{V}_i)_{(1:m-1)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It is easy to show by direct computation that the first $m - 1$ ultimate bound components are then minimised to their minimum possible values. ■

2) *Minimising m ultimate bounds:* As stated in Theorem 3.4, for an n -order m -input system, it is possible to minimise $m - 1$ ultimate bounds. From (23) and (28), it follows that there are some arbitrary eigenvalues whose assignment could be explored to attempt to minimise another ultimate bound.

Theorem 3.5: Given a system of order n with m inputs, if n_z is the number of zero rows of $A^{-1}B$, minimisation of m ultimate bounds is achievable if there exist $r \triangleq n - m - n_z$ stable eigenvalues $\{\lambda_i, i = 1, \dots, r\}$ that satisfy, for some $k \in \{1, 2, \dots, n\}$,

$$[\text{adj}(\lambda_i I_n - A)]_{(k,:)} b_j = 0 \quad \text{for } j = 1, \dots, m. \quad (30)$$

Proof: Assume, without loss of generality, that we want to minimise the first m ultimate bounds. Thus, the first m rows of the eigenvector matrix should have $n - 1$ zeros and a nonzero element associated with a zero eigenvalue, which means there should be m zero eigenvalues. If $A^{-1}B$ has n_z zero rows, we need to set another n_z zero eigenvalues so that there are $m - n_z$ eigenvectors and n_z pairs of eigenvector-generalised eigenvector associated with zero eigenvalues.

First, place $m + n_z$ eigenvalues at zero and using (20)-(21) and (25)-(26), assign $m - n_z$ eigenvectors and n_z pairs of eigenvector-generalised eigenvector such that of their first m elements, there is just one nonzero at different places, i.e.

$$\lambda = \begin{bmatrix} \underbrace{0 \ \dots \ 0}_{2n_z} & \underbrace{0 \ \dots \ 0}_{m-n_z} & \underbrace{\lambda_1 \ \dots \ \lambda_r}_{n-m-n_z} \end{bmatrix} \quad (31)$$

$$V = \begin{bmatrix} V_1 \dots V_{n_z} & \bar{V}_1 \dots \bar{V}_{m-n_z} & \hat{V}_1 \dots \hat{V}_r \end{bmatrix} \quad (32)$$

where

$$(V_i)_{(1:m,:)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & \bar{v}_{i,i} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad (\bar{V}_i)_{(1:m,:)} = \begin{bmatrix} 0 \\ \vdots \\ v_{n_z+i, 2n_z+i} \\ \vdots \\ 0 \end{bmatrix},$$

$$(\hat{V}_i)_{(1:m,:)} = \begin{bmatrix} v_{1,m+n_z+i} \\ \vdots \\ v_{m-1,m+n_z+i} \\ v_{m,m+n_z+i} \end{bmatrix}.$$

Next, we want to assign the remaining $r = n - m - n_z$ eigenvalues λ_i , $i \in \{1, \dots, r\}$, such that their corresponding eigenvectors have their first m elements equal to zero. From Lemma 3.3, for each eigenvalue λ_i we can assign its eigenvector to have $m - 1$ zeros, that is,

$$(\hat{V}_i)_{(1:m,:)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_{m,m+n_z+i} \end{bmatrix} \quad (33)$$

for $i = 1, \dots, r$. Then, the problem is to find r eigenvalues such that

$$\begin{bmatrix} v_{m,m+n_z+1} & \dots & v_{m,m+n_z+r} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}. \quad (34)$$

For an m -input system, the m -th component of the eigenvector associated with eigenvalue λ_i is

$$v_{m,i} = [(\lambda_i I_n - A)^{-1}]_{(m,:)} B \alpha_i = \frac{[\text{adj}(\lambda_i I_n - A)]_{(m,:)} B \alpha_i}{\det(\lambda_i I_n - A)}. \quad (35)$$

Since α_i is determined from the previous step (so that the associated eigenvector has $m - 1$ zeros), in order for $v_{m,i} = 0$ we need to search for r eigenvalues that satisfy (30) (for $k = m$). ■

Note that the condition of Theorem 3.5 becomes more restrictive as the number of inputs increases.

C. Example

Given a 4th order 3-input system with matrices

$$A = \begin{bmatrix} -0.5519 & 0.5610 & -0.2265 & -0.1513 \\ 0.3357 & 0.3507 & 0.8320 & -0.0782 \\ 0.6888 & -0.9866 & -0.9977 & 0.5403 \\ -0.3111 & 0.2043 & -0.0751 & -0.3551 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.6899 & -0.4837 & -0.4798 \\ 0.6176 & -0.2681 & -0.2660 \\ -0.2681 & 0.8120 & -0.1865 \\ -0.2660 & -0.1865 & 0.8150 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = 1,$$

we are interested in minimising ultimate bounds of the first and second states, hence $\mathcal{M} = \{1, 2\}$.

Since $m = 3$, we can minimise $m - 1 = 2$ bounds without any restriction. In order to minimise two ultimate bounds, we should set two closed-loop eigenvalues to zero and assign their eigenvectors to satisfy (18).

First, set $\lambda_1 = 0$ and define $\mathcal{M}_1 = \{2\}$. Thus, it is needed to have $v_{1,1} \neq 0$ and $v_{2,1} = 0$. But since $[(-A)^{-1}B]_{(1,:)} = [0 \ 0 \ 0]$, it means that zero is a common root of polynomials (19) for $j = 1$ and it is impossible to find α_1 satisfying $v_{1,1} \neq 0$. Thus, another eigenvalue should be set to zero and an eigenvector-generalised eigenvector assignment should be performed. For the eigenvector-generalised eigenvector pair

$$\begin{bmatrix} v_1 & \bar{v}_1 \end{bmatrix} = [(-A)^{-1}B\alpha_1 \quad (-A)^{-1}B\alpha_1 - (-A)^{-2}B\beta_1],$$

we want to have $v_{2,1} = 0$, $\bar{v}_{1,1} \neq 0$ and $\bar{v}_{2,1} = 0$. The eigenvector-generalised eigenvector assignment

$$\alpha_1 = [-1.2395 \ 0.6326 \ 0.2518]^T \in \mathcal{N}\{[(-A)^{-1}B]_{(2,:)}\}$$

$$\beta_1 = [-1.3066 \ 0.5313 \ -0.1021]^T \in \mathcal{N}\{[(-A)^{-2}B]_{(2,:)}\}$$

yields

$$v_1 = [0 \ 0 \ 1.2891 \ 0.9015]^T, \bar{v}_1 = [0.1457 \ 0 \ 0.4125 \ 1.6824]^T.$$

Next, another eigenvalue needs to be placed at zero in order to have a nonzero element in the second row of the eigenvector matrix. Hence, for $\lambda_3 = 0$, we require the associated eigenvector $v_3 = (\lambda_3 I_4 - A)^{-1}B\alpha_3$ to have $v_{1,3} = 0$ and $v_{2,3} \neq 0$, i.e. $\mathcal{M}_2 = \{1\}$ (see (20)–(21)). Choosing $\alpha_3 = [1 \ 0 \ 1]^T \in \mathcal{N}\{[(-A)^{-1}B]_{(1,:)}\}$ gives

$$v_3 = [0 \ 2.4830 \ -1.1664 \ 3.2215]^T.$$

Finally, as we have accomplished to have two nonzero elements in the first two rows of the eigenvector matrix (rows with indices in $\mathcal{M}$), the last eigenvalue can be selected arbitrarily and its associated eigenvector should be assigned to have zero in the first two elements. For $\lambda_4 = 0.1524$ and its associated eigenvector $v_4 = (\lambda_4 I_4 - A)^{-1}B\alpha_4$, eigenvector assignment satisfying $\alpha_4 = [-0.4821 \ 0.8439 \ -0.2355]^T \in \mathcal{N}\{[(\lambda_4 I_4 - A)^{-1}B]_{(\mathcal{M},:)}\}$ yields

$$v_4 = [0 \ 0 \ 0.5065 \ -0.5106]^T.$$

The eigenvalue and eigenvector matrices are therefore

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1524 \end{bmatrix}, V = \begin{bmatrix} 0 & 0.1457 & 0 & 0 \\ 0 & 0 & 2.4830 & 0 \\ 1.2891 & 0.4125 & -1.1664 & 0.5065 \\ 0.9015 & 1.6824 & 3.2215 & -0.5106 \end{bmatrix},$$

and the ultimate bound vector is $\mathbf{b} = [1 \ 1 \ 27.4144 \ 32.0309]^T$, where the first two bounds are minimised to their least possible values.

IV. CONCLUSION

This paper has analysed the ultimate bound minimisation problem for LTI discrete-time systems with persistent bounded disturbances. For single-input systems, we have shown that the minimum possible ultimate bound can always be achieved for one state if the transfer function from the control input to that state has relative degree one and is minimum phase. For systems with m inputs, we have presented an eigenstructure assignment procedure to simultaneously achieve minimum ultimate bounds for $m - 1$ states. Finally, we presented additional conditions under which m minimum ultimate bounds may also be simultaneously achieved.

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