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On the Zero-freeness of Tall Multirate Linear Systems

Mohsen Zamani, Giulio Bottegal and Brian D. O. Anderson

Abstract—In this paper, tall discrete-time linear systems with multirate outputs are studied. In particular, we focus on their zeros. In systems and control literature zeros of multirate systems are defined as those of their corresponding time-invariant systems obtained through blocking of the original multirate systems. We assume that blocked systems are tall i.e. have more outputs than inputs. It is demonstrated that, for generic choice of the parameter matrices, linear systems with multirate outputs generically have no finite nonzero zeros. However, they may have zeros at the origin or at infinity depending on the choice of blocking delay and the input, state and output dimensions.

I. INTRODUCTION

Multirate linear systems have been studied in different sub-disciplines, such as sampled-data control [6], signal processing [21], [1], [16] and econometric modeling [7] for some decades. The study of these systems has been in progress in different branches of systems and control, e.g., adaptive control [27], fault detection [15] and system identification [14].

The identification of multirate systems is a difficult task in general, see e.g., [23] and [9]. It has been shown [8] that when a linear time-invariant system is *zero-free*, identification of the system from measured output data and with unmeasured white noise input becomes far simpler than a normal system that in general may have zeros. For this reason, it is interesting to study the zero properties of linear dynamic systems. In reference [2], the authors have shown that tall linear time-invariant (single rate) systems are generically zero-free. A corresponding demonstration till now has been lacking for the multirate case, and the central task of this paper is to address that shortcoming. We specifically show that tall multirate linear systems are generically zero-free, perhaps excluding zeros at infinity and the origin, which require special treatment. This result is relevant in econometric problems of generalized dynamic factor models [10], where the models are typically both multirate (some data being quarterly, other data being monthly) and tall.

While our prime motivation has been to demonstrate a property which implies, as noted above, substantial simplification in the identification or modelling task, we comment that the

result may have separate importance from a control design perspective. In fact, zeros which are unstable or stable but close to a stability boundary can provide obstructions to the existence of inverses of linear systems and more generally, the design of high performance controllers. The results of this paper suggest that, when one is dealing with a system whose parameters are assuming generic values, the controller design may then be easier if one can add extra sensors to make the system have more outputs than inputs, and thereby suppress occurrence of any zeros at all, apart possibly from zeros at zero or infinity.

What is meant by ‘generic values’? The term is admittedly a loose one. In the context of this paper, it means (as is usual) the parameter values do not lie in a proper semialgebraic set. By way of example, if A is a square matrix, we can say it is nonsingular for generic values of its entries, i.e. for all those values which do not lie in the set defined by $|A| = 0$. In whatever context the phrase is used, it means that all values are generic except maybe some in a set of measure zero. The particular semialgebraic set in the light of the measure zero property is often considered unimportant, and not necessarily explicitly described. Thus one can say that a 2×2 matrix with generic entries is nonsingular, without having to add the detail that it is nonsingular unless $a_{11}a_{22} - a_{12}a_{21} = 0$. The fact that this equality describes the singular set is most times nowhere near as relevant as the fact that this is a set of measure zero.

In the literature, zeros of tall multirate systems are defined as those of their associated blocked systems [4]. Hence, in this paper, we consider blocked systems resulting from blocking of multirate systems with generic parameter matrices. Even though results for zeros of unblocked linear time-invariant systems with a generic choice of parameter matrices are well-known [13], extension of these available results to the blocked case is not straightforward. This is partly because the parameter matrices of the blocked system are structured and many entries of system matrices cannot independently assume generic values.

Since the analysis of zeros for tall blocked systems is quite involved, we consider three cases separately, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero. In the next section, we shall introduce the problem of interest. Section III studies tall blocked systems for the choice of finite nonzero zeros. It is explicitly established that tall blocked systems generically have no finite nonzero zeros. As a byproduct in this section, we also establish results on the generic rank of a system matrix resulting from blocking a multi-rate system. Following this, in Section IV zeros of tall blocked systems are examined at $Z = 0$ and $Z = \infty$. It is shown when tall blocked systems have a zero at $Z = 0$ or $Z = \infty$ and when they are zero-free at those aforementioned points. Finally, Section V offers concluding remarks.

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II. ZEROS OF MULTIRATE SYSTEMS

We consider the following multirate linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \quad k = 1, 2, \dots \\ y^f(k) &= C^f x(k) + D^f u(k) \quad k = 1, 2, \dots \\ y^s(k) &= C^s x(k) + D^s u(k) \quad k = \tau, N + \tau, \dots \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ the input. The signal $y^f(k) \in \mathbb{R}^{p_1}$ is measured at all k , fast output, and $y^s(k) \in \mathbb{R}^{p_2}$ is only measured at $k = \tau, N + \tau, \dots$, where $N > 1$ is a positive integer denoting the rate of the slow output and $\tau \in \{1, 2, \dots, N\}$ is a time displacement. We assume that *all* entries of parameter matrices in (1) are generic, and so in particular are nonzero and in principle could be freely or randomly chosen. The system defined by (1) can be seen as a periodic system. Periodic systems have been extensively studied in the literature of systems and control, see e.g. [3] for a survey. In this paper, we are interested in analyzing zeros of the system (1). In the literature [4] zeros of (1) are defined as those of its corresponding blocked or lifted systems, and definitions are reviewed below. Our key contributions, which are novel, will be to show that generically, tall multirate systems have no finite nonzero zeros, and to provide details concerning the structure of zeros at zero and infinity.

The well-known technique of blocking or lifting [4] transfers the system (1) into a linear time-invariant system. We obtain actually N distinct alternative ways to block this system, depending on the value assumed by τ . Even though these N different systems share the same poles, their zeros are not identical in the whole complex plane (see [4], pages 173-179). We index these systems with the integer τ , and define

$$U(k) \triangleq \begin{bmatrix} u(k+1) \\ u(k+2) \\ \vdots \\ u(k+N) \end{bmatrix}, \quad Y_\tau(k) \triangleq \begin{bmatrix} y^f(k+1) \\ y^f(k+2) \\ \vdots \\ y^f(k+N) \\ y^s(k+\tau) \end{bmatrix} \quad (2)$$

and $X(k) \triangleq x(k+1)$, where now $k = 0, N, 2N, \dots$. Then the blocked system $\sum_\tau$ is defined by

$$\begin{aligned} X(k+N) &= A_\tau X(k) + B_\tau U(k) \\ Y_\tau(k) &= C_\tau X(k) + D_\tau U(k), \end{aligned} \quad (3)$$

where

$$\begin{aligned} A_\tau &\triangleq A^N, \quad B_\tau \triangleq \begin{bmatrix} A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix}, \\ C_\tau &\triangleq \begin{bmatrix} C^f{}^T & A^T C^f{}^T & \dots & A^{(N-1)T} C^f{}^T & A^{(\tau-1)T} C^s{}^T \end{bmatrix}^T, \\ D_\tau &\triangleq \begin{bmatrix} D^f & 0 & \dots & 0 \\ C^f B & D^f & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-2}B & C^f A^{N-3}B & \dots & D^f \\ \hline & D_\tau^s & & \end{bmatrix}, \end{aligned} \quad (4)$$

where $D_\tau^s = [C^s A^{\tau-2}B \dots C^s B \ D^s \ 0 \dots 0]$ for $\tau > 1$ with $N - \tau$ zero blocks of size $p_2 \times m$, and when $\tau = 1$, it

is given by $D_\tau^s = [D^s \ 0 \dots 0]$ where there are $N - 1$ zero blocks of size $p_2 \times m$.

For completeness, we recall the following standard definition [13].

Definition 2.1: The finite zeros of the transfer function $V_\tau(Z) = C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau \ \forall \tau \in \{1, 2, \dots, N\}$ with minimal realization $\{A_\tau, B_\tau, C_\tau, D_\tau\}$ are defined to be the finite values of Z for which the rank of the following system matrix falls below its normal rank

$$M_\tau(Z) = \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix}.$$

Further, $V_\tau(Z)$ is said to have an infinite zero when $n + \text{rank}(D_\tau)$ is less than the normal rank of $M_\tau(Z)$, or equivalently the rank of D_τ is less than the normal rank of $V_\tau(Z)$.

We also provide the following definition for the geometric multiplicity of a zero.

Definition 2.2: The geometric multiplicity of a finite zero $Z_0 \in \mathbb{C}$ is $\text{normal rank}(M_\tau(Z)) - \text{rank}(M_\tau(Z_0))$. Moreover, the geometric multiplicity of a zero at infinity is $\text{normal rank}(M_\tau(Z)) - n - \text{rank}(D_\tau)$.

In this paper, we use the term multiplicity to refer to the geometric multiplicity.

Remark 2.1: Those zeros defined through the system matrix are referred to as **invariant zeros** in the literature [19]. There also exists the notion of **transmission zeros** which are obtainable from the Smith-McMillan form of a transfer function matrix. It is worthwhile noting that when a realization is minimal invariant zeros and transmission zeros coincide. However, when it is not minimal the invariant zeros includes the transmission zeros. Furthermore, all unreachable and unobservable modes known as **input decoupling zeros** and **output decoupling zeros** are also invariant zeros when the system has square structure. The latter may not necessary hold for systems with non-square structure.¹ This paper uses an algebraic approach to study zeros. The geometric approach has also been used to handle zeros, see e.g. [24], [18], [11] and [20].

Remark 2.2: Though an alternative approach to the study of zeros is by means of the Smith-McMillan form of the transfer function associated with the system (3), it appears that, for periodic systems, the study of the system matrix is preferable (see [5], [22] and [11]).

Remark 2.3: It is also worthwhile mentioning that due to genericity assumption for system matrices of the unblocked system (1), the reachability of the pair (A_τ, B_τ) is equivalent to its controllability.

Motivated by the application in econometrics, we investigate the case where the blocked system attains tall structure. This happens only when either $p_1 > m$ or $p_1 \leq m$, $Np_1 + p_2 > Nm$. One should note that we assume $p_1, p_2 > 0$. The case $p_1 > m$ becomes a trivial extension of results in [26] and is not included here. Thus, we focus on the case $p_1 \leq m$, $Np_1 + p_2 > Nm$ which is more involved.

¹For more detailed study about different notions of zeros and their relationships, interested readers can refer to [17], [19] and [12].

III. BLOCKED SYSTEMS WITH GENERIC PARAMETERS-FINITE NONZERO ZEROS

The main result of this section is to show that $\sum_{\tau}$ with $p_1 \leq m$, $Np_1 + p_2 > Nm$ is generically zero-free. In the conference paper [25], we treated this case under a restrictive assumption, namely that the system matrix of the blocked system had full-column rank, and we shall drop this assumption here.

In the upcoming parts, we first characterize the normal rank of $M_{\tau}(Z)$; following that, we return to the question of zeros' existence. Before proving our result on the normal rank of $M_{\tau}(Z)$, we need to introduce the following preliminary results.

Proposition 3.1: Consider the system $\sum_{\tau} \forall \tau \in \{1, 2, \dots, N\}$, with $p_1 \leq m$, $Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then

- 1) if $n \leq (\tau - 1)(m - p_1)$, the matrix D_{τ} has rank equal to $(N - 1)p_1 + m + n$;
- 2) if $n > (\tau - 1)(m - p_1)$, the matrix D_{τ} has rank equal to $(N - \tau)p_1 + \tau m$.

Proof: Refer to the appendix for a proof. ■

Proposition 3.2: The normal rank of the system matrix $M_{\tau}(Z)$ is the same for every value of $\tau \in \{1, 2, \dots, N\}$.

Proof: From [4] page 176, one can easily conclude that for $Z_0 \in \mathbb{C} - \{0, \infty\}$ and Z_0 not being one of the poles of the $V_{\tau}(Z)$, $\text{rank}(V_{\tau}(Z_0)) = \text{rank}(V_{\tau+1}(Z_0))$. Hence, we can conclude that $V_{\tau+1}(Z)$ and $V_{\tau}(Z)$ have the same normal rank and so do their associated system matrices i.e. $M_{\tau+1}(Z)$ and $M_{\tau}(Z)$. ■

Now the general result on the normal rank of $M_{\tau}(Z)$ is as follows.

Theorem 3.1: Consider the system $\sum_{\tau} \forall \tau \in \{1, 2, \dots, N\}$ with $p_1 < m$, $Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then the normal rank of the system matrix $M_{\tau}(Z)$ is equal to

- 1) $n + Nm$, if $n \geq (N - 1)(m - p_1)$;
- 2) $(N - 1)p_1 + m + 2n$, if $n < (N - 1)(m - p_1)$.

Proof: Here, we focus on the system matrix $M_N(Z)$; every result on its normal rank can be easily extended to any value of $\tau \in \{1, \dots, N - 1\}$ using Proposition 3.2.

Consider the matrix D_N and define $r \triangleq \text{rank}(D_N)$; note that due to the tallness assumption we have $r \leq Nm$. We note that $\text{normal rank}(M_N(Z)) = \text{normal rank}(ZI - A_N) + \text{normal rank}(C_N(ZI - A_N)^{-1}B_N + D_N) \geq n + \text{rank}(\lim_{Z \rightarrow \infty}[C_N(ZI - A_N)^{-1}B_N + D_N]) = n + \text{rank}(D_N)$. Hence, the normal rank of $M_N(Z)$ is greater than or equal to $r + n$. There are two cases in the theorem statement. We now treat the first one. Suppose $n \geq (N - 1)(m - p_1)$. Recalling Proposition 3.1, $r = Nm$; hence $\text{normal rank}(M_N(Z)) = n + Nm$ and $M_N(Z)$ is full normal rank.

We now turn to the second case, i.e. $n < (N - 1)(m - p_1)$. In this case, from Proposition 3.1 we have $r = (N - 1)p_1 + m + n$, hence $\text{normal rank}(M_N(Z)) \geq (N - 1)p_1 + m + 2n$. Now, consider the submatrix formed by the first $n + (N - 1)p_1$ rows of $M_N(Z)$. Such a submatrix is full normal rank, since it can be viewed also as a submatrix of the system matrix associated with the fast outputs only. Such a system matrix generically

has full-row normal rank [26]. Now consider the remaining rows of $M_N(Z)$, i.e. the matrix

$$\Pi = \begin{bmatrix} C^f A^{N-1} & C^f A^{N-2} B & \dots & C^f B & D^f \\ C^s A^{N-1} & C^s A^{N-2} B & \dots & C^s B & D^s \end{bmatrix}$$

which can be factorized as

$$\Pi = \begin{bmatrix} C^f & D^f \\ C^s & D^s \end{bmatrix} \begin{bmatrix} A^{N-1} & A^{N-2} B & \dots & B & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix} \triangleq H \bar{R}.$$

Since A is generically full rank, then also A^{N-1} is full rank and thus the matrix $\bar{R}$ is full-row rank, namely $n + m$. Thus, the rank of Π depends on the rank of H , which, for generic choice of matrices C^s, D^s, C^f, D^f , is equal to $\alpha \triangleq \min\{p_1 + p_2, m + n\}$. Then $\text{normal rank}(M_N(Z)) \leq n + (N - 1)p_1 + \alpha$. Recall that due to the tallness condition $p_2 > N(m - p_1)$ and by assumption $n < (N - 1)(m - p_1)$; hence, we have $n + m < (N - 1)(m - p_1) + m = N(m - p_1) + p_1 < p_2 + p_1$, and so $\alpha = n + m$. Thus, we obtain $\text{normal rank}(M_N(Z)) \leq (N - 1)p_1 + m + 2n$. Combining this bound with the lower bound found previously, we conclude that $\text{normal rank}(M_N(Z)) = (N - 1)p_1 + m + 2n$. ■

We now return to the main task of studying the zeros of the blocked system (3). We state the following result from our earlier work [25].

Theorem 3.2: Consider the system $\sum_{\tau} \forall \tau \in \{1, 2, \dots, N\}$, with $p_1 \leq m$, and $Np_1 + p_2 > Nm$. Then for generic values of the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ the system matrix $M_{\tau}(Z)$ has rank equal to its normal rank for all finite nonzero values of Z , and accordingly $\sum_{\tau}$ has no finite nonzero zero.

IV. BLOCKED SYSTEMS WITH GENERIC PARAMETERS-ZEROS AT THE ORIGIN AND INFINITY

In the previous section, zeros of tall blocked systems with generic parameters for the choice of finite nonzero zeros were studied. In this section, zeros of these systems are investigated for choices of zeros at zero and infinity. We first state the following result which, perhaps surprisingly, relates zeros of the system $\sum_{\tau}$ at infinity to zeros of the system $\sum_{N-\tau+1}$ at the origin and conversely.

Lemma 4.1: Consider the family of systems $\sum_{\tau} \forall \tau \in \{1, 2, \dots, N\}$, where the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ assume generic values. Then the following fact holds: $\sum_{\tau}$ has κ zeros at $Z = 0$ and μ zeros at $Z = \infty$ if and only if $\sum_{N-\tau+1}$ has μ zeros at $Z = 0$ and κ zeros at $Z = \infty$.

Proof: See Appendix. ■

We first study zeros of tall blocked systems at infinity. According to Definition 2.1, the rank of matrix D_{τ} plays a crucial role in the determination of the zeros at infinity. We now use the result of Proposition 3.1 to determine the multiplicity of zeros at infinity.

Theorem 4.1: Consider the system $\sum_{\tau} \forall \tau \in \{1, 2, \dots, N\}$, with $p_1 \leq m$ and $Np_1 + p_2 > Nm$. Assume that the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ take generic values. Then $M_{\tau}(Z)$ has zeros at $Z = \infty$ with multiplicity equal to

- 1) 0 if $n \leq (\tau - 1)(m - p_1)$;
- 2) $n - (\tau - 1)(m - p_1)$ if $(\tau - 1)(m - p_1) < n \leq (N - 1)(m - p_1)$;
- 3) $(N - \tau)(m - p_1)$ if $n > (N - 1)(m - p_1)$.

Proof: Denote by σ the multiplicity of zeros at infinity. Then, by Definition 2.2 we have $\sigma = \text{normal rank } M_\tau(Z) - n - \text{rank } D_\tau$. Consider the following cases.

- 1) $n \leq (\tau - 1)(m - p_1)$. From Theorem 3.1 we have that $\text{normal rank } M_\tau(Z) = (N - 1)p_1 + m + 2n$, while Proposition 3.1 yields that $\text{rank } D_\tau = (N - 1)p_1 + m + n$. Then we easily conclude that $\sigma = 0$.
- 2) $(\tau - 1)(m - p_1) < n \leq (N - 1)(m - p_1)$. From Theorem 3.1 we still have that $\text{normal rank } M_\tau(Z) = (N - 1)p_1 + m + 2n$, while now Proposition 3.1 yields $\text{rank } D_\tau = (N - \tau)p_1 + \tau m$. Hence, in this case we obtain $\sigma = n - (\tau - 1)(m - p_1)$.
- 3) $n > (N - 1)(m - p_1)$. In this case, from Theorem 3.1 we have that the system matrix is full-column normal rank, namely $n + Nm$, while, according to Proposition 3.1, the rank of $\text{rank } D_\tau$ is still $(N - \tau)p_1 + \tau m$. Then we can conclude that $\sigma = (N - \tau)(m - p_1)$.

■

The following corollary studies zeros at the origin.

Corollary 4.1: Consider the system $\sum_\tau \forall \tau \in \{1, \dots, N\}$, with $p_1 \leq m$ and $Np_1 + p_2 > Nm$. Assume that the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ take generic values. Then $M_\tau(Z)$ has zeros at $Z = 0$ with multiplicity equal to

- 1) 0 if $n \leq (N - \tau)(m - p_1)$;
- 2) $n - (N - \tau)(m - p_1)$ if $(N - \tau)(m - p_1) < n \leq (N - 1)(m - p_1)$;
- 3) $(\tau - 1)(m - p_1)$ if $n > (N - 1)(m - p_1)$.

Proof: The result can be obtained by combining Theorem 4.1 with Lemma 4.1. ■

Remark 4.1: The above results reveal that, assuming A, B , etc. generic with $p_1 < m$ and $Np_1 + p_2 > Nm$, when $\tau = N$ all zeros are at the origin and no zero is at infinity. Conversely, when $\tau = 1$ all zeros are at infinity and there are no zeros at the origin. Furthermore, when $\tau = N$ there is always at least one zero at the origin, while when $\tau = 1$ there is always at least one zero at infinity (unless one considers a system with no dynamics, i.e. a system with $n = 0$).

Remark 4.2: When $p_1 = m$, the conditions given in Theorem 4.1 and the subsequent corollary imply that the system $\sum_\tau$ does not have any zeros at $Z = 0$ and $Z = \infty$.

Remark 4.3: In some special cases depending on the state, input and output dimensions, $\sum_\tau$ may have zeros at the origin or at infinity for some values of τ but be completely zero-free for other values of τ . For example, consider a system $\sum_\tau$ such that $n = 5$, $m = 5$, $p_1 = 3$, $p_2 = 24$ and $N = 8$ which has zeros for all values of τ , except for $\tau = 4, 5$. In these particular cases, the system $\sum_\tau$ is totally zero-free. This can be easily checked by using Theorem 4.1 and the subsequent corollary.

The following example is instrumental to summarize the findings of this paper.

Example 4.1: Let

$$A = \begin{bmatrix} 0 & 1 \\ 0.24 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C^f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \\ C^s = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, D^s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that $n = m = p_2 = 2$, $p_1 = 1$. Assume $N = 2$ and choose $\tau = N$. Then we have $Np_1 + p_2 = Nm$, i.e., the resulting blocked system is square. It can be checked using computational methods (see e.g. [22]) that the blocked system $\sum_N$ has one zero at 0 and one zero at $Z = 1.804$. Now, add the rows $c^{s,3} = [0 \ 2]$, $d^{s,3} = [0 \ 1]$ to C^s and D^s , respectively. Computing again the zeros, one can see that the nonzero zero has now vanished but there remains one zero at zero, as expected from Corollary 4.1. Note that this system satisfies the third condition of Corollary 4.1. Note also that the nonzero zero will remain only if $c^{s,3}$ and $d^{s,3}$ are such that the row vector $[c^{s,3}A \ c^{s,3}B \ d^{s,3}]$ lies in the row space of

$$H = \begin{bmatrix} C^f & D^f & 0 \\ C^f A & C^f B & D^f \\ C^s A & C^s B & D^s \end{bmatrix},$$

a condition that is nongeneric, since $H \in \mathbb{R}^{Nm \times (n + Nm)}$.

V. CONCLUSIONS

Zeros of tall discrete-time multirate linear systems were addressed in this paper, with the zeros of multirate linear systems being defined as those of their corresponding blocked systems. The system matrix of tall blocked systems was investigated for generic choice of parameter matrices. It was specifically shown that tall blocked systems generically have no finite nonzero zeros. However, we showed that there are situations in which these systems present zeros at $Z = 0$ or $Z = \infty$ or both. Such situations can be characterized in terms of the relevant integer parameters (input, state, and output dimensions and ratio of sampling rates). As part of the investigation, we also identified the generic rank assumed by the system matrix of a blocked system and the transfer function of that system.

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APPENDIX

A. Proof of Proposition 3.1

We first need to introduce the following lemma.

Lemma A.1: Consider a generic pair of matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Then, given $\nu \in \mathbb{N}$, the matrix

$$C = \begin{bmatrix} B & AB & \dots & A^{\nu-1}B \end{bmatrix} \quad (5)$$

is always full rank, i.e. its rank is equal to:

- 1) its number of rows, n , if $n \leq \nu m$,

- 2) its number of columns, νm , if $n > \nu m$.

Proof: The first part is immediate from the generic controllability of the pair (A, B) [24] and the Cayley-Hamilton theorem. The statement of the second part can be proven by finding a pair (A, B) such that the matrix C attains full rank, since it means that this happens for any generic pair of such matrices. One can verify that for the following choices of A and B the theorem's statement holds.

$$A = \begin{bmatrix} 0_{m \times (n-m)} & I_m \\ I_{n-m} & 0_{(n-m) \times m} \end{bmatrix} \quad B = \begin{bmatrix} I_m \\ 0_{(n-m) \times m} \end{bmatrix}, \quad (6)$$

We can now prove Proposition 3.1. For the sake of brevity, we treat only the case $n \geq m$, since the case $n < m$ is virtually the same. Fix τ and first assume $n \leq (\tau-1)(m-p_1)$. One can find a particular nongeneric choice of matrices A, B and etc., such that $\text{rank}(D_\tau) = (N-1)p_1 + m + n$. Hence, for a generic choice of these matrices $\text{rank}(D_\tau) \geq (N-1)p_1 + m + n$. We now seek an upper bound and show that it indeed coincides with the lower bound just found.

Define $\bar{D}_\tau$ that is obtained by some row permutations from D_τ . In particular $\bar{D}_\tau$ has the structure $\begin{bmatrix} \Delta_1 & 0 \\ * & \Delta_2 \end{bmatrix}$ where

$$\Delta_1 \triangleq \begin{bmatrix} D^f & 0 & \dots & 0 \\ \vdots & & & \ddots \\ C^f A^{\tau-2} B & \dots & & D^f \\ C^s A^{\tau-2} B & \dots & & D^s \end{bmatrix} \in \mathbb{R}^{\tau p_1 + p_2 \times \tau m} \text{ and}$$

$$\Delta_2 \triangleq \begin{bmatrix} D^f & \dots & 0 \\ \vdots & \ddots & \\ C^f A^{N-\tau-2} B & \dots & D^f \end{bmatrix} \in \mathbb{R}^{(N-\tau)p_1 \times (N-\tau)m}$$

. Note that since D^f is fat, Δ_2 has full-row rank, namely $\text{rank}(\Delta_2) = (N-\tau)p_1$. Hence, we have $\text{rank}(\bar{D}_\tau) = \text{rank}(D_\tau) = \text{rank}(\Delta_1) + \text{rank}(\Delta_2)$. We now decompose Δ_1 as $\Delta_1 = \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix}$ with $\Delta_b \triangleq \begin{bmatrix} C^f A^\tau B & \dots & C^f B & D^f \\ C^s A^\tau B & \dots & C^s B & D^s \end{bmatrix}$ and Δ_a capturing the remaining components of Δ_1 . Due to the genericity of D^f , one can verify that $\text{rank}(\Delta_a) = (\tau-1)p_1$. Moreover, Δ_b can be written as $\Delta_b = \mathcal{H}\mathcal{R}$, with

$$\mathcal{H}\mathcal{R} = \begin{bmatrix} C^f & D^f \\ C^s & D^s \end{bmatrix} \begin{bmatrix} A^\tau B & A^{\tau-1}B & \dots & B & 0 \\ 0 & 0 & \dots & 0 & I_m \end{bmatrix}.$$

Since by the assumption $n \leq (\tau-1)(m-p_1)$, with the help of Lemma A.1 we conclude that the matrix $\mathcal{R}$ is full-row rank. Hence, $\text{rank}(\Delta_b) = \text{rank}(\mathcal{H})$. From inequalities $p_2 > N(m-p_1)$ and $n \leq (\tau-1)(m-p_1)$ and the genericity assumption of parameter matrices, one obtains $\text{rank}(\Delta_b) = n + m$, $\text{rank}(\Delta_1) \leq (\tau-1)p_1 + n + m$ and $\text{rank}(D_\tau) \leq (N-1)p_1 + n + m$ that is the same as the lower bound found earlier. To complete the proof, we now consider $n > (\tau-1)(m-p_1)$. Note that from the previous case when $n = (\tau-1)(m-p_1)$, $\text{rank}(D_\tau) = \tau m + (N-\tau)p_1$ that is an upper bound for $\text{rank}(D_\tau)$. Then the case $n > (\tau-1)(m-p_1)$ becomes immediate as one can easily find a set of generic matrices which achieve this permissible upper bound.

B. Proof of Lemma 4.1

Consider a reverse-time description of the system (1), namely

$$\begin{aligned} x(k) &= A^{-1}x(k+1) - A^{-1}Bu(k), \quad k = 1, 2, \dots \\ y^f(k) &= C^f x(k) + D^f u(k), \quad k = 1, 2, \dots \\ &= C^f A^{-1}x(k+1) + (D^f - C^f A^{-1}B)u(k) \\ y^s(k) &= C^s x(k) + D^s u(k); \quad k = \tau, N+\tau, \dots \\ &= C^s A^{-1}x(k+1) + (D^s - C^s A^{-1}B)u(k) \end{aligned} \quad (7)$$

and define the following matrices

$$\begin{aligned} \tilde{A} &\triangleq A^{-1} & \tilde{B} &\triangleq -A^{-1}B \\ \tilde{C}^f &\triangleq C^f A^{-1} & \tilde{D}^f &\triangleq D^f - C^f A^{-1}B \\ \tilde{C}^s &\triangleq C^s A^{-1} & \tilde{D}^s &\triangleq D^s - C^s A^{-1}B \end{aligned} \quad (8)$$

which are still in a generic setting since the genericity of $\{A, B, C^f, D^f, C^s, D^s\}$ is assumed. Note that the matrix A^{-1} is well-defined, since A is generically full rank. Recall the blocking procedure introduced in (2) for a given value of τ ; we can obtain the blocked time-invariant system associated with the system (7) (again a reverse-time system) as

$$\begin{aligned} x_\tau(k) &= \tilde{A}_\tau x_\tau(k+N) + \tilde{B}_\tau U_\tau(k) \\ Y_\tau(k) &= \tilde{C}_\tau x_\tau(k+N) + \tilde{D}_\tau U_\tau(k), \end{aligned} \quad (9)$$

where $k = 0, N, \dots$, and

$$\begin{aligned} \tilde{A}_\tau &\triangleq \tilde{A}^N, \quad \tilde{B}_\tau \triangleq \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{N-2}\tilde{B} & \tilde{A}^{N-1}\tilde{B} \end{bmatrix}, \\ \tilde{C}_\tau &\triangleq \begin{bmatrix} \tilde{A}^{(N-1)^T} \tilde{C}^f{}^T & \dots & \tilde{C}^f{}^T & \tilde{A}^{(\tau-1)^T} \tilde{C}^s{}^T \end{bmatrix}^T, \\ \tilde{D}_\tau &\triangleq \left[\begin{array}{ccc|c} \tilde{D}^f & \dots & \tilde{C}^f \tilde{A}^{N-3} \tilde{B} & \tilde{C}^f \tilde{A}^{N-2} \tilde{B} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \tilde{D}^f & \tilde{C}^f \tilde{B} \\ 0 & \dots & 0 & \tilde{D}^f \end{array} \right] \frac{\tilde{D}^s}{\tilde{D}_\tau^s}. \end{aligned} \quad (10)$$

In the latter expression, when $\tau < N$ the matrix $\tilde{D}_\tau^s$ is equal to $\begin{bmatrix} 0 & \dots & 0 & \tilde{D}^s & \dots & \tilde{C}^s \tilde{A}^{N-\tau-1} \tilde{B} \end{bmatrix}$, with $\tau-1$ zero blocks of size $p_2 \times m$, while, when $\tau = N$, it becomes $\begin{bmatrix} 0 & \dots & 0 & \tilde{D}^s \end{bmatrix}$. Now let us introduce the N -step backward operator ζ , such that $\zeta x(k+N) = x(k)$. Then the transfer function $\tilde{V}_\tau(\zeta) \triangleq \tilde{C}_\tau(\zeta I - \tilde{A}_\tau)^{-1} \tilde{B}_\tau + \tilde{D}_\tau$ is associated with the blocked system (9). It can be easily checked through simple computations that this transfer function is connected to the transfer function $V_\tau(Z)$ associated with the system $\sum_\tau$ at the points zero and infinity through the equalities

$$\tilde{V}_\tau(0) = \lim_{Z \rightarrow \infty} V_\tau(Z) \quad \lim_{\zeta \rightarrow \infty} \tilde{V}_\tau(\zeta) = V_\tau(0). \quad (11)$$

Define the system matrix associated with the system (9) as

$$\tilde{M}_\tau(\zeta) \triangleq \begin{bmatrix} \zeta I - \tilde{A}_\tau & -\tilde{B}_\tau \\ \tilde{C}_\tau & \tilde{D}_\tau \end{bmatrix}. \quad (12)$$

For our purpose in this paper, we define the following equalities

$$\begin{aligned} \text{rank} \left(\lim_{Z \rightarrow \infty} M_\tau(Z) \right) &\triangleq n + \text{rank}(D_\tau) \\ \text{rank} \left(\lim_{\zeta \rightarrow \infty} \tilde{M}_\tau(\zeta) \right) &\triangleq n + \text{rank}(\tilde{D}_\tau) \end{aligned} \quad (13)$$

Then using the equation (11) one can write

$$\begin{aligned} \text{rank} \left(\lim_{Z \rightarrow \infty} M_\tau(Z) \right) &= \text{rank}(\tilde{M}_\tau(0)) \\ \text{rank} \left(\lim_{\zeta \rightarrow \infty} \tilde{M}_\tau(\zeta) \right) &= \text{rank}(M_\tau(0)). \end{aligned} \quad (14)$$

Again, note that the above equalities are well-defined since, due the genericity assumption of the matrix A , the matrices A_τ and $\tilde{A}_\tau$ do not have any eigenvalues at the origin. Now, by comparing (4) and (10), one can verify that there exist permutation matrices Q_1 and Q_2 such that $Q_1 M_\tau(\zeta) Q_2 = \Psi_\tau(\zeta)$, and $\Psi_\tau(\zeta)$ is exactly $M_{N-\tau+1}(Z)$ when $\tilde{A}, \tilde{B}, \tilde{C}^f, \tilde{C}^s, \tilde{D}^f, \tilde{D}^s, \zeta$ are replaced by $A, B, C^f, D^f, C^s, D^s, Z$, accordingly. Since the parameter matrices A, B, C, D assume generic values, we have the following equalities

$$\begin{aligned} \text{rank} \left(\lim_{\zeta \rightarrow \infty} \tilde{M}_\tau(\zeta) \right) &= \text{rank} \left(\lim_{Z \rightarrow \infty} M_{N-\tau+1}(Z) \right) \\ \text{rank}(\tilde{M}_\tau(0)) &= \text{rank}(M_{N-\tau+1}(0)) \end{aligned} \quad (15)$$

Then, by combining equations (14) and (15) we obtain

$$\text{rank} \left(\lim_{Z \rightarrow \infty} M_\tau(Z) \right) = \text{rank}(M_{N-\tau+1}(0)) \quad (16)$$

and

$$\text{rank}(M_\tau(0)) = \text{rank} \left(\lim_{Z \rightarrow \infty} M_{N-\tau+1}(Z) \right). \quad (17)$$

Thus, by using equations (16), (17) and the fact that the normal rank of $M_\tau(Z)$ does not depend on τ (see Proposition 3.2), the conclusion of the lemma readily follows.