NOVA

University of Newcastle Research Online

nova.newcastle.edu.au

Seron, M. M., and De Doná, J. A.; "On invariant sets and closed-loop boundedness of Lure-type nonlinear systems by LPV-embedding". Published in International Journal of Robust and Nonlinear Control Vol. 26, Issue 5, p. 1092-1111 (2016)

Available from: <http://dx.doi.org/10.1002/rnc.3354>

This is the peer reviewed version of the following article: Seron, M. M., and De Doná, J. A. (2016) On invariant sets and closed-loop boundedness of Lure-type nonlinear systems by LPV-embedding. Int. J. Robust. Nonlinear Control, 26: 1092–1111, which has been published in final form at <http://dx.doi.org/10.1002/rnc.3354>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Self-Archiving

Accessed from: <http://hdl.handle.net/1959.13/1319670>

On Invariant Sets and Closed-loop Boundedness of Lure-type Nonlinear Systems by LPV-embedding

Maria M. Seron

José A. De Doná

Priority Research Centre for Complex Dynamic Systems and Control
School of Electrical Engineering and Computer Science
The University of Newcastle, Australia
{maria.seron,jose.dedona}@newcastle.edu.au

Abstract

We address the problem of achieving trajectory boundedness and computing ultimate bounds and invariant sets for Lure-type nonlinear systems with a sector bounded nonlinearity. Our first contribution is to compare two systematic methods to compute invariant sets for Lure systems. In the first method, a linear-like bound is considered for the nonlinearity and this bound is used to compute an invariant set by regarding the nonlinear system as a linear system with a nonlinear perturbation. In the second method, the sector bounded nonlinearity is treated as a time-varying parameterised linear function with bounded parameter variations, and then invariant sets are computed by embedding the nonlinear system into a convex polytopic LPV system. We show that under some conditions on the system matrices these approaches give identical invariant sets, the LPV-embedding method being less conservative in the general case. The second contribution of the paper is to characterise a class of Lure systems for which an appropriately designed linear state feedback achieves bounded trajectories of the closed-loop nonlinear system and allows for the computation of an invariant set via a simple, closed-form expression. The third contribution is to show that, for disturbances that are ‘aligned’ with the control input, arbitrarily small ultimate bounds on the system states can be achieved by assigning the eigenvalues of the linear part of the system with ‘large enough’ negative real part. We illustrate the results via examples of a pendulum system, a Josephson junction circuit and the well-known Chua circuit.

1 Introduction

The concepts of boundedness, ultimate boundedness and set invariance provide a useful framework for the analysis and control of the behaviour of dynamical systems affected by persistent disturbances [5]. For example, these concepts have proven to be very valuable analysis and control design tools in: (i) Model Predictive Control, where they have been instrumental in establishing stability and robustness properties; see, e.g., [8, 19, 22]; and, (ii) Fault Tolerant Control systems, where they have allowed to derive conditions—based on the separation of sets characterising system behaviour under healthy operation from sets describing faulty operation—for the correct detection and isolation of component faults and the associated controller reconfiguration; see, e.g., [25, 20, 24]. Some constructive methods to compute ultimate bounds and invariant sets for linear systems, and procedures to obtain different approximations, can be found in [3, 14, 15, 16, 21].

For nonlinear systems, on the other hand, the computation of invariant sets is typically a very difficult task and constructive procedures leading to closed-form expressions are in general not known and, even for particular cases, are scarce. For general classes of nonlinear systems invariant sets can, in theory, be obtained as level sets of Lyapunov functions (see, e.g., [13]); however the difficulty still remains in the task of finding a suitable Lyapunov function, for which systematic procedures are not available in the general nonlinear case. For some classes of systems such as those containing saturation-like or piecewise affine nonlinearities, some interesting numerical procedures have been proposed in [2], [1] and [26].

In this paper we consider the problem of achieving trajectory boundedness and computing ultimate bounds and invariant sets for a class of nonlinear systems, namely Lure systems, which consist of linear dynamics with nonlinear internal feedback. Lure systems constitute an important class of nonlinear systems useful for representing many practically relevant systems across a range of domains including mechanics, electronics, electrical circuits, electrical machines, to name such a few (we provide a number of examples of such systems in the second part of the paper). Stability of Lure systems has been traditionally analysed in the framework of absolute stability theory, which assumes sector conditions for the nonlinearity combined with passivity-like conditions on the linear part of the system, see e.g., [13]. This theory, however, is not directly applicable to compute invariant sets for the Lure nonlinear system, except possibly via level sets of the associated Lyapunov functions. In this paper we depart from absolute stability and propose a methodology to compute invariant sets for Lure systems with a sector bounded nonlinearity leading to a simple, closed-form expression. Our first contribution is then to compare two systematic methods (relying on results from [9]) to compute invariant sets for Lure systems. In the first method, which we call ‘nonlinearity bounding’ method, a linear-like bound is considered for the nonlinearity and this bound is used to compute an invariant set by regarding the nonlinear system as a linear system with a nonlinear perturbation bounded by an affine function of the state. In the second method, which we call ‘LPV-embedding’ method, the sector bounded nonlinearity is treated as a time-varying parameterised linear function with bounded parameter variations, and then invariant sets are computed by embedding the nonlinear system into a convex polytopic LPV system. We show that under some conditions on the system matrices these approaches can give identical invariant sets, the LPV-embedding method being less conservative in the general case. The second contribution of the paper is to characterise a class of Lure systems for which a simple linear state feedback that assigns the eigenvalues of the linear part of the nonlinear system to appropriate locations, achieves bounded trajectories of the closed-loop nonlinear system and allows for the computation of an invariant set via a closed-form expression. The third contribution is to show that, for disturbances that are ‘aligned’ with the control input, arbitrarily small ultimate bounds on the system states can be achieved by assigning the eigenvalues of the linear part with ‘large enough’ negative real part.

The paper proceeds as follows. The problem statement is presented in Section 2. Section 3 reviews two techniques for establishing trajectory boundedness and computing invariant sets for a Lure-type nonlinear system, namely the ‘nonlinearity bounding’ and ‘LPV-embedding’ techniques. Conditions for these two methods to yield identical invariant sets are derived in Section 4. In Section 5 we show that a state feedback can be designed that achieves bounded trajectories and allows for the computation of an invariant set. Section 6 shows that arbitrarily small ultimate bounds can be achieved by appropriate eigenvalue assignment for ‘control-aligned’ disturbances. In Section 7 we analyse the case when the nonlinearity in the model is not matched with the control. Section 8 presents three application examples and Section 9 concludes the paper.

2 Problem Statement

Consider the Lure-type nonlinear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Gf(Hx(t)) + Ew(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}$ is the control input; $f : \mathbb{R} \rightarrow \mathbb{R}$ is a scalar nonlinear function; $w(t) \in \mathbb{R}^r$ is a disturbance componentwise¹ bounded as $|w(t)| \leq \bar{w}$ for all $t \geq 0$ and some nonnegative vector $\bar{w}$; and the matrices A , B , G , H and E have compatible dimensions.

Introduce the parameterisation²

$$r = r(x) \doteq \frac{f(Hx)}{Hx}, \quad (2)$$

and assume the nonlinear function $f(\cdot)$ is such that r is bounded in an interval as

$$r \in [r_1, r_2], \quad (3)$$

where r_1 and r_2 are finite real numbers such that $r_1 < r_2$.

It will be convenient to redefine the nonlinear function $f(Hx)$ in (1) by adding to it a linear term of the form cHx (and then subtracting this term from the linear part of the equation), with $c \neq 0$ a real number selected to achieve certain property to be discussed later in Section 4. We hence rewrite the nonlinear system (1) as

$$\dot{x}(t) = (A - cGH)x(t) + Bu(t) + G\varphi(Hx(t)) + Ew(t), \quad (4)$$

where

$$\varphi(Hx) \doteq f(Hx) + cHx, \quad (5)$$

A state feedback $u(t) = Kx(t)$ is now designed so that the closed-loop matrix $A_K \doteq A - cGH + BK$ is Hurwitz.³ The closed-loop system takes the form

$$\dot{x}(t) = A_K x(t) + G\varphi(Hx(t)) + Ew(t). \quad (6)$$

The purpose of this work is threefold. Firstly, for the closed-loop system (6), we will study the computation of an invariant set $\mathcal{S}$ of the form

$$\mathcal{S} = \{x \in \mathbb{R}^n : |V^{-1}x| \leq b\}, \quad (7)$$

for some transformation matrix V and vector b . The invariant set $\mathcal{S}$ is also required to be a *global ultimate-bound* set for the trajectories of the closed-loop system (6), that is, for any initial condition $x(0)$ the corresponding trajectory $x(t)$ satisfies

$$\limsup_{t \rightarrow \infty} |V^{-1}x(t)| \leq b. \quad (8)$$

We will compare two methods for computing an invariant, ultimate-bound set of the form (7). In the first method, a linear-like bound is considered for the nonlinearity and this bound is

---

¹In this paper, the bars $|\cdot|$ denote elementwise magnitude (absolute value) and the inequalities and max operations are interpreted elementwise.

²The dependence of the system variables on time t will be omitted when clear from the context.

³A square real matrix is Hurwitz if all its eigenvalues have negative real part.

used to compute an invariant set by regarding the nonlinear system as a linear system with a nonlinear perturbation bounded by an affine function of the state. In the second method, using the description (2) of the nonlinearity as a time-varying parameterised linear function $f(Hx) = rHx$ with bounded parameter variations, then invariant sets can be computed by embedding the nonlinear system into a convex polytopic LPV system. These techniques are described in Section 3 below.

Secondly, we give conditions on the system matrices so that by appropriate assignment of the eigenvalues of the closed-loop matrix $A_K = A - cGH + BK$ the closed-loop system (6) has bounded trajectories and an invariant ultimate-bound set $\mathcal{S}$ of the form (7) can be computed.

The third objective is to show that ultimate bounds on the state of the form

$$\limsup_{t \rightarrow \infty} |x| \leq |V|b \quad (9)$$

can be made arbitrarily small when the disturbance w in (6) is *aligned* with the control input, that is, $E = eB$ for some $e \neq 0$.

3 Ultimate Boundedness and Invariance Tools

In this section we briefly review two techniques for establishing trajectory ultimate boundedness and computing invariant sets for a Lure-type nonlinear system with dynamics of the form (6). Details and proofs can be found in [9].

Nonlinearity bounding. Assume the nonlinear function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$|\varphi(Hx)| \leq L|Hx| \quad (10)$$

for all $x \in \mathbb{R}^n$ and a scalar $L > 0$. Let V be a (possibly complex) transformation and define

$$\Lambda \doteq \mathcal{M}(V^{-1}A_KV), \quad (11)$$

where the operator $\mathcal{M}$ is such that

$$M = \mathcal{M}(N) \text{ has entries } M_{i,j} = \mathbb{R}e\{N_{i,j}\} \text{ if } i = j \text{ and } M_{i,j} = |N_{i,j}| \text{ if } i \neq j, \quad (12)$$

and

$$Q \doteq |V^{-1}G|L|H||V|. \quad (13)$$

Then, if $\Lambda + Q$ is Hurwitz, an ultimate-bound invariant set of the form (7) exists with

$$b = b_{NL} \doteq (-\Lambda_{NL})^{-1}\bar{e}, \quad \Lambda_{NL} \doteq \Lambda + Q, \quad (14)$$

where $\bar{e} \doteq |V^{-1}E|\bar{w}$.

LPV embedding. Consider the parameterisation (2)–(3) and define

$$\rho \doteq \frac{\varphi(Hx)}{Hx} = r + c \in [\rho_1, \rho_2] \doteq [r_1 + c, r_2 + c]. \quad (15)$$

We then rewrite (6) as

$$\dot{x} = (A_K + GH\rho)x + Ew, \quad (16)$$

where $\mathcal{A}(\rho) \doteq (A_K + GH\rho)$ belongs to the convex hull of $\{\mathcal{A}(\rho_1), \mathcal{A}(\rho_2)\}$. Let V be a (complex) transformation such that

$$\Lambda_{LPV} \doteq \max_{\rho \in [\rho_1, \rho_2]} \mathcal{M}(V^{-1}\mathcal{A}(\rho)V) = \max_{\rho \in [\rho_1, \rho_2]} \mathcal{M}(V^{-1}(A_K + GH\rho)V), \quad (17)$$

is Hurwitz. Then an ultimate-bound invariant set of the form (7) exists with

$$b = b_{LPV} \doteq (-\Lambda_{LPV})^{-1}\bar{e}, \quad (18)$$

where $\bar{e} \doteq |V^{-1}E|\bar{w}$.

Remark 3.1. *Once the system is embedded in the LPV representation (16), parametric Lyapunov techniques can also be used to synthesise a stabilising feedback gain, and invariant sets can be obtained as level sets of the associated Lyapunov function. The advantage of the methodology proposed in this paper is that we provide conditions, which can be checked a priori, for the feedback gain to be stabilising and, further, we give an explicit formula for the computation of ultimate-bound invariant sets. This is in contrast with the available Lyapunov techniques for LPV systems, which typically rely on numerical solutions based on linear matrix inequalities. Relations between the Lyapunov approach and the methodology explained above have been explored in [9], where it was shown that a quadratic Lyapunov function can be computed based on the required transformation V (cf. (17)). Finally, both techniques (Lyapunov-based and the methodology proposed in this paper) can be combined to obtain tighter bounds than could be obtained by either methodology applied individually.* $\circ$

Remark 3.2. *The results derived in this paper apply to any Lure-type nonlinear system with a nonlinearity that can be expressed as a time-varying parameterised linear function $\varphi(Hx) = \rho Hx$ with the parameter ρ in a bounded interval (cf. (15)). More generally, extensions of the results are currently being explored in two main directions: (i) when the parameter ρ is not globally bounded, in which case a regional/local treatment is required; and, (ii) when multiple nonlinearities appear in the model, in which case several parameters may need to be considered in the analysis.* $\circ$

4 Achieving Identical Ultimate-Bound Invariant Sets via Both Methods

In this section we will select a specific value of the ‘shifting’ constant c in (4)–(5), which will allow to identify conditions for the two methods described above to yield identical ultimate-bound invariant sets.

We assume in this section that the nonlinearity is ‘matched’ with the control input, that is $G = gB$, $g \neq 0$ (the case of unmatched nonlinearity is discussed later in Section 7). Also, for convenience we consider that the matrices A_K , B and H have the (canonical) form

$$A_K = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & \dots & -a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}, \quad H = [h_1 \quad h_2 \quad \dots \quad h_n]. \quad (19)$$

We observe that if the system is not natively described in the canonical form (19), say its state-space description is $\dot{\xi} = \bar{A}\xi + \bar{B}u + \bar{G}f(\bar{H}\xi) + \bar{E}w$ with the matrices having arbitrary form, then provided the pair $(\bar{A}, \bar{B})$ is controllable and $\bar{G} = \bar{B}g$ with $g \neq 0$, a transformation $\xi = Ux$ can easily be obtained (see e.g., [17]) that takes the system to the form (6)–(19), where $A_K = U^{-1}(\bar{A} - c\bar{G}\bar{H} + \bar{B}\bar{K})U$, $K = \bar{K}U$, $B = U^{-1}\bar{B}$, $G = U^{-1}\bar{G}$, $H = \bar{H}U$ and $E = U^{-1}\bar{E}$. In the original coordinates, the ultimate-bound invariant set (7) and associated state bound (9) take the form

$$\bar{\mathcal{S}} = \{\xi \in \mathbb{R}^n : |(UV)^{-1}\xi| \leq b\}, \quad (20)$$

$$|\xi| \leq |UV|b. \quad (21)$$

We propose to select

$$c \doteq -\frac{r_1 + r_2}{2}, \quad (22)$$

which makes the interval of variation of ρ in (15) symmetric around zero, that is,

$$\rho_2 = r_2 + c = \frac{r_2 - r_1}{2} > 0, \quad \rho_1 = r_1 + c = -\rho_2 = \frac{r_1 - r_2}{2} < 0. \quad (23)$$

Note from (15) that, with this choice, the nonlinearity bounding constant L in (10) can be taken as

$$L \doteq \rho_2 = \frac{r_2 - r_1}{2}. \quad (24)$$

We will show below that under certain conditions the choice (22) implies that the sets of the form (7) computed by both methods are identical when using a particular transformation V , the same for both methods.

To this end, suppose the eigenvalues of the closed-loop matrix $A_K \doteq A - cGH + BK$ are distinct numbers $\lambda_1, \dots, \lambda_n$ (in conjugate pairs if complex), and define

$$V = [v_1 \quad \dots \quad v_n] \doteq \begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}, \quad D_K \doteq V^{-1}A_KV = \text{diag}\{\lambda_1, \dots, \lambda_n\}, \quad (25)$$

that is, the transformation V takes the matrix A_K into its diagonal form D_K . For future use, consider the following notation for the last column of V^{-1} :

$$V^{-1} = \begin{bmatrix} \dots & \mu_1 \\ \dots & \mu_2 \\ \ddots & \vdots \\ \dots & \mu_n \end{bmatrix}, \quad (26)$$

where, by direct computation of the inverse of the Vandermonde matrix V , the entries μ_i are given by

$$\mu_i = \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1}, \quad i = 1, \dots, n. \quad (27)$$

With V as in (25), consider the bounds (14) and (18) with associated sets of the form (7). From the form of the bounds b_{NL} and b_{LPV} we need to compare the corresponding matrices

$$\Lambda_{NL} = \Lambda + Q = \mathbb{R}e(D_K) + |V^{-1}G|L|H||V|, \quad (28)$$

$$\Lambda_{LPV} = \max_{\rho \in [\rho_1, \rho_2]} \mathcal{M}(V^{-1}(A_K + GH\rho)V) = \mathbb{R}e(D_K) + \max_{\rho \in [\rho_1, \rho_2]} \mathcal{M}(V^{-1}GH\rho V), \quad (29)$$

where we have used (11)–(13), (17) and the fact that D_K defined in (25) is diagonal.

The following result compares the matrices Λ_{NL} and Λ_{LPV} in (28) and (29) and gives conditions for them to be equal.

Theorem 4.1. *For the controlled system (6) with (10), (15) and c given in (22), consider the matrices Λ_{NL} and Λ_{LPV} given in (28)–(29), with V and D_K defined in (25). Then*

$$\Lambda_{NL} \geq \Lambda_{LPV}, \quad (30)$$

with equality if the closed-loop eigenvalues $\lambda_1, \dots, \lambda_n$ are real and

$$\Delta H_i \doteq |H||v_i| - |Hv_i| = 0, \quad i = 1, \dots, n, \quad (31)$$

where v_i are the columns of V and H is the nonlinearity internal matrix defined in (19).

Proof. Using the form of G in (19) and of V and V^{-1} in (25) and (26), we have

$$V^{-1}GHV = \begin{bmatrix} \mu_1 g H v_1 & \mu_1 g H v_2 & \dots & \mu_1 g H v_n \\ \mu_2 g H v_1 & \mu_2 g H v_2 & \dots & \mu_2 g H v_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n g H v_1 & \mu_n g H v_2 & \dots & \mu_n g H v_n \end{bmatrix}. \quad (32)$$

The second term in (28) then satisfies

$$|V^{-1}G|L|H||V| = L \begin{bmatrix} |\mu_1 g||H||v_1| & |\mu_1 g||H||v_2| & \dots & |\mu_1 g||H||v_n| \\ |\mu_2 g||H||v_1| & |\mu_2 g||H||v_2| & \dots & |\mu_2 g||H||v_n| \\ \vdots & \vdots & \ddots & \vdots \\ |\mu_n g||H||v_1| & |\mu_n g||H||v_2| & \dots & |\mu_n g||H||v_n| \end{bmatrix}, \quad (33)$$

that is, the (i, j) entry, $i, j = 1, \dots, n$, has the form (using (24) and the definition of ΔH_i in (31))

$$L|\mu_i g||H||v_j| = \rho_2 |\mu_i g|(|Hv_j| + \Delta H_j). \quad (34)$$

From (32), (23), and using the definition of the operator $\mathcal{M}$ in (12), the second term in (29) has diagonal entries satisfying

$$\begin{aligned} \max_{\rho \in [\rho_1, \rho_2]} \operatorname{Re}(g\mu_i H v_i) \rho &= \begin{cases} \rho_2 \operatorname{Re}(\beta_i) = \rho_2 |\operatorname{Re}(\beta_i)| & \text{if } \operatorname{Re}(\beta_i) \geq 0 \\ \rho_1 \operatorname{Re}(\beta_i) = |\rho_1| |\operatorname{Re}(\beta_i)| & \text{if } \operatorname{Re}(\beta_i) < 0 \end{cases} \\ &= \rho_2 |\operatorname{Re}(\beta_i)| = \rho_2 |\operatorname{Re}(g\mu_i H v_i)| \leq \rho_2 |\mu_i g||Hv_i| = \rho_2 |\beta_i|, \end{aligned} \quad (35)$$

where we have used $|\rho_1| = \rho_2$ and the notation $\beta_i \doteq g\mu_i H v_i$. Similarly, the off-diagonal entries of the second term in (29) have the form (using the definition of the operator $\mathcal{M}$ in (12) and $|\rho_1| = |\rho_2| = \rho_2$)

$$\max\{|\rho_1|, |\rho_2|\} |\mu_i g||Hv_j| = \rho_2 |\mu_i g||Hv_j|. \quad (36)$$

Comparing (34) with (35)–(36) and noting that $\Delta H_i \geq 0$ (see (31)), we conclude that all entries of the matrix in the second term of (28) are greater than or equal to the corresponding entries of the matrix in the second term of (29), proving (30). If the eigenvalues are real (which implies that μ_i in (27) and v_i in (25) are real) and condition (31) holds then (30) holds with equality, thus completing the proof. $\square$

Theorem 4.1 shows that under condition (31), if the closed-loop eigenvalues $\lambda_1, \dots, \lambda_n$ are real, we have

$$\Lambda_{NL} = \Lambda_{LPV} = \Lambda + M, \quad (37)$$

where M has entries

$$m_{ii} = \rho_2 |\operatorname{Re}(\mu_i g H v_i)|, \quad m_{ij} = \rho_2 |\mu_i g| |H v_j|, \quad i, j = 1, \dots, n. \quad (38)$$

From (14) and (18) we then have $b_{NL} = b_{LPV}$.

If condition (31) does not hold and/or the closed-loop eigenvalues are not all real, then $\Lambda_{LPV} = \Lambda + M$ still holds but, from (34)–(36), Λ_{NL} has the general form

$$\Lambda_{NL} = \Lambda_{LPV} + \Gamma, \quad (39)$$

where

$$\Gamma \doteq \rho_2 \operatorname{diag} \{(|\beta_1| - |\operatorname{Re}(\beta_1)|), \dots, (|\beta_n| - |\operatorname{Re}(\beta_n)|)\} + \rho_2 |g| \begin{bmatrix} |\mu_1| \\ |\mu_2| \\ \vdots \\ |\mu_n| \end{bmatrix} [\Delta H_1 \quad \dots \quad \Delta H_n]. \quad (40)$$

We then have the following result relating $(-\Lambda_{LPV})^{-1}$ and $(-\Lambda_{NL})^{-1}$.

Lemma 4.2. *Suppose Λ_{NL} of the form (39) is Hurwitz. Then Λ_{LPV} is Hurwitz and*

$$(-\Lambda_{NL})^{-1} \geq (-\Lambda_{LPV})^{-1}. \quad (41)$$

Proof. Note that Λ_{LPV} is Metzler⁴ and $\Gamma \geq 0$. Then $\Lambda_{NL} = \Lambda_{LPV} + \Gamma$ is also Metzler. Since Λ_{NL} is then Hurwitz (by assumption) and Metzler, and $\Lambda_{LPV} + \Gamma$ is a regular splitting, then part (iii) of Lemma A.3 (see Appendix A at the end of the paper) shows that Λ_{LPV} is Hurwitz and $\Pi = (-\Lambda_{LPV})^{-1} \Gamma$ has spectral radius less than one. Also, Λ_{LPV} Metzler and Hurwitz implies $(-\Lambda_{LPV})^{-1} > 0$ (see part (ii) of Lemma A.3). Thus, $\Pi \geq 0$, which, together with Π having spectral radius less than one imply $(I - \Pi)^{-1} \geq 0$ (see Lemma A.4). We then write, using the matrix inversion lemma

$$(-\Lambda_{NL})^{-1} = (-\Lambda_{LPV} - \Gamma)^{-1} = (-\Lambda_{LPV})^{-1} + (-\Lambda_{LPV})^{-1} \Gamma (I - \Pi)^{-1} (-\Lambda_{LPV})^{-1}. \quad (42)$$

Inequality (41) then follows since the second term on the right hand side above is nonnegative. $\square$

Comparing (14) and (18) we conclude that (41) implies $b_{NL} \geq b_{LPV}$. Hence, in the general case when $\Gamma \neq 0$ in (39)–(40), equal or smaller ultimate-bound invariant sets are obtained using the LPV embedding approach.

Remark 4.3. *The sufficient condition of Theorem 4.1 is not necessary. Indeed, there are cases where assigning some eigenvalues to be complex can still achieve (30) with equality, that is, $\Gamma = 0$ in (39)–(40). For example, consider $n = 4$, $H = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ in (19), and $\lambda_1 = -3.3593 + j$, $\lambda_2 = -3.3593 - j$, $\lambda_3 = -2$, $\lambda_4 = -3$ in (25). With the corresponding V as in (25), and μ_i as defined in (26), it can be verified by direct calculation that condition (31) holds and, for any $g \neq 0$, $|\beta_i| = |g \mu_i H v_i| = |\operatorname{Re}(\beta_i)|$ for $i = 1, \dots, 4$. Thus, $\Gamma = 0$ in (39)–(40) for this particular combination of complex and real eigenvalues. $\circ$*

⁴A real matrix is Metzler if its off-diagonal entries are nonnegative. Metzler matrices are also called *exponentially positive* matrices since Λ is Metzler if and only if $e^{\Lambda t} \geq 0$ for all $t \geq 0$.

5 Sufficient Condition for the Existence of an Invariant Set

We consider only the LPV embedding approach since, as shown above, it gives equal or smaller sets than the NL bounding approach. An ultimate-bound invariant set of the form (7) can be computed with bound $b = b_{LPV}$ given in (18) if the matrix Λ_{LPV} given by (11), (37)–(38) is Hurwitz. We next show that this is always possible to achieve by appropriate assignment of the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A_K = A - cGH + BK$ in (6) so that the magnitude of their (negative) real part is large enough to satisfy a certain condition.

Lemma 5.1. *The matrix Λ_{LPV} given by (11), (37)–(38) is Hurwitz if the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A_K = A - cGH + BK$ in (6) have negative real part and satisfy*

$$\frac{|\operatorname{Re}(\lambda_i)| \prod_{j \neq i} |\lambda_i - \lambda_j|}{\sum_{j=1}^n |Hv_j|} > \rho_2 |g|, \quad i = 1, \dots, n. \quad (43)$$

Proof. Combining (38) and (27) we have that the entries of each i -th row of the matrix M in (37) satisfy

$$\sum_{j=1}^n |m_{ij}| \leq \sigma_i \doteq \rho_2 |g| \left(\prod_{j \neq i} |\lambda_i - \lambda_j| \right)^{-1} \sum_{j=1}^n |Hv_j|, \quad i = 1, \dots, n. \quad (44)$$

We will next use a matrix perturbation result from Chapter 6 of [12] (specifically, Section 6.3), which states that the eigenvalues of the matrix (37)–(38), regarded as a ‘perturbation’ of the diagonal matrix $\Lambda = \operatorname{Re}(D_K)$, with D_K defined in (25), are contained in the discs

$$\left\{ z \in \mathbb{C} : |z - \operatorname{Re}(\lambda_i)| \leq \sum_{j=1}^n |m_{ij}| \right\}, \quad i = 1, \dots, n. \quad (45)$$

From (44)–(45) it follows that a sufficient condition for the ‘perturbed’ matrix (37)–(38) to be Hurwitz, that is, for its eigenvalues to have negative real part, is $\operatorname{Re}(\lambda_i) < -\sigma_i$, for $i = 1, \dots, n$, which is equivalent to (43) if $\operatorname{Re}(\lambda_i) < 0$. $\square$

We next select a seed eigenvalue configuration $\{c_i\}_{i=1}^n$ of n distinct and stable eigenvalues where complex eigenvalues appear in complex conjugate pairs, and then define the closed-loop eigenvalues as scaled versions of this seed configuration. More precisely, we have the following definition.

Definition 5.2. *Let $c_i \in \mathbb{C}$, $i = 1, \dots, n$, satisfy $c_i \neq c_j$ for $i \neq j$, $\operatorname{Re}(c_i) < 0$, and if $c_i \notin \mathbb{R}$, then either $c_{i-1} = \bar{c}_i$ or $c_{i+1} = \bar{c}_i$. We then define the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A_K = A - cGH + BK$ in (25) as $\lambda_i = \kappa c_i$, where $\kappa > 0$ is a scaling factor.* $\circ$

Theorem 5.3. *Suppose the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A_K = A - cGH + BK$ in (6) are scaled as in Definition 5.2. Then there exists $\kappa^* > 0$ such that for all $\kappa > \kappa^*$ the matrix Λ_{LPV} in (37)–(38) is Hurwitz and hence an ultimate-bound invariant set of the form (7) with $b = b_{LPV}$ given in (18) can be computed for the nonlinear closed-loop system (6).*

Proof. Using the eigenvalue scaling described in the statement of the theorem and noting from (19) and (25) that $Hv_j = h_1 + h_2 \lambda_j + \dots + h_n \lambda_j^{n-1}$, we have that the fraction on the left hand side of (43) satisfies

$$\frac{|\operatorname{Re}(\lambda_i)| \prod_{j \neq i} |\lambda_i - \lambda_j|}{\sum_{j=1}^n |Hv_j|} = \frac{\kappa |\operatorname{Re}(c_i)| \kappa^{n-1} \prod_{j \neq i} |c_i - c_j|}{\sum_{j=1}^n |h_1 + h_2 c_j \kappa + \dots + h_n (c_j \kappa)^{n-1}|}. \quad (46)$$

For large κ the denominator of the last fraction in (46) grows at most as κ^{n-1} and the numerator grows proportionally to κ^n . The fraction then grows monotonically to infinity when $\kappa \rightarrow \infty$ and, hence, condition (43) can always be achieved by taking κ large enough, that is, there exists $\kappa^* > 0$ such that condition (43) will be satisfied for all $\kappa > \kappa^*$ in the eigenvalue assignment of Definition 5.2. The result then follows from Lemma 5.1 and the invariant set computation techniques in [9]. $\square$

6 Arbitrarily Small Ultimate Bounds

In this section we show that when the disturbances are ‘aligned’ with the control input, i.e., $E = Be = [0 \ \dots \ 0 \ e]^T$, for some constant $e \neq 0$, and considering the eigenvalue scaling of Definition 5.2, then the ultimate bounds on the state can be made as small as desired by increasing the scaling factor κ .

To see this, we first note that using the eigenvalue scaling of Definition 5.2, the transformation matrix V in (25) satisfies

$$V = \text{diag}\{1, \kappa, \dots, \kappa^{n-1}\}V_c \implies V^{-1} = V_c^{-1}\text{diag}\{1, 1/\kappa, \dots, 1/\kappa^{n-1}\}, \quad (47)$$

where

$$V_c \doteq \begin{bmatrix} 1 & \dots & 1 \\ c_1 & \dots & c_n \\ \vdots & \ddots & \vdots \\ c_1^{n-1} & \dots & c_n^{n-1} \end{bmatrix}. \quad (48)$$

Also, from (37)–(38), (25) and (27), an asymptotic analysis for large κ as performed in the proof of Theorem 5.3 shows that $\Lambda = \mathbb{R}e(D_K)$ will dominate over M , and thus the inverse of $\Lambda_{LPV} = \Lambda + M$ will asymptotically behave like $\Lambda^{-1} = \text{diag}\{1/\mathbb{R}e(c_1), \dots, 1/\mathbb{R}e(c_n)\}/\kappa < 0$. We then have from (18) that the ultimate bound (9) for $|x|$ satisfies

$$|x| \leq |V|b_{LPV} = |V|(-\Lambda_{LPV})^{-1}|V^{-1}E|\bar{w} \xrightarrow{\kappa \rightarrow \infty} b_x \doteq |V|(-\Lambda)^{-1}|V^{-1}E|\bar{w}, \quad (49)$$

and, using (47), (48) and the form of E , yields

$$\begin{aligned} b_x &= \underbrace{\text{diag}\{1, \kappa, \dots, \kappa^{n-1}\}|V_c|}_{|V|} \underbrace{\frac{1}{\kappa} \text{diag}\{1/|\mathbb{R}e(c_1)|, \dots, 1/|\mathbb{R}e(c_n)|\}}_{-\Lambda^{-1}} \underbrace{\left|V_c^{-1} \begin{bmatrix} 0 \\ \vdots \\ e/\kappa^{n-1} \end{bmatrix}\right|}_{|V^{-1}E|} \bar{w}, \\ &= \text{diag}\left\{\frac{1}{\kappa^n}, \frac{1}{\kappa^{n-1}}, \dots, \frac{1}{\kappa}\right\} \underbrace{|V_c| \text{diag}\{1/|\mathbb{R}e(c_1)|, \dots, 1/|\mathbb{R}e(c_n)|\}}_{\doteq w_c} \left|V_c^{-1} \begin{bmatrix} 0 \\ \vdots \\ e \end{bmatrix}\right| \bar{w}, \end{aligned} \quad (50)$$

where w_c is a constant vector independent of κ . Thus, all components of the ultimate bound for $|x|$ decrease at least as $1/\kappa$.

7 Un-matched Nonlinearities

We briefly discuss the case when the nonlinearity in the model is not matched with the control input. To this end, we consider that the matrix G in (1) has a single nonzero entry but instead of it being the last (n -th) entry as in (19), we let $G = [0 \ \dots \ g \ \dots 0]^T$, with $g \neq 0$ in the r -th entry, with $1 \leq r \leq n-1$.

Consider for simplicity the eigenvalue scaling as in Definition 5.2, and the factorisation of V given in (47), (48). Let $[\nu_{r,1} \ \nu_{r,2} \ \dots \ \nu_{r,n}]^T$ be the r -th column of V_c^{-1} . Proceeding as in the proof of Theorem 4.1, we find that Λ_{LPV} in (29) can be expressed as

$$\Lambda_{LPV} = \Lambda + M_r, \quad (51)$$

where M_r has entries

$$m_{r,ii} = \rho_2 |\operatorname{Re}(\nu_{r,i} g H v_i)| / \kappa^{r-1}, \quad m_{r,ij} = \rho_2 |\nu_{r,i} g| |H v_j| / \kappa^{r-1}, \quad i, j = 1, \dots, n, \quad (52)$$

and $H v_j = h_1 + h_2 c_j \kappa + \dots + h_n c_j^{n-1} \kappa^{n-1}$. Using, as in Lemma 5.1, the fact that the eigenvalues of Λ_{LPV} are contained in discs of the form (45), with $m_{r,ij}$ replacing m_{ij} , a sufficient condition for Λ_{LPV} to be Hurwitz is

$$\begin{aligned} \kappa |\operatorname{Re}(c_i)| &> \rho_2 |\nu_{r,i} g| / \kappa^{r-1} \sum_{j=1}^n |h_1 + h_2 c_j \kappa + \dots + h_n c_j^{n-1} \kappa^{n-1}| \iff \\ |\operatorname{Re}(c_i)| / (\rho_2 |\nu_{r,i} g|) &> \sum_{j=1}^n |h_1 + h_2 c_j \kappa + \dots + h_n c_j^{n-1} \kappa^{n-1}| / \kappa^r, \quad i = 1, \dots, n. \end{aligned} \quad (53)$$

Since for un-matched nonlinearities $r \leq n-1$ then the right hand side of (53) cannot be made arbitrarily small by increasing κ unless $h_j = 0$ for $j = r+1, \dots, n$.

8 Examples

8.1 Pendulum equation

We illustrate the results of Section 4 on achieving equal ultimate-bound invariant sets with the nonlinearity bounding and the LPV embedding methods, through an example of a pendulum system of the form (1) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ (g_r/\ell) \end{bmatrix}, \quad H = [1 \ 0],$$

where g_r and ℓ are the gravity acceleration constant and the length of pendulum respectively. The states of the system, x_1 and x_2 , are the angle and angular velocity respectively. Note that the system matrices are already in the canonical form (19). The nonlinear function in the model is $f(Hx) = f(x_1) = \sin x_1$. By direct inspection, for $f(x_1) = \sin x_1$ the parameter $r = \sin x_1 / x_1$ defined in (2) satisfies (3) with $r_1 = -0.22$ and $r_2 = 1$. Following (22) we take $c = -(r_1 + r_2)/2 = -0.3900$, which yields $\rho_2 = r_2 + c = 0.61$.

We see that condition (31) holds for this system since $H v_i = 1$ for $i = 1, 2$ (cf. the form of H above and of V in (25)). Then, Theorem 4.1 shows that the assignment of real eigenvalues

Figure 1: Invariant sets for the pendulum system.

yields $\Lambda_{NL} = \Lambda_{LPV}$ and hence both methods give the same ultimate-bound invariant set of the form (7). To select the eigenvalues, we let $c_1 = -1$, $c_2 = -2$ and use the scaling of Definition 5.2 by letting $\lambda_i = \kappa c_i$, $i = 1, 2$, where $\kappa > 0$. To compute the ultimate-bound invariant set, we use Lemma 5.1 and Theorem 5.3 to find a value of κ such that condition (43) holds. The latter condition takes the form $\kappa^2 > 2\rho_2|g_r/\ell|$ for $i = 1$ and $\kappa^2 > \rho_2|g_r/\ell|$ for $i = 2$. For $g_r = 9.8\text{ms}^{-2}$ and $\ell = 0.3\text{m}$, $\kappa = 8$ satisfies both inequalities and thus the linear feedback gain $K = -[140.7400 \quad 24.00]$, which assigns the eigenvalues of $A_K = A - cGH + BK$ to $\lambda_1 = -8$ and $\lambda_2 = -16$, ensures closed-loop stability of the pendulum system and the possibility of computing the ultimate-bound invariant set (7). To obtain this set, we use V of the form (25) with the selected eigenvalues, and then we compute $\Lambda_{LPV} = \Lambda_{NL}$ from (17) and $b_{LPV} = b_{NL}$ from (18) (taking $\bar{w} = 1$) as

$$\Lambda_{LPV} = \begin{bmatrix} -5.5092 & 2.4908 \\ 2.4908 & -13.5092 \end{bmatrix}, \quad b_{LPV} = \begin{bmatrix} 0.0293 \\ 0.0147 \end{bmatrix}.$$

The resulting set $\mathcal{S}_{LPV} = \mathcal{S}_{NL}$ of the form (7) is shown in the left plot of Figure 1. For comparison, we also assigned complex eigenvalues $\lambda_i = 8c_i$, for $i = 1, 2$, where $c_1 = (-1+j)\sqrt{2}/2$ and $c_2 = (-1-j)\sqrt{2}/2$, and computed the sets $\mathcal{S}_{NL}$ (using (14)) and $\mathcal{S}_{LPV}$ (using (17) and (18)). In this case $\mathcal{S}_{NL}$ contains $\mathcal{S}_{LPV}$, as depicted in the right plot of Figure 1.

8.2 Josephson junction

We consider a controlled shunted linear resistive-capacitive-inductive Josephson junction model, known as an RCLSJ model, adapted from [7] and described by the following equations:

$$\dot{\xi} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\gamma/\beta_C & 1/\beta_C \\ 0 & 1/\beta_L & -1/\beta_L \end{bmatrix}}_A \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_B u + \underbrace{\begin{bmatrix} 0 \\ -1/\beta_C \\ 0 \end{bmatrix}}_G f(\underbrace{[1 \ 0 \ 0]}_{\bar{H}} \xi) + \underbrace{\begin{bmatrix} 0 \\ 1/\beta_C \\ 0 \end{bmatrix}}_{\bar{E}} \underbrace{i_0}_w, \quad (54)$$

Figure 2: Time evolution of the Josephson system states (top) and corresponding second-third state trajectory on the ξ_2 - ξ_3 plane (bottom), for $u = 0$ and the initial condition $\xi = [1 \ 3 \ 0]^T$.

where $\beta_C = 0.707$, $\beta_L = 2.6$, $\gamma = 0.2135$ (taken as an average of the 2 values in the switched resistance model used in [7]), $i_0 = 1.2$ and $f(\bar{H}\xi) = f(\xi_1) = \sin \xi_1$. Loosely speaking, β_C , β_L and γ are normalised capacitance, inductance and conductance, respectively, and i_0 is a normalised external dc current. We refer the reader to [7] for the modelling details and precise physical interpretation of the parameters.

Without control ($u = 0$), the above model and parameters exhibit bounded oscillations in the ξ_2 - ξ_3 variables while the ξ_1 variable shows an oscillation mounted on a ramp; these trajectories are illustrated in Figure 2 for a particular initial condition (other parameter values can cause chaotic behaviour, see e.g., [7]). We will apply the proposed methodology to design a linear feedback gain that can stabilise the system to an (arbitrarily small) invariant set around the origin of the state space.

We first transform the system (54) to the canonical form (6)–(19) via $\xi = Ux$ with

$$U = \begin{bmatrix} 0.3846 & 1 & 0 \\ 0 & 0.3846 & 1 \\ 0 & 0.3846 & 0 \end{bmatrix}. \quad (55)$$

This yields

$$A = U^{-1}\bar{A}U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.6602 & -0.6866 \end{bmatrix}, \quad B = U^{-1}\bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$G = (-1/\beta_C)B, \quad E = (1/\beta_C)B, \quad H = \bar{H}U = [0.3846 \ 1 \ 0].$$

Note that, since $h_1 > 0$, $h_2 > 0$ and each v_i in (25) has a negative second element $\lambda_i < 0$, then condition (31) does not hold and hence the LPV embedding approach will give equal or tighter sets than the nonlinearity bounding method.

By direct inspection, for $f(\xi_1) = \sin \xi_1$ the parameter r defined in (2) satisfies (3) with $r_1 = -0.22$ and $r_2 = 1$. Following (22) we take $c = -(r_1 + r_2)/2 = -0.3900$, which yields $\rho_2 = r_2 + c = 0.61$.

Figure 3: LHS and RHS of inequality (43) as a function of $|\lambda|$, for $\lambda_1 = \lambda < 0$, $\lambda_2 = 2\lambda$, $\lambda_3 = 4\lambda$.

The next step is to choose the eigenvalues λ_1 , λ_2 and λ_3 of $A_K = A - cGH + BK$ —and design the corresponding gain K —such that condition (43) is satisfied. We take $\lambda_1 = \lambda < 0$, $\lambda_2 = 2\lambda$ and $\lambda_3 = 4\lambda$. The left hand side (LHS) and right hand side (RHS) of inequality (43), for $i = 1, 2, 3$, as a function of $|\lambda|$ are plotted in Figure 3. Taking $\lambda = -1.33$ condition (43) is satisfied, and thus a linear feedback gain K that assigns the eigenvalues of $A_K = A - cGH + BK$ to $\lambda_1 = -1.33$, $\lambda_2 = -2.66$ and $\lambda_3 = -5.32$ ensures closed-loop stability of the controlled Josephson system and the possibility of computing an invariant set and ultimate bounds using the proposed LPV embedding method. The gain that assigns the desired eigenvalues is

$$K = - \begin{bmatrix} 48.3832 & 8.6234 & -73.1825 \end{bmatrix},$$

and the transformation matrix (25) takes the form

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -1.33 & -2.66 & -5.32 \\ 1.7689 & 7.0756 & 28.3024 \end{bmatrix}. \quad (56)$$

The resulting closed-loop system's trajectories starting from the initial condition $\xi = [1 \ 3 \ 0]^T$ and with $u = 0$ until $t = 20$, time when the linear controller $u = K\xi$ is switched on, are shown in Figure 4. The top plot shows all three states as a function of time with a zoom showing the convergence to the ultimate bounds. The bottom plot shows the ξ_2 , ξ_3 state trajectory, with a zoom showing the convergence of the trajectory inside a polytopic set centred at zero. This set is the projection onto the ξ_2 - ξ_3 plane of an ultimate-bound invariant set $\tilde{S}$ of the form (20)

To obtain this set, we start with a set of the form (7) (in the ‘canonical’ x -coordinates) and compute Λ_{LPV} from (17) and b_{LPV} from (18) as

$$\Lambda_{LPV} = \begin{bmatrix} -1.1763 & 0.3699 & 0.8024 \\ 0.2306 & -2.1051 & 1.2036 \\ 0.0769 & 0.1850 & -4.9188 \end{bmatrix}, \quad b_{LPV} = \begin{bmatrix} 0.4005 \\ 0.3004 \\ 0.0501 \end{bmatrix}.$$

Figure 4: Closed-loop time evolution of the Josephson system states (top) and corresponding second-third state trajectory on the ξ_2 - ξ_3 plane (bottom), for $u = K\xi$ switched on at $t = 20$, and the initial condition $\xi = [1 \ 3 \ 0]^T$.

Substituting, in (20), U from (55), V from (56) and $b = b_{LPV}$ as given above yields the ultimate-bound invariant set $\bar{S}$ in the original coordinates. The bound (21) is given by

$$|\xi| \leq \bar{b} \doteq |UV|b_{LPV} = \begin{bmatrix} 1.3093 \\ 3.6364 \\ 0.6147 \end{bmatrix},$$

which, although verified in the simulation (see the zoomed area in Figure 4), suffers from conservatism due to the double transformation UV . Note that $\bar{b}$ scales with the magnitude of the excitation signal $\bar{w} = i_0$.

As shown in Section 6, the ultimate bound $\bar{b}$ can be made as small as desired by increasing the seed-eigenvalue-magnitude $|\lambda|$. Figure 5 confirms this property via a plot of the components of the ultimate bound vector $\bar{b}$ as a function of $|\lambda|$ in logarithmic scale. Thus, by choosing a larger value of $|\lambda|$ smaller ultimate bound estimates are achieved, albeit at the cost of higher feedback gains K . This will cause more aggressive control actions leading to large transients before the trajectories finally decay towards the ultimate bound set. Indeed, note that the first state's magnitude can be considerably large (as seen, for example, from the top plots of Figures 2 and 4) if the controller is applied after a certain time of operation in open loop, and any controller that feeds back this state proportionally, especially if the gain is large, will cause (bounded but) large transient responses.

We observe that other controllers which have been proposed for the Josephson junction system, such as the backstepping designs of [11, 27], only feed back the first state through the bounded nonlinear function $f(\xi_1)$ and, although they thus avoid the large transients, they do not regulate the first state, which remains fixed at a large value. Moreover, these backstepping designs rely on exact cancellation of the system nonlinearities and of the excitation current i_0 and do not guarantee any performance if these terms are not exactly known. Our design, on the other hand, achieves regulation of all three system's states without cancellation, is valid for any bounded

Figure 5: Components of the ultimate bound vector $\bar{b}$ as a function of $|\lambda|$ in logarithmic scale.

excitation current i_0 (not necessarily constant) and applies, in fact, to any uncertain nonlinearity as long as it satisfies the LPV parameter bounds.

8.3 Chua circuit

Consider the Chua controlled circuit as described in [10]:

$$\dot{\xi} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -\alpha_1 & \alpha_1 & 0 \\ \alpha_2 & -\alpha_2 & R\alpha_2 \\ 0 & -\alpha_3 & -R_0\alpha_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (u + w) + \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} f(\xi_1), \quad (57)$$

where ξ_1 and ξ_2 are the voltages across the capacitors C_1 and C_2 , ξ_3 is the current through the inductance L , R_0 and R are linear resistors, $\alpha_1 = 1/(RC_1)$, $\alpha_2 = 1/(RC_2)$, $\alpha_3 = 1/L$, $g = -R\alpha_1$ and $f(\xi_1)$ is the current through a nonlinear resistor, which is a piecewise affine function expressed as

$$f(\xi_1) = \begin{cases} \gamma_b \xi_1 + (\gamma_a - \gamma_b) & \text{if } \xi_1 > 1. \\ \gamma_a \xi_1 & \text{if } |\xi_1| \leq 1, \\ \gamma_b \xi_1 - (\gamma_a - \gamma_b) & \text{if } \xi_1 < -1. \end{cases} \quad (58)$$

The numerical values of the parameters are $\alpha_1 = 10$, $\alpha_2 = 1$, $R = 1$, $\alpha_3 = 15$, $R_0 = 0.0385/15$, $\gamma_b = -0.69$, $\gamma_a = -1.28$. The input disturbance w is assumed bounded as $|w| \leq \bar{w} = 0.1$.

Without control ($u = 0$), the above model and parameters exhibit chaotic trajectories, as illustrated in Figure 6 for the initial condition $\xi = 0.15 \times [1 \ 1 \ 1]^T$. For the simulation the input disturbance w was taken as a band-limited white noise with amplitude saturated at $\bar{w}$. We will apply the proposed methodology to design a linear feedback gain that can stabilise the system to an (arbitrarily small) invariant set around the origin of the state space.

Figure 6: Time evolution of the Chua system states (top) and corresponding first-second state trajectory on the ξ_1 - ξ_2 plane (bottom), for $u = 0$, $|w| \leq 0.1$, and the initial condition $\xi = 0.15 \times [1 \ 1 \ 1]^T$.

It can be readily verified that the transformation $\xi = Ux$ with

$$U = \begin{bmatrix} (R + R_0)\alpha_2\alpha_3 & \alpha_2 + R_0\alpha_3 & 1 \\ R_0\alpha_2\alpha_3 & \alpha_2 & 0 \\ -\alpha_2\alpha_3 & 0 & 0 \end{bmatrix} \quad (59)$$

takes the system (57) to the canonical form (19), i.e.,

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -R\alpha_1\alpha_2\alpha_3 & -R_0\alpha_1\alpha_3 - (R + R_0)\alpha_2\alpha_3 & -\alpha_1 - \alpha_2 - R_0\alpha_3 \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B=E} (u+w) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}}_G f(Hx), \quad (60)$$

where $H = [1 \ 0 \ 0]U = [(R + R_0)\alpha_2\alpha_3 \ \alpha_2 + R_0\alpha_3 \ 1]$. Note that, since $H > 0$ and each v_i in (25) has a negative second element $\lambda_i < 0$, then condition (31) does not hold and hence the LPV embedding approach will give tighter sets than the nonlinearity bounding method.

Defining the time-varying parameter r as in (2) for the nonlinear function (58) we have, by direct calculation, that r satisfies (3) with $r_1 = \gamma_a$ and $r_2 = \gamma_b$. Following (22) we take $c = -(r_1 + r_2)/2 = 0.985$, which yields $\rho_2 = r_2 + c = 0.295$.

We next choose the eigenvalues λ_1 , λ_2 and λ_3 of $A_K = A - cGH + BK$ —and design the corresponding gain K —such that condition (43) is satisfied. We take $c_1 = -1$, $c_2 = (-1+j)\sqrt{2}/2$ and $c_3 = (-1-j)\sqrt{2}/2$ and use the scaling of Definition 5.2 by letting $\lambda_i = \kappa c_i$, $i = 1, 2, 3$, where $\kappa > 0$. The left hand side (LHS) and right hand side (RHS) of inequality (43), for $i = 1, 2, 3$, as a function of κ are plotted in Figure 7 (note that the plots for $i = 2$ and $i = 3$ coincide). By taking $\kappa = 14.6$ condition (43) is satisfied, and thus the linear feedback gain

$$K = -[34.0590 \quad 474.0492 \quad -171.9879],$$

Figure 7: LHS and RHS of inequality (43) as a function of κ , for $\lambda_i = \kappa c_i$, $i = 1, 2, 3$.

which assigns the eigenvalues of $A_K = A - cGH + BK$ as described above ensures closed-loop stability of the controlled Chua system and the possibility of computing an ultimate-bound invariant set using the proposed LPV embedding method. The transformation matrix (25) associated with the assigned eigenvalues is

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -14.60 & -10.32(1-j) & -10.32(1+j) \\ 213.16 & -213.16i & 213.16i \end{bmatrix}. \quad (61)$$

To obtain an ultimate-bound invariant set of the form (20), we compute Λ_{LPV} from (17) and b_{LPV} from (18) as

$$\Lambda_{LPV} = \begin{bmatrix} -9.5670 & 4.7838 & 4.7838 \\ 2.7239 & -9.2822 & 2.5890 \\ 2.7239 & 2.5890 & -9.2822 \end{bmatrix}, \quad b_{LPV} = \begin{bmatrix} 0.2504 \\ 0.1666 \\ 0.1666 \end{bmatrix} \times 10^{-3}.$$

Substituting, in (20), U from (59), V from (61) and $b = b_{LPV}$ as given above yields an ultimate-bound invariant set in the original coordinates $\bar{S}$ of the form (20).

The ultimate bound (21) is given by

$$|\xi| \leq \bar{b} \doteq |UV|b_{LPV} = \begin{bmatrix} 0.1208 \\ 0.0085 \\ 0.0088 \end{bmatrix}.$$

The resulting closed-loop system's trajectories starting from the initial condition $\xi = 0.15 \times [1 \ 1 \ 1]^T$ and with $u = 0$ until $t = 40$, time when the linear controller $u = K\xi$ is switched on, are shown in Figure 8. The top plot shows all three states as a function of time with a zoom showing the convergence to the ultimate bounds. The bottom plot shows the ξ_1, ξ_2 state trajectory, with a zoom showing the convergence of the trajectory inside a polytopic set centred at zero. This set is the projection onto the ξ_1 - ξ_2 plane of a bounding box for the ultimate-bound invariant set $\bar{S}$ described above.

Figure 8: Closed-loop time evolution of the Chua system states (top) and corresponding first-second state trajectory on the ξ_1 - ξ_2 plane (bottom), for $u = K\xi$ switched on at $t = 40$, and the initial condition $\xi = 0.15 \times [1 \ 1 \ 1]^T$.

9 Conclusions

In this paper we have addressed the problem of trajectory boundedness and the computation of ultimate bounds and invariant sets for Lure-type nonlinear systems with a sector bounded nonlinearity. We first compared two systematic methods to compute ultimate-bound invariant sets for Lure systems, namely, a ‘nonlinearity bounding’ method and an ‘LPV-embedding’ method. We showed that under some conditions on the system matrices these approaches can give identical sets, the LPV-embedding method being less conservative in the general case. Secondly, we characterised a class of Lure systems for which an appropriately designed linear state feedback can achieve bounded trajectories of the closed-loop nonlinear system and allow for the computation of an ultimate-bound invariant set via a closed-form expression. Finally, we showed that for disturbances that are ‘aligned’ with the control input, arbitrarily small ultimate bounds on the system states can be achieved by assigning eigenvalues with ‘large enough’ negative real part. We illustrated the results via examples of a pendulum system, a Josephson junction circuit and the Chua circuit. Future work includes the extension of the results to more general nonlinearities, and the application of the proposed techniques in model predictive control and fault tolerant control.

A Some Results on Metzler and Nonnegative Matrices

Theorem A.1 (Perron Frobenius, see e.g., [12, 18]). *Let $\Gamma \in \mathbb{R}^{n \times n}$ be nonnegative. Then there exist (nonzero) eigenvectors $v \geq 0$ and $w \geq 0$ such that $\Gamma v = \text{sr}(\Gamma)v$ and $\Gamma^T w = \text{sr}(\Gamma)w$, where $\text{sr}(\Gamma)$ is the spectral radius of Γ , i.e., the largest magnitude of the eigenvalues of Γ . $\circ$*

Theorem A.2 (Spectral bound property, see e.g., [18]). *Let $\Lambda \in \mathbb{R}^{n \times n}$ be Metzler. Then there*

exist (nonzero) eigenvectors $v \geq 0$ and $w \geq 0$ such that $\Lambda v = \text{sb}(\Lambda)v$ and $\Lambda^T w = \text{sb}(\Lambda)w$, where $\text{sb}(\Lambda)$ is the spectral bound of Λ , i.e., the largest real part of the eigenvalues of Λ . $\circ$

The following result is adapted from [6], see also [23].

Lemma A.3 (Properties of Metzler matrices). *Let $\Lambda \in \mathbb{R}^{n \times n}$ be Metzler, then the following are equivalent:*

- (i) Λ is Hurwitz;
- (ii) $-\Lambda^{-1} > 0$;
- (iii) if $\Lambda = \bar{\Lambda} + \Gamma$, where $\bar{\Lambda}$ is Metzler and $\Gamma \geq 0$ (a ‘regular splitting’ of Λ), then $\bar{\Lambda}$ is Hurwitz and $-\bar{\Lambda}^{-1}\Gamma$ has spectral radius less than one.

Proof. We only show the implications that are used in the paper.

[(i) $\implies$ (ii)] Let $y > 0$. Since Λ is Hurwitz we can define $x = \int_0^\infty e^{\Lambda t} y dt > 0$. Then

$$\Lambda x = \Lambda \int_0^\infty e^{\Lambda t} y dt = \int_0^\infty (d e^{\Lambda t} / dt) y dt = -y.$$

Hence $\Lambda x = -y$ is solvable for all $y > 0$ with solution $x > 0$, which implies that Λ is nonsingular and $-\Lambda^{-1} > 0$.

[(i), (ii) $\implies$ (iii)] Let $x > 0$ be as above, so that

$$\Lambda x = \bar{\Lambda} x + \Gamma x < 0. \tag{62}$$

Since $\Gamma x \geq 0$ then necessarily $\bar{\Lambda} x < 0$. Since, by assumption, $\bar{\Lambda}$ is Metzler, then from Theorem A.2, $\bar{\Lambda}^T w = \text{sb}(\bar{\Lambda})w$, with $w \geq 0$. Then $0 > w^T \bar{\Lambda} x = w^T x \text{sb}(\bar{\Lambda})$. Since $w^T x > 0$ then necessarily $\text{sb}(\bar{\Lambda}) < 0$ and hence $\bar{\Lambda}$ is Hurwitz. $\bar{\Lambda}$ Metzler and Hurwitz implies $-\bar{\Lambda}^{-1} > 0$ from (ii) above. Then, premultiplying (62) by $-\bar{\Lambda}^{-1} > 0$ yields

$$-\bar{\Lambda}^{-1} \Lambda x = (-I - \bar{\Lambda}^{-1} \Gamma) x = (\Pi - I) x < 0,$$

where $\Pi \doteq -\bar{\Lambda}^{-1} \Gamma \geq 0$. By the Perron-Frobenius Theorem A.1, $\Pi^T w = \text{sr}(\Pi)w$, with $w \geq 0$. Then, $0 > w^T (\Pi - I) x = w^T x [\text{sr}(\Pi) - 1]$. Since $w^T x > 0$ then necessarily $\text{sr}(\Pi) < 1$ and the result follows. $\square$

Lemma A.4 (Lemma 2.1 in [4]). *The nonnegative matrix $\Pi \in \mathbb{R}^{n \times n}$ is convergent, that is, the spectral radius of Π is less than one, if and only if $(I - \Pi)^{-1}$ exists and*

$$(I - \Pi)^{-1} = \sum_{k=0}^{\infty} \Pi^k \geq 0. \tag{63}$$

$\circ$

References

- [1] T. Alamo, A. Cepeda, M. Fiacchini, and E.F. Camacho. Convex invariant sets for discrete-time Lur'e systems. *Automatica*, 45:1066–1071, 2009.
- [2] T. Alamo, A. Cepeda, and D. Limon. Improved computation of ellipsoidal invariant sets for saturated control systems. In *Proceedings of the 44th IEEE Conference on Decision and Control and 2005 European Control Conference*, Seville, Spain, December 2005.
- [3] Z. Artstein and S. V. Raković. Feedback and invariance under uncertainty via set-iterates. *Automatica*, 44(2):520–525, 2008.
- [4] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM. Society for Industrial and Applied Mathematics, Philadelphia, 1994.
- [5] F. Blanchini and S. Miani. *Set-theoretic methods in control*. Birkhauser, 2007.
- [6] T. Damm. Applied Linear Algebra. Lecture notes on a course held in Padova. Preprint, 2008.
- [7] S.K. Dana, D.C. Sengupta, and K.D. Edoh. Chaotic dynamics in Josephson junction. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 48(8):990–996, 2001.
- [8] G.C. Goodwin, M.M. Seron, and J.A. De Doná. *Constrained Control and Estimation. An Optimisation Approach*. Springer, 2005.
- [9] H. Haimovich and M.M. Seron. Bounds and invariant sets for a class of switching systems with delayed-state-dependent perturbations. *Automatica*, 49(3):748–754, 2013. Extended version available at <http://arxiv.org/abs/1202.0455>.
- [10] L-Y. Hao and G-H. Yang. Fault tolerant control for a class of uncertain chaotic systems with actuator saturation. *Nonlinear dynamics*, 73:2133–2147, 2013.
- [11] A.M. Harb and B.A. Harb. Controlling chaos in Josephson-junction using nonlinear backstepping controller. *IEEE Transactions on Applied Superconductivity*, 16(4):1988–1998, 2006.
- [12] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
- [13] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, third edition, 2002.
- [14] E. Kofman, H. Haimovich, and M.M. Seron. A systematic method to obtain ultimate bounds for perturbed systems. *International Journal of Control*, 80(2):167–178, 2007.
- [15] E. Kofman, M.M. Seron, and H. Haimovich. Control design with guaranteed ultimate bound for perturbed systems. *Automatica*, 44(7):1815–1821, 2008.
- [16] I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4:317–367, 1998.
- [17] D. Luenberger. Canonical forms for linear multivariable systems. *IEEE Transactions on Automatic Control*, 12:290–293, 1967.

- [18] D.G. Luenberger. *Introduction to dynamic systems: theory, models and applications*. John Wiley & Son, 1979.
- [19] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.
- [20] S. Olaru, J.A. De Doná, M.M. Seron, and F. Stoican. Positive invariant sets for fault tolerant multisensor control schemes. *International Journal of Control*, 83(12):2622–2640, 2010.
- [21] S.V. Raković, E.C. Kerrigan, K.I. Kouramas, and D.Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.
- [22] J.B. Rawlings and D.Q. Mayne. *Model predictive control: Theory and design*. Nob Hill Publishing, 2009.
- [23] H. Schneider. Positive operators and an inertia theorem. *Numerische Mathematik*, 7:11–17, 1965.
- [24] M.M. Seron, J.A. De Doná, and J. Richter. Invariant-set-based fault diagnosis in Lure systems. *International Journal of Robust and Nonlinear Control*, 24(16):2405–2422, November 2014.
- [25] M.M. Seron, X.W. Zhuo, J.A. De Doná, and J.J. Martínez. Multisensor switching control strategy with fault tolerance guarantees. *Automatica*, 44(1):88–97, 2008.
- [26] S. Tarbouriech, I. Queinnec, T. Alamo, Fiacchini, and E.F. Camacho. Ultimate bounded stability and stabilization of linear systems interconnected with generalized saturated functions. *Automatica*, 47:1473–1481, 2011.
- [27] A. Ucar U.E. Vincent and, J.A. Laoye, and S.O. Kareem. Control and synchronization of chaos in RCL-shunted Josephson junction using backstepping design. *Physica C*, pages 374–382, 2008.