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Approaching the Capacity of AWGN Channels using Multi-Layer Raptor Codes and Superposition Modulation

Mahyar Shirvanimoghaddam and Sarah J. Johnson

School of Electrical Engineering and Computer Science, The University of Newcastle, NSW, Australia

e-mail: {mahyar.shirvanimoghaddam; sarah.johnson}@newcastle.edu.au.

Abstract—We propose a capacity approaching coding strategy for additive white Gaussian noise (AWGN) channels by using multi-layer Raptor codes and superposition modulation. Each AWGN channel is divided into several binary-input AWGN (BI-AWGN) channels at a very low signal to noise ratio (SNR), where a capacity approaching Raptor code can be used to encode the message over each layer. A single capacity approaching degree distribution is then used for the Raptor codes over all the layers. A well-known multi-stage decoder is used for successive interference cancellation and decoding the multi-layer Raptor code, where each decoding stage is modeled by a BI-AWGN channel of a fixed SNR. This allows the development of a capacity approaching code for an AWGN channel at every SNR by using a single Raptor code with a fixed degree distribution as the component code.

Index Terms—multi-stage decoder, Raptor codes, superposition modulation, successive interference cancellation.

I. INTRODUCTION

Superposition modulation (SM) is a low complexity alternative to conventional mapping schemes like amplitude shift keying (ASK), quadrature amplitude modulation (QAM), and phase shift keying (PSK), which can provide Gaussian like signals by superimposing weighted antipodal symbols, without signal shaping [1]. A SM scheme is constructed by multiple superimposed layers, where each layer is protected by a near-capacity binary channel code and the coded layers are appropriately weighted by the layer specific amplitude scaling factor before their linear superposition takes place [2]. The Gaussian distributed channel input of SM is desirable from an information theoretic point of view, as it is capable of approaching the Shannon capacity and achieving the shaping gain, which is the mutual information difference between a Gaussian input and the discrete conventional modulation input system [2]. The main challenge to guaranteeing near capacity performance of SM is the design of capacity approaching codes for each layer of SM. In fact, in SM the channel code and modulation strategy should be jointly designed to guarantee near capacity performance.

Recently, Raptor codes [3] were studied in the low signal to noise ratio (SNR) regime in [4], and it has been shown that a single optimal degree distribution exists in very low SNRs which enables Raptor codes to achieve rate efficiencies larger than 95% in binary input additive white Gaussian noise channels (BI-AWGNs). Motivated by this, we propose a multi-

level coding scheme with Raptor component codes and SM [2] to approach the capacity of AWGN channel at any SNR. Coded symbols from each level are BPSK modulated and multiplied by a weight coefficient, where the sum of weighted modulated coded symbols of all layers is sent through the channel. A simple multi stage decoder (MSD) and successive interference cancellation (SIC) are then used for decoding the layers. The weight coefficients of the multi-layer code are designed in such a way to have the same equivalent SNR over all layers, subject to the successful decoding of previous stages. The number of coding layers is chosen such that the equivalent SNR for each decoding stage is very low; thus, each equivalent channel can be modeled by a BI-AWGN channel in the low SNR regime. This allows the same Raptor code with the optimal degree distribution in the low SNR regime [4], to be used as the component code in all layers.

We show that if the component Raptor code has an efficiency η at very low SNRs, the multi-level code with an appropriate number of layers, will also have an efficiency of η at every SNR. This means, by using a simple Raptor code in low SNRs, a transmission strategy can be appropriately designed (i.e., finding the number of layers and determining the weight coefficients) for a given channel SNR, which approaches the capacity of the AWGN channel with a constant efficiency determined by the component Raptor code. This will significantly reduce the complexity of the communication system as each decoding stage is treated as a BI-AWGN channel. Moreover, the same decoder can be used for all decoding stages, as the same graph structure can be used for all component Raptor codes. We show that the accuracy of the BI-AWGN assumption for each layer of the multi-layer code improves when the effective SNR of each channel goes to zero. Moreover, we show that the proposed code closely approaches the capacity of a Gaussian channel across a wide range of SNRs.

The rest of the paper is organized as follows. An overview of Raptor codes in the low SNR regime is presented in Section II. In Section III, the encoder/decoder structure of the proposed multi-layer Raptor code is presented. The weight coefficient design is then presented in Section IV, where we also study the accuracy of the Gaussian assumption for each layer of the proposed multi-layer structure. Simulation results are shown in Section V. Finally, Section VI concludes the paper.

II. AN OVERVIEW OF RAPTOR CODES IN THE LOW SNR REGIME

A Raptor code [3] is a simple concatenation of a high-rate low density parity check (LDPC) code and a Luby transform (LT) code [5]. A k -bit information sequence is first encoded by using a high-rate LDPC code to generate k' LDPC coded symbols, also referred to as input symbols. Using an LT code, a potentially limitless number of coded (output) symbols can then be generated.

The encoding process of LT codes contains two important steps. First, an integer d , called degree, is obtained from a predefined probability distribution function, called a degree distribution. Second, d distinct input symbols are uniformly selected at random and then XORed to generate one output symbol. The encoding process will be terminated when the sender receives an acknowledgement from the destination or a pre-determined number of coded symbols are sent. Fig. 1 shows the bipartite graph representation of an LT code of message length k , which is truncated at codeword length m .

The performance of Raptor codes are mainly determined by the degree distribution. Let Ω_d denote the probability that the degree is d . Then, the degree distribution can be represented in a polynomial form as follows:

$$\Omega(x) = \sum_{d=1}^D \Omega_d x^d, \quad (1)$$

where D is the maximum code degree.

A sum product algorithm (SPA) is usually used for the decoding of Raptor codes, where log-likelihood ratios (LLRs) are passed as messages along edges from variable to check nodes and vice versa in an iterative manner. More details of this decoder can be found in [6]. The design of Raptor codes over AWGN channels in the low SNR regime has been studied in [4], where an exact expression for the degree distribution polynomial in the low SNR regime was found. More specifically, the asymptotic degree distribution polynomial in the low SNR regime when the maximum code degree goes to infinity is given by [4]:

$$\Omega(x) = \frac{1}{4 \ln(2)} \int_0^x \varphi^{-1}(t) dt, \quad x \in [0, 1], \quad (2)$$

where $\varphi^{-1}(x)$ is the inverse of $\varphi(x)$, which is defined as follows for $x > 0$:

$$\varphi(x) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} \tanh\left(\frac{u}{2}\right) e^{-\frac{(u-x)^2}{4x}} du. \quad (3)$$

A set of practical degree distributions with limited maximum degree were also designed in [4], which have shown an excellent rate performance in very low SNRs. More specifically, a rate efficiency of 0.95 was achieved for a Raptor code with maximum degree of 300 for SNRs below -10 dB. Here, the rate efficiency is defined as the ratio of the realized rate of the Raptor code and the channel capacity. This is an interesting property for Raptor codes, where a single degree distribution can be used for all SNRs below -10 dB to achieve a near capacity performance over BI-AWGN channels.

Fig. 1. The factor graph of an LT code truncated at code length m .

III. CODE DESIGN FOR AN ARBITRARY AWGN CHANNEL

In this section, we show how a capacity approaching code can be developed for an arbitrary AWGN channel by using a Raptor code with the degree distribution optimized for a very low SNR as a single component code.

A. The Encoder Structure

The main idea is to divide the original Gaussian channel into a large number of BI-AWGN channels with very low SNRs. For this aim, we use the multi layer encoding structure as shown in Fig. 2. A potentially limitless number of coded symbols are generated by each coding layer, where each coded symbol is BPSK modulated. The weighted sum of modulated coded symbols over all layers is transmitted over the AWGN channel.

Let x_i denote the output of the BPSK modulator of the i^{th} layer. Then, the received signal, y , can be shown as follows:

$$y = \sum_{i=1}^L w_i x_i + z, \quad (4)$$

where $w_i > 0$ is the weight coefficient for the i^{th} layer and z is zero mean additive white Gaussian noise with variance σ_z^2 . We assume that $\sum_{i=1}^L w_i^2 = 1$; thus, the channel SNR can be defined as $\gamma = 1/\sigma_z^2$.

B. The Decoder Structure

For the decoding of the proposed multi-layer Raptor code, we use the well-known multistage decoding (MSD) and successive interference cancellation (SIC). More specifically, the decoder in the i^{th} stage removes all coded symbols from all previous stages, and treats the coded symbols of stages $i+1$ to L as additive noise. Let y_i denote the effective input of the i^{th} stage of the decoder, it is given by:

$$y_i = w_i x_i + z_i, \quad (5)$$

where $z_i = \sum_{j>i} w_j x_j + z$ is the effective noise of the i^{th} decoder stage. It can be easily verified that z_i has zero mean and its variance, denoted by σ_i^2 , can be calculated as follows:

$$\sigma_i^2 = \sum_{j=i+1}^L w_j^2 + \sigma_z^2. \quad (6)$$

The effective SNR for the i^{th} stage of the decoder is then defined as follows:

$$\gamma_i = \frac{w_i^2}{\sum_{j=i+1}^L w_j^2 + \sigma_z^2}. \quad (7)$$

Parameter L is chosen in such a way that the maximum effective SNR is less than γ_0 . The maximum effective SNR belongs to the L^{th} stage of the decoder and is given by $\gamma_L = w_L^2/\sigma_z^2$.

Fig. 2. Encoder and decoder structure of the proposed code with Raptor component codes.

C. Channel Adapters

It is important to note that for each component code the channel output must satisfy the symmetry condition. Generally, a binary input channel is symmetric if [7]

$$p(Y = y|C = 0) = p(Y = -y|C = 1)$$

where c and y are the input and output of the binary-input channel, respectively. We follow [7] and include channel adapters to ensure that each layer of the multi-layer Raptor code satisfy the symmetry condition. As shown in [7], by using i.i.d. channel adapters, we can force the symmetry of the equivalent binary-input component channels. This way the decoding error probability is independent of the codeword. As shown in Fig. 2, each i.i.d. channel adapter will generate binary symbols $t_i \in \{0, 1\}$, $i = 1, \dots, L$, according to an i.i.d. equiprobable distribution. Then, the coded symbol from i^{th} layer, c_i , will be XORed with t_i , i.e., $d_i = c_i \oplus t_i$. The sign of the channel output will be adjusted by $v_i = u_i(1 - 2t_i)$, where u_i is the LLR of the APP module output, and v_i is the input of the i^{th} stage decoder. More specifically, the APP module at stage i , will subtract the coded symbols from the previous stages from the channel output y , and calculate the output LLR, u_i , of the equivalent BI-AWGN channel (5) as follows:

$$u_i = \frac{2y_i}{\sigma_i^2}. \quad (8)$$

As shown in [7], the capacity of the binary input channel with an i.i.d. equiprobable input distribution is equal to the capacity of the equivalent binary-input output symmetric channel with the i.i.d. channel adapter. This means that no rate loss is introduced in the system due to use of the channel adapter. More details on the capacity analysis and relative coded modulation schemes can be found in [7].

IV. ANALYSIS OF THE MULTI-LAYER RAPTOR CODE

A. Achievable Rate

Let C_i denote the coded symbols in the i^{th} layer of the multi-layer Raptor code (see Fig. 2). With the well-known

multi-stage decoding and the mutual information chain rule, we have

$$I(C_1, \dots, C_L; Y) = I(C_1; Y) + I(C_2; Y|C_1) + \dots + I(C_L; Y|C_1, \dots, C_{L-1}), \quad (9)$$

which shows that the transmission of vector $(c_1, \dots, c_L)$ can be separated into the parallel transmission of c_i over L equivalent binary input channels [7]. More specifically, the mutual information for the i^{th} equivalent binary input channel can be shown as follows:

$$I(C_i; Y|C_1, \dots, C_{i-1}) \approx \frac{1}{2} \log_2(1 + \gamma_i), \quad (10)$$

which is valid since we assume that the equivalent SNR for all layers is very small. By using (10) and (7), (9) can be rewritten as follows:

$$\begin{aligned} I(C_1, \dots, C_L; Y) &= \frac{1}{2} \sum_{i=1}^L \log_2 \left(1 + \frac{w_i^2}{\sum_{j=i+1}^L w_j^2 + \sigma_z^2} \right) \\ &= \frac{1}{2} \log_2 \left(\prod_{i=1}^L \frac{\sum_{j=i}^L w_j^2 + \sigma_z^2}{\sum_{j=i+1}^L w_j^2 + \sigma_z^2} \right) \\ &= \frac{1}{2} \log_2 \left(\frac{\sum_{j=1}^L w_j^2 + \sigma_z^2}{\sigma_z^2} \right) \\ &= \frac{1}{2} \log_2(1 + \gamma). \end{aligned} \quad (11)$$

By using (10) and the fact that the equivalent SNR for all layers is γ_0 , (10) can be rewritten as follows:

$$I(C_1, \dots, C_L; Y) \approx \frac{L}{2} \log_2(1 + \gamma_0). \quad (12)$$

This shows that each AWGN channel can be divided into a large number of very low SNR BI-AWGN channels, where the same Raptor codes is used over all sub-channels. In fact, the code design problem for an arbitrary AWGN channel is translated to the design of a capacity approaching Raptor code in the low SNR regime, where a single degree distribution exists. Fig. 3 shows the equivalent realization of an arbitrary AWGN channel of SNR $\gamma = 1/\sigma_z^2$ using L BI-AWGN sub-channels at low SNRs.

Fig. 3. Dividing an arbitrary AWGN channel into L BI-AWGN sub-channels at low SNR.

B. Designing the Weight Coefficients

The following lemma shows how to design the weight coefficients of the multi-layer Raptor code in such a way that the effective SNR for all layers is the same.

Lemma 1: Let γ_0 denote the target effective SNR for all layers of the multi-layer Raptor code. Then for an AWGN channel of SNR γ , the number of layers is given by:

$$L = \left\lceil \frac{\ln(1+\gamma)}{\ln(1+\gamma_0)} \right\rceil, \quad (13)$$

and the weight coefficient of the i^{th} layer of the multi-layer code can be calculated as follows:

$$w_{L-i}^2 = (1 + \tilde{\gamma}_0)^i \frac{\tilde{\gamma}_0}{\gamma}, \quad (14)$$

where $\tilde{\gamma}_0 = \sqrt[L]{1+\gamma} - 1$ and $\lceil \cdot \rceil$ is the ceil operator.

Proof: The number of layers is calculated such that the total achievable rate of the layered code of target SNR γ_0 is equal to the channel capacity at SNR γ . Since the weight coefficients are designed such that the effective SNR of each decoding stage is γ_0 and by using (11) and (12), the achievable rate of the layered code is given by:

$$L \log_2(1 + \gamma_0)/2, \quad (15)$$

which results in (13), when it equals to $0.5 \log_2(1 + \gamma)$. To avoid any rate loss due to the ceil operation over the number of layers, we update the target SNR, which is defined as $\tilde{\gamma}_0 = \sqrt[L]{1+\gamma} - 1$.

It is clear that for the L^{th} layer, we have $w_L^2 = \tilde{\gamma}_0/\gamma$, which proves (14) for $i = 0$. We assume that (14) is held for $i = 0, \dots, j$ for $j \geq 0$, we then show that it also holds for $j + 1$. As we assume that the effective SNR for all layers is $\tilde{\gamma}_0$, then for layer $L - j - 1$ we have:

$$\begin{aligned} \tilde{\gamma}_0 &= \frac{w_{L-j-1}^2}{\sum_{i=L-j}^L w_i^2 + \sigma_z^2} = \frac{w_{L-j-1}^2}{\sum_{i=0}^j w_{L-i}^2 + \sigma_z^2} \\ &= \frac{w_{L-j-1}^2}{\sum_{i=0}^j (1 + \tilde{\gamma}_0)^i \frac{\tilde{\gamma}_0}{\gamma} + \sigma_z^2} \\ &= \frac{\gamma w_{L-j-1}^2}{(1 + \tilde{\gamma}_0)^{j+1}}, \end{aligned}$$

which is equivalent to (14) for $j + 1$. ■

Fig. 4. The KL distance between the pdf of the equivalent noise of each layer and ideal Gaussian distribution with the same variance, when $L = 10$.

C. Accuracy of the Proposed Model

It is important to note that each sub-channel in the proposed multi-layer structure can be modelled as BI-AWGN only if its equivalent noise closely follows the Gaussian distribution. In other words, if the equivalent noise of each sub-channel does not follow a Gaussian distribution, the degree distribution will not be optimal anymore as it was originally designed for BI-AWGN channels. This will lead to a significant rate loss over each layer. Here, we use the Kullback-Liebler (KL) distance to measure the difference between the probability distribution function (pdf) of the equivalent noise of each layer and that of the Gaussian distribution with the same variance.

For the i^{th} layer of the multi-layer Raptor code in (5), it is easy to show that the pdf of its equivalent noise, z_i , can be calculated as follows:

$$p_i(z_i = \zeta) = \sum_{\mathbf{x} \in \{-1,1\}^{L-i}} \frac{2^{i-L}}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{1}{2\sigma_z^2}(\zeta - \mathbf{x}\mathbf{w}_i)^2}, \quad (16)$$

where $\mathbf{w}_i = [w_j]_{j \geq i+1}$. The KL-distance between $p_i(x)$ and the Gaussian distribution with zero mean and variance σ_i^2 , can be found as follows:

$$D_{\text{KL}}(p_i||q_i) = \int_{-\infty}^{\infty} p_i(x) \log \frac{p_i(x)}{q_i(x)} dx, \quad (17)$$

where $q_i(x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{x^2}{2\sigma_i^2}\right)$ and σ_i^2 is given in (6).

It is clear that the distribution of equivalent noise of different layers become Gaussian with sufficiently small γ_0 and are more Gaussian for higher number of layers. We show, numerically, in the following that the sub-channel noise distributions are indeed very close to Gaussian at the γ_0 values that we typically employ (i.e. γ_0 between -10 and -30 dB). Fig. 4 shows the KL divergence of each layer of a 10-layer multi-layer Raptor code with different number of layers for different target SNR values. As can be seen in this figure, as γ_0 decreases, the KL divergence value decreases, which shows that the equivalent noise of each layer becomes more Gaussian. This shows that each layer of the proposed scheme can be finely approximated by an BI-AWGN channel model; thus, the optimal degree distributions designed for BI-AWGN channels can be used over all the channels.

More generally, it can be shown that $D_{\text{KL}}(p_i||q_i)$ is a decreasing function of the layered number i and is an increasing function of γ_0 . This is mainly due to the fact that with decreasing the SNR, the weight coefficients become very smaller than the noise variance and thus have less effect on the shape of the pdf of the equivalent noise of each layer. We omit the exact proof here due to space limitations.

V. SIMULATION RESULTS AND DISCUSSIONS

For simulations, we consider the optimal degree distribution for a Raptor code with the maximum degree 300 [4], which is given by $\Omega(x) = 0.0174x + 0.3488x^2 + 0.2309x^3 + 0.0695x^4 + 0.0873x^5 + 0.0002x^6 + 0.0805x^7 + 0.0004x^8 + 0.0191x^{11} + 0.0518x^{12} + 0.0123x^{23} + 0.0310x^{24} + 0.0220x^{59} + 0.0020x^{60} + 0.0268x^{300}$. The message length of each layer is assumed to be $k = 10^5$ bits. This code shows an excellent performance in the low SNR regime, where the average rate efficiency of 96% is achieved at -20 dB. Moreover, the minimum rate efficiency of the same code at the same SNR is 94%, which shows that the Raptor code provides near capacity performance in low SNRs [4]. Fig. 5 shows the achievable rate of the proposed multi-layer Raptor code over an AWGN channel at different SNRs. As can be seen in this figure, the multi-layer Raptor code shows near capacity performance over a large range of SNR values. This is achieved by using a single degree distribution function and the same code structure over all the layers. This will significantly reduce the complexity of the code design, as for any SNR only the number of layers should be changed, while the code structure and the degree distribution function remain the same. On the other hand, the same decoder is used over all the layers. It is important to note that all the layers in the multi-layer Raptor code can be jointly decoded, which significantly increase the decoding complexity as the number of layers is very large in high SNRs and the joint decoder needs many iterations between the layers to converge. For this reason, we consider SIC and MSD to limit the decoding complexity to that of the single layer code.

It is also clear from Fig. 5 and Fig. 6 that the multi-layer Raptor code with a lower target SNR γ_0 achieves higher rates, which is due to the fact that the degree distribution used in component Raptor codes was originally optimized for very low SNR regime; therefore it shows a better rate performance in lower SNRs. Furthermore, as we showed in Section IV, with decreasing the target SNR, the Gaussian assumption for the equivalent noise of each layer becomes more accurate. This means that the equivalent channel for each layer of the multi-layer Raptor code can be finely approximated by a BI-AWGN channel with a negligible error, which then allows the Raptor code over each layer shows a near capacity performance at the target SNR γ_0 . More specifically, the Raptor code shows a minimum rate efficiency of 94% in very low SNRs, and the same rate efficiency can be achieved by the proposed multi-layer Raptor code at all SNRs (see Fig. 6), when $\gamma_0 = -20$ dB. The performance can be further improved when a joint decoding algorithm is used for decoding the layers, which is considered as the future work.

Fig. 5. Rate performance of the proposed multi-layer Raptor codes versus the channel SNR for different values of γ_0 .

Fig. 6. Rate efficiency of the multi-layer Raptor code versus SNR for different values of γ_0 .

VI. CONCLUSIONS

In this paper, we proposed a multi-layer Raptor code with superposition modulation, which can approach the capacity of AWGN channels. Each arbitrary AWGN channel was divided into several BI-AWGN channels at very low SNR, where the same Raptor code with the same degree distribution function was used over all the layers. We showed that when the equivalent SNR of the layers is very small, the corresponding effective noise of each layer closely follows a Gaussian distribution with the same variance, which validated the accuracy of the Gaussian assumption behind proposed multi-layer structure. Simulation results showed that the proposed code achieves very high rate efficiencies over all SNRs using a Raptor code with a fixed degree distribution function as the component code.

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