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Ultimate Bounds and Regions of Attraction for Two-Inverter Microgrids with Primary and Secondary Frequency Control Loops

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Abstract—In this paper we study frequency stabilisation in inverter-based microgrid systems with primary droop controller and three types of secondary control strategies. Particularising the microgrid to the case of two inverter buses, we propose structured models for which frequency regulation, ultimate bounds and an estimate of the region of attraction associated with the ultimate bound set are obtained in closed-form analytical expressions.

I. INTRODUCTION

The analysis of controllable inverters interconnected in microgrids has attracted substantial research focus in recent years. A microgrid is a small-scale power system consisting of a collection of distributed generation units, loads and local storage, operating together with energy management, control and protection devices and associated software [10].

Control strategies are indispensable to provide stability in microgrids [11]. Recently, hierarchical control for microgrids, consisting of three main control layers, has been proposed to achieve voltage and frequency stability and regulation, and power flow and economic optimisation [6]. This paper focuses on the primary-secondary control layers aiming to stabilise the frequency of the microgrid.

The primary control layer locally controls the distributed generation sources interfaced with the AC grid via inverters. For inductive lines, the frequency/active-power droop control [4], emulating the droop characteristic of synchronous generators, is conventionally implemented in the primary layer as a decentralised controller. Due to the proportional nature of the droop control, it induces frequency steady-state errors that are then compensated in the secondary control layer. This layer can operate in a centralised, decentralised or distributed fashion [3], [12], [13].

Stability and convergence properties of droop-controlled networks of inverters and loads have only very recently started to be analysed in detail [2], [3], [14]. In [14] for instance, the authors proposed a distributed secondary control scheme to dynamically regulate the network frequency to a nominal value without requiring time-scale separation between primary and secondary control loops, which is a common assumption in more conventional analyses [1], [2]. For this configuration, they then presented a necessary and sufficient condition for the existence of a unique and locally exponentially stable steady-state equilibrium.

In this paper we analyse frequency regulation and ultimate boundedness of the state trajectories in a two-inverter microgrid under the centralised, distributed, and decentralised secondary control layers considered in [5]. We follow an approach that we presented in [8], which exploits a structured nonlinear model for inverter-based microgrids with embedded primary and secondary control layers. A key feature of this approach is that stability properties beyond local considerations around an equilibrium point can be analysed.

The main contribution of the present paper is the derivation of closed-form analytical expressions for the regions of attraction (RoAs) and state ultimate bounds (UBs) for the two-inverter microgrid system under each of the three secondary control mechanisms considered. This extends the analysis of [8] when particularised to the case of two inverters, since in [8] only the decentralised controller was considered and closed-loop expressions were not derived. The expressions for the RoAs and UBs derived in the present paper are functions of the line impedance, and the mismatch between generation and demand in each bus. Interestingly, the expressions are identical for all three controllers considered. Thus, for the case of two inverters, the same ‘certificates’ for the quality of frequency regulation—in the sense of a predictable bound on the bus angle difference starting from a prespecified set of initial conditions—apply to all three secondary control strategies. The estimated RoAs and UBs are compared in a numerical example. Preliminary studies show that these results may apply to more general microgrids. This is the subject of our current investigations.

II. PRELIMINARY RESULTS

Consider a linear system under non-vanishing, state-dependent, possibly nonlinear perturbation

$$\dot{x}(t) = \Lambda x(t) + Hw(x(t)) \quad (1)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $\Lambda \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{n \times m}$ and the perturbation $w(\cdot)$ is bounded above by a CI¹ function

$$|w(x)| \leq W(x). \quad (2)$$

Lemma II.1 (Ultimate boundedness [7, Theorem 3])

Consider the system (1) with the perturbation bound of the form (2), where Λ is a diagonal Hurwitz matrix and

¹A nonnegative vector function $T : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$ is said to be componentwise increasing (CI) if $z_1, z_2 \in \mathbb{R}_{+0}^n$ and $z_1 \leq z_2$, imply $T(z_1) \leq T(z_2)$. $\mathbb{R}_{+0}^n$ denotes the set of real n -vectors with nonnegative components.

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the bounding function $W(x)$ is CI. Define the nonlinear mapping $T : \mathbb{R}_{+0}^n \rightarrow \mathbb{R}_{+0}^n$ as

$$T(x) = -\Lambda^{-1}|H|W(x). \quad (3)$$

Suppose there exists $\beta \in \mathbb{R}_{+0}^n$ satisfying $T(\beta) < \beta$. Then,

- (a) For every $k \in \{0, 1, 2, \dots\}$, $T^{k+1}(\beta) \leq T^k(\beta)$ and $\lim_{k \rightarrow \infty} T^k(\beta) = b \geq 0$.
(b) If $|x(0)| \leq \beta$, then $\limsup_{t \rightarrow \infty} |x(t)| \leq b$.

Lemma II.1 introduces an UB set, namely, the set $\{x \in \mathbb{R}^n : |x| \leq b\}$, as a set where the trajectories of the system starting within a certain “originating” region of the state space will ultimately converge and remain inside. The originating region of the state space, that is, $\{x \in \mathbb{R}^n : |x| \leq \beta\}$, where the trajectories should start within is an *estimate of the RoA* to the UB set.

Applying Lemma II.1 requires finding a vector $\beta > 0$ satisfying $T(\beta) < \beta$. If such a vector exists, then the UB can be computed by iterating $T(x)$ from $x = \beta$.

In the following sections we will propose a model for the microgrid system that has the form (1)–(2). For this system, the mapping $T(x)$ defined in (3) can be taken as a cubic function. In the following lemma we present a property of these functions that will be required later in the paper.

Lemma II.2 Consider the function

$$T(x) = \frac{2}{3}x^3 + \Omega \quad (4)$$

where $0 < \Omega < \frac{\sqrt{2}}{3}$. Then, $T(x) < x$ for all $x \in (x_{\min}, x_{\max})$, where,

$$T(x_{\min}) = x_{\min} < (3/2)\Omega, \quad (5)$$

$$0 < T(x_{\max}) = x_{\max} > (3/2)[\Omega + \sqrt{3}(\sqrt{2}/3 - \Omega)]. \quad (6)$$

Proof: See Lemma 2.2 in [9]. ■

III. DROOP CONTROL MODEL

For a microgrid with inductive lines, the standard $\omega - P$ (primary) droop controller proportionally relates the deviation in the frequency to the active power balance at the inverter bus as follows [5]:

$$d_i \dot{\theta}_i = P_i^* - P_{e,i}, \quad i = 1, \dots, n \quad (7)$$

where $d_i > 0$ is the droop coefficient, θ is the bus phase angle, $P_i^* \doteq P_{ref,i} - P_{L,i}$ is the inverter power mismatch between the inverter nominal injection setpoint $P_{ref,i}$ and the bus load $P_{L,i}$, and $P_{e,i}$ is the bus power injection. The frequency of the voltage signal at the i -th inverter is given by $\omega_i = \omega^* + \dot{\theta}_i$, with ω^* as the nominal frequency and $\dot{\theta}_i$ as the deviation from the nominal frequency. The power injection to each bus in a lossless microgrid is

$$P_{e,i} = \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j), \quad (8)$$

with $a_{ij} \doteq y_{ij} E_i E_j$, E_i denoting the bus voltage magnitude, y_{ij} denoting the pure imaginary ij -th line admittance. We adopt the standard decoupling approximation in the frequency control context where all voltage magnitudes E_i are considered constant.

Due to its proportional nature, the droop controller (7) results in a static error in the steady state frequency. In [1], it is shown that as long as the network trajectories remain in a specified region, then the controller (7) ensures network synchronisation to the average frequency error [2]

$$\omega_{sync} = \left(\sum_{i=1}^n d_i \dot{\theta}_i \right) / \left(\sum_{i=1}^n d_i \right). \quad (9)$$

When just the droop controller (7) acts on the system and for purely inductive lines where $\sum_{i=1}^n P_{e,i} = 0$, ω_{sync} takes the form $\frac{\sum_{i=1}^n P_i^*}{\sum_{i=1}^n d_i}$. In this case, it can be seen that $\omega_{sync} = 0$ if and only if $\sum_{i=1}^n P_i^* = 0$ or equivalently $\sum_{i=1}^n P_{ref,i} = \sum_{i=1}^n P_{L,i}$, that is, the power is balanced at all buses. However, since the load demand is generally unknown and variable, the above balance cannot be achieved [5]. Hence, a complementary control action is required to eliminate or reduce the frequency error ω_{sync} ; for example, by including a correction term u_i to each inverter bus as [5]

$$d_i \dot{\theta}_i = P_i^* - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j) - u_i, \quad (10)$$

where u_i is determined in a secondary control loop. In the following sections, we study three different mechanisms considered in [5] for the secondary control loop: centralised, distributed and decentralised, and analyse their effect on the stability of the system for a microgrid with two parallel inverters. We focus on obtaining a guaranteed UB set and a precise estimate of the RoA to this set.

IV. MAIN RESULTS

The main results of the paper are Theorem IV.2 and IV.3 below, where we analyse the average frequency error in (9) when the control strategy is given by (10), and provide closed-form expressions for RoAs and UBs for the difference between inverter bus angles. These expressions hold for each of the three secondary control mechanisms analysed in the paper where the correction term u_i in (10) is obtained as:

- *Centralised control:*

$$u_i = p_i, \quad k_i \dot{p}_i = \frac{\sum_{j=1}^n d_j \dot{\theta}_j}{\sum_{j=1}^n d_j}, \quad (11)$$

where $k_i > 0$ are the secondary controller coefficients. As we explain below, we will follow [5], where the all-to-all communication approach required in (11) — in which the secondary loop of each inverter receives information from all other inverters to generate the correction signal u_i — is implemented in a centralised fashion.

- *Distributed control:*

$$u_i = p_i, \quad k_i \dot{p}_i = d_i \dot{\theta}_i + \sum_{j=1}^n L_{ij} \left(\frac{p_i}{d_i} - \frac{p_j}{d_j} \right), \quad (12)$$

where L_{ij} is the ij -th entry of the Laplacian matrix L . In this case, the secondary loop at each inverter collects

² $L = BY^BT$ where $Y = \text{diag}\{\{a_{ij}\}_{i,j \in \{1, \dots, n\}}\}$ and $B \in \mathbb{R}^{n \times m}$ is the Incidence matrix of the graph such that $B_{ij} = 1$ if the node i is the source of the edge j and $B_{ij} = -1$ if the node i is the sink node of the edge j ; all other entries are zero.

information from other inverters, but not necessarily all inverters, to generate the correction signal u_i .

- *Decentralised control:*

$$u_i = p_i, \quad k_i \dot{p}_i = \dot{\theta}_i - \epsilon_i p_i, \quad (13)$$

where $k_i > 0$ and $\epsilon_i > 0$ are the controller coefficients.

We will focus on the configuration of Figure 1 consisting of two parallel inverters connected by an inductive line as a special case of an inverter-based microgrid.

Fig. 1. Microgrid with two inverter buses

For this case with $n = 2$, the controlled microgrid system (10) with $u_i = p_i$ for the centralised, distributed and decentralised secondary mechanisms, can be written, respectively, as

$$\begin{bmatrix} d_1 \dot{\theta}_1 \\ d_2 \dot{\theta}_2 \\ k_1 \dot{p}_1 \\ k_2 \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -a \sin(\theta_1 - \theta_2) - p_1 \\ a \sin(\theta_1 - \theta_2) - p_2 \\ -(p_1 + p_2)/(d_1 + d_2) \\ -(p_1 + p_2)/(d_1 + d_2) \end{bmatrix} + \begin{bmatrix} P_1^* \\ P_2^* \\ (P_1^* + P_2^*)/(d_1 + d_2) \\ (P_1^* + P_2^*)/(d_1 + d_2) \end{bmatrix}, \quad (14)$$

$$\begin{bmatrix} d_1 \dot{\theta}_1 \\ d_2 \dot{\theta}_2 \\ k_1 \dot{p}_1 \\ k_2 \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -a \sin(\theta_1 - \theta_2) - p_1 \\ a \sin(\theta_1 - \theta_2) - p_2 \\ -a \sin(\theta_1 - \theta_2) - p_1 + L_{12}(\frac{p_1}{d_1} - \frac{p_2}{d_2}) \\ a \sin(\theta_1 - \theta_2) - p_2 + L_{21}(\frac{p_2}{d_2} - \frac{p_1}{d_1}) \end{bmatrix} + \begin{bmatrix} P_1^* \\ P_2^* \\ P_1^* \\ P_2^* \end{bmatrix}, \quad (15)$$

$$\begin{bmatrix} d_1 \dot{\theta}_1 \\ d_2 \dot{\theta}_2 \\ k_1 \dot{p}_1 \\ k_2 \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -a \sin(\theta_1 - \theta_2) - p_1 \\ a \sin(\theta_1 - \theta_2) - p_2 \\ -a \sin(\theta_1 - \theta_2) - (1 + \epsilon_1 d_1) p_1 \\ a \sin(\theta_1 - \theta_2) - (1 + \epsilon_2 d_2) p_2 \end{bmatrix} + \begin{bmatrix} P_1^* \\ P_2^* \\ P_1^* \\ P_2^* \end{bmatrix}, \quad (16)$$

where $a = a_{12} = a_{21}$ and $a = yE_1E_2$ (see Figure 1).

Assumption IV.1 (Control coefficients) In (11), (12) and (13) we consider $k_i = k$, $d_i = d$ and $\epsilon_i = \epsilon$ for $i = 1, 2$.

Theorem IV.2 (Boundedness of ω_{sync}) Consider the closed-loop microgrid systems (14)–(16), under Assumption IV.1. Then,

- The average frequency error ω_{sync} in (9) is bounded if the power injection errors P_1^* and P_2^* are bounded.
- When u_i is computed using either the centralised controller (11), or the distributed controller (12), ω_{sync} converges to 0 when $(P_1^* + P_2^*)$ is constant.
- When u_i is computed using the decentralised controller (13) and $(P_1^* + P_2^*)$ is constant, ω_{sync} converges to

$$\omega_{sync_{ss}} = (P_1^* + P_2^*)\epsilon/(2 + 2\epsilon d). \quad (17)$$

Theorem IV.3 (Boundedness of $\theta_1 - \theta_2$) Consider the microgrid systems (14)–(16), under Assumption IV.1. Suppose that the inverter power mismatches satisfy $|P_1^* - P_2^*| < \frac{4\sqrt{2}a}{3}$ and let $\Omega \doteq \frac{|P_1^* - P_2^*|}{4a}$. Then, the system trajectories starting within a region of attraction whose projection on the bus-angle difference is

$$|\theta_1 - \theta_2| \leq 3\Omega + 3\sqrt{3}(\sqrt{2}/3 - \Omega), \quad (18)$$

are ultimately bounded and, in particular,

$$\limsup_{t \rightarrow \infty} |\theta_1(t) - \theta_2(t)| \leq 3\Omega. \quad (19)$$

The proofs of these results for each type of secondary controller are provided in the rest of the section.

A. Centralised Secondary Control

Using Assumption IV.1, the dynamics of the secondary controller variables p are identical and thus, (11) can be replaced by

$$u_i = d_i p, \quad k_c \dot{p} = \frac{\sum_{j=1}^n (P_j^* - d_j p)}{\sum_{j=1}^n d_j} = -p + \frac{\sum_{j=1}^n P_j^*}{\sum_{j=1}^n d_j} \quad (20)$$

where $k_c = k_i d_i$ and we have used (10). Thus, the closed-loop system (14) can be simplified by letting $p_1 = p_2 = p$ and replacing the last two equations by (20). Let $f(x) \doteq \sin(x) - x$ and define the matrices $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $L = a \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Considering Assumption IV.1, the closed-loop system with centralised control can be expressed as

$$\dot{x}_\theta = -\frac{1}{d} L x_\theta - p \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{a}{d} B f(\theta_1 - \theta_2) + \frac{1}{d} \bar{P} \quad (21)$$

$$k_c \dot{p} = -p + (P_1^* + P_2^*)/(2d) \quad (22)$$

where $x_\theta = [\theta_1 \ \theta_2]^T$, $\bar{P} = [P_1^* \ P_2^*]^T$, and B and L are as defined in footnote 2. We next observe that the matrix $-L/d$ in (21) can be diagonalised as $\Lambda_c = V_c^{-1}(-L/d)V_c$ where

$$\Lambda_c = \text{diag}\{0, -2a/d\}, \quad V_c = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (23)$$

Using the eigenvector matrix V_c in (23), consider the transformation $x_\theta = V_c z$. The transformed system (21) is

$$\dot{z} = \Lambda_c z - \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -a/d \end{bmatrix} f(\theta_1 - \theta_2) + \begin{bmatrix} (P_1^* + P_2^*)/(2d) \\ (P_1^* - P_2^*)/(2d) \end{bmatrix}. \quad (24)$$

We study stability of the closed-loop system (24), (22) in terms of frequency regulation and ultimate boundedness of the system states in the following two subsections.

1) *Boundedness of ω_{sync} :* Here we prove Theorem IV.2 for the case of a centralised secondary controller defined by (20). Firstly, from (22), it can be seen that p is bounded if (P_1^*, P_2^*) are bounded. Then, from (24) we have $\dot{z}_1 = -p + (P_1^* + P_2^*)/(2d)$ and thus, $\dot{z}_1$ remains bounded for bounded (P_1^*, P_2^*) . From $z = V_c^{-1} x_\theta$ it follows that

$$z_1 = (\theta_1 + \theta_2)/2, \quad z_2 = (\theta_1 - \theta_2)/2. \quad (25)$$

Using (25) we have

$$\dot{z}_1 = \underbrace{(\dot{\theta}_1 + \dot{\theta}_2)/2}_{\omega_{sync}} = -p + (P_1^* + P_2^*)/(2d). \quad (26)$$

Then ω_{sync} also remains bounded if (P_1^*, P_2^*) are bounded. Moreover, if $(P_1^* + P_2^*)$ is constant, p in (22) converges to $\frac{P_1^* + P_2^*}{2d}$, thus, from (26), ω_{sync} converges to 0. ■

2) *Ultimate boundedness of $(\theta_1 - \theta_2)$:* We now prove Theorem IV.3 for the centralised secondary controller (20). From (25), we have $\theta_1 - \theta_2 = 2z_2$ and hence the dynamics of the bus angle difference can be studied by just considering the bottom row of the system (24), which is decoupled from the p -subsystem (22). From (23)–(25) we have

$$\dot{z}_2 = -\frac{2a}{d} z_2 - \frac{a}{d} f(2z_2) + \frac{P_1^* - P_2^*}{2d}. \quad (27)$$

The above system has the form (1) with $\Lambda = -2a/d$ Hurwitz. Thus, following Lemma II.1, the trajectories of the system (27) starting inside a region of the state space are ultimately bounded if the nonlinear perturbation term is bounded by a CI function and is such that the associated

mapping $T_c(z_2)$ of the form (3) (to be defined below) satisfies the contractivity condition $T_c(\bar{z}_2) < \bar{z}_2$ for some $\bar{z}_2 > 0$.

Using the inequality $|f(x)| = |\sin(x) - x| \leq |x|^3/6$ to obtain the bound $|f(2z_2)| \leq (2z_2)^3/6$, we can compute

$$T_c(z_2) \doteq \frac{d}{2a} \left[\frac{a(2z_2)^3}{6} + \frac{|P_1^* - P_2^*|}{2d} \right] = \frac{2}{3} z_2^3 + \frac{|P_1^* - P_2^*|}{4a}. \quad (28)$$

With the definition $\Omega \doteq \frac{|P_1^* - P_2^*|}{4a}$, the CI function (28) is equal to (4) and hence, using Lemma II.2, the contractivity condition $T_c(\bar{z}_2) < \bar{z}_2$ holds true if $\Omega < \frac{\sqrt{2}}{3}$ or equivalently, $|P_1^* - P_2^*| < \frac{4\sqrt{2}a}{3}$. Then, $\bar{z}_2$ can be taken as any point in the interval $(x_{\min}, x_{\max})$ and the UB set and an estimate of its corresponding RoA for the z_2 state are determined by $x_{\min}$ and $x_{\max}$ in Lemma II.2, respectively. Equations (19) and (18) then follow from (5) and (6), respectively. ■

B. Distributed Secondary Control

Consider the closed-loop system with the primary-secondary controller (10)–(12) presented in (15). Considering $f(x) \doteq \sin(x) - x$ and Assumption IV.1, (15) takes the form

$$\dot{x} = A_d x + H_d f(\theta_1 - \theta_2) + \bar{P}_d \quad (29)$$

where $x = [x_\theta^T \ x_p^T]^T$, $x_\theta = [\theta_1 \ \theta_2]^T$, $x_p = [p_1 \ p_2]^T$, and

$$A_d = \begin{bmatrix} -\frac{L}{d} & -\frac{1}{d} I_2 \\ -\frac{L}{k} & -\frac{1}{k} (I_2 + \frac{L}{d}) \end{bmatrix}, H_d = \begin{bmatrix} -\frac{aB}{d} \\ -\frac{aB}{k} \end{bmatrix}, \bar{P}_d = \begin{bmatrix} \bar{P} \\ \bar{P} \end{bmatrix}, \quad (30)$$

with $\bar{P} = [P_1^* \ P_2^*]^T$. We next determine the eigenstructure of the linear part of the system (29)–(30) for later use.

Lemma IV.4 For system (29)–(30), the eigenvalues λ_i and eigenvectors v_i of the matrix A_d have the form

$$\lambda_1 = 0, \quad \lambda_2 = -1/k, \quad \lambda_{3,4} = -(\mu + \gamma \mp R)/2 \quad (31)$$

$$R \doteq \sqrt{(\mu + \gamma)^2 - 4\mu(\gamma - 1/k)}$$

$$\mu = 2a/d, \quad \gamma = (2a + d)/(dk)$$

$$V_d = \begin{bmatrix} 1 & k/d & k(\lambda_3 + \gamma)/(2a) & k(\lambda_4 + \gamma)/(2a) \\ 0 & 1 & -k(\lambda_3 + \gamma)/(2a) & -k(\lambda_4 + \gamma)/(2a) \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \quad (32)$$

In addition, $\lambda_2 < 0$, $\lambda_{3,4} < 0$.

Proof: Immediate by verifying the identity $V_d \Lambda_d = A_d V_d$, where $\Lambda_d \doteq \text{diag}\{\lambda_1, \dots, \lambda_4\}$. The signs of the eigenvalues follow from the fact that $a, d, k > 0$ yielding $\mu, \gamma, (\gamma - 1/k), \mu/k > 0$. ■

From (31) and $\Lambda_d \doteq V_d^{-1} A_d V_d = \text{diag}\{\lambda_1, \dots, \lambda_4\}$, consider the state transformation $x = V_d z$. Then, from (29)–(30), the state z satisfies

$$\dot{z} = \Lambda_d z + V_d^{-1} H_d f(\theta_1 - \theta_2) + V_d^{-1} \bar{P}_d \quad (33)$$

where, by direct computation

$$V_d^{-1} H_d = \frac{a}{kR} \begin{bmatrix} 0 \\ \lambda_3 \\ \lambda_3 \\ -\lambda_4 \end{bmatrix}, V_d^{-1} \bar{P}_d = \frac{1}{2kR} \begin{bmatrix} (P_1^* + P_2^*)R \\ (P_2^* - P_1^*)\lambda_3 \\ (P_2^* - P_1^*)\lambda_3 \\ -(P_2^* - P_1^*)\lambda_4 \end{bmatrix}. \quad (34)$$

We can rewrite (33)–(34) as the two subsystems

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/k \end{bmatrix} \frac{P_2^* - P_1^*}{2} \quad (35)$$

$$\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \lambda_3 \\ -\lambda_4 \end{bmatrix} \frac{1}{kR} \left(a f(\theta_1 - \theta_2) + \frac{P_2^* - P_1^*}{2} \right). \quad (36)$$

Lemma IV.5 The two subsystems (35) and (36) are decoupled from each other.

Proof: Clearly, the only possible connection between the two subsystems is through $f(\theta_1 - \theta_2)$. From $x = V_d z$ and (32), the angle difference satisfies

$$\theta_1 - \theta_2 = \underbrace{\left[\frac{k}{2a}(\gamma - \mu + R) \right]}_{\alpha} z_3 + \underbrace{\left[\frac{k}{2a}(\gamma - \mu - R) \right]}_{\beta} z_4, \quad (37)$$

and hence, it only depends on the variables z_3 and z_4 . ■

The system (36) then takes the form

$$\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \lambda_3 \\ -\lambda_4 \end{bmatrix} \frac{a}{kR} \left[f(\alpha z_3 + \beta z_4) + \frac{P_2^* - P_1^*}{2a} \right]. \quad (38)$$

We now proceed to the stability analysis of the transformed closed-loop system (33)–(34) (equivalently, (35), (38)).

1) *Boundedness of ω_{sync} :* We prove Theorem IV.2 for the distributed secondary controller (12). From $z = V_d^{-1} x$ it follows that

$$z_1 = (\theta_1 + \theta_2)/2 - k z_2/d, \quad z_2 = (p_1 + p_2)/2. \quad (39)$$

Since from (35), $\dot{z}_2 = \lambda_2 z_2 + (P_1^* + P_2^*)/(2k)$ with $\lambda_2 = -1/k < 0$, then z_2 and $\dot{z}_2$ remain bounded for bounded P_1^* and P_2^* . Furthermore, from (35) and (39) we have

$$\dot{z}_1 = \underbrace{(\dot{\theta}_1 + \dot{\theta}_2)/2}_{\omega_{sync}} - k \dot{z}_2/d = 0. \quad (40)$$

Consequently, the average frequency error in (9) is also bounded for bounded (P_1^*, P_2^*) . Moreover, from (35), z_2 converges to a constant if $(P_1^* + P_2^*)$ is constant and hence $\dot{z}_2 \rightarrow 0$. Then, from (40) ω_{sync} also converges to zero. ■

2) *Ultimate boundedness of $(\theta_1 - \theta_2)$:* We prove Theorem IV.3 for the distributed secondary controller. As discussed before, ultimate boundedness of $(\theta_1 - \theta_2)$ can be investigated by analysing the system (38), which has the form (1) with $\Lambda = \text{diag}\{\lambda_3, \lambda_4\}$ Hurwitz (according to Lemma IV.4). Next, following Lemma II.1, we analyse the nonlinear perturbation term.

Using the inequality $f(x) = |\sin(x) - x| \leq |x|^3/6$ to obtain $|f(\alpha z_3 + \beta z_4)| \leq (|\alpha||z_3| + |\beta||z_4|)^3/6$, we can compute the mapping (3) as

$$T(z_3, z_4) \doteq \begin{bmatrix} |\lambda_3| & 0 \\ 0 & |\lambda_4| \end{bmatrix}^{-1} \begin{bmatrix} |\lambda_3| \\ |\lambda_4| \end{bmatrix} T_d(z_3, z_4) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} T_d(z_3, z_4).$$

with

$$T_d(z_3, z_4) \doteq \frac{2a}{kR} \left[\frac{(|\alpha||z_3| + |\beta||z_4|)^3}{12} + \frac{|P_2^* - P_1^*|}{4a} \right] \quad (41)$$

satisfying

$$\frac{2a}{kR} \left| \frac{f(\alpha z_3 + \beta z_4)}{2} + \frac{(P_2^* - P_1^*)}{4a} \right| \leq T_d(z_3, z_4). \quad (42)$$

To find $(\bar{z}_3, \bar{z}_4)$ such that the contractivity condition

$$T(\bar{z}_3, \bar{z}_4) < (\bar{z}_3, \bar{z}_4) \quad (43)$$

holds componentwise, we select $(\bar{z}_3, \bar{z}_4) = (\bar{z}, \bar{z})$ rendering

$$\bar{T}_d(\bar{z}) \doteq T_d(\bar{z}, \bar{z}) = \frac{2a}{kR} \left[\frac{(|\alpha| + |\beta|)^3 \bar{z}^3}{12} + \frac{|P_2^* - P_1^*|}{4a} \right] < \bar{z}. \quad (44)$$

From (31), it follows that $R = \sqrt{(\gamma - \mu)^2 + 4\mu/k} > |\gamma - \mu|$ and hence, from (37), $\alpha > 0$ and $\beta < 0$, yielding $|\alpha| + |\beta| = \alpha - \beta = kR/a$. We can then write (44) as

$$\frac{2a}{kR} \left[\frac{2}{3} \left(\frac{kR}{2a} \bar{z} \right)^3 + \frac{|P_2^* - P_1^*|}{4a} \right] < \bar{z}, \quad (45)$$

or equivalently, defining $x \doteq kR\bar{z}/(2a)$ and $\Omega \doteq |P_2^* - P_1^*|/(4a)$, (45) can be expressed as

$$G_d(x) \doteq (2/3)x^3 - x + \Omega < 0. \quad (46)$$

Using Lemma II.2, the contractivity condition $\bar{T}_d(x) < x$ is then true if $\Omega < \frac{\sqrt{2}}{3}$ or equivalently, $|P_2^* - P_1^*| < \frac{4\sqrt{2}a}{3}$.

Then, $x = kR\bar{z}/(2a)$ satisfying the contractivity condition can be taken as any point in the interval $(x_{\min}, x_{\max})$ and the UB set and the RoA estimate for the (z_3, z_4) trajectory are determined by $2ax_{\min}/(kR)$ and $2ax_{\max}/(kR)$ in Lemma II.2, respectively. From the fact that $|z_3| < \bar{z}$ and $|z_4| < \bar{z}$, it can be implied that

$$|\theta_1 - \theta_2| \leq |\alpha|\bar{z} + |\beta|\bar{z} = (\alpha - \beta)\bar{z} = (kR/a)\bar{z}. \quad (47)$$

Equations (18), (19) follow from (5), (6), respectively. ■

C. Decentralised Secondary Control

Consider the the closed-loop system with the primary-secondary controller (10)–(13) presented in (16). As before, letting $f(x) \doteq \sin(x) - x$, and considering Assumption IV.1, the primary-secondary controller (10) and (13) takes the form

$$\dot{x} = A_s x + H_s f(\theta_1 - \theta_2) + \bar{P}_s \quad (48)$$

where $x = [x_\theta^T x_p^T]^T$, $x_\theta = [\theta_1 \ \theta_2]^T$, $x_p = [p_1 \ p_2]^T$ and

$$A_s = - \begin{bmatrix} \frac{1}{d}L & \frac{1}{d}I_2 \\ \frac{1}{dk}L & \frac{e}{dk}I_2 \end{bmatrix}, H_s = - \begin{bmatrix} \frac{1}{d}Ba \\ \frac{1}{dk}Ba \end{bmatrix}, \bar{P}_s = \begin{bmatrix} \frac{1}{d}\bar{P} \\ \frac{1}{dk}\bar{P} \end{bmatrix} \quad (49)$$

with $\bar{P} = [P_1^* \ P_2^*]^T$ and $e \doteq 1 + \epsilon d$. The eigenstructure of A_s in (48) can be directly computed as shown in the following.

Lemma IV.6 For system (48)–(49), the eigenvalues λ_i and eigenvectors v_i of the matrix A_s have the form

$$\lambda_1 = 0, \lambda_2 = -e/(dk), \lambda_{3,4} = -(ak + e/2 \mp R)/(dk) \quad (50)$$

$$V_s = \begin{bmatrix} \frac{1}{d}k/e & (e+dk\lambda_3)/(2a) & (e+dk\lambda_4)/(2a) \\ \frac{1}{d}k/e & -(e+dk\lambda_3)/(2a) & -(e+dk\lambda_4)/(2a) \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (51)$$

where $R \doteq \sqrt{(ak - e/2)^2 + 2ak}$. Moreover, $\lambda_{2,3,4} < 0$.

Proof: Similar to Lemma IV.4. ■

Let $\Lambda_s \doteq V_s^{-1}A_sV_s = \text{diag}\{\lambda_1, \dots, \lambda_4\}$ and consider the transformation $x = V_s z$. Then, from (48), the state z satisfies

$$\dot{z} = \Lambda_s z + V_s^{-1}H_s f(z) + V_s^{-1}\bar{P}_s \quad (52)$$

where, by direct computation

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 - 1/e \\ 1/k \end{bmatrix} \frac{P_2^* - P_1^*}{2d} \quad (53)$$

$$\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \lambda_3 \\ -\lambda_4 \end{bmatrix} \frac{1}{2R} \left(af(\theta_1 - \theta_2) + \frac{P_2^* - P_1^*}{2} \right). \quad (54)$$

Lemma IV.7 The two subsystems (53) and (54) are decoupled from each other.

Proof: From $x = V_s z$ and (51), $\theta_1 - \theta_2$ satisfies

$$\theta_1 - \theta_2 = \underbrace{[e/(2a) - k + R/a]}_{\alpha} z_3 + \underbrace{[e/(2a) - k - R/a]}_{\beta} z_4, \quad (55)$$

and hence, it only depends on the variables z_3 and z_4 . ■

The system (54) then takes the form

$$\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \lambda_3 \\ -\lambda_4 \end{bmatrix} \frac{a}{R} \left[\frac{f(\alpha z_3 + \beta z_4)}{2} + \frac{P_2^* - P_1^*}{4a} \right]. \quad (56)$$

We now proceed to the stability analysis of the transformed closed-loop system (53)–(54).

1) *Boundedness of ω_{sync} :* We show Theorem IV.2 for the decentralised secondary control using the subsystem (53). From $z = V_s^{-1}x$ it can be shown that

$$z_1 = (\theta_1 + \theta_2)/2 - kz_2/e, \quad z_2 = (p_1 + p_2)/2. \quad (57)$$

Since from (53) $\dot{z}_2 = \lambda_2 z_2 + (P_1^* + P_2^*)/(2dk)$ with $\lambda_2 = -e/(dk) < 0$, then z_2 and $\dot{z}_2$ remain bounded for bounded P_1^* and P_2^* . Furthermore, from (53) and (57) we have

$$\dot{z}_1 = (\dot{\theta}_1 + \dot{\theta}_2)/2 - k\dot{z}_2/e = (e-1)(P_1^* + P_2^*)/(2de). \quad (58)$$

Then ω_{sync} in (9), also remains bounded for bounded inverter power mismatches P_1^* and P_2^* . Moreover, from (53), z_2 converges to a constant if $(P_1^* + P_2^*)$ is constant and $\dot{z}_2$ goes to zero. Then, from (58) ω_{sync} converges to (17). ■

2) *Ultimate boundedness of $(\theta_1 - \theta_2)$:* We prove Theorem IV.3 for the decentralised secondary control case by applying Lemma II.1 to the decoupled subsystem (56), which has the form (1) with $\Lambda = \text{diag}\{\lambda_3, \lambda_4\}$ Hurwitz. Using the bound $|f(\alpha z_3 + \beta z_4)| \leq (|\alpha||z_3| + |\beta||z_4|)^3/6$, we can compute the mapping (3) as

$$T(z_3, z_4) \doteq \begin{bmatrix} |\lambda_3| & 0 \\ 0 & |\lambda_4| \end{bmatrix}^{-1} \begin{bmatrix} |\lambda_3| \\ |\lambda_4| \end{bmatrix} T_s(z_3, z_4) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} T_s(z_3, z_4).$$

with

$$T_s(z_3, z_4) \doteq \frac{a}{R} \left[\frac{(|\alpha||z_3| + |\beta||z_4|)^3}{12} + \frac{|P_2^* - P_1^*|}{4a} \right] \quad (59)$$

satisfying

$$\frac{a}{R} \left| \frac{f(\alpha z_3 + \beta z_4)}{2} + \frac{(P_2^* - P_1^*)}{4a} \right| \leq T_s(z_3, z_4). \quad (60)$$

For the contractivity condition $T(\bar{z}_3, \bar{z}_4) < (\bar{z}_3, \bar{z}_4)$ to hold componentwise, we select $(\bar{z}_3, \bar{z}_4) = (\bar{z}, \bar{z})$ and obtain

$$\bar{T}_s(\bar{z}) \doteq T_s(\bar{z}, \bar{z}) = \frac{a}{R} \left[\frac{(|\alpha| + |\beta|)^3 \bar{z}^3}{12} + \frac{|P_2^* - P_1^*|}{4a} \right] < \bar{z}. \quad (61)$$

From (50) and $a, k > 0$, it follows that $R/a > |e/(2a) - k|$ and hence, from (55), $\alpha > 0$ and $\beta < 0$, yielding $|\alpha| + |\beta| = \alpha - \beta = 2R/a$. Therefore, (62) takes the form

$$\bar{T}_s(\bar{z}) = \frac{a}{R} \left[\frac{2}{3} \left(\frac{R}{a} \bar{z} \right)^3 + \frac{|P_2^* - P_1^*|}{4a} \right] < \bar{z}. \quad (62)$$

Comparing with (45) we see that (62) is a special case corresponding to $k = 2$, and hence the result is obtained from the analysis following (45). ■

Remark IV.8 Considering (24), (33) and (52) for centralised, distributed and decentralised configurations of the

inverters, respectively, the equilibrium points are identical and calculated to be $\sin(\theta_1 - \theta_2) = 2\Omega$. Then, assuming that the angle difference is small and hence $\sin(\theta_1 - \theta_2) \approx (\theta_1 - \theta_2)$, the equilibrium points are well inside the ultimate bound which is equal to 3Ω as in (19).

V. EXAMPLE

Consider the two-inverter microgrid system of Figure 1 with the following parameters:

- Grid: $f = 60\text{Hz}$, $E = [120, 122]\text{V}$, $y = 3.8$ Siemens.
- Controller coefficients: $d = 4\text{kW.s}$, $k = 0.1\text{s}$, $\epsilon = 0.001$.
- Inverter settings: $P_{ref} = [2, 3]$ kW.
- Load demands: The initial values are $P_L = [1.5, 2]$ kW until $t = 6\text{s}$ when they change to $P_L = [1.8, 1]$ kW.

Figure 2(a) illustrates frequency regulation for the three types of secondary controller. As it can be seen, in the decentralised approach ω_{sync} converges to (17) in Theorem IV.2 whereas the other two yield zero frequency error.

For the initial load demand values, we have $P^* = P_{ref} - P_L = [0.5, 1]^T$ kW and using $a = y_{12}E_1E_2 = 5.5477 \times 10^4$, the condition $\Omega = |P_1^* - P_2^*|/(4a) < \sqrt{2}/3$ is satisfied as $0.0023 < 0.4714$. Similar calculations show that the condition also holds for the load demand values after $t = 6\text{s}$. Hence, according to Theorem IV.3, the difference between bus angles starting within the RoA ultimately lie and remain inside the UB set for both load conditions.

From (18), the projection of the system RoA on the bus angle difference contains the set $|\theta_1 - \theta_2| \leq 2.4445$. Similarly, from (19) the UB set is contained in the set $|\theta_1 - \theta_2| \leq b$, where $b = 0.0068$ for $t \in [0, 6]$ and $b = 0.0243$ for $t \in [6, 10]$. Starting inside the RoA, satisfaction of the UB can be seen from Figure 2(b) where the bus angle differences are shown to lie inside the desired region of the state space.

Fig. 2. (a) Average frequency error, (b) Bus angle difference ultimate boundedness

VI. CONCLUSIONS

The paper has applied an analytical methodology from [8] to analyse ultimate boundedness of the states of the parallel interconnection of two inverters. The inverters are linked

by an inductive line, and regulate their frequency using primary and secondary droop control mechanisms. Closed-form expressions are derived for the ultimate state bounds and their associated RoA under three types of secondary frequency control mechanisms: (1) centralised, where a central controller receives information from each inverter to generate common control signals; (2) distributed, where each inverter is independently controlled using information from other buses (but not necessary all of them); and (3) decentralised, where each inverter is independently controlled using exclusively locally available information. While the analysis of [8] applies to any number of inverter buses, only the decentralised controller case was considered and closed-loop expressions were not derived.

The expressions for the RoAs and UBs are functions of the line impedance, and the mismatch between generation and demand in each bus. Interestingly, and perhaps surprisingly, the expressions are identical for all three controllers considered. Preliminary simulation studies show that similar results may apply to more general microgrids. This is the subject of our current investigations.

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