On the normalized turbulent energy dissipation rate

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This paper revisits values of the normalized energy dissipation rate (C_ϵ) in different flows (two-dimensional wakes, grid turbulence, and homogeneous shear flow). Previously published as well as new data are considered over a relatively wide range of the Taylor-microscale Reynolds number R_λ . C_ϵ exhibits wide scatter (in the range 0.5–2.5 for $R_\lambda > 50$) although, for a given flow and initial conditions, it is independent of R_λ when the latter is sufficiently large. An alternative definition [B. R. Pearson, P.-Å. Krogstad, and W. van de Water, “Measurements of the turbulent energy dissipation rate,” *Phys. Fluids* **14**, 1288 (2002)] of C_ϵ has been checked in the same flows but has failed to yield a universal value for the coefficient. © 2005 American Institute of Physics. [DOI: 10.1063/1.2055529]

The dependence of the turbulent energy dissipation rate $\langle \epsilon \rangle$ (here, angular brackets denote averaging with respect to time) on the Taylor-microscale Reynolds number R_λ has been studied extensively. Frisch¹ noted that the constancy of $C_\epsilon (\equiv \langle \epsilon \rangle L / \langle u^2 \rangle^{3/2})$, where $3\langle u^2 \rangle / 2$ is the mean turbulent kinetic energy and L the integral length scale) at large enough R_λ is one of the two experimental laws of fully developed turbulence. Others^{2,3} refer to it as the zeroth law of turbulence. Kolmogorov implicitly assumed that, for large values of R_λ , the mean energy dissipation rate should be independent of the viscosity. Thus, for a fixed turbulent kinetic energy, C_ϵ should attain a constant value as the viscosity is decreased, i.e., as R_λ is increased. Batchelor⁴ tried to verify this hypothesis in grid turbulence, albeit for a restricted range of low R_λ . While there is general consensus that C_ϵ becomes constant for moderately large R_λ (≥ 50 , e.g., Ref. 5), the actual asymptotic values of C_ϵ in different turbulent flows such as wakes, jets, grid turbulence, and in numerical simulations of box turbulence display a large scatter, e.g., Refs. 6–8. Even for grid turbulence, which is to a reasonable approximation homogeneous and isotropic, differences arise between different types of grids.⁶ A unique value of C_ϵ would imply that the small scales are independent of the viscosity, in the limit of large R_λ .

In this paper, we consider new as well as published data of C_ϵ for different R_λ and in several flow types. This collection of data represents an update of the data in Ref. 6 (which are not repeated here) and aims at assessing the influence of initial conditions on the values of C_ϵ , a point left open by Sreenivasan.^{6,7} The other purpose is to evaluate a recent suggestion for an alternate definition of the dissipation rate coefficient. Pearson *et al.*⁹ proposed that C_ϵ should be redefined using the integral length scale (L_p) derived from the inverse of the wave number ($L_p \equiv 1/k_{1p}$) at which the one-dimensional (1D) longitudinal energy spectrum premultiplied by the longitudinal wave number (i.e., $k_1 E_{11}$) is maximum. According to these authors, the resulting coefficient,

$C_\epsilon^p (\equiv \langle \epsilon \rangle L_p / \langle u^2 \rangle^{3/2})$, is independent of the flow and is close to 0.5, when $R_\lambda \gtrsim 300$. We recall that, in experiments, the integral length scale (L), which features in C_ϵ , is usually defined as

$$L \equiv \int_0^{r_0} \rho_{uu}(r_1) dr_1, \quad (1)$$

where $\rho_{uu}(r_1) \equiv \langle u(x_1)u(x_1+r_1) \rangle / \langle u^2 \rangle$ is the autocorrelation coefficient of the longitudinal velocity fluctuation, the integration being usually extended to the first zero crossing (r_0) of ρ_{uu} . This limit avoids the difficulty associated with unsteadiness at low wave numbers of the velocity signal, which may prevent the autocorrelation from reaching zero at large separations, much larger than the integral length scale. In isotropic turbulence and with r_0 in Eq. (1) equal to ∞ , this definition is equivalent to

$$L = \frac{\pi E_{11}(0)}{2\langle u^2 \rangle}, \quad (2)$$

since E_{11} and ρ_{uu} are a Fourier pair, i.e., $E_{11}(k_1) = 2/(\pi \langle u^2 \rangle) \int_0^\infty \rho_{uu}(r_1) \cos(k_1 r_1) dr_1$. A different definition,

$$L \equiv \frac{\pi}{2\langle u^2 \rangle} \int_0^{k_{\max}} k^{-1} E(k) dk, \quad (3)$$

is almost always applied to direct numerical simulations (DNS) data for which the three-dimensional (3D) energy spectrum $E(k)$ is known, up to a maximum value $k_{\max}$ of the wave number. For homogeneous isotropic turbulence, (2) and (3) are equivalent (e.g., Ref. 10), which can be readily shown by considering the isotropic relation

$$E(k) = \frac{1}{2} k^3 \frac{d}{dk} \left(\frac{1}{k} \frac{dE_{11}(k_1)}{dk_1} \right) \bigg|_{k_1=k}$$

and integrating by parts.

The measurement of the mean energy dissipation rate is fraught with uncertainties, especially in connection with probe resolution and electronic noise. Further, if the turbulence is not isotropic, the 1D estimate $\langle \epsilon \rangle_{\text{iso}} = 15\nu \langle (\partial u / \partial x)^2 \rangle$

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*et al.*⁹ (their Fig. 1). It seems that the latter authors, when including the data of Ref. 12, assumed a *constant* value of L_p (≈ 0.1 m), irrespectively of R_λ (caption of their Fig. 1). Pearson *et al.*'s reason for preferring L_p over L_u is that there is a Strouhal number, associated with a "resemblance of a vortex street,"⁹ which is related to the peak in the distribution of $k_1 E_{11}$. This is probably relevant to the far wake of a bluff body, where coherent vortices can still be discerned.^{16,17} In the initial period of decay of grid turbulence, a large scale organization would not be expected. In this case, the peak of the spectrum (after multiplying by k_1) cannot be interpreted as a signature of periodicity in the flow. It thus seems misleading to try to reconcile the differences in the values of C_ϵ by resorting to a large scale periodicity that would apply to all flow types.

A further comment needs to be made with regard to the conclusion of Pearson *et al.* These authors used Eq. (3) for their DNS data of homogeneous isotropic turbulence. For this flow, L is not necessarily equal to L_p . To show this, one needs to consider a form of $E(k)$, such as¹⁸

$$E(k) = \frac{\langle u^2 \rangle L C_p (kL)^p}{[1 + (kL/k_e L)^2]^{p/2+5/6}}, \quad (4)$$

where the constants C_p and $k_e L$ can be expressed in terms of p (the exponent of the spectrum for $k \rightarrow 0$) only, i.e.,

$$C_p = \left[\frac{4}{\pi B(\frac{5}{6}, \frac{1}{2}p)} \right]^{1+p} \left[\frac{B(\frac{1}{3}, \frac{1}{2}(p+1))}{3} \right]^p,$$

and

$$k_e L = \frac{3\pi}{4} \frac{B(\frac{5}{6}, \frac{1}{2}p)}{B(\frac{1}{3}, \frac{1}{2}(p+1))},$$

where B is the beta function. The 1D energy spectrum, $E_{11}(k_1)$, can be calculated from

$$E_{11}(k_1) = \int_{k_1}^{\infty} \frac{E(k)}{k} \left(1 - \frac{k_1^2}{k^2} \right) dk. \quad (5)$$

Using (4), the integral can be expressed in terms of the incomplete beta function, $\tilde{B}$, for $p \geq 3$,

$$\begin{aligned} \frac{E_{11}(k_1)}{\langle u^2 \rangle L} &= C_p (k_e L)^p \frac{1}{2} \tilde{B}\left(\frac{5}{6}, \frac{1}{2}p; z\right) \\ &\quad - C_p (k_e L)^{p-2} (k_1 L)^2 \frac{1}{2} \tilde{B}\left(\frac{11}{6}, \frac{p}{2} - 1; z\right) \end{aligned}$$

with

$$z = \frac{1}{1 + \left[k_1 L \frac{4}{3\pi} \frac{B(\frac{1}{3}, \frac{1}{2}(p+1))}{B(\frac{5}{6}, \frac{1}{2}p)} \right]^2}.$$

It can be shown that the condition $\max[k_1 E_{11}(k_1)]$, verified for $k_1 = k_{1\max} = 1/L_p$, implies

FIG. 3. Distribution of C_ϵ as a function of R_λ for different numerical simulations of decaying and forced homogeneous isotropic turbulence (adapted from Ref. 7). ■, Jimenez *et al.* (Ref. 22); ▲, ◇, Wang *et al.* (Ref. 23); ◆, Yeung and Zhou (Ref. 24); ★, Cao *et al.* (Ref. 25); ●, Gotoh *et al.* (Ref. 20); ►, Kaneda *et al.* (Ref. 19).

$$\tilde{B}\left(\frac{5}{6}, \frac{1}{2}p; z\right) = 3 \frac{(k_{1\max} L)^2}{(k_e L)^2} \tilde{B}\left(\frac{11}{6}, \frac{p}{2} - 1; z\right),$$

which is an implicit equation for $k_{1\max}$. For other values of $p < 3$, a numerical integration of Eq. (5) provides $E_{11}(k_1)$, from which the location of the maximum of $k_1 E_{11}(k_1)$ can be determined. For $p = 1, 2, 3, 4$, L_p/L is equal to 0.79, 0.97, 1.05, 1.09, respectively. For $p = 4$, the value in the von Kármán spectrum,¹⁰ the difference between the two definitions is 10%. This suggests that L and L_p are not interchangeable in homogeneous isotropic turbulence, as implied by Pearson *et al.*,⁹ who assimilated L_p to L in their DNS data, even though the difference between the two definitions is relatively small in this case.

To summarize, we have reported values of the normalized energy dissipation rate in different flows and for different initial conditions. We have used two different definitions for C_ϵ and shown that a universal value for this coefficient is not tenable. The flow type and initial conditions (for any given flow type) seem to have a persistent influence even in the fully developed region of the flows. Part of the scatter in the published asymptotic values of C_ϵ is ascribable to the uncertainty in measuring the energy dissipation rate and/or to errors in estimating the integral length scale. In direct numerical simulations of box turbulence,^{7,19,20} where all the quantities can arguably be evaluated more reliably, C_ϵ is much less dispersed and seems to converge to 0.5 for $R_\lambda \gtrsim 200$ (see Fig. 3).

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¹U. Frisch, *Turbulent Flows* (Cambridge University Press, Cambridge, 1995).

²G. Eyink, *Zeroth Law of Turbulence, Program on Physics of Hydrodynamic Turbulence*, P. Constantin, I. Procaccia, B. Shraiman, and K. Sreenivasan, coordinators (2000); available from <http://online.itp.ucsb.edu/online/hydt00/eyink/>

- ³B. R. Pearson, T. A. Yousef, N. E. L. Haugen, A. Brandenburg, and P.-Å. Krogstad, The “zeroth law” of turbulence in steady isotropic turbulence, in *15th Australasian Fluid Mechanics Conference*, edited by M. Behnia, W. Lin, and G. D. McBain, paper N. AFMC00040 (The University of Sydney, 2004).
- ⁴G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, 1953).
- ⁵J. L. Lumley, “Some comments on turbulence,” *Phys. Fluids A* **4**, 203 (1992).
- ⁶K. R. Sreenivasan, “On the scaling of the turbulence energy dissipation rate,” *Phys. Fluids* **27**, 1048 (1984).
- ⁷K. R. Sreenivasan, “An update on the energy dissipation rate in isotropic turbulence,” *Phys. Fluids* **10**, 528 (1998).
- ⁸R. A. Antonia and B. R. Pearson, “Effect of initial conditions on the mean energy dissipation rate and the scaling exponent,” *Phys. Rev. E* **62**, 8086 (2000).
- ⁹B. R. Pearson, P.-Å. Krogstad, and W. van de Water, “Measurements of the turbulent energy dissipation rate,” *Phys. Fluids* **14**, 1288 (2002).
- ¹⁰S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000).
- ¹¹R. A. Antonia, T. Zhou, and G. P. Romano, “Small-scale turbulence characteristics of two-dimensional bluff body wakes,” *J. Fluid Mech.* **459**, 67 (2002).
- ¹²L. Mydlarski and Z. Warhaft, “On the onset of high-Reynolds-number grid-generated wind tunnel turbulence,” *J. Fluid Mech.* **320**, 331 (1996).
- ¹³H. S. Kang, S. Chester, and C. Meneveau, “Decaying turbulence in an active-grid-generated flow and comparisons with large-eddy simulation,” *J. Fluid Mech.* **480**, 129 (2003).
- ¹⁴X. Shen and Z. Warhaft, “The anisotropy of the small scale structure in high Reynolds number ($R_\lambda \sim 1000$) turbulent shear flow,” *Phys. Fluids* **12**, 2976 (2000).
- ¹⁵M. Ferchichi and S. Tavoularis, “Reynolds number effects on the fine structure of uniformly sheared turbulence,” *Phys. Fluids* **12**, 2942 (2000).
- ¹⁶A. K. M. F. Hussain and M. Haykawa, “Eduction of large-scale organised structures in a plane turbulent wake,” *J. Fluid Mech.* **180**, 193 (1987).
- ¹⁷R. A. Antonia, L. W. B. Browne, D. K. Bisset, and L. Fulachier, “A description of the organized motion in the turbulent far-wake of a cylinder at low Reynolds number,” *J. Fluid Mech.* **184**, 423 (1987).
- ¹⁸H. Wang and W. K. George, “The integral scale in homogeneous isotropic turbulence,” *J. Fluid Mech.* **459**, 429 (2002).
- ¹⁹Y. Kaneda, T. Ishihara, M. Yokokawa, K. Itakura, and A. Uno, “Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box,” *Phys. Fluids* **15**, L21 (2003).
- ²⁰T. Gotoh, D. Fukayama, and T. Nakano, “Velocity field statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation,” *Phys. Fluids* **14**, 1065 (2002).
- ²¹J. V. Larssen and W. J. Devenport, “The generation of high Reynolds number homogeneous turbulence,” AIAA Paper 2002-2861 (2002).
- ²²J. Jimenez, A. A. Wray, P. G. Saffman, and R. S. Rogallo, “The structure of intense vorticity in homogeneous isotropic turbulence,” *J. Fluid Mech.* **255**, 65 (1993).
- ²³L. P. Wang, S. Chen, J. G. Brasseur, and J. C. Wyngaard, “Examination of hypotheses in the Kolmogorov refined turbulence theory through high resolution simulations. Part 1. Velocity field,” *J. Fluid Mech.* **309**, 113 (1996).
- ²⁴P. K. Yeung and Y. Zhou, “On the universality of the Kolmogorov constant in numerical simulations of turbulence,” *Phys. Rev. E* **56**, 1746 (1997).
- ²⁵N. Cao, S. Chen, and G. D. Doolen, “Statistics and structures of pressure in isotropic turbulence,” *Phys. Fluids* **11**, 2235 (1999).