Spatial frequency antiwindup strategy for cross-directional control problems

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Abstract: A MIMO antiwindup strategy for cross-directional control of web forming processes is presented. The scheme relies on a basis function expansion of the system interaction matrix which allows one to analyse the problem in terms of the spatial frequency components of the web profile. Antiwindup protection is achieved by prioritising the control directions in which the available control authority is employed. In particular, disturbance compensation is first sought in those spatial frequency components associated with the largest gain, that is, where the required control effort is smaller. Stability of the scheme is analysed for the case in which complete knowledge of the input and output rotation matrices is assumed. Simulation results show that the steady-state performance attained with the proposed strategy is superior to that achieved by traditional MIMO antiwindup schemes and is close to an optimal QP solution to the problem.

1 Introduction

Web forming processes can be found in many practical applications, including paper making [1–3], flat metal rolling [4–6] and plastics film extrusion [7, 8]. The control objective is usually twofold: to regulate the sheet property profile along the machine, referred to as the machine-direction (MD), and to maintain a flat profile across the machine web, referred to as the cross-direction (CD).

Some of the key difficulties associated with this type of processes are as follows.

- The system is usually of high dimension, with the number of actuators ranging from 30 up to possibly 300, and sensor measurements from 200 to 1000 [9].
- Actuators are usually positioned in an array along the cross-direction of the web. The control action exerted by one actuator does not only affect the measurement of the sensor in a direct downstream position, but may also influence several other sensors placed nearby. Depending on the degree of coupling between the steady-state effect of the actuators, poor conditioning of the interaction matrix can result.
- The actuators usually have limited authority. Hence, actuator saturation is very likely to occur. (From this point of view, web forming processes can be thought of as being similar to under-actuated systems, since there exist spatial components of the output space that are not controllable in any significant practical sense due to input constraints).

In many applications of cross-directional control, the process is modelled with an interaction matrix $\mathbf{M}$, together with a decoupled linear and time-invariant dynamical system $\mathbf{G}(s)$. This suggests that, in the absence of actuator saturation, a satisfactory control strategy can be obtained by including the inverse of $\mathbf{M}$ in the controller definition and using a decentralised PID control law. However, as might be expected, this simplified scheme will fail to operate effectively in the presence of significant actuator saturation.

The robustness issue associated with poor conditioning of cross-directional systems has been extensively studied in earlier contributions [10–13]. Work by Heath [7] and Duncan *et al.* [14] has shown that there exists a maximum set of basis functions (the spatial bandwidth) that can be controlled. This bandwidth corresponds to the rank of the interaction matrix $\mathbf{M}$. All spatial frequencies beyond the spatial bandwidth are uncontrollable [7]. Spatial frequency modes which have considerable uncertainty in the sign of their gain are also uncontrollable. In fact, if the sign of a certain spatial-mode gain is not known with confidence, then no internal model controller can stabilise the system [11].

The input constraint problem has received less attention within the cross-directional control literature. Var Antwerp *et al.* [10], for example, maintain that the use of a robust control strategy can avoid the necessity of an explicit handling of input constraints. However, constraints are ubiquitous in all practical control systems and they may affect the closed-loop performance to a great extent, especially in cross-directional control systems where, as stated, the actuators usually have limited authority. Whenever the input constraint problem has been addressed, the standard approach is usually model predictive control [10]. However, the main drawback of the method is that MPC can be computationally demanding, especially when dealing with a system with possibly up to 200 actuators and sensor measurements. This places a lower limit on the controller update time, making this implementation unsuitable for rapidly changing processes with small time constants [15, 16].

An alternative approach to handling input constraints is the design of a linear controller incorporating antiwindup compensation. This is indeed a common practice in industry; its main advantage being that it is a simple technique to implement. Nevertheless, a MIMO antiwindup controller cannot be exempted from the inherent difficulty associated with the control of poorly conditioned systems, such as cross-directional control systems. In fact, this type of system usually has several control directions that are hard to control, in the sense that a considerable control effort is necessary to compensate for a disturbance acting in one or more of those directions. Therefore in the presence of input constraints, the manipulated variables are likely to be operating on the constraint boundary during significant time periods. This effect commonly manifests itself in the observed phenomenon of picketing, with each actuator being driven to alternate saturation levels. As a result, important performance degradation may be expected.

We present and analyse an antiwindup scheme specifically directed at underactuated cross-directional control problems. The main objective is to develop a control strategy limited in complexity but, at the same time, able to address the input constraint problem in web forming processes with improved performance results compared with other antiwindup solutions. The proposed controller still builds on the ideas common to many antiwindup schemes described in the literature [17–22], but takes special consideration of the specific features of cross-directional control problems.

We describe our scheme in a general setting. However, we give particular emphasis in the analysis of the method to the special case of the circulant symmetric interaction matrix $\mathbf{M}$. This is characteristic of blown bubble film extrusion processes. A key feature of these processes is that the eigenvectors of $\mathbf{M}$ can always be represented by the same set of unitary-invariant basis functions, namely the columns of a Fourier matrix [23, 24]. All MIMO problems of this nature can be decomposed into a set of SISO control problems. This platform is chosen for the analysis of the proposed antiwindup scheme owing to its simple structure. We also argue that this gives a good approximation in the noncirculant case provided the number of actuator–sensor pairs is large.

2 Background to cross-directional control problems

It is commonly assumed that a web forming process can be modelled by separating its spatial response from its dynamical response using a model of the following form [24]:

$$\mathbf{y}(s) = g(s)\mathbf{M}\mathbf{u}(s) \quad (1)$$

where $\mathbf{y}(s) \in \mathbb{R}^N$ is the vector of sensor measurements and $\mathbf{u}(s) \in \mathbb{R}^N$ is the vector of control signals. $\mathbf{M} \in \mathbb{R}^{N \times N}$ is a constant and real-valued matrix, known as the normalised steady-state interaction matrix. As its name suggests, this matrix represents the spatial influence of each actuator on the system outputs.

In this paper, the number of sensors and actuators is assumed equal. In practice, the sensors will usually outnumber the actuators. In that case, the sensor data can be filtered in the spatial domain to reduce its order while minimising the loss of controllable information [3]. However, care must be taken as this procedure can produce performance degradation when model uncertainty and actuator constraints are taken into account [25].

The system model (1) assumes that the process dynamics are identical across the machine, since all the dynamics are represented by the scalar transfer function $g(s)$. In general the transfer function $g(s)$ is assumed to be a low order, stable and minimum-phase system, possibly having a transport delay $\theta > 0$ [24, 26]

$$g(s) = \frac{e^{-\theta s}}{\tau s + 1} \quad (2)$$

where $\tau > 0$. More sophisticated models are possible when considering different time constants along the cross-direction or when incorporating extra features such as multiple poles or zeros.

The control objective of cross-directional control systems is usually stated in terms of the steady-state variance of the profile across the web, that is

$$\min_{\mathbf{u} \in \mathbb{U}} \|\bar{\mathbf{y}}\|_2^2, \quad \bar{\mathbf{y}} = \lim_{t \rightarrow \infty} \mathbf{y}(t) \quad (3)$$

where the vector of manipulated variables $\mathbf{u}(t)$ is forced to belong to the constraint polytope $\mathbb{U}$

$$\mathbb{U} = \{\mathbf{u}(t) \in \mathbb{R}^N \mid \|\mathbf{u}(t)\|_\infty \leq \Delta\} \quad (4)$$

$\Delta > 0$ represents the saturation limit imposed on every element of the control vector $\mathbf{u}(t)$.

The simple model structure (1) suggests that, if an inverse for the interaction matrix $\mathbf{M}$ exists, then, by introducing $\mathbf{M}^{-1}$ into the controller, we should be able to obtain a decoupled MIMO control system. However, $\mathbf{M}$ will not in general be a square matrix and if it is, it is likely to be ill-conditioned due to actuator interaction [14], as described in Section 1. To deal with these difficulties a singular value decomposition (SVD) of the interaction matrix $\mathbf{M}$ is usually considered [24]

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (5)$$

where $\mathbf{S}, \mathbf{U}, \mathbf{V} \in \mathbb{R}^{N \times N}$. $\mathbf{S}$ is a diagonal matrix with positive singular values arranged in decreasing order; $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices whose columns generate an orthonormal basis for the system output and input space respectively, that is $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I}$. Featherstone and Braatz [11] have suggested an alternative decomposition of $\mathbf{M}$ equivalent to (5), in which the entries of $\mathbf{S}$ are allowed to have a sign. They are then referred to as pseudosingular values [11, 27–29]. This description allows the extension to the MIMO case of some known results in the SISO case. For example, in the scalar case, it is clear that the sign of the plant DC gain needs to be correctly identified to ensure that the corresponding closed loop is stable; otherwise this would be equivalent to having positive feedback in the control loop instead of negative feedback. For large-scale systems a similar conclusion can be drawn in terms of the associated pseudosingular values. Indeed, if the sign of any pseudosingular value of the system is not known with confidence, then no internal model controller can stabilise the system even if complete knowledge of matrices $\mathbf{U}$ and $\mathbf{V}$ is assumed [11].

Closely related to the SVD of $\mathbf{M}$ in (5) is the eigenvalue–eigenvector decomposition of a symmetric matrix (indeed, if $\mathbf{M}$ is symmetric and positive definite, both decompositions are equivalent [30]). This approach is especially insightful when $\mathbf{M}$ is circulant, since any circulant matrix $\mathbf{C} \in \mathbb{C}^{N \times N}$ has a complete set of independent eigenvectors. Moreover, every circulant matrix $\mathbf{C}$ of order N can be

diagonalised by the same eigenvector matrix, namely the Fourier matrix $\mathbf{W}$ given by [31]

$$\mathbf{W}^H = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{N-2} & \cdots & w \end{bmatrix} \quad (6)$$

where $w = e^{j(2\pi/N)}$. Hence

$$\mathbf{C} = \mathbf{W}\mathbf{\Lambda}_C\mathbf{W}^H \quad (7)$$

where $\mathbf{\Lambda}_C$ is a diagonal matrix containing the eigenvalues of $\mathbf{C}$. When a circulant matrix $\mathbf{C}$ is symmetric, the analysis can be simplified further, since the associated eigenvector matrix can be chosen to be the so-called real Fourier matrix $\mathbf{R}$ [24, 31, 32]. Hence

$$\mathbf{C} = \mathbf{R}\mathbf{\Lambda}_C\mathbf{R}^T \quad (8)$$

This is a result of the fact that the eigenvalues of $\mathbf{C}$ appear in pairs, thus the corresponding eigenvectors can be chosen to be real.

Remark 2.1: Note that both matrices $\mathbf{W}$ and $\mathbf{R}$ are independent of the actual entries in the matrix $\mathbf{M}$; indeed, both can be easily computed based only on knowledge of the dimension N of $\mathbf{M}$. This is a very useful feature of circulant (symmetric) matrices which allows a completely decoupled controller design and synthesis, even when model uncertainties are present. However, large-scale circulant systems are still subject to robustness issues due to poor conditioning. In fact, as mentioned, a nonreliable estimation of the sign associated with the smallest pseudosingular values of $\mathbf{M}$ can lead to instability in the corresponding control loop.

The diagonalisation of $\mathbf{C}$ through $\mathbf{W}$ and $\mathbf{R}$ in (7) and (8) suggests a direct connection between the singular values (or the absolute value of the eigenvalues) of a circulant (symmetric) interaction matrix $\mathbf{M}$ and the discrete Fourier transform of the weighting sequence $\{m_0, m_1, \dots, m_{N-1}\}$ representing the steady-state effect of one single actuator over the entire set of outputs. This connection is described in the following lemma:

Lemma 2.1: Consider a circulant interaction matrix $\mathbf{M}$, and the associated set of singular values $\{\sigma_0, \sigma_1, \dots, \sigma_{N-1}\}$. Also, consider the DFT of the sequence $m_{0,l}$, $l=0, \dots, N-1$ which, in turn, is the first row of $\mathbf{M}$, given by

$$\hat{m}_l = \sum_{k=0}^{N-1} m_{0,k} w^{-lk} \quad l=0, \dots, N-1 \quad (9)$$

Then

$$\sigma_l = |\lambda_l| = |\hat{m}_l| \quad \forall l \quad (10)$$

Proof: Since $\mathbf{M}$ is circulant, its eigenvalue-eigenvector decomposition is given by (7), using the Fourier matrix $\mathbf{W}$ defined in (6). Therefore we can write

$$\mathbf{W}^H \mathbf{M} \mathbf{W} = \mathbf{\Lambda}_M \quad (11)$$

$\mathbf{W}$ can be expressed in terms of its column vectors (or eigenvectors of $\mathbf{M}$)

$$\mathbf{W} = \frac{1}{\sqrt{N}} [\mathbf{w}_0 \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_{N-1}] \quad (12)$$

where

$$\mathbf{w}_l = [1 \ w^{-l} \ w^{-2l} \ \cdots \ w^{-(N-1)l}]^T \quad (13)$$

If we analyse in more detail the product $\mathbf{M}\mathbf{W}$ we can easily see from (9) and (12) that the first row of the matrix $\mathbf{M}\mathbf{W}$ contains the DFT of the sequence $m_{0,l}$. We also note that, since $\mathbf{M}$ is circulant, its i th row is equivalent to its first row shifted i positions right, and since the entries of the column vector $\mathbf{w}_l$ in (13) are also circulant with the same period N , we can write

$$\begin{aligned} [\mathbf{M}\mathbf{W}]_{(i,l)} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} m_{i,k} w^{-lk} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} m_{0,k} w^{-(l(k+i))} \\ &= \frac{1}{\sqrt{N}} w^{-il} \sum_{k=0}^{N-1} m_{0,k} w^{-lk} = \frac{1}{\sqrt{N}} w^{-il} \hat{m}_l \end{aligned} \quad (14)$$

Since w^{-il} is the i th entry of the column vector $\mathbf{w}_l$ in (13), we can express the product $\mathbf{M}\mathbf{W}$ as follows:

$$\mathbf{M}\mathbf{W} = \frac{1}{\sqrt{N}} [\hat{m}_0 \mathbf{w}_0 \ \hat{m}_1 \mathbf{w}_1 \ \cdots \ \hat{m}_{N-1} \mathbf{w}_{N-1}] \quad (15)$$

We know that $\mathbf{W}$ is a unitary matrix i.e., $\mathbf{W}^H \mathbf{W} = \mathbf{W}\mathbf{W}^H = \mathbf{I}$, which is equivalent to

$$\mathbf{w}_i^H \mathbf{w}_j = \begin{cases} N & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (16)$$

Therefore evaluating the product $\mathbf{W}^H \mathbf{M}\mathbf{W}$ in (11) using (15) and (16) we finally obtain

$$\mathbf{W}^H \mathbf{M}\mathbf{W} = \text{diag}\{\hat{m}_0, \hat{m}_1, \dots, \hat{m}_{N-1}\} = \mathbf{\Lambda}_M \quad (17)$$

which shows that the eigenvalues of $\mathbf{M}$ are equal to the DFT of the sequence $\{m_{0,l}\}$. $\square$

Using the result of lemma 2.1 we can easily compute the singular values of a circulant matrix by simply performing a DFT of the weighting sequence $m_{0,l}$. In this way we are able to use a discrete frequency analysis approach to estimate the qualitative behaviour of the singular values of $\mathbf{M}$. The following example illustrates this idea.

Example 2.2: Consider the model (1) of the cross-directional dynamic behaviour of a blown-film process [27]. This type of process has an interaction matrix $\mathbf{M}$ with a circulant symmetric structure. Let $N=2n+1$, $n \in \mathbb{R}$, and assume that the effect of one actuator over the system outputs is given by the following profile:

$$m_{0,l} = \begin{cases} e^{-\alpha l} & \text{for } l=0, 1, \dots, n, \\ e^{-\alpha(N+1-l)} & \text{for } l=n+1, \dots, 2n. \end{cases} \quad (18)$$

where $\alpha > 0$. Since $\mathbf{M}$ is circulant and symmetric, the associated eigenvalues are real and equal to the singular values of $\mathbf{M}$. Based on the result of lemma 2.1 the value of σ_l is equal to the l th coefficient of the DFT of $m_{0,l}$ in (18)

$$\begin{aligned} \sigma_l &= 1 + e^{-\alpha} w^l + \cdots + e^{-\alpha n} w^{ln} + e^{-\alpha n} w^{l(n+1)} \\ &\quad + \cdots + e^{-\alpha} w^{2ln} \end{aligned} \quad (19)$$

Recall that w^k with $k=0, 1, \dots, N-1$ are the roots of $w^N = 1$ and they are evenly distributed over the unity circle. As a result they are the complex conjugate of each other in pairs i.e.

$$w^k = w^{-(2n+1-k)} \quad \text{for } k=1, \dots, n \quad (20)$$

Hence (19) can be rewritten as

$$\begin{aligned} \sigma_l &= 1 + \sum_{k=1}^n e^{-\alpha k} w^{lk} + \sum_{k=1}^n e^{-\alpha k} w^{-lk} \\ &= 1 + \frac{e^{-\alpha} w^l e^{-\alpha(n+1)} w^{l(n+1)}}{1 - e^{-\alpha} w^l} + \frac{e^{-\alpha} w^{-l} e^{-\alpha(n+1)} w^{-l(n+1)}}{1 - e^{-\alpha} w^{-l}} \end{aligned} \quad (21)$$

Fig. 1 Singular values of matrix $\mathbf{M}$ for example 2.2

Using (21) it can be shown that the singular values of $\mathbf{M}$ repeat themselves in pairs, that is $\sigma_l = \sigma_{N+1-l}$ for $l = 1, \dots, n$.

It is illustrative to plot the values of σ_l in a logarithmic scale against the values of the spatial frequencies $\{0, w_N, 2w_N, \dots, nw_N\}$, where $w_N = 2\pi/N$. This is shown in Fig. 1 where $\alpha = 0.1$ and $N = 401$ are considered. Such a large number of actuators and measurements is not unusual in paper making applications [9]. The result is clearly close to the estimated spectrum of the continuous time signal $f(t) = e^{-\alpha|t|}$, obtained from a sampled sequence of N points with normalised sampling period $T_s = 1[s]$ (dashed line in Fig. 1). The difference results from the symmetry about the sampling frequency inherent in the DFT.

3 Spatial frequency antiwindup strategy

The singular value decomposition of the interaction matrix $\mathbf{M}$ suggests that this type of system can be effectively dealt with by using a decoupling MIMO controller having the following structure:

$$\mathbf{C}(s) = \mathbf{V}\mathbf{C}^*(s)\mathbf{U}^T \\ = \mathbf{V} \text{diag}\{c_1^*(s), c_2^*(s), \dots, c_N^*(s)\}\mathbf{U}^T \quad (22)$$

Each scalar controller $c_i^*(s)$ $i = 1, \dots, N$ can be designed separately, by means of any appropriate control design strategy. For example, in the case of the simple dynamics associated with web forming processes, one may adopt a PI control law of the form

$$c_i^*(s) = \frac{\hat{\tau}s + 1}{\hat{k}_i s} \quad (23)$$

If special handling of the time delay T_d is needed, then a Smith predictor can be a possible choice.

A controller such as the one described in (22) and (23) will perform adequately, provided system (1) is well behaved and the manipulated variable $\mathbf{u}(t)$ is not subject to any constraint. However, as described in Section 1, this is usually not the case for high-dimensional systems such as those analysed here.

In terms of the poor conditioning of $\mathbf{M}$ and the associated uncertainty about the value and sign of the smallest pseudosingular values of the interaction matrix $\mathbf{M}$, the

generally accepted advice is to avoid controlling the plant in directions associated with uncertain gains. These, in turn, are referred to as 'uncontrollable modes' [7, 14]. Therefore if only a limited subset $\{\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_r\}$ with $r \leq N$ of the pseudosingular values of $\mathbf{M}$ are known with reasonable confidence, a robust controller structure would be similar to (22) with $\mathbf{C}^*(s)$ defined as follows [11]:

$$\mathbf{C}^*(s) = \mathbf{C}_r^*(s) = \text{diag}\{c_1^*(s), \dots, c_r^*(s), 0, \dots, 0\} \quad (24)$$

where we see that the control action is effectively disabled in the 'uncontrollable' directions by setting the corresponding controller gains to zero; of course this does not explicitly account for controller saturation.

In the presence of input constraints there are several control strategies that may be adopted. Among them, antiwindup controllers have the advantage of being a simple, yet effective option compared with other methods, like MPC which can be computationally more demanding. Antiwindup schemes can be constructed by rearranging the control law so that all of the controller's dynamics are driven by the real (i.e. constrained) actuator signals. This concept is common to many apparently dissimilar controllers incorporating antiwindup protection [17].

To describe the idea in more detail, assume, without loss of generality, that $\mathbf{C}^*(s)$ in (22) is biproper (if this is not the case, additional 'fast' zeros can be appended). Thus, we can write

$$\mathbf{C}^*(s)^{-1} = \mathbf{C}_\infty^{-1} + \tilde{\mathbf{C}}^*(s) \quad (25)$$

where $\mathbf{C}_\infty^*$ is the high-frequency gain matrix i.e. $\mathbf{C}_\infty^* = \lim_{s \rightarrow \infty} \mathbf{C}^*(s)$, and $\tilde{\mathbf{C}}^*(s)$ is a strictly proper transfer function. We also assume that the controller has a stable inverse, so that $\tilde{\mathbf{C}}^*(s)$ in (25) is stable. We can then implement $\mathbf{C}^*(s)$ using the feedback scheme shown in Fig. 2. The scheme is clearly equivalent to $\mathbf{C}(s)$ if the limiting circuit in the forward path is chosen to be the identity matrix. This is indeed the case when $\mathbf{u}(t) = \mathbf{V}\mathbf{u}^*(t)$ is inside the constraint polytope. On the contrary, if the demanded control vector $\hat{\mathbf{u}}^*(t)$ causes $\mathbf{u}(t)$ to be outside the constraint polytope, then the limiting circuit should operate in such a way that $\mathbf{V}\mathbf{u}^*(t)$ is always equal to the actual control vector applied to the system. This is the key feature that makes the configuration of Fig. 2 incorporate antiwindup protection [17].

Fig. 2 General antiwindup controller implementation in the singular-value space

In the MIMO case the design of the limiting circuit in Fig. 2 presents some subtleties since there are, in general, many different ways of bringing $\mathbf{u}(t)$ back inside the constraint polytope. For example, three simple strategies are [17] as follows.

- *Input saturation or actuator clipping:* The nonlinear element in this case is chosen as N individual saturation blocks. However, it is well known that saturating individual actuator inputs does not respect the directionality of the input vector $\mathbf{u}(t)$. This aggravates loss of decoupling, rendering this approach far from effective.
- *Input scaling or directional compensation:* This scheme performs a limiting operation on the controller output vector $\hat{\mathbf{u}}^*(t)$ shown in Fig. 2. To maintain $\mathbf{u}(t)$ inside the constraint polytope the limiting circuit determines a certain scalar value $\beta \in [0, 1]$ such that $\mathbf{u}^*(t) = \beta \hat{\mathbf{u}}^*(t)$ and $\mathbf{u}(t) \in \mathbb{U}$. Note that in this case the limiting circuit preserves the direction of the control vector $\mathbf{u}(t)$.
- *Error scaling:* This scheme is very similar to the previous scheme with the difference being that it operates on the error signal vector $\mathbf{e}(t)$. In fact $\mathbf{e}(t)$ is scaled to $\beta \mathbf{e}(t)$ so that $\mathbf{u}(t) \in \mathbb{U}$. This approach works well for saturation effects arising from transient behaviour [17]. However, if steady-state saturation errors occur, the error signal will be scaled back to zero, losing control over all control directions.

The common disadvantage of the antiwindup strategies described is that none of them account for the issue of widely different gains encountered in cross-directional control problems. In fact, the condition number of the interaction matrix $\mathbf{M}$ is usually big i.e. the difference between the biggest and smallest singular value is considerable. As a result, the system has control directions that are ‘harder’ to control than others since the associated gain i.e. the value of the associated singular value is small.

If we adopt the frequency analysis framework introduced in Section 2 to describe the behaviour of the singular values of $\mathbf{M}$, we can say that the smallest gains are typically associated with the ‘highest’ spatial frequencies while the biggest gains are associated with the lower spatial frequencies. Consequently, the control energy necessary to achieve disturbance compensation in the highest spatial frequencies is bigger than the one required in lower spatial frequencies. If this characteristic of cross-directional control problems is not adequately taken into consideration, the controller risks wasting all the available control energy trying to compensate for disturbances in the highest spatial frequencies, that is, in those directions hardest to control.

This discussion leads to the conclusion that a possibly successful approach would be to use the limited control authority available in $\mathbf{u}(t)$ to first compensate for the disturbance components in the control directions ‘easiest’ to control, that is, in those directions that require less control energy.

We propose an antiwindup scheme specifically tailored for cross-directional control problems, aimed at overcoming the difficulty of having widely different gains and

incorporating the ‘philosophy’ described. We call this approach spatial frequency antiwindup (SFAW). The strategy fits into the general structure depicted in Fig. 2 for a controller incorporating antiwindup protection in the singular value space. In particular, we note that the singular value decomposition of $\mathbf{M}$ in (5) generates a natural ordering among the plant input directions in terms of the control effort necessary to compensate for a disturbance acting in the corresponding output direction. Indeed, in (5) the control directions are ordered starting from the easiest to control (associated with the biggest σ_i) and ending with the hardest to control (associated with the smallest σ_i). As a result, the SFAW strategy acts in the singular-value space by scaling the manipulated variables $\hat{\mathbf{u}}^*(t)$, starting from the highest frequency component and descending towards the lowest frequency components until actuator saturation is avoided. In this way, rather than fixing the number of active directions *a priori* as would be the case when using a controller structure such as (24), we allow the selection to be made online. We envisage two key advantages of using the SFAW strategy

- the choice of active directions is matched to the particular characteristic of the current disturbance
- additional robustness features accrue due to the fact that there is an inherent feedback mechanism involved in making the choice.

The limiting circuit associated with the SFAW strategy defines a diagonal weighting matrix $\mathbf{P}_\gamma(t) \in \mathbb{R}^{N \times N}$ as follows:

$$\mathbf{u}^*(t) = \mathbf{P}_\gamma(t) \hat{\mathbf{u}}^*(t) \quad (26)$$

where

$$\mathbf{P}_\gamma(t) = \begin{cases} \text{diag}\{p_1, \dots, p_r, p_{r+1}, 0, \dots, 0\}, & \text{when } \|\mathbf{V}\hat{\mathbf{u}}^*(t)\|_\infty \geq \Delta \\ \mathbf{I}_{N \times N}, & \text{when } \|\mathbf{V}\hat{\mathbf{u}}^*(t)\|_\infty < \Delta \end{cases} \quad (27)$$

$\Delta \in \mathbb{R}$ is the saturation level imposed on every entry of $\mathbf{u}(t)$ in (4). $\mathbf{P}_\gamma(t)$ is parameterised in terms of the variable $\gamma(t) \in \mathbb{R}$ which is allowed to evolve only in the interval $[0, N]$. $\gamma(t)$ describes the spatial components that are active at any given time t . If $r \in \mathbb{Z}^+$ is the biggest integer such that $r \leq \gamma$, then the entries of $\mathbf{P}_\gamma(t)$ in (27) are

$$p_1 = p_2 = \dots = p_r = 1 \quad (28)$$

$$p_{r+1} = \gamma - r \quad (29)$$

The SFAW strategy computes the value of $\gamma(t)$ in such a way that if a constraint violation occurs, the control vector $\mathbf{u}(t)$ is brought back on the boundary of the constraint polytope $\mathbb{U}$, that is

$$\|\mathbf{u}(t)\|_\infty = \|\mathbf{V}\mathbf{P}_\gamma(t)\hat{\mathbf{u}}^*(t)\|_\infty = \Delta$$

Depending on the time evolution of the closed loop, the SFAW strategy uses a variable set of basis functions that can increase or decrease in number so as to keep the input $\mathbf{u}(t)$ within actuator saturation limits while maintaining good feedback control. The resulting control action achieves zero steady-state error in a subset of the spatial frequency modes of $\mathbf{M}$.

Remark 3.1: By utilising the SFAW strategy we overcome the problem of losing decoupling in the singular value space when the system is subject to input saturation. This is because the control vector $\mathbf{u}(t)$ is always kept on the boundary of the constraint polytope no matter what control signal the controller is actually requesting.

4 Robust stability analysis

It is obvious, by construction, that the proposed antiwindup scheme leads to a bounded-input bounded-output closed-loop system, since the plant is assumed open-loop stable. However, it is instructive to examine stronger forms of stability. In particular, we first assume that the diagonalising pre and post compensators $\mathbf{U}$ and $\mathbf{V}$ for the interaction matrix $\mathbf{M}$ are known. The stability analysis of the closed-loop system, in this case, is simplified since it reduces to the analysis of N decoupled scalar systems. This is because, in the singular value space, the controller $\mathbf{C}^*(s)$ is assumed diagonal, see (22). Fig. 3 shows the i th control loop in the singular value space. The DC gain of the plant is given by the value of the i th pseudosingular value $\tilde{\sigma}_i$ of $\mathbf{M}$. We consider a PI control law as in (23). We also replace the limiting circuit of Fig. 2 by the corresponding gain of the weighting matrix $\mathbf{P}_i(t)$ in (27).

For simplicity, we assume that the time delay θ is negligible, but that the plant gain and time constant are uncertain. We then have the following result:

Theorem 4.1: Consider the web-forming process defined by (1), and apply the spatial frequency antiwindup control strategy. Assume that the diagonalising pre and post compensating matrices, $\mathbf{U}$ and $\mathbf{V}$, are known. Then each of the decoupled SISO control loops are globally asymptotically stable if the following conditions are satisfied:

$$\begin{aligned} \frac{\tilde{\sigma}_i}{\hat{k}_i} &> \frac{1}{\hat{\tau}} \\ \frac{\tilde{\sigma}_i}{\hat{k}_i} &> \frac{\tau^2}{\hat{\tau}^3} \end{aligned} \quad (30)$$

Proof: We see from Fig. 3, that every decoupled control loop which is part of the whole multivariable closed-loop system, can be expressed in the form of a Lur'e type system i.e. a closed-loop system having a linear transfer function in the forward path and a static nonlinearity in the feedback path. From Fig. 3 we see that the linear transfer function in the forward path is given by

$$L(s) = \frac{(\tilde{\sigma}_i \tau^2 - \hat{k}_i \tau)s + \tilde{\sigma}_i \tau - \hat{k}_i}{\hat{k}_i (\hat{\tau}s + 1)(\tau s + 1)} \quad (31)$$

The static nonlinearity p_i is a time varying gain belonging to the sector $[0, \infty]$. The result then follows from the application of the circle criterion [33], which states that a sufficient condition for global asymptotic stability of the control loop in Fig. 3 is that the Nyquist plot of the transfer function $L(s)$ in (31) lies completely inside the open right half-plane. The conditions given in (30) guarantee this requirement. $\square$

In the case when the true plant time constant τ is also known, theorem 4.1 reduces to the following corollary:

Fig. 3 Scheme of single loop of decoupled system

Corollary 4.2: Consider the same conditions described in theorem 4.1, except that now we assume that the plant time constant τ is known i.e. $\hat{\tau} = \tau$. Then a sufficient condition for the scalar control loop of Fig. 3 to be globally asymptotically stable is

$$\hat{k}_i \tilde{\sigma}_i > 0 \quad (32)$$

$$|\hat{k}_i| < \tau |\tilde{\sigma}_i| \quad (33)$$

Remark 4.1: Corollary 4.2 establishes that a sufficient condition for the closed-loop to be globally asymptotically stable is that there exists no sign mismatch between the plant pseudosingular value $\tilde{\sigma}_i$ and the gain $\hat{k}_i$ used in the controller, provided also that the inequality (33) holds.

Remark 4.2: The result strictly holds only for the case when the diagonalising pre and post decoupling matrices are known. This is the case when the coupling matrix is circulant e.g. in the case of blown bubble film extrusion. Under these conditions the diagonalising pre and post compensators $\mathbf{U}$ and $\mathbf{V}$ are simply the real Fourier matrix as discussed in Section 2 [31]. Additionally, one might anticipate that similar robustness results would hold more generally for the noncirculant case. For example, when the number of actuator and sensors pairs is large, the eigenvectors of $\mathbf{M}$ will be close to the invariant Fourier matrix [24]. This suggests that the performance will be insensitive to the specific form of $\mathbf{U}$ and $\mathbf{V}$.

Remark 4.3: A case of interest not covered by theorem 4.1 is when the sign of the gain of the diagonalised system is unknown. In this case, we still have bounded input bounded output stability. However, stronger properties apply. This is because the scheme automatically turns-off the diagonal controller gains which contribute to input saturation. Thus the scheme has a stabilising effect when the gain sign is incorrect.

5 Simulation results

This section compares the performance of the proposed SFAW strategy with other known MIMO antiwindup schemes. In particular, we consider the following three control strategies:

- Antiwindup controller with input clipping
- Antiwindup controller with input scaling
- SFAW strategy.

We assume that the three approaches use the same controller (22) with the same tuning parameters for every scalar controller $c_i^*(s)$ in (23).

We also want to test the performance of each strategy when there is a mismatch between the model structure used to synthesise the controller of Fig. 2 and the actual structure of the system interaction matrix $\mathbf{M}$. When this is the case we lose decoupling in the spatial space i.e. the scalar controllers $c_i^*(s)$ interact with each other.

Say that the dimension N of the matrix $\mathbf{M}$ is equal to $2n + 1$ with $n \in \mathbb{Z}^+$. We then adopt the following possible structures for the system interaction matrix $\mathbf{M}$:

(a) Model M1: Circulant-symmetric structure with exponential profile i.e.

$$m_{i,j} = \begin{cases} e^{-\alpha|i-j|} & \text{for } |i-j| \leq n \\ e^{-\alpha(N-|i-j|)} & \text{for } |i-j| > n \end{cases} \quad (34)$$

Fig. 4 Steady-state cross-directional profile with different control strategies and complete knowledge of the system

a Steady-state response profile in output space

b Steady-state response profile in singular-value space

(b) Model M2: Circulant-symmetric structure with gaussian profile i.e.

$$m_{i,j} = \begin{cases} e^{-\alpha|i-j|^2} & \text{for } |i-j| \leq n \\ e^{-\alpha(N-|i-j|)^2} & \text{for } |i-j| > n \end{cases} \quad (35)$$

(c) Model M3: Toeplitz (noncirculant) symmetric structure with exponential profile i.e.

$$m_{i,j} = e^{-\alpha|i-j|}, \quad \text{for } i, j = 1, \dots, N. \quad (36)$$

(d) Model M4: Toeplitz (noncirculant) symmetric structure with gaussian profile i.e.

$$m_{i,j} = e^{-\alpha|i-j|^2}, \quad \text{for } i, j = 1, \dots, N. \quad (37)$$

For all the models consider $\alpha = 0.2$. Also, let the dynamical response of the system be characterised by a time constant $\tau = 1[s]$ in (2) and assume, for simplicity, that there is no time delay i.e. $\theta = 0$. We consider an interaction matrix $\mathbf{M}$ with dimension $N=21$ and we force the manipulated variable vector $\mathbf{u}(t)$ to belong to the constraint polytope $\mathbb{U}$ in (4) with $\Delta = 1$.

We are interested in analysing the performance of the SFAW strategy when applied to the four different models of $\mathbf{M}$ listed. In each case the SFAW controller is the same, designed based on model M1 i.e. a circular symmetric structure with exponential profile.

The cross-directional control system is perturbed with a given output disturbance having the same profile in each test. In Fig. 4, we present the steady-state response of the system using the different antiwindup MIMO controllers considered in the analysis, for case (a) summarised in

Table 1. In case (a), there is no mismatch between the model and the plant since both have structure M1. In Fig. 4, the thicker line corresponds to the applied disturbance profile. In Fig. 4a we see that the SFAW strategy significantly outperforms the other two antiwindup approaches, achieving substantial steady-state disturbance compensation. As seen in Table 1, the output profile obtained using the SFAW strategy has a l_2 norm of 0.98 compared to 2.56 obtained using input clipping antiwindup and 2.4 obtained using input scaling antiwindup.

The same performance result is shown in Fig. 4b where the steady state profiles have been plotted in the spatial frequency domain. The disturbance profile is concentrated in the low spatial frequencies (especially in the third mode) with some components also present in the high spatial frequencies. The result in Fig. 4b clearly shows how the SFAW strategy is able to cancel the disturbance in the third mode while the other two strategies can only achieve limited compensation for that mode. To be more precise, the SFAW strategy achieves zero steady-state error in the first seven spatial frequency components and partial compensation of the disturbance in the eighth mode. No control action is taken in the rest of the spatial frequency components. Despite this, we see in Fig. 4b that even though input clipping antiwindup and input scaling antiwindup do compensate the disturbance in higher frequency components, the resulting attenuation is not substantial.

It is illustrative to see how the strategy performs when there is mismatch between the model and the actual interaction matrix $\mathbf{M}$ of the system. Fig. 5 shows the result when the plant has the structure M4, i.e. a Toeplitz (non circulant) symmetric structure with gaussian profile.

Table 1: Comparison of l_2 norm of steady-state profile achieved by different antiwindup control strategies

	Model	Plant	Input clipping	Input scaling	SFAW	QP
Case (a)	M1	M1	2.56	2.40	0.98	0.74
Case (b)	M1	M2	2.98	2.41	0.77	0.68
Case (c)	M1	M3	0.80	0.62	0.31	0.15
Case (d)	M1	M4	1.89	1.51	0.39	0.36

Fig. 5 Steady-state cross-directional profile with different control strategies and model-plant mismatch

a Steady-state response profile in output space

b Steady-state response profile in singular-value space

In Fig. 5*a* we notice that all the strategies perform well, however the SFAW control is still better than the other two. This is further clarified in Fig. 5*b* where it is also interesting to note that despite the mismatch between the model's rotation matrices U and V and those of the plant, the SFAW strategy is still able to reduce considerably the error in the low frequencies components to almost zero.

Table 1 summarises all the cases considered in the simulations described. The entries represent the l_2 norm of the corresponding steady-state profiles. We note that the variance of the output profiles do not differ markedly between the case when complete knowledge of the plant is assumed (case (a)) and when model mismatch occurs (cases (b)–(d)). Moreover, in all the cases considered the variance of the output profile obtained using the SFAW strategy is conclusively better than any other MIMO antiwindup approach.

We also present, in Table 1, the l_2 norm of the optimal steady-state output profile obtained minimising (3) subject to the input constraint $\mathbf{u}(t) \in \mathcal{U}$. This ideal QP solution provides a lower bound for all steady-state solutions and can be viewed as a benchmark for the performance. In all of the cases, the SFAW strategy result is very close to the ideal QP solution.

6 Conclusions

The problem of MIMO design with saturating actuators has been discussed for the particular problem of cross-directional control in web-forming processes. A spatial frequency antiwindup (SFAW) strategy has been described to deal with this issue.

Conditions guaranteeing global closed-loop asymptotic stability for this approach have been presented, for the case where the set of unitary-invariant orthogonal basis functions is known.

Several simulation examples have been presented to demonstrate that the strategy yields superior steady-state performance, compared to two other commonly used antiwindup methods. In addition, the performance of the SFAW strategy is close to an optimal QP solution to the problem, based on the true, but unknown plant.

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8 References

- DUMONT, G.A.: 'Application of advanced control methods in the pulp paper industry – a survey', *Automatica*, 1986, **22**, pp. 143–153.
- SMOOK, G.A.: 'Handbook for pulp and paper Technologies' (Tappi, Atlanta, GA, 1982).
- STEWART, G.E., GORINEVSKY, D.M., and DUMONT, G.A.: 'Robust GMV cross directional control of paper machines'. Proceedings of the American Control Conference, Philadelphia, PA, USA, June 1998 pp. 3002–3007.
- GOODWIN, G.C., CARNEY, B., and EDWARDS, W.J.: 'Analysis of thermal camber control in rolling mills'. Presented at the IFAC World Congress, Tallinn, USSR, August 1990.
- CARNEY, B., EDWARDS, W.J., and GOODWIN, G.C.: 'Achieving functional control of camber in rolling mills'. Presented at the 4th Australian Conference of Control Engineering, August 1990.
- EDWARDS, W.J., GOMEZ, G., and GOODWIN, G.C.: 'A review of thickness control in reversing cold rolling mills'. Proceedings of Control'95, Melbourne, Australia, 1995, pp. 129–134.
- HEATH, W.P.: 'Orthogonal functions for cross-directional control of web-forming processes', *Automatica*, 1996, **32**, (2), pp. 183–198.
- LEVY, S., and CARLEY, J.F.: 'Plastics extrusion technology handbook' (Industrial Press, New York, 1989, 2nd edn.).
- STEWART, G.E., GORINEVSKY, D.M., and DUMONT, G.A.: 'Spatial loopshaping: a case study on cross-directional profile control'. Proceedings of the American Control Conference, San Diego, California, June 1999, pp. 3098–3103.
- VAN ANTWERP, J.G., FEATHERSTONE, A.P., and BRAATZ, R.D.: 'Robust cross-directional control of large-scale sheet and film processes', *J. Process Control*, 2001, **11**, pp. 149–177.
- FEATHERSTONE, A.P., and BRAATZ, R.D.: 'Integrated robust identification and control of large-scale processes', *Ind. Eng. Chem. Res.*, 1998, **37**, pp. 97–106.
- DUNCAN, S.R.: 'The design of robust cross-directional control systems for paper making'. Proceedings of the American Control Conference, Seattle, Washington, June 1995, pp. 1800–1805.
- LAUGHLIN, D.L., MORARI, M., and BRAATZ, R.D.: 'Robust performance of cross-directional basis-weight control in paper machines', *Automatica*, 1993, **29**, (6), pp. 1395–1410.
- DUNCAN, S.R., and BRYANT, G.F.: 'The spatial bandwidth of cross-directional control systems for web processes', *Automatica*, 1997, **33**, (2), pp. 139–153.
- BRAATZ, R.D., TYLER, M.L., MORARI, M., PRANCKH, F.R., and SARTOR, L.: 'Identification and cross-directional control of coating processes', *AIChE J.*, 1992, **38**, (9), pp. 1329–1339.
- MICHAEL HEAVEN, E., JONSSON, I.M., MAX KEAN, T., MANNESS, M.A., and VYSE, R.N.: 'Recent advances in cross machine profile control', *IEEE Control Syst. Mag.*, 1994, **14**, (5), pp. 35–46.

- 17 GOODWIN, G.C., GRAEBE, S.F., and SALGADO, M.E.: 'Control system design' (Prentice Hall, Upper Saddle River, NJ, 2001)
- 18 GLATTFELDER, A.H., and SCHAUFELBERGER, W.: 'Stability analysis of single-loop control systems with saturation and antireset-windup circuits', *IEEE Trans. Autom. Control*, 1983, **28**, pp. 1074–1081
- 19 SHIM, J.H., GOODWIN, G.C., and GRAEBE, S.F.: 'MIMO design with saturating actuators'. Presented at CESA'96, Lille, France, 1996
- 20 HIPPE, P., and WURMTHALER, C.: 'Controller and plant windup prevention in mimo loops with input saturation'. Proceedings 4th European Control Conference, Brussels, Belgium, 1–4 July 1997
- 21 CHEN, C., and PERNG, M.: 'Optimization based anti-windup design for MIMO control systems', *Int. J. Control*, 1998, **69**, (3), pp. 393–418
- 22 PENG, Y., VRANČIĆ, D., HANUS, R., and WELLER, S.S.R.: 'Anti-windup designs for multivariable controllers', *Automatica*, 1998, **34**, (12), pp. 1559–1565
- 23 FEATHERSTONE, A.P., VAN ANTWERP, J.G., and BRAATZ, R.D.: 'Identification and control of sheet and film processes' (Springer Verlag, London, 2000)
- 24 STEWART, G.E.: 'Two-dimensional loop shaping controller design for paper machine cross-directional processes August 2000, PhD, University of British Columbia Department of Electrical and Computer Engineering, Vancouver, Canada
- 25 CORSCADDEN, K.W., and DUNCAN, S.R.: 'The use of basis function expansions to analyse the robustness of cross-directional control systems'. Proceedings of the American Control Conference, Albuquerque, New Mexico, 1997, pp. 1478–1482
- 26 BERGH, L.G., and MACGREGOR, J.F.: 'Spatial control of film and sheet forming processes', *Canad. J. Chem. Eng.*, 1987, **65**, pp. 148–155
- 27 FEATHERSTONE, A.P., and BRAATZ, R.D.: 'Control-oriented modeling of sheet and film processes', *AIChE J.*, 1997, **43**, (8), pp. 1989–2001
- 28 HOVD, M., BRAATZ, R.D., and SKOGESTAD, S.: 'SVD controllers for H_2 -, H_∞ - and μ -optimal control', *Automatica*, 1997, **33**, (3), pp. 433–439
- 29 FEATHERSTONE, A.P., and BRAATZ, R.D.: 'Control relevant identification of sheet and film processes'. Proceedings of the American Control Conference, Seattle, Washington, June 1995, pp. 2692–2696
- 30 STRANG, G.: 'Linear algebra and its applications' (Harcourt Brace, Orlando, FL, USA, 1988, 3rd edn.)
- 31 DAVIS, P.J.: 'Circulant matrices' (Wiley, New York, 1979)
- 32 HOVD, M., and SKOGESTAD, S.: 'Control of symmetrically interconnected plants', *Automatica*, 1994, **30**, (6), pp. 957–973
- 33 VIDYASAGAR, M.: 'Nonlinear systems analysis' (Prentice Hall, Englewood Cliffs, NJ, 1978)