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Easily Applied Tests of Fit for the Rayleigh Distribution

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SUMMARY

In this note two easily applied moment tests of the Rayleigh hypothesis are compared with the Anderson-Darling test, some smooth tests and tests based on the empirical Laplace transform.

Some key words: Anderson-Darling test; Chernoff-Lehman X^2 test; Empirical Laplace transform; Orthogonal polynomials; Parametric bootstrap; Smooth tests.

1. INTRODUCTION

The Rayleigh distribution was originally introduced by the physicist Lord Rayleigh in connection with an acoustics problem. More recently it has been used to model data that is skewed to the right; such data occurs, for example, in life testing and reliability contexts. A random variable X follows the Rayleigh distribution if and only if it has probability density function

$$f(x; \theta) = (x/\theta^2)\exp\{-x^2/(2\theta^2)\}, 0 \leq x < \infty, \text{ in which } \theta > 0.$$

The distribution of $Y = X/\theta$ is the Rayleigh distribution with parameter unity, which has mean $\mu = \sqrt{(\pi/2)}$ and variance $\mu_2 = (4 - \pi)/2$.

In section 2 below we discuss maximum likelihood (ML) estimation and method of moments (MOM) estimation of θ and define the tests of fit based on the empirical Laplace transform and the Anderson-Darling statistic. We also introduce two new easily applied moment-based tests of fit. In section 3 we present a small size and power study. Section 4 applies these tests of fit for the Rayleigh distribution to the data given in the following paragraph while section 5 introduces smooth tests of fit for the Rayleigh distribution. Section 6 is a brief conclusion.

In recent years there has been much interest in alternative methods of electricity production, including the use of wind turbines. The distribution of average daily wind speed is important in examining the feasibility of installing wind turbines and the Rayleigh distribution is a possible candidate distribution. In section 4 we consider how well the Rayleigh distribution models 30 average daily wind speeds (in kilometres/hour) for the month of November 2007 at Elanora Heights, a north-eastern suburb of Sydney, Australia. The data follow:

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2.7	3.2	2.1	4.8	7.6	4.7	4.2	4.0	2.9	2.9
4.6	4.8	4.3	4.6	3.7	2.4	4.9	4.0	7.7	10.0
5.2	2.6	4.2	3.6	2.5	3.3	3.1	3.7	2.8	4.0

2. DEFINITIONS

Suppose we have n data points $x_1, x_2, \dots, x_n$ and we wish to test the fit of the Rayleigh model to these data. It is easy to show that the maximum likelihood estimator of θ , $\hat{\theta}$, and the method of moments estimator, $\tilde{\theta}$, are given by

$$\hat{\theta} = \sqrt{\sum_{i=1}^n x_i^2 / (2n)} \text{ and } \tilde{\theta} = \sqrt{(2/\pi)} \left(\sum_{i=1}^n x_i \right) / n.$$

This defines the method of moments estimator as the solution obtained by equating sample and population first moments. If sample and population second moments about the origin are equated the maximum likelihood estimator is obtained. Thus in a sense, for this distribution, the maximum likelihood estimator is a particular case of method of moments.

The classical Chernoff-Lehmann X_{CL}^2 test, based on the test statistic X_{CL}^2 , is still commonly used to test the fit of the Rayleigh model but has the disadvantage that information is lost when the data are grouped. See, for example, Rayner and Best (1989) or Rayner et al. (2009). In the power comparisons of the next section we used class intervals (0, 0.5), (0.5, 1), (1, 1.5), (1.5, 2), (2, ∞). A recently suggested test of fit which does not have this problem is based on the empirical Laplace transform statistic, L say, and was suggested by Meintanis and Iliopoulos (2003) who defined

$$L = \frac{n}{a} + \frac{\sqrt{2}}{n} \sum_{j=1}^n \sum_{k=1}^n \left\{ \frac{1}{\hat{Y}_j + \hat{Y}_k + a} + \frac{(\hat{Y}_j + \hat{Y}_k)}{(\hat{Y}_j + \hat{Y}_k + a)^2} + \frac{2(\hat{Y}_j \hat{Y}_k + 2)}{(\hat{Y}_j + \hat{Y}_k + a)^3} + \frac{6(\hat{Y}_j + \hat{Y}_k)}{(\hat{Y}_j + \hat{Y}_k + a)^4} + \frac{24}{(\hat{Y}_j + \hat{Y}_k + a)^5} \right\} \\ - 2\sqrt{2} \sum_{j=1}^n \left\{ \frac{1}{(\hat{Y}_j + a)} + \frac{\hat{Y}_j}{(\hat{Y}_j + a)^2} + \frac{2}{(\hat{Y}_j + a)^3} \right\}$$

where a is a parameter to be chosen before sighting the data and where $\hat{Y}_j = X_j / \hat{\theta}$. This definition of L has been adjusted by a factor of $\sqrt{2}$ to agree with our definition of a Rayleigh variable which is $\sqrt{2}$ times the Rayleigh variable defined by Meintanis and Iliopoulos (2003).

The Anderson-Darling test gives powerful tests for many important distributions. Suppose $F(x; \theta)$ is the cumulative distribution function corresponding to $f(x; \theta)$ and put $z_{(i)} = F(x_{(i)}; \hat{\theta})$ where $x_{(i)}$ and hence $z_{(i)}$ are in ascending order. The Anderson-Darling test is based on A^2 , which is given by

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \{ \log(z_{(i)}) + \log(1 - z_{(n+1-i)}) \}.$$

The critical values for L and A^2 are not known and the parametric bootstrap could be used to obtain p-values. However we note that for very large samples the bootstrap p-value can take a little time to calculate due to the double summation in L and the need to order the data in A^2 .

We now introduce U and W , two new easily applied moment statistics for testing the fit of the Rayleigh model. First define $m_1 = \sum_{i=1}^n x_i / n$ and $m_r = \sum_{i=1}^n (x_i - m_1)^r / n$ for $r = 2, 3, \dots$, and then take

$$W = m_1 - \mu\hat{\theta} = m_1 - \hat{\theta}\sqrt{(\pi/2)} \text{ and}$$

$$U = m_2 - \mu_2\tilde{\theta}^2 = m_2 - \tilde{\theta}^2(4 - \pi)/2$$

where the hat indicates θ is estimated by the MLE and the tilde indicates estimation by MOM. Observe that we do not consider a statistic $m_1 - \tilde{\theta}\mu$ which is identically zero. Now $W^2/\text{var}(W)$ and $U^2/\text{var}(U)$ are each asymptotically χ_1^2 distributed where

$$n \text{ var}(W) = 0.0365044 \hat{\theta}^2 \text{ and } n \text{ var}(U) = 0.371831 \tilde{\theta}^4.$$

These asymptotic formulae for $\text{var}(W)$ and $\text{var}(U)$ can be derived by the delta method or otherwise. It can be shown that the asymptotic correlation between U and W is -1.0 and so we expect U and W to perform similarly. We note that Y , which has the Rayleigh distribution with parameter unity, has third and fourth central moments $\mu_3 = (\pi - 3) \sqrt{(\pi/2)}$, and $\mu_4 = (32 - 3\pi^2)/4$ respectively. These moments are needed to calculate $\text{var}(U)$ using the delta method.

3. SIZE AND POWER STUDY

In Table 1 for a test size of 0.05 we study the approach, as the sample size n increases, to the asymptotic critical value of 3.84 for $W^2/\text{var}(W)$ and $U^2/\text{var}(U)$. Table 1 suggests that these two statistics approach their asymptotic distribution fairly quickly, and so we can obtain good approximate p-values simply by using the χ_1^2 distribution. For small samples the approximation can be verified by use of the parametric bootstrap. Gürtler and Henze (2000) give a discussion of the parametric bootstrap in a goodness of fit context.

Table 1. Critical values and test sizes of $W^2/\text{var}(W)$ and $U^2/\text{var}(U)$ for $\alpha = 0.05$, n as shown and 100,000 Monte Carlo simulations; sizes are based on the asymptotic critical values

n	$W^2/\text{var}(W)$	$U^2/\text{var}(U)$	Size $W^2/\text{var}(W)$	Size $U^2/\text{var}(U)$
20	3.53	3.43	0.040	0.038
50	3.63	3.56	0.047	0.045
100	3.71	3.74	0.049	0.048
500	3.82	3.81	0.049	0.049
1,000	3.83	3.83	0.050	0.050
∞	3.84	3.84	0.050	0.050

The powers in Table 2 were found using random numbers generated using IMSL (1995). For some of the alternatives the distribution was inverted and random numbers found by transforming a uniform $U(0, 1)$ variate. The powers were calculated using a parametric bootstrap with 1,000 simulations in both the inner and outer loops. We used similar alternatives to those employed by Meintanis and Iliopoulos (2003). Specifically for $\theta > 0$ and $x \geq 0$ unless otherwise noted, these alternative distributions are the

- (i) Weibull with probability density function $\theta x^{\theta-1} \exp(-x^\theta)$, denoted by $W(\theta)$
- (ii) gamma with probability density function $\{ \Gamma(\theta) \}^{-1} x^{\theta-1} \exp(-x)$, denoted by $G(\theta)$
- (iii) log normal with probability density function $(\theta x \sqrt{2\pi})^{-1} \exp\{-(\log x)/(2\theta^2)\}$, denoted by $LN(\theta)$

- (iv) Gompertz with distribution function $1 - \exp\{(1 - e^x)/\theta\}$, denoted by $GO(\theta)$
- (v) power distribution with probability density function $\theta^{-1} x^{\{(1 - e^x)/\theta\}}$ for $0 \leq x \leq 1$, denoted by $PW(\theta)$
- (vi) exponential power with distribution function $1 - \exp\{(1 - \exp[x^\theta])\}$, denoted by $EP(\theta)$
- (vii) Poisson-exponential, denoted by $PE(\theta)$ and defined as $(E_1 + \dots + E_N)$ where each E_i has an exponential density with parameter unity and N has a Poisson distribution with parameter θ .

Table 2. 100×powers of some tests of fit for the Rayleigh distribution for $n = 20$ and $\alpha = 0.05$

Alternative	L	A^2	$W^2/\text{var}(W)$	$U^2/\text{var}(U)$	X_{CL}^2
W(1.0)	96	95	95	96	78
W(2.0)	5	5	5	5	5
W(3.0)	51	48	54	42	27
G(1.5)	75	73	74	77	47
G(2.0)	46	45	45	52	23
LN(0.8)	74	76	74	77	54
GO(0.5)	80	78	76	80	53
GO(1.5)	46	47	35	40	28
PW(1.0)	27	40	16	20	28
PW(2.0)	98	98	94	95	86
EP(1.0)	61	60	54	59	35
EP(2.0)	25	26	28	21	19
PE(3.0)	83	80	79	82	58
PE(4.0)	54	55	49	53	34

In Table 2 we calculated A^2 , X_{CL}^2 and L using $\hat{\theta}$. We took $a = 2\sqrt{2}$ in the definition of L so that our powers for the test based on L are comparable with those given by Meintanis and Iliopoulos (2003) for their test with index 2.0. We see that $W^2/\text{var}(W)$ and $U^2/\text{var}(U)$ have powers comparable to those of A^2 and L . However $W^2/\text{var}(W)$ and $U^2/\text{var}(U)$ are arithmetically easier to calculate and have an approximate χ_1^2 distribution. Thus they are more easily applied.

4. APPLICATION TO AVERAGE WIND SPEED DATA

If we group the average wind speed data into four intervals (0, 3), (3, 4.5), (4.5, 6) and (6, ∞) we obtain counts of 8, 12, 7 and 3. Using the ungrouped data we find $\hat{\theta} = 3.18$ and $\tilde{\theta} = 3.33$. Using $\hat{\theta}$ we calculated the four expected frequencies 10.78, 8.20, 5.96 and 5.06 giving a value of 3.49 for X_{CL}^2 , the Chernoff-Lehmann X^2 statistic. Using the χ_2^2 distribution gives an approximate p-value of 0.17. Although we would not recommend grouping the data we give the above details because it is a common approach.

We also find $W^2/\text{var}(W) = 2.78$ and $U^2/\text{var}(U) = 2.43$ with approximate p-values based on χ_1^2 of 0.10 and 0.12. These p-values are a little more critical of the Rayleigh hypothesis but perhaps not importantly so. However in general tests based on ungrouped data would be expected to be more powerful. Of course there is also some ambiguity in how to choose the classes for the Chernoff-Lehmann X^2 statistic. See Figure 1.

Figure 1. Histogram of the wind speed data and a fitted Rayleigh probability density function.

5. SMOOTH TESTS OF FIT FOR THE RAYLEIGH DISTRIBUTION

Smooth tests of fit have been shown to be competitive in terms of power with other tests of fit. See, for example, Rayner and Best (1989) and Rayner et al. (2009). They are also useful in suggesting alternative models when an hypothesised model is discredited. We will now show that W and U are special cases of smooth test statistics and derive a higher order smooth test statistic.

To construct smooth tests we need polynomials $\{g_r(y)\}$ orthonormal on the standardised Rayleigh $f(y; 1)$. The first three polynomials orthogonal on the standardised Rayleigh are

$$g_0(y) = 1 \text{ for all } y, g_1(y) = \{y - \sqrt{(\pi/2)}\}, \text{ and} \\ g_2(y) = \{y - \sqrt{(\pi/2)}\}^2 - (\mu_3/\mu_2)\{y - \sqrt{(\pi/2)}\} - \mu_2.$$

The r th component of the smooth test statistic, which is a sum of squares of such components, is

$$V_r = \sum_{i=1}^n g_r(Y_i) / \sqrt{nd_r} \text{ for } r = 1, 2, \dots$$

in which the d_r are normalising constants. It is not difficult to see that

$$W = \hat{V}_1 \sqrt{d_1/n} \text{ and } U = \tilde{V}_2 \sqrt{d_2/n}.$$

Parallel to these test statistics define

$$T = \tilde{V}_3 \sqrt{d_3/n}.$$

Thus

$$T = m_3 - 1.177343 m_2 \tilde{\theta} + 0.32786 \tilde{\theta}^3 \text{ with } \text{var}(T) = 0.38723/(n \tilde{\theta}^6).$$

To calculate $\text{var}(T)$ via the delta method we need

$$\mu_5 = (\pi^2 + 5\pi - 25) \sqrt{(\pi/2)} \text{ and } \mu_6 = 48 + 15\pi - 15\pi^2/2 - 5\pi^3/8.$$

We note that $d_2 = 0.340207$ and $d_3 = \text{var}(T)$ respectively. For some distributions, such as the exponential, these normalization constants and variances are equal. However in general there is no reason to expect this to be so.

It is routine to show that $\text{cov}(T, U)$ is asymptotically zero and so asymptotically

$$S = T^2/\text{var}(T) + U^2/\text{var}(U) = \tilde{V}_2^2/1.09296 + V_3^2$$

should have an approximate χ_2^2 distribution. Table 3 shows the approach of $T^2/\text{var}(T)$ and S to their asymptotic distributions while Table 4 gives powers to compare with those in Table 2. We do not consider $\hat{V}_2 \sqrt{d_2/n}$ as it can be shown that $\hat{V}_2 \sqrt{d_2/n} = 2\mu W/(\pi - 4)$. Higher order V_r can be calculated using the recurrence relation in Rayner et al. (2008).

The critical values in Table 3 approach their asymptotic limits at about the same rate as do those in Table 1.

Table 3. Critical values and test sizes of $T^2/\text{var}(T)$ and S for $\alpha = 0.05$, n as shown and 100,000 Monte Carlo simulations; sizes are based on the asymptotic critical values

n	$T^2/\text{var}(T)$	S	Size $T^2/\text{var}(T)$	Size S
20	3.15	5.60	0.037	0.044
50	3.54	5.96	0.044	0.050
100	3.74	5.98	0.047	0.052
500	3.82	5.99	0.049	0.050
1,000	3.83	5.99	0.050	0.050
∞	3.84	5.99	0.050	0.050

Table 4. 100 $\times$ powers of tests based on $T^2/\text{var}(T)$ and S for $n = 20$ and $\alpha = 0.05$

Alternative	$T^2/\text{var}(T)$	S
W(1.0)	70	96
W(2.0)	5	5
W(3.0)	1	24
G(1.5)	49	76
G(2.0)	33	48
LN(0.8)	65	78
GO(0.5)	49	80
GO(1.5)	31	45
PW(1.0)	49	40
PW(2.0)	72	99
EP(1.0)	39	63
EP(2.0)	1	7
PE(3.0)	54	82

From Table 4 we see that the test based on $T^2/\text{var}(T)$ is generally not as powerful as the Table 2 tests, but the test based on S is. Further, the test based on $T^2/\text{var}(T)$ has power less than the size for two alternatives and for these alternatives the test based on S has less power than the Table 2 tests. However the test based on S seems better than the Table 2 tests for the PW(1.0) alternative.

To explain the bias of the test based on $T^2/\text{var}(T)$ we need to give some detail about moment tests. We present these as smooth tests of goodness of fit, for which see Rayner et al. (2009). First suppose we wish to test for the probability density function $f(x; \beta)$ where β is a vector of nuisance parameters. This probability density function is embedded in a smooth alternative

$$g_r(x; \theta_r, \beta) = C(\theta_r, \beta) \exp\{\theta_r h_r(x; \beta)\}$$

in which $\{h_r(x; \beta)\}$ are orthonormal on the weight function $f(x; \beta)$ and where $C(\theta_r, \beta)$ is a normalising function ensuring that $g_r(x; \theta_r, \beta)$ integrates (sums) to one. Equivalently we may work with $g_r(x; \theta_r, \beta) = \{1 + \theta_r h_r(x; \beta)\} f(x; \beta)$.

Given a random sample $X_1, \dots, X_n$ we test for $f(x; \beta)$ by testing $H_0: \theta_r = 0$ against $K: \theta_r \neq 0$ with the model $g_r(x; \theta_r, \beta)$ above. If $\hat{\beta}$ is an estimator of β (usually maximum likelihood or method of moments) then writing $\hat{V}_r = \sum_{j=1}^n h_r(X_j; \hat{\beta}) / \sqrt{n}$ the score test statistic is $\hat{V}_r^2$. In testing for normality with $\{h_r(x; \beta)\}$ the set of Hermite polynomials $\hat{V}_3^2$ is a standardised version of the coefficient of skewness and $\hat{V}_4^2$ is a standardised version of the coefficient of kurtosis. In general when $\{h_r(x; \beta)\}$ is the set of orthonormal polynomials $\hat{V}_r$ may be expressed as a linear combination of sample moments up to the r th. Hence the $\hat{V}_r^2$ may be considered to be the r th standardised moment test statistic, assessing departures of the sample from the hypothesised distribution in moments up to the r th. However see the more complete discussion throughout Rayner et al. (2009): moments up to the $2r$ th may affect testing.

Another general discussion is given in Henze and Klar (1996) while Spinelli and Stephens (1997) discusses the Poisson case, Rayner et al. (1995) the normal case, and Thas et al. (2009) the logistic case. Henze and Klar (1996) discuss diagnostic statistics and this is a particular case of the bias we are discussing. They suggest using a different variance estimate but this can result in low power.

In a similar setting Neyman (1937) used his generalised fundamental lemma to derive tests that were asymptotically locally uniformly most powerful unbiased symmetric size α . We might therefore reasonably expect that asymptotically the test based on $\hat{V}_r^2$ is unbiased. Thus these tests may be biased in finite samples, but the expectation must be that the bias is minimal in samples of moderate size. This means that for some $\theta_r \neq 0$ the test's power may be less than the test size, the nominal significance level. This makes no conclusion about θ_s with $r \neq s$. Thus there may be an extensive region of the parameter space where the test power is less than the test size. This region may be small in extent corresponding to θ_r , but may be considerable in extent with regard to the other parameters. In summary $\hat{V}_r^2$ gives protection against unbiasedness with regard to θ_r , but no other θ_s .

The alternatives considered in our power studies project into the parameter space described in terms of $\theta_1, \theta_2, \dots$. Whether or not power less than the test size is observed for a particular alternative depends on whether or not that alternative projects into a region of bias, that can be expected to be narrow in terms of θ_r .

Similarly a test based on $\hat{V}_2^2 + \hat{V}_3^2$ for example, gives protection against bias with regard to θ_3 and θ_4 , but no other θ_s . So the region where the power is less than the test size can be expected to be narrow in terms of both θ_2 and θ_3 .

Recalling our definition above allowed $\hat{V}_r$ to be calculated using either maximum likelihood or method of moments estimators we see that in our former notation $\hat{V}_2^2 = U^2/\text{var}(U)$, $\hat{V}_3^2 = T^2/\text{var}(T)$ and $S = \hat{V}_2^2 + \hat{V}_3^2$. The Appendix considers the bias of U for a specific alternative – the rectangular distribution. We conclude from our discussion of moment tests and bias that bias is less likely for the test based on S than for, say the tests based on $U^2/\text{var}(U)$ and $T^2/\text{var}(T)$ individually, which is in agreement with our choice of S as a test of fit for the Rayleigh. In practice our experience is that tests based on two moment tests like S will hardly ever be biased.

For the data in section 4 we find $S = 7.35$ with p-value 0.03 based on the χ_2^2 approximation. Thus we might conclude that the Rayleigh model is not appropriate and suggest that the model and the data disagree in respect to a linear combination of second and third moments. Certainly Figure 1 shows that the data are less dispersed than expected from the Rayleigh model. Further tests of fit, including data-driven choice of statistic, are discussed in Rayner et al. (2009).

6. CONCLUSION

It seems that all of the statistics we have considered may be the basis of reasonable tests of fit for the Rayleigh distribution. Although not evident in Table 2, it should be possible to find alternatives such that the tests based on $W^2/\text{var}(W)$ and $U^2/\text{var}(U)$ have powers less than the size of the test, as was the case with $T^2/\text{var}(T)$ in Table 4. Although we suggest this problem of the test being biased may be rare in practice, use of the test based on S would lessen the chances of this occurring and so the test based on S is our powerful but easily applied test of choice for use as a test of fit of the Rayleigh distribution. Use of asymptotic critical values was verified for sample sizes as small as 20.

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APPENDIX

Asymptotically $U \sqrt{(n/0.37183)}$ has a normal distribution with mean $\sqrt{(n/0.37183)}E[U]$ and variance V . For the null case $E[U] = 0$ and $V = 1$. Asymptotically, for any alternative and using the delta method,

$$E[U] = \mu_2 - (4/\pi - 1) \mu^2 \text{ and} \\ V = f_x^2 \mu_2 + \mu_4 - \mu_2^2 + 2 f_x \mu_3.$$

Here $f_x = -(8/\pi - 2)\mu$, μ is the mean and μ_r , $r = 2, 3, \dots$ are the central moments of the alternative distribution. The asymptotic power of U for test size $\alpha = 0.05$ and an alternative with $E[U] = 0$ is

$$P(Z > 1.645/\sqrt{(V/0.37183)} \mid Z \text{ is } N(0, 1)).$$

Suppose the alternative is a rectangular distribution with support (a, b) , $R(a, b)$ say, in which $a = \sqrt{(\pi/2)} - \sqrt{\{3(2 - \pi/2)\}}$ and $b = 2\sqrt{(\pi/2)} - a$. Then $E[R] = \sqrt{(\pi/2)}$ and $\text{var}(R) = 2 - \pi/2$, equal to the corresponding moments of the Rayleigh distribution with $\theta = 1$, so that asymptotically $E[U] = 0$ and $V = 0.3487$. Thus the asymptotic power of U is

$$P(Z > 1.699 \mid Z \text{ is } N(0, 1)) = 0.045.$$

Thus asymptotically the test based on U is biased, albeit that the bias is only slight.

The following table shows how the power of the test based on U approaches its asymptotic limit as $n \rightarrow \infty$. The finite sample powers for the test based on U in Table 5 are calculated by parametric bootstrap with 1,000 simulations in both the inner and outer loops.

Table 5. Power of the test based on U for various n and $\alpha = 0.05$

n	Power of the test based on U
20	0.047
50	0.049
100	0.049
200	0.048
∞	0.045