A New and Self-Contained Proof of Borwein's Norm Duality Theorem

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Abstract. Borwein's norm duality theorem establishes the equality between the outer (inner) norm of a sublinear mapping and the inner (outer) norm of its adjoint mappings. In this note we provide an extended version of this theorem with a new and self-contained proof relying only on the Hahn-Banach theorem. We also give examples showing that the assumptions of the theorem cannot be relaxed.

Key words: convex process, sublinear mapping, norm duality, inner norm, outer norm.

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Borwein's norm duality theorem was proved in [3] in a paper involving a general study of convex processes. A revised version of it was published later in [6] with more details and some applications. The proof of the theorem given in both papers is somewhat involved and relying on other results scattered in the literature, such as a Lagrange multiplier theorem given in [5, Theorem 3.1]. In view to the fundamental importance of this result in variational analysis and beyond, we feel that the availability of a self-contained and detailed proof would benefit everyone involved in research and teaching in this area. In this paper we provide such a proof simultaneously extending the original version of the theorem.

For mappings acting in Euclidean spaces, two different proofs are provided in [4, Theorem 5.4.10] and [10, Theorem 11.29]. The first one explicitly uses the finite dimensions. The second one is based on support functions and their properties. Because of the complexity of the proof and the use of a number of other results all stated in finite dimensions, the author is not in a position to determine whether the proof in [10] can be extended to infinite-dimensional spaces. Here we present a proof in normed spaces the only prerequisite for which is the Hahn-Banach theorem. The proof follows partially the original Borwein's proof in [3].

Throughout the paper X and Y are real linear normed spaces whose norms are both denoted by $\|\cdot\|$. For a multivalued mapping acting from X into the subsets of Y , denoted by $F : X \rightrightarrows Y$, we define its *graph* and its *domain*, respectively, by

$$\text{gph } F = \{(x, y) \in X \times Y \mid y \in F(x)\} \quad \text{and} \quad \text{dom } F = \{x \in X \mid F(x) \neq \emptyset\},$$

and the associated *inverse* mapping by

$$F^{-1}(y) = \{x \in X \mid y \in F(x)\}, \quad \text{for } y \in Y.$$

A mapping $F : X \rightrightarrows Y$ is said to be *positively homogeneous* when $0 \in F(0)$ and $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$ and $x \in X$, or equivalently, when $\text{gph } F$ is a cone in $X \times Y$. F is said to be *sublinear* when it is positively homogeneous and, in addition, $F(x) + F(x') \subset F(x + x')$ for all $x, x' \in X$; equivalently, when $\text{gph } F$ is a convex cone in $X \times Y$. A sublinear mapping whose domain is the whole space X and which is single-valued on X is a *linear* function. Sublinear mappings are introduced by R. T. Rockafellar in [8, 9] under the name *convex processes*.

For any sublinear mapping $F : X \rightrightarrows Y$, the *outer norm* $\|F\|^+$ and the *inner norm* $\|F\|^-$ are defined by

$$\|F\|^+ = \sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \|y\| \quad \text{and} \quad \|F\|^- = \sup_{x \in \mathbb{B}} \inf_{y \in F(x)} \|y\|.$$

Here we follow the convention adopted in [10] setting $\inf_{y \in A} \|y\| = \infty$ and $\sup_{y \in A} \|y\| = -\infty$ if $A = \emptyset$. Another equivalent way to define these quantities (see [3] and [6]) is

$$\|F\|^+ = \inf\{r > 0 \mid F(\mathbb{B}) \subset r\mathbb{B}\} \quad \text{and} \quad \|F\|^- = \inf\{r > 0 \mid F(x) \cap r\mathbb{B} \neq \emptyset, \forall x \in \mathbb{B}\}.$$

When $\text{dom } F = X$ and F is single-valued on X both norms agree. For F linear and bounded, both norms reduce to the operator norm $\|F\|$. Neither $\|F\|^+$ nor $\|F\|^-$ satisfy the conditions in the definition of a norm, since sublinear mappings do not form a vector space.

Since the infimum over a nonempty set is greater or equal than the supremum, for any sublinear mapping with $\text{dom } F = X$ one has $\|F\|^+ \geq \|F\|^-$. But when $\text{dom } F \neq X$, and thus $\|F\|^- = \infty$, we can have $\|F\|^+ < \|F\|^-$. In fact, for a sublinear multivalued mapping, the inner and the outer norms cannot be finite simultaneously (see [1] for details).

Robinson gave in [7] a definition of the inner norm restricted to the domain, namely,

$$\|F\|_d^- = \sup_{x \in \mathbb{B} \cap \text{dom } F} \inf_{y \in F(x)} \|y\|.$$

This restriction to the domain changes significantly the inner norm so that the duality theorem does not hold anymore (for more, see [1]).

The norm duality theorem relates inner and outer norms of a mapping and its adjoint in the sense of Rockafellar [8]. Denoting X^* and Y^* the dual spaces of X and Y , the *upper adjoint* of a positively homogeneous mapping $F : X \rightrightarrows Y$ is a mapping $F^{*+} : Y^* \rightrightarrows X^*$ defined as

$$x^* \in F^{*+}(y^*) \iff \langle x^*, x \rangle \leq \langle y^*, y \rangle \text{ for all } (x, y) \in \text{gph } F,$$

while the *lower adjoint* is a mapping $F^{*-} : Y^* \rightrightarrows X^*$ having

$$x^* \in F^{*-}(y^*) \iff \langle x^*, x \rangle \geq \langle y^*, y \rangle \text{ for all } (x, y) \in \text{gph } F,$$

that is, $\text{gph } F^{*+} = -\text{gph } F^{*-}$.

Following [10] we say that a mapping $F : X \rightrightarrows Y$ is *inner semicontinuous* at $\bar{x} \in \text{dom } F$ if for every $y \in F(\bar{x})$ and every neighborhood V of y one can find a neighborhood U of $\bar{x}$ with

$$F(x) \cap V \neq \emptyset, \quad \text{for all } x \in U.$$

The version of Borwein's theorem we prove is attached next, where, unlike in [3], we do not assume that X is a Banach space.

Theorem 1 (Norm Duality). *Let $F : X \rightrightarrows Y$ be a sublinear mapping between normed spaces X and Y . Then*

$$(1) \quad \|F\|^+ = \|F^{*-}\|^- = \|F^{*+}\|^-.$$

Even more, if Y is a Banach space and $\text{gph } F$ is closed, then

$$(2) \quad \|F\|^- = \|F^{*-}\|^+ = \|F^{*+}\|^+.$$

In the proof we use the Hahn-Banach extension theorem (see, e.g., [11, Theorem 3.2]).

Theorem 2 (Hahn-Banach). *Let X be a real linear space, and let $p : X \rightarrow \mathbb{R}$ be a function such that $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$, and $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \geq 0$. Let M be a real linear subspace of X and let $t : M \rightarrow \mathbb{R}$ be a linear function such that $t(x) \leq p(x)$ for all $x \in M$. Then there exists a linear function $T : X \rightarrow \mathbb{R}$ such that $T(x) = t(x)$ for all $x \in M$, and $T(x) \leq p(x)$ for all $x \in X$.*

We also use the following corollary of the extension Theorem 2 (see, e.g., [2, Theorem 2.14]).

Corollary 3 (Separation). *Let X be a real normed space and let $C \subset X$ be a nonempty closed convex set. If $x_0 \notin C$, then there is $x^* \in X^*$ such that*

$$\langle x^*, x_0 \rangle > \sup_{x \in C} \langle x^*, x \rangle.$$

Proof of Theorem 1. First, note that $F^{*+}(y^*) = -F^{*-}(-y^*)$ for any $y^* \in Y^*$. Then

$$\|F^{*-}\|^- = \|F^{*+}\|^- \quad \text{and} \quad \|F^{*-}\|^+ = \|F^{*+}\|^+.$$

By definition,

$$\|F\|^+ = \sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \|y\| = \sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \sup_{y^* \in \mathbb{B}} \langle y^*, y \rangle,$$

and

$$\|F^{*-}\|^- = \sup_{y^* \in \mathbb{B}} \inf_{x^* \in F^{*-}(y^*)} \|x^*\|.$$

To prove the equality $\|F\|^+ = \|F^{*-}\|^-$ it is enough to show that

$$\sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \langle y^*, y \rangle = \inf_{x^* \in F^{*-}(y^*)} \|x^*\| \quad \text{for all } y^* \in \mathbb{B}.$$

In fact, this holds true for any $y^* \in Y^*$. Fix $y^* \in Y^*$.

STEP 1. If $\inf_{x^* \in F^{*-}(y^*)} \|x^*\| < r$ for some $r > 0$, then there exist $x^* \in F^{*-}(y^*)$ such that $\|x^*\| < r$. For any $\tilde{x} \in \mathbb{B}$ and $\tilde{y} \in F(\tilde{x})$ we have

$$\langle y^*, \tilde{y} \rangle \leq \langle x^*, \tilde{x} \rangle \leq \sup_{x \in \mathbb{B}} \langle x^*, x \rangle = \|x^*\| < r,$$

and then $\sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \langle y^*, y \rangle \leq r$. Hence

$$(3) \quad \sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \langle y^*, y \rangle \leq \inf_{x^* \in F^{*-}(y^*)} \|x^*\|.$$

STEP 2. To prove the inequality opposite to (3), assume that

$$\sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \langle y^*, y \rangle < r$$

for some $r > 0$. Pick $0 < d < r$ such that

$$(4) \quad \sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \langle y^*, y \rangle \leq d.$$

Define the mapping $H : X \rightrightarrows \mathbb{R}$ as

$$H(x) := \{\langle y^*, y \rangle \mid y \in F(x + \mathbb{B})\}, \quad \text{for } x \in X.$$

First, observe that $\text{gph } H$ is convex. Indeed, if $(x_1, z_1), (x_2, z_2) \in \text{gph } H$ and $0 < \lambda < 1$, then there exist $y_i \in Y$ and $w_i \in \mathbb{B}$ with $z_i = \langle y^*, y_i \rangle$ and $y_i \in F(x_i + w_i)$, for $i = 1, 2$. Since F is sublinear, we get

$$\lambda y_1 + (1 - \lambda)y_2 \in F(\lambda(x_1 + w_1) + (1 - \lambda)(x_2 + w_2)) \subset F(\lambda x_1 + (1 - \lambda)x_2 + \mathbb{B}),$$

and thus,

$$\lambda(x_1, z_1) + (1 - \lambda)(x_2, z_2) = (\lambda x_1 + (1 - \lambda)x_2, \langle y^*, \lambda y_1 + (1 - \lambda)y_2 \rangle) \in \text{gph } H.$$

We will show next that H is inner semicontinuous at 0. Take $\tilde{z} \in H(0)$ and $\varepsilon > 0$. Let $\tilde{z} = \langle y^*, \tilde{y} \rangle$, for $\tilde{y} \in F(\tilde{w})$ and $\tilde{w} \in \mathbb{B}$. Since $\langle y^*, \cdot \rangle$ is continuous, there is some $\gamma > 0$ such that $|\langle y^*, y \rangle - \tilde{z}| \leq \varepsilon$ when $\|y - \tilde{y}\| \leq \gamma$. Choose $\delta \in (0, 1)$ such that $\delta\|\tilde{y}\| \leq \gamma$. If $\|x\| \leq \delta$, we have

$$\|(1 - \delta)\tilde{w} - x\| \leq \|(1 - \delta)\tilde{w}\| + \|x\| \leq 1,$$

and hence $(1 - \delta)\tilde{w} - x \in \mathbb{B}$. Since F is sublinear,

$$(1 - \delta)\tilde{y} \in F((1 - \delta)\tilde{w}) = F(x + ((1 - \delta)\tilde{w} - x)) \subset F(x + \mathbb{B}),$$

whenever $\|x\| \leq \delta$. Moreover,

$$\|(1 - \delta)\tilde{y} - \tilde{y}\| = \delta\|\tilde{y}\| \leq \gamma,$$

and then

$$|\langle y^*, (1 - \delta)\tilde{y} \rangle - \tilde{z}| \leq \varepsilon.$$

Therefore, for all $x \in \delta\mathbb{B}$, we have $\langle y^*, (1 - \delta)\tilde{y} \rangle \in H(x) \cap \mathbb{B}_\varepsilon(\tilde{z})$, and hence H is inner semicontinuous at 0 as desired.

Let us now define a mapping $K : X \rightrightarrows \mathbb{R}$ by

$$\text{gph } K := \text{cone}(\text{gph}(d - H)).$$

Clearly K is a sublinear mapping. Since H is inner semicontinuous at 0, there is some neighborhood U of 0 with $U \subset \text{dom } H$, and therefore $\text{dom } K = X$. Consider

$$k(x) := \inf\{z \mid z \in K(x)\} \quad \text{for } x \in X.$$

Since K is sublinear and $d - H(0) \subset \mathbb{R}_+$, we have

$$(5) \quad K(x) + K(-x) \subset K(0) \subset \mathbb{R}_+.$$

This inclusion implies in particular that any point in $-K(-x)$ is a lower bound for the set of values $K(x)$, for any $x \in X$. Indeed, let $x \in X$ and $y \in -K(-x)$. Then (5) yields $K(x) - y \subset \mathbb{R}_+$, and thus

$$y \leq z \quad \text{for all } z \in K(x).$$

Therefore $k(x)$ is finite for all $x \in X$, and hence the function $k : X \rightarrow \mathbb{R}$ is well defined. Also, from the sublinearity of K and the properties of the infimum, we have

$$k(x + y) \leq k(x) + k(y) \quad \text{and} \quad k(\alpha x) = \alpha k(x), \quad \text{for all } x, y \in X \text{ and } \alpha \geq 0.$$

Consider the subspace $M = \{0\} \subset X$ and define $t : M \rightarrow \mathbb{R}$ by $t(0) := k(0) = 0$. Applying the Hahn-Banach extension Theorem 2 to t , we find a linear functional $T : X \rightarrow \mathbb{R}$ such that $T(0) = 0$ and $T(x) \leq k(x)$ for all $x \in X$.

We will show now that T is continuous at 0 and hence it is continuous on the whole X . Continuity at 0 means that for all $\varepsilon > 0$ there is $\delta > 0$ such that $(T(x) + \mathbb{R}_+) \cap \varepsilon\mathbb{B} \neq \emptyset$, whenever $x \in \delta\mathbb{B}$. Let $z \in d - H(0)$. Take $0 < \lambda < 1$ and a neighborhood V of z such that $\lambda V \subset \varepsilon\mathbb{B}$. Since H is inner semicontinuous at 0, there is some $\delta > 0$ such that

$$(d - H(x)) \cap V \neq \emptyset, \quad \text{for all } x \in (\delta/\lambda)\mathbb{B}.$$

Since $d - H(x) \subset k(x) + \mathbb{R}_+$ and $k(x) \geq T(x)$, we have $d - H(x) \subset T(x) + \mathbb{R}_+$,

$$(T(x) + \mathbb{R}_+) \cap V \neq \emptyset, \quad \text{for all } x \in (\delta/\lambda)\mathbb{B},$$

and then

$$(T(\lambda x) + \mathbb{R}_+) \cap \lambda V \neq \emptyset, \quad \text{for all } x \in (\delta/\lambda)\mathbb{B},$$

which yields

$$(T(x) + \mathbb{R}_+) \cap \varepsilon\mathbb{B} \neq \emptyset, \quad \text{for all } x \in \delta\mathbb{B}.$$

This means that for all $x \in \delta\mathbb{B}$ there exists some $z \geq T(x)$ with $|z| \leq \varepsilon$. Since T is linear, $T(-x) = -T(x)$, and therefore $|T(x)| \leq \varepsilon$ for all $x \in \delta\mathbb{B}$. This shows the continuity of T .

The inclusion $d - H(x) - T(x) \subset \mathbb{R}_+$ is equivalent to

$$d - \langle y^*, y \rangle - T(x) \geq 0, \quad \text{whenever } x \in X \text{ and } y \in F(x + \mathbb{B}).$$

Let $x^* \in X^*$ be such that $\langle x^*, x \rangle = -T(x)$ for all $x \in X$. Then

$$\langle y^*, y \rangle - \langle x^*, x \rangle \leq d, \quad \text{for all } x \in X \text{ and } y \in F(x + \mathbb{B}).$$

Pick $y \in F(x)$ and $\lambda > 0$. Then $\lambda y \in F(\lambda x)$ and

$$\langle y^*, \lambda y \rangle - \langle x^*, \lambda x \rangle \leq d,$$

or equivalently,

$$\langle y^*, y \rangle - \langle x^*, x \rangle \leq d/\lambda.$$

Passing to the limit with $\lambda \rightarrow \infty$, we obtain $x^* \in F^{*-}(y^*)$. Let now $x \in \mathbb{B}$. Since $0 \in F(0)$, we have $0 \in F(-x + \mathbb{B})$, and hence

$$\langle y^*, 0 \rangle - \langle x^*, -x \rangle \leq d.$$

Therefore $\|x^*\| \leq d < r$ and then $\inf_{x^* \in F^{*-}(y^*)} \|x^*\| < r$. This completes Step 2 and the proof of (1).

We will complete the proof of (2) by showing that $\|F\|^- = \|F^{*+}\|^+$ when Y is a Banach space and $\text{gph } F$ is closed.

STEP 3. Let $\|F\|^- < r$ for some $r > 0$. Then for any $\tilde{x} \in \mathbb{B}$ there is some $\tilde{y} \in F(\tilde{x})$ with $\|\tilde{y}\| < r$. Given some $y^* \in \mathbb{B}$ and $x^* \in F^{*+}(y^*)$ we have

$$\langle x^*, \tilde{x} \rangle \leq \langle y^*, \tilde{y} \rangle \leq \|\tilde{y}\| < r.$$

Since this last inequality is valid for any $\tilde{x} \in \mathbb{B}$, we obtain $\|x^*\| \leq r$, and therefore $\|F^{*+}\|^+ \leq r$. Hence

$$\|F\|^- \geq \|F^{*+}\|^+.$$

STEP 4. To prove $\|F\|^- \leq \|F^{*+}\|^+$, suppose that $\|F^{*+}\|^+ < r$. Pick $s > 0$ with

$$\sup_{x^* \in F^{*+}(\mathbb{B})} \|x^*\| = \|F^{*+}\|^+ \leq s < r.$$

Thus, $F^{*+}(\mathbb{B}) \subset s\mathbb{B}$. We will show that $(F^{-1}(\mathbb{B}))^\circ \subset F^{*+}(\mathbb{B})$, and hence

$$(6) \quad (F^{-1}(\mathbb{B}))^\circ \subset s\mathbb{B}.$$

Let $x^* \in (F^{-1}(\mathbb{B}))^\circ$. Then

$$\langle x^*, x \rangle \leq 1 \quad \text{for all } x \in F^{-1}(\mathbb{B}),$$

or, equivalently,

$$\sup_{y \in \mathbb{B}} \sup_{x \in F^{-1}(\mathbb{B})} \langle x^*, x \rangle \leq 1.$$

This latter inequality is analogous to (4), with $d = 1$ and F replaced by F^{-1} and thus, with y and y^* replaced by x and x^* , respectively. By repeating the argument in the proof of Step 2, we find some $y^* \in (F^{-1})^{*-}(x^*) = (F^{*+})^{-1}(x^*)$ with $\|y^*\| \leq 1$. But then $x^* \in F^{*+}(\mathbb{B})$ and (6) follows.

Now we will show that (6) implies

$$(7) \quad s^{-1}\mathbb{B} \subset \text{cl } F^{-1}(\mathbb{B}).$$

Indeed, if $x \notin \text{cl } F^{-1}(\mathbb{B})$, then from the separation Corollary 3, there exists $\tilde{x}^* \in X^*$ with

$$\langle \tilde{x}^*, x \rangle > \sup_{z \in \text{cl } F^{-1}(\mathbb{B})} \langle \tilde{x}^*, z \rangle \geq \langle \tilde{x}^*, 0 \rangle = 0.$$

Let $\lambda > 0$ be such that

$$\sup_{z \in \text{cl } F^{-1}(\mathbb{B})} \langle \tilde{x}^*, z \rangle < \lambda^{-1} < \langle \tilde{x}^*, x \rangle.$$

Then we have

$$\langle \lambda \tilde{x}^*, x \rangle > 1 > \sup_{z \in \text{cl } F^{-1}(\mathbb{B})} \langle \lambda \tilde{x}^*, z \rangle.$$

In particular, this implies that $\lambda \tilde{x}^* \in (F^{-1}(\mathbb{B}))^\circ \subset s\mathbb{B}$. Hence,

$$s \geq \|\lambda \tilde{x}^*\| \geq \langle \lambda \tilde{x}^*, x / \|x\| \rangle > \frac{1}{\|x\|},$$

and therefore $x \notin s^{-1}\mathbb{B}$. Thus, (6) holds.

We will show next that

$$(8) \quad \text{int } s^{-1}\mathbb{B} \subset \text{int } F^{-1}(\mathbb{B}).$$

Let $x \in \text{int } s^{-1}\mathbb{B}$. Then from (6) we have $x \in \text{int } \text{cl } F^{-1}(\mathbb{B})$, and hence, there is some $\varepsilon > 0$ such that

$$\mathbb{B}_\varepsilon(x) \subset \text{cl } F^{-1}(\mathbb{B}).$$

Since $\text{cl } F^{-1}(\mathbb{B}) \subset F^{-1}(\mathbb{B}) + (\varepsilon/2)\mathbb{B}$, we have

$$x + \varepsilon\mathbb{B} \subset F^{-1}(\mathbb{B}) + \frac{\varepsilon}{2}\mathbb{B}.$$

Multiplying the last inclusion by $1/2$ and adding $x/2$ to both sides we obtain

$$(9) \quad \mathbb{B}_{\varepsilon/2}(x) \subset \frac{1}{2}F^{-1}(\mathbb{B}) + \frac{1}{2}\mathbb{B}_{\varepsilon/2}(x).$$

We will show next that $\mathbb{B}_{\varepsilon/2}(x) \subset F^{-1}(\mathbb{B})$. From (9) we have

$$\mathbb{B}_{\varepsilon/2}(x) \subset \frac{1}{2}F^{-1}(\mathbb{B}) + \frac{1}{4}F^{-1}(\mathbb{B}) + \dots + \frac{1}{2^k}F^{-1}(\mathbb{B}) + \frac{1}{2^{k+1}}\mathbb{B}_{\varepsilon/2}(x),$$

for all $k \in \mathbb{N}$. Let $z \in \mathbb{B}_{\varepsilon/2}(x)$. Then, there is $z_1, \dots, z_k \in F^{-1}(\mathbb{B})$ such that

$$z \in \frac{1}{2}z_1 + \frac{1}{4}z_2 + \dots + \frac{1}{2^k}z_k + \frac{1}{2^{k+1}}\mathbb{B}_{\varepsilon/2}(x).$$

Hence, by induction, for every $n \in \mathbb{N}$, there is $z_n \in F^{-1}(\mathbb{B})$ such that $z = \sum_{n \in \mathbb{N}} 2^{-n}z_n$. Thus, for each $n \in \mathbb{N}$, there is $y_n \in \mathbb{B}$ with $(z_n, y_n) \in \text{gph } F$. Consider the sequences

$$\tilde{z}_n := \sum_{\kappa=1}^n \frac{1}{2^\kappa} z_\kappa \quad \text{and} \quad \tilde{y}_n := \sum_{\kappa=1}^n \frac{1}{2^\kappa} y_\kappa,$$

for $n \in \mathbb{N}$. Clearly, $\tilde{z}_n$ converges to z . Moreover, since

$$\|\tilde{y}_n - \tilde{y}_{n-1}\| = \frac{1}{2^n} \|y_n\| \leq \frac{1}{2^n}, \quad \text{for all } n \in \mathbb{N},$$

we conclude that $\tilde{y}_n$ is a Cauchy sequence in Y . Then, since Y is a Banach space, it converges to some $y \in Y$ whose norm satisfies

$$\|y\| \leq \sum_{n \in \mathbb{N}} \frac{1}{2^n} \|y_n\| \leq 1.$$

Therefore, the sequence $(\tilde{z}_n, \tilde{y}_n)$ converges to (z, y) , and because of the closedness of $\text{gph } F$, we have $z \in F^{-1}(y) \subset F^{-1}(\mathbb{B})$.

Summarizing, we have shown that for any $x \in \text{int } s^{-1}\mathbb{B}$ there exists some $\varepsilon > 0$ such that $\mathbb{B}_{\varepsilon/2}(x) \subset F^{-1}(\mathbb{B})$. This yields

$$\text{int } s^{-1}\mathbb{B} \subset \text{int } F^{-1}(\mathbb{B}).$$

Then

$$r^{-1}\mathbb{B} \subset \text{int } s^{-1}\mathbb{B} \subset F^{-1}(\mathbb{B}),$$

and thus, $\|F\|^- \leq r$. This completes the proof of Step 4 and hence we have shown that $\|F\|^- = \|F^{*+}\|^+$, concluding the proof of (2). $\square$

Given a sublinear mapping $F : X \rightrightarrows Y$, where X is a normed space and Y is a Banach space, we can define a sublinear mapping $\bar{F} : X \rightrightarrows Y$ by $\text{gph } \bar{F} := \text{cl gph } F$, where $\text{cl } A$ denotes the *closure* of the set A . Since $\text{gph } F \subset \text{gph } \bar{F}$, it is clear that $\text{gph } \bar{F}^{*+} \subset \text{gph } F^{*+}$. On the other hand, suppose that $(y^*, x^*) \in \text{gph } F^{*+}$. Let $(x, y) \in \text{gph } \bar{F}$. Then there is a sequence $(x_n, y_n) \in \text{gph } F$ with $(x_n, y_n) \rightarrow (x, y)$, and hence $\langle x^*, x_n \rangle \leq \langle y^*, y_n \rangle$. Taking limits, we have $\langle x^*, x \rangle \leq \langle y^*, y \rangle$, and since this is true for all $(x, y) \in \text{gph } \bar{F}$, we obtain $(y^*, x^*) \in \text{gph } \bar{F}^{*+}$. Therefore

$$(10) \quad \bar{F}^{*+} = F^{*+}.$$

Also, given $x \in X$, if $y \in F(x)$, then $(x, y) \in \text{gph } F \subset \text{gph } \bar{F}$, and hence $y \in \bar{F}(x)$. Thus, applying (10) and Theorem 1,

$$(11) \quad \|F\|^- = \sup_{x \in \mathbb{B}} \inf_{y \in F(x)} \|y\| \geq \sup_{x \in \mathbb{B}} \inf_{y \in \bar{F}(x)} \|y\| = \|\bar{F}\|^- = \|\bar{F}^{*+}\|^+ = \|F^{*+}\|^+.$$

In fact, this inequality follows directly from Step 3 of Theorem 1, where Y does not need to be a Banach space.

Corollary 4. *Let $F : X \rightrightarrows Y$ be a sublinear mapping between normed spaces X and Y . Then*

$$(12) \quad \|F\|^- \geq \|F^{*+}\|^+ = \|F^{*-}\|^+.$$

The natural question now is the following: is the equality (2) valid without the closedness assumption? The answer is *no*, even for a single-valued mapping acting between Banach spaces, as the following example shows.

Consider the Banach space ℓ_1 whose elements are the scalar-valued sequences $x = \{x_n\}_{n \in \mathbb{N}}$ satisfying $\sum_{n \in \mathbb{N}} |x_n| < \infty$, endowed with the norm

$$\|x\|_1 = \sum_{n \in \mathbb{N}} |x_n|.$$

Let $F : \ell_1 \rightarrow \ell_1$ defined by

$$F(x) = \begin{cases} x & \text{if } x_n \neq 0 \text{ for a finite number of } n \\ \emptyset & \text{otherwise,} \end{cases}$$

for $x = \{x_n\}_{n \in \mathbb{N}}$. It is easy to check that F is a sublinear mapping having $\text{dom } F \neq X$ (e.g., for $x = \{1/2^n\}_{n \in \mathbb{N}} \in \ell_1$ we have $F(x) = \emptyset$). Thus $\|F\|^- = \infty$. On the other hand, given $x = \{x_n\}_{n \in \mathbb{N}} \in \ell_1$, we can define $x^k = \{x_n^k\}_{n \in \mathbb{N}} \in \ell_1$ by

$$x_n^k := \begin{cases} x_n & \text{for } n \leq k, \\ 0 & \text{for } n > k. \end{cases}$$

Then

$$\lim_{k \rightarrow \infty} \|x - x^k\| = \lim_{k \rightarrow \infty} \left(\sum_{n=k+1}^{\infty} |x_n| \right) = \lim_{k \rightarrow \infty} \left(\sum_{n \in \mathbb{N}} |x_n| - \sum_{n=1}^k |x_n| \right) = 0,$$

and thus $x^k \rightarrow x$. Since $F(x^k) = x^k$ for all $k \in \mathbb{N}$, we have $\bar{F}(x) = x$ for all $x \in \ell_1$. Then $\|F^{*+}\|^+ = \|\bar{F}^{*+}\|^+ = \|\bar{F}\|^- = 1$, but $\|F\|^- = \infty$.

A modification of this example serve us to show that the first equality in (2) is not valid in general when Y is not complete. Consider the Banach space c_0 whose elements are the scalar-valued sequences $x = \{x_n\}_{n \in \mathbb{N}}$ satisfying $\lim_{n \rightarrow \infty} x_n = 0$, endowed with the norm

$$\|x\|_{\infty} = \sup_{n \in \mathbb{N}} |x_n|.$$

Let c_{00} be the subspace of c_0 consisting of all the sequences with a finite number of nonzero elements. It is well known that c_{00} is a normed space which is not complete. Consider the mapping $F : c_0 \rightarrow c_{00}$ defined by

$$F(x) = \begin{cases} x & \text{if } x \in c_{00}, \\ \emptyset & \text{otherwise,} \end{cases}$$

for $x = \{x_n\}_{n \in \mathbb{N}}$. It is not difficult to check that F is a sublinear mapping with $\text{dom } F \neq c_0$, and hence $\|F\|^- = \infty$. In this case $\text{gph } F$ is closed, but $\|F^{*+}\|^+ = 1$. Indeed, given $(y^*, x^*) \in (c_{00}^*, c_0^*)$, we have

$$(y^*, x^*) \in \text{gph } F^{*+} \iff \langle x^*, x \rangle = \langle y^*, x \rangle \quad \text{for all } x \in c_{00}.$$

Fix $y^* \in \mathbb{B}$ and $x^* \in F^{*+}(y^*)$. Then

$$\|x^*\| = \sup_{x \in c_0 \cap \mathbb{B}} \langle x^*, x \rangle \geq \sup_{x \in c_{00} \cap \mathbb{B}} \langle x^*, x \rangle = \sup_{x \in c_{00} \cap \mathbb{B}} \langle y^*, x \rangle = \|y^*\|.$$

Moreover, we must have $\|x^*\| = \|y^*\|$. Otherwise, there is some $\varepsilon > 0$ such that $\|x^*\| > \|y^*\| + \varepsilon$, and thus, there exists some $\tilde{x} \in c_0 \cap \mathbb{B}$ with

$$\langle x^*, \tilde{x} \rangle > \|y^*\| + \varepsilon = \sup_{y \in c_{00} \cap \mathbb{B}} \langle y^*, y \rangle + \varepsilon.$$

This implies

$$(13) \quad \langle x^*, \tilde{x} \rangle > \langle y^*, y \rangle + \varepsilon = \langle x^*, y \rangle + \varepsilon \quad \text{for any } y \in c_{00} \cap \mathbb{B}.$$

If $\tilde{x} = \{\tilde{x}_n\}_{n \in \mathbb{N}}$, consider the sequence $z^n := \{\tilde{x}_1, \dots, \tilde{x}_n, 0, \dots\} \in c_{00} \cap \mathbb{B}$, for $n \in \mathbb{N}$. Since

$$\|z^n - \tilde{x}\|_\infty = \sup_{k > n} |\tilde{x}_k|,$$

and $\tilde{x} \in c_0$, we have that z^n converges to $\tilde{x}$. But (13) implies

$$\langle x^*, \tilde{x} \rangle > \langle x^*, z^n \rangle + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Then, passing to the limit with $n \rightarrow \infty$, we obtain a contradiction. Hence, we have $\|x^*\| = \|y^*\| \leq 1$, and therefore $\|F^{*+}\|^+ = 1 \neq \|F\|^- = \infty$.

Now the next question would be whether the closedness condition is necessary in order to have (2). The answer turns out to be again negative. Indeed, let $F : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$F(x) = \begin{cases} (x, \infty) & \text{for } x \neq 0, \\ [0, \infty) & \text{for } x = 0. \end{cases}$$

The graph of this mapping is not closed, but it has $\|F\|^- = \|F^{*+}\|^+ = \|F^{*-}\|^+ = 1$.

In [3] Borwein considered another sufficient condition for having the equality (2): the inner semicontinuity of the mapping F (there called lower semicontinuity). As we can see from the following example, neither the closedness nor the inner semicontinuity conditions are necessary for having (2).

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a sublinear mapping defined by

$$F(x) = \begin{cases} (0, \infty) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ \emptyset & \text{for } x < 0. \end{cases}$$

Here F does not have the graph closed and it is not inner semicontinuous at 0, but it has $\|F\|^- = \|F^{*+}\|^+ = \infty$.

These examples leads us to the next result.

Corollary 5. *Let $F : X \rightrightarrows Y$ be a sublinear mapping between normed spaces X and Y . Let $\bar{F}$ be the closed sublinear mapping defined by $\text{gph } \bar{F} = \text{cl gph } F$. If $\text{dom } \bar{F} \neq X$ or F has the property*

$$(14) \quad \bar{F}(x) = \text{cl } F(x) \quad \text{for all } x \neq 0,$$

then

$$(15) \quad \|F\|^- = \|F^{*+}\|^+ = \|F^{*-}\|^+.$$

Proof. Because of (11), we just have to prove that $\|F\|^- = \|\bar{F}\|^-$. If $\text{dom } \bar{F} \neq X$, then

$$\|F\|^- \geq \|\bar{F}\|^- = \infty,$$

and thus we obtain (15). Now assume that F verifies (14). We will show that

$$(16) \quad \inf_{y \in F(x)} \|y\| = \inf_{y \in \bar{F}(x)} \|y\|, \quad \text{for all } x \in X.$$

For $x = 0$ this is true, since $(0, 0) \in \text{gph } F$. Consider now $x \in X \setminus \{0\}$. Because of (14), proving (16) is equivalent to show that

$$\inf_{y \in F(x)} \|y\| = \inf_{y \in \text{cl } F(x)} \|y\|,$$

which is always valid. Taking the relevant suprema we complete the proof. $\square$

We will now show that the property (14) is weaker than both closedness and inner semicontinuity. Therefore, replacing the second part of Theorem 1 with the statement of Corollary 5 gives us a true generalization of Borwein's theorem. Indeed, if F is a sublinear mapping with closed graph it is clear that $\bar{F} = F$ and $\text{cl } F(x) = F(x)$ for all $x \in X$. This implies (14). Now suppose that F is inner semicontinuous at 0. Since always $\text{cl } F(x) \subset \bar{F}(x)$ for any $x \in X$, we just need to prove the opposite inclusion. Let $y \in \bar{F}(x)$, for some $x \in X$. Then there is a sequence $(x_n, y_n) \in \text{gph } F$ such that $(x_n, y_n) \rightarrow (x, y)$. Let $V_k := (1/k)\mathbb{B}$ for $k \in \mathbb{N}$. Since F is inner semicontinuous at 0 and $0 \in F(0)$, for every $k \in \mathbb{N}$ there is a neighborhood U_k of 0 such that

$$(17) \quad F(x) \cap V_k \neq \emptyset, \quad \text{for any } x \in U_k.$$

Since $x - x_n \rightarrow 0$, for every $k \in \mathbb{N}$ there is some $n_k \geq k$ with $x - x_{n_k} \in U_k$. Then from (17) there is some $w_k \in (1/k)\mathbb{B} \cap F(x - x_{n_k})$, for every $k \in \mathbb{N}$. Hence $w_k \rightarrow 0$ and $(x - x_{n_k}, w_k) \in \text{gph } F$, and therefore,

$$(x, y_{n_k} + w_k) = (x_{n_k}, y_{n_k}) + (x - x_{n_k}, w_k) \in \text{gph } F,$$

having $y_{n_k} + w_k \xrightarrow{k} y$. Thus $y \in \text{cl } F(x)$ and this implies (14).

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