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A Fault Tolerant Multisensor Switching Scheme for State Estimation

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Abstract: A multisensor estimation scheme with the ability to accommodate multiple sensor faults is presented. A switching strategy is employed such that, at each sampling time, a sensor-estimator pair is selected to provide the best state estimate as measured by an optimisation criterion. We show that, if a set of conditions on the system parameters (such as bounds on the sensors noises, disturbances, operating conditions, etc.) is satisfied then the switching estimation scheme is able to guarantee fault tolerant capabilities under multiple sensor failures.

Keywords: Fault tolerant systems, Sets, Multisensor integration, State estimation

1. INTRODUCTION

Sensors are used to obtain qualitative and quantitative information of variables of interest in a dynamical system. Each type of sensor typically has a good performance under specific operating or environmental conditions. However, during certain periods of time one or more sensors can fail or operate outside their specific operating range. Thus, in some applications a single sensor is not adequate to provide reliable information due to changes in the environment, failure and other limitations. A sensible approach is then to use multiple sensors of different characteristics so as to improve the performance attained with individual sensors. Several methods have been developed to combine the data provided by multiple sensors into a single integrated measurement. One of the more popular approaches is known as *sensor fusion*. Most of the existing multisensor fusion literature focuses on the problem of obtaining better and more reliable estimates from the availability of multiple sensors, e.g., Luo et al. [2002], Sun and Deng [2004]. However, less attention has been given to the problem of sensor failure, e.g., Fernandez and Durrant-Whyte [1994].

In this paper we propose a novel methodology for combining measurements from multiple sensors, which consists of a sensor-switching strategy. We investigate the properties of the resulting state estimate in the presence of measurement noises and process disturbances. We show that the estimation errors remain bounded, and that they converge to zero in the absence of disturbances and measurement noises. An important practical issue that we address is that of sensor failure. We investigate the case of complete loss of information (i.e., when faulty sensors provide at their outputs only noise uncorrelated to the process variables). A set of conditions is provided such that the switching scheme never selects failed sensors and, hence, by choosing only among healthy sensors, is able to keep the estimation errors bounded with bounds given by the healthy sensors.

We provide a simulation example that illustrates the performance of the switching estimation scheme under sensor failures, and compare it with a conventional approach for multisensory data estimation. To perform this comparison, we have chosen a well-established technique based on *fusion Kalman filters* (Sun and Deng [2004]) equipped with a *fault detection and isolation* (FDI) module based on hypothesis testing. The simulation example shows that the multisensor switching strategy performs, under healthy operation of all sensors, in a comparable way to the sensor fusion strategy. However, the main advantages of the switching scheme can be appreciated in the presence of faulty sensors. The simulations of Section 4 illustrate the superior performance, in the presence of sensor faults, of the multisensor switching scheme when compared to the sensor fusion/FDI scheme.

Experimental results for the proposed methodology, in the context of closed-loop control of a magnetic levitation system, can be found in Yetendje et al. [2009].

2. MULTISENSOR SWITCHING SCHEME

2.1 Plant and Sensors Models

The switching estimation scheme is shown in Figure 1, where the plant P is given by the following linear discrete-time model:

$$x(k+1) = Ax(k) + Bu(k) + Ew(k), \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is a known input, and $w(k) \in \mathbb{R}^r$ is the process disturbance.

We assume a family of N sensors S_i , $i = 1, \dots, N$, each of which measures a (possibly different) linear combination of the system states according to the following equations²

$$y_i(k) = C_i x(k) + v_i(k), \quad i = 1, \dots, N, \quad (2)$$

² To simplify the exposition we use *static* sensors, which provide a good approximation when the sensor dynamics are much faster than the process dynamics. Sensors with dynamics can also be utilised and all the analyses carry through with very slight modifications.

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Fig. 1. Multisensor switching estimator with plant P , sensors S_i and estimators E_i ($i = 1, \dots, N$).

where $y_i(k) \in \mathbb{R}^{p_i}$ is the measurement provided by each sensor and $v_i(k) \in \mathbb{R}^{p_i}$ is the measurement noise.

Assumption 1. The pairs (A, C_i) are detectable for $i = 1, \dots, N$. $\circ$

Definition 2. (Healthy sensor). A sensor is “healthy”, or operational, when its measured output is given by (2). $\circ$

2.2 Decentralised State Estimators

We consider a family of N state estimators E_i , $i = 1, \dots, N$, where each estimator is designed to estimate the state of the system using the measured output of one sensor only. The estimators are described by the following dynamic and update equations:

$$\hat{x}_i(k+1) = A\hat{x}_i(k) + Bu(k) + L_i[y_i(k) - C_i\hat{x}_i(k)], \quad (3)$$

$$\hat{x}_i^{\text{UP}}(k) = \hat{x}_i(k) + M_i[y_i(k) - C_i\hat{x}_i(k)], \quad (4)$$

$i = 1, \dots, N$.

Assumption 3. The gains L_i are such that the matrices

$$A_{L_i} = A - L_i C_i, \quad (5)$$

for $i = 1, \dots, N$, have all eigenvalues strictly inside the unit circle (this is always possible by Assumption 1), and M_i are arbitrary matrices of appropriate dimensions. $\circ$

Remark 4. The state estimation errors, defined as

$$\tilde{x}_i(k) \triangleq x(k) - \hat{x}_i(k), \quad i = 1, \dots, N, \quad (6)$$

satisfy, using (1), (2), (3) and (5),

$$\tilde{x}_i(k+1) = A_{L_i}\tilde{x}_i(k) + Ew(k) - L_i v_i(k). \quad (7)$$

$\circ$

Note that the state estimation errors $\tilde{x}_i(k)$ defined in (6) cannot be computed from measurements since the state $x(k)$ is not available for measurement. Hence, we define the *innovations* of the state estimators as

$$\hat{z}_i(k) = y_i(k) - C_i\hat{x}_i(k), \quad i = 1, \dots, N, \quad (8)$$

which can be readily computed from measurements.

2.3 Switching Criterion

We propose a switching estimator that at each time instant selects one of the N sensor-estimator pairs by performing the following optimisation

$$\ell = \arg \min_i \left\{ \hat{z}_i(k)^T \Theta_i \hat{z}_i(k) : i = 1, \dots, N \right\}, \quad (9)$$

where $\Theta_i \in \mathbb{R}^{p_i \times p_i}$, $i = 1, \dots, N$, are positive definite weighting matrices that constitute a design choice. (From

a number of simulation studies we have, heuristically, concluded that a choice of constant $\Theta_i = \Theta > 0$ for all $i = 1, \dots, N$ provides a good discrimination between sensors with different measurement noise levels. Thus, a reasonable choice is the identity matrix, $\Theta_i = I_{p_i \times p_i}$, $i = 1, \dots, N$.) Suppose that the result of the optimisation (9) is the ℓ th sensor-estimator pair, $\ell \in \{1, \dots, N\}$, then the switching state estimate $\hat{x}_{\text{sw}}(k)$ is taken as the corresponding updated state estimate (4), that is,

$$\hat{x}_{\text{sw}}(k) = \hat{x}_\ell^{\text{UP}}(k) = \hat{x}_\ell(k) + M_\ell[y_\ell(k) - C_\ell\hat{x}_\ell(k)]. \quad (10)$$

In the remaining sections we investigate the properties of the proposed switching estimator.

3. OPERATION OF THE SWITCHING SCHEME

In this section we establish the main properties of the proposed switching estimation scheme. Namely, we show that, *under sensor failure*, the switching state estimation error defined as

$$\tilde{x}_{\text{sw}}(k) \triangleq x(k) - \hat{x}_{\text{sw}}(k), \quad (11)$$

with $\hat{x}_{\text{sw}}(k)$ defined by (10), remains bounded with *bounds determined by the healthy sensors*; and that it converges to zero in the absence of disturbances and measurement noise in the healthy sensors. We achieve these results by providing conditions that guarantee that the switching scheme never selects faulty sensors.

3.1 Assumptions and Preliminary Results

$|M|$ denotes the elementwise magnitude of a matrix M and $x \leq y$ ($x < y$) denotes the set of elementwise (strict) inequalities between the components of the real vectors x and y , and similarly for $x \geq y$ ($x > y$). Furthermore, if $\nu(k) \in \mathbb{R}^m$ is a discrete-time signal and $\bar{\nu} \in \mathbb{R}^m$, $\bar{\nu} \geq 0$, then $|\nu| \leq \bar{\nu}$ denotes the elementwise bound $|\nu(k)| \leq \bar{\nu}$ for all times $k \geq k^*$, for some $k^* \geq 0$. For some scalar $\gamma > 0$ we denote $\mathbb{B}(\gamma) \triangleq \{x \in \mathbb{R}^n : \|x\| \leq \gamma\}$, where $\|\cdot\|$ is the vector 2-norm. The following definition describes the failure model, which considers abrupt faults that lead to the outage of the sensor.

Definition 5. (Faulty Sensor). A sensor fails when its output ceases to carry information about the states in the plant. The sensor measured output is in this case

$$y_j(k) = v_j^F(k), \quad (12)$$

where $v_j^F(k)$ is a measurement noise uncorrelated with the states of the plant. $\circ$

Assumption 6. The disturbance in (1), the measurement noises of the *healthy* sensors (2), and the failure noises (12) are bounded; $|w(k)| \leq \bar{w}$, $|v_i(k)| \leq \bar{v}_i$, and $|v_j^F(k)| \leq \bar{v}_j^F$, respectively; where the bounds $\bar{w} > 0$, $\bar{v}_i > 0$ and $\bar{v}_j^F > 0$, $i, j = 1, \dots, N$, are problem data. $\circ$

Remark 7. In order to establish the main properties of the proposed switching estimation scheme we do not assume any particular probabilistic description for the noises and disturbances, and only require them to be bounded according to Assumption 6. This boundedness property is realistic in many practical applications, since it is often the case that disturbances and noises are generated by processes with limited capacity (e.g., quantisation errors are often modeled as bounded noises). $\circ$

Remark 8. It follows, from Assumption 3 and equation (7) that the individual state estimation errors $\tilde{x}_i(k)$, defined

in (6), corresponding to healthy sensors are bounded whenever $w(k)$ and $v_i(k)$ are bounded as in Assumption 6. Similarly, it follows that in the absence of disturbances and measurement noise in the healthy sensors ($w(k) \equiv 0$ and $v_i(k) \equiv 0$), the state estimation errors $\tilde{x}_i(k)$ corresponding to healthy sensors converge to zero. $\circ$

Proposition 9. Under the conditions of Assumptions 3 and 6, and provided only healthy sensors are selected by the switching criterion (9), the switching state estimation error $\tilde{x}_{\text{sw}}(k)$ defined by (11) is bounded. Moreover, provided only healthy sensors are selected, $\tilde{x}_{\text{sw}}(k)$ converges to zero in the absence of disturbances and measurement noise in the healthy sensors. $\circ$

Proof. It follows from (2), (6), (10) and (11) that

$$\tilde{x}_{\text{sw}}(k) = \tilde{x}_\ell(k) - M_\ell [C_\ell \tilde{x}_\ell(k) + v_\ell(k)]. \quad (13)$$

From Assumption 6 the measurement noises $v_\ell(k)$ corresponding to healthy sensors are bounded and, as noted in Remark 8, the estimation errors $\tilde{x}_\ell(k)$ corresponding to healthy sensors are also bounded. It then follows from (13) that $\tilde{x}_{\text{sw}}(k)$ is bounded. Finally, in the absence of disturbances ($w(k) \equiv 0$) and measurement noise in the healthy sensors ($v_\ell(k) \equiv 0$), $\tilde{x}_{\text{sw}}(k)$ converges to zero since, as noted in Remark 8, so does $\tilde{x}_\ell(k)$. $\square$

Assumption 10. The known input is bounded as $|u(k) - u_0| \leq \bar{u}$, where the bound $\bar{u} > 0$ and the offset u_0 are problem data, and such that the state of system (1) remains bounded as $|x(k) - x_0| \leq \bar{x}$. We assume that the bound $\bar{x} > 0$ and the offset x_0 are known and problem data. (This is trivially achieved if matrix A is stable, where x_0 is related to u_0 through the steady-state gain, i.e., $x_0 = (I - A)^{-1}Bu_0$. When A is unstable, it is assumed that $u(k)$ is generated by a stabilising feedback control loop.) $\circ$

Remark 11. It follows from Assumptions 6 and 10 that the measurements provided by healthy sensors are bounded as $|y_i(k) - y_{i0}| \leq \bar{y}_i$, where the bounds $\bar{y}_i > 0$ and offsets y_{i0} are readily computed from (2) and the known offset and bound on $x(k)$ and bounds on $v_i(k)$. In addition, it follows from (3)–(4) and Assumption 3 that the individual state estimates corresponding to healthy sensors, $\hat{x}_i(k)$ and $\hat{x}_i^{UP}(k)$, are also bounded. $\circ$

We present below a theorem from Kofman et al. [2007] that will allow us to compute ultimate bounds for the state estimates and estimation errors.

Theorem 12. Consider the system $x(k+1) = Ax(k) + Bv(k)$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and A has eigenvalues strictly inside the unit circle. Let $V\Lambda V^{-1}$ be the Jordan matrix decomposition of A . Assume that $|\nu(k)| \leq \bar{\nu}$ for all $k \geq 0$, where $\bar{\nu} \in \mathbb{R}^m$, $\bar{\nu} > 0$. For $\epsilon \in \mathbb{R}^n$, $\epsilon \geq 0$, define

$$S_\epsilon \triangleq \left\{ x \in \mathbb{R}^n : |V^{-1}x| \leq (I - |\Lambda|)^{-1} |V^{-1}B| \bar{\nu} + \epsilon \right\}. \quad (14)$$

Then:

- (1) For any $\epsilon \geq 0$, the set S_ϵ is *robust positively invariant* (RPI). That is, if $x(0) \in S_\epsilon$ then, for any sequence $\nu(k)$ such that $|\nu(k)| \leq \bar{\nu}$, $x(k) \in S_\epsilon$ for all $k \geq 0$.
- (2) Given $\epsilon \in \mathbb{R}^n$, $\epsilon > 0$, then for any $x(0) \in \mathbb{R}^n$ there exists $k^* \geq 0$ such that $x(k) \in S_\epsilon$ for all $k \geq k^*$. $\circ$

Remark 13. If the eigenvalues of $A = V\Lambda V^{-1}$ are real, then the sets S_ϵ in (14) are polyhedral sets. $\circ$

Finally, we present a procedure and two results from Oлару et al. [2008] that will be used in Sections 3.2 and 3.3 below to analyse the behaviour of the multisensor switching scheme. The first result allows to find tighter ultimate bounds than those of Theorem 12 above. The second result allows to construct an RPI set that includes a given set of initial conditions. To this end, we denote the polytopic constraint set for the disturbance $\nu(k)$ in Theorem 12 as:

$$\Delta \triangleq \{\nu : |\nu| \leq \bar{\nu}\}. \quad (15)$$

We then have, from Oлару et al. [2008]:

Algorithm 1:

Given the set Δ in (15), a scalar $\gamma > 0$, and matrices A and B as in Theorem 12:

- 1) Compute the Jordan decomposition of A ;
- 2) Initialise the algorithm with $\Phi_0 = S_0$ (i.e., the set S_ϵ in (14) with $\epsilon = 0$);
- 3) Find $s \in \mathbb{N}^+$ such that $A^{s+1}\Phi_0 \subset \mathbb{B}(\gamma/2)$;
- 4) For $k = 0$ to $k = s$ compute the sets Φ_k as follows: $\Phi_{k+1} = A\Phi_k \oplus B\Delta$, where $\oplus$ denotes Minkowski sum. $\circ$

The following result follows from Theorem 5 of Oлару et al. [2008].

Theorem 14. The final set Φ_{s+1} , constructed with Algorithm 1, is an RPI outer γ -approximation of the *minimal robust positive invariant* set Ω ; that is, $\Omega \subset \Phi_{s+1} \subset \Omega \oplus \mathbb{B}(\gamma)$. $\circ$

For the next result, we consider a given bounded set $P \subset \mathbb{R}^n$ of initial conditions for the dynamics of system $x(k+1) = Ax(k) + Bv(k)$, as in Theorem 12. We construct the set $\Psi_0 = \mu\Phi_0$, where $\Phi_0 = S_0$ (i.e., the set S_ϵ in (14) with $\epsilon = 0$) and where the scaling factor $\mu \geq 1$ is such that $P \subset \Psi_0$ (see Oлару et al. [2008] for the details), and construct the set sequence:

$$\Psi_{k+1} = \text{Conv.Hull}\{P, A\Psi_k \oplus B\Delta\}. \quad (16)$$

The following result follows from Theorems 8 and 10 of Oлару et al. [2008].

Theorem 15. The set sequence constructed as in (16) is monotonically non-increasing $\Psi_{k+1} \subset \Psi_k$ and inclusion preserving $P \subset \Psi_k$, $\forall k \in \mathbb{N}$. The sets Ψ_k are convex, compact and RPI for system $x(k+1) = Ax(k) + Bv(k)$. For all $\gamma > 0$ there exists $s \in \mathbb{N}^+$ such that the following RPI outer γ -approximation exists: $\bar{\Omega}_P \subset \Psi_{s+1} \subset \bar{\Omega}_P \oplus \mathbb{B}(\gamma)$. $\circ$

The set $\bar{\Omega}_P$ in Theorem 15 is the *minimal convex RPI* set that preserves the inclusion $P \subset \bar{\Omega}_P$ (see Oлару et al. [2008] for the details).

3.2 Dynamics and ultimate bounds for healthy sensors

Applying Theorem 12 (with $\epsilon = 0$) to the estimation error subsystems (7) corresponding to healthy sensors, and using the bounds of Assumption 6, we obtain the following attractive, invariant sets:

$$\tilde{S}_i \triangleq \{\tilde{x} \in \mathbb{R}^n : \quad (17)$$

$$|V_i^{-1}\tilde{x}| \leq (I - |\Lambda_i|)^{-1} |V_i^{-1}[E - L_i]| \begin{bmatrix} \bar{w} \\ \bar{v}_i \end{bmatrix} \},$$

for $i = 1, \dots, N$, where $A_{L_i} = V_i\Lambda_i V_i^{-1}$ is the Jordan decomposition of A_{L_i} . Then, applying Algorithm 1, initialised with $\tilde{S}_i$ from (17), to the dynamics (7) we obtain

tighter bound sets that (as a consequence of the disturbances in Assumption 6 being bounded by symmetric constraints) can be expressed as:

$$\tilde{\Phi}_i \triangleq \{\tilde{x} \in \mathbb{R}^n : |\tilde{\alpha}_i \tilde{x}| \leq \tilde{\beta}_i\}, \quad (18)$$

for $i = 1, \dots, N$, where $\tilde{\alpha}_i$ and $\tilde{\beta}_i > 0$ are, respectively, matrices and vectors that are readily obtained from the initial sets (17) and as a result of the iteration 4) of Algorithm 1 for $k = 0$ to s . (The integer s is obtained as explained in Step 3 of Algorithm 1.)

From (3) and (5) we can write the dynamics of the estimators corresponding to healthy sensors as:

$$\hat{x}_i(k+1) = A_{L_i} \hat{x}_i(k) + Bu(k) + L_i y_i(k), \quad (19)$$

$i = 1, \dots, N$. Applying Theorem 12 (with $\epsilon = 0$) to the estimator dynamics (19) and using the bounds and offsets of Assumption 10 and Remark 11 we obtain the following invariant sets:

$$\hat{S}_i \triangleq \{\hat{x} \in \mathbb{R}^n : \quad (20)$$

$$|V_i^{-1}(\hat{x} - \hat{x}_{i0})| \leq (I - |\Lambda_i|)^{-1} |V_i^{-1} [B \ L_i]| \begin{bmatrix} \bar{u} \\ \bar{y}_i \end{bmatrix} \Big\},$$

for $i = 1, \dots, N$, where the offsets $\hat{x}_{i0}$ are given by:

$$\hat{x}_{i0} = (I - A_{L_i})^{-1} (Bu_0 + L_i y_{i0}). \quad (21)$$

Similarly as in (18), we obtain tighter bounds for dynamics (19) using Algorithm 1, resulting in the invariant sets:

$$\hat{\Phi}_i \triangleq \{\hat{x} \in \mathbb{R}^n : |\hat{\alpha}_i(\hat{x} - \hat{x}_{i0})| \leq \hat{\beta}_i\}, \quad (22)$$

$i = 1, \dots, N$, where $\hat{\alpha}_i$ and $\hat{\beta}_i > 0$ are, respectively, matrices and vectors obtained from the iteration of Algorithm 1.

3.3 Dynamics and ultimate bounds for faulty sensors

Substituting (12) into (3) we have that the dynamics of the estimator associated with a failed j sensor satisfies:

$$\hat{x}_j(k+1) = A_{L_j} \hat{x}_j(k) + Bu(k) + L_j v_j^F(k), \quad (23)$$

for $j \in \{1, \dots, N\}$. Following similar steps as described in Section 3.2 and using the bounds and offsets in Assumptions 6 and 10 we obtain the following *under-fault* invariant sets, and corresponding tighter bound sets:

$$\hat{S}_j^F \triangleq \{\hat{x} \in \mathbb{R}^n : \quad (24)$$

$$|V_j^{-1}(\hat{x} - \hat{x}_{j0}^F)| \leq (I - |\Lambda_j|)^{-1} |V_j^{-1} [B \ L_j]| \begin{bmatrix} \bar{u} \\ \bar{v}_j^F \end{bmatrix} \Big\},$$

$$\hat{\Phi}_j^F \triangleq \{\hat{x} \in \mathbb{R}^n : |\hat{\alpha}_j^F(\hat{x} - \hat{x}_{j0}^F)| \leq \hat{\beta}_j^F\}, \quad (25)$$

for $j = 1, \dots, N$, where the offsets $\hat{x}_{j0}^F$ are given by:

$$\hat{x}_{j0}^F = (I - A_{L_j})^{-1} Bu_0. \quad (26)$$

Finally introduce, for future use, the following definition.

Definition 16. We define RPI sets $\hat{\Psi}_j^F$, $j = 1, \dots, N$, corresponding to the *under-fault* estimator dynamics (23), each of which contains the respective set $\hat{\Phi}_j$ given by (22). The sets $\hat{\Psi}_j^F$, $j = 1, \dots, N$, can be readily computed using iteration (16) for $k = 0, \dots, s$ (see Theorem 15). $\circ$

The sets $\hat{\Psi}_j^F$ constitute invariant sets for the *transient behavior* of estimators corresponding to sensors that have become faulty after working under healthy operation for a sufficiently long period of time.

3.4 Conditions for Healthy Sensor Selection

In this section we provide conditions that guarantee that the switching scheme never selects faulty sensors to compute the switching state estimate (10). To this end, we analyse the switching criterion (9). Using (2), (6) and (8) we have that the innovations $\hat{z}_i(k)$ in (9) corresponding to *healthy sensors* are given by:

$$\hat{z}_i(k) = C_i \tilde{x}_i(k) + v_i(k). \quad (27)$$

Using (8) and (12) we obtain that the innovations $\hat{z}_j(k)$ in (9) corresponding to *faulty sensors* are:

$$\hat{z}_j(k) = -C_j \hat{x}_j(k) + v_j^F(k). \quad (28)$$

Similarly to (15), using the bounds of Assumption 6 we define, for $i, j = 1, \dots, N$, the sets:

$$\Delta_i \triangleq \{v_i : |v_i| \leq \bar{v}_i\}, \quad (29)$$

$$\Delta_j^F \triangleq \{v_j^F : |v_j^F| \leq \bar{v}_j^F\}. \quad (30)$$

The ultimate bound invariant sets obtained in Sections 3.2 and 3.3 above allow us to define the following sets where the variables $\hat{z}_i(k)$ in (27), corresponding to healthy sensors, and the variables $\hat{z}_j(k)$ in (28), corresponding to faulty sensors, *live*. For convenience, we scale these variables with a factor of $\Theta_i^{1/2}$ (the symmetric square root of Θ_i), to take into account the weighting matrices Θ_i in the switching criterion (9). The resulting sets are defined as

$$\hat{\Pi}_i \triangleq \Theta_i^{1/2} \{C_i \tilde{\Phi}_i \oplus \Delta_i\}, \quad (31)$$

$$\hat{\Pi}_j^F \triangleq \Theta_j^{1/2} \{-C_j \hat{\Psi}_j^F \oplus \Delta_j^F\}, \quad (32)$$

$i, j = 1, \dots, N$, where the sets Δ_i and Δ_j^F are given, respectively, by (29) and (30), the sets $\tilde{\Phi}_i$ are given by (18) and the sets $\hat{\Psi}_j^F$ are defined in Definition 16.

We are ready to provide conditions such that the switching criterion (9) never selects faulty sensors to compute the switching state estimate. To this end we introduce the variables $\hat{s}_i \in \mathbb{R}^{p_i}$, $\hat{s}_i \triangleq \Theta_i^{1/2} \hat{z}_i$, and define the indices

$$J_i = \max \left\{ \|\hat{s}_i\|^2 : \hat{s}_i \in \hat{\Pi}_i \right\}, \quad (33)$$

$$J_j^F = \min \left\{ \|\hat{s}_j\|^2 : \hat{s}_j \in \hat{\Pi}_j^F \right\}, \quad (34)$$

$i, j = 1, \dots, N$, where $\|\cdot\|$ is the vector 2-norm. Furthermore, we impose the following “set separation” condition.

Assumption 17. Suppose that the following condition:

$$\max_i \{J_i : i \in \{1, \dots, N\}, i \neq j\} < J_j^F, \quad (35)$$

holds for all $j = 1, \dots, N$. $\circ$

Proposition 18. Under condition (35) of Assumption 17 and provided there exists at least one healthy sensor with state estimation error dynamics (7) inside the RPI set $\tilde{\Phi}_i$ defined in (18) and all faulty sensors have estimator dynamics (23) inside the RPI sets $\hat{\Psi}_j^F$ defined in Definition 16, the multisensor switching mechanism based on criterion (9) selects exclusively among healthy sensors. $\circ$

Proof. Note that the switching criterion (9) selects, at each time instant, the estimator that provides the smallest value of $\hat{z}_i^T \Theta_i \hat{z}_i = \|\hat{s}_i\|^2$, where $\hat{s}_i \triangleq \Theta_i^{1/2} \hat{z}_i$. The state estimation error dynamics (7) associated to a healthy sensor, say the i th sensor, that is inside the corresponding RPI set $\tilde{\Phi}_i$ will remain inside this set. It then follows

from (27) and the definition $\hat{s}_i \triangleq \Theta_i^{1/2} \hat{z}_i$ that the variable $\hat{s}_i$ corresponding to the i th healthy sensor will remain inside the set $\hat{\Pi}_i$ defined in (31). Likewise, the estimator dynamics (23) associated to all the j th faulty sensors, $j \neq i$, that are inside the RPI sets $\hat{\Psi}_j^F$ will remain inside these sets. It then follows from (28) and the definition $\hat{s}_j \triangleq \Theta_j^{1/2} \hat{z}_j$ that the variables $\hat{s}_j$ corresponding to the j th faulty sensors will remain inside the corresponding sets $\hat{\Pi}_j^F$ defined in (32). Finally, it follows from the analysis above and the conditions (33)–(35), that the variable $\hat{s}_i$ associated to the healthy i th sensor (for any $i \in \{1, \dots, N\}$) will always have smaller switching criterion value $\|\hat{s}_i\|^2$ than the variables $\hat{s}_j$ associated to any of the faulty j th sensors ($j \neq i$). It then follows that, under these conditions, estimations based on faulty sensors are precluded by the switching criterion (9). $\square$

We next explain how the set separation condition of Assumption 17 can be achieved in practice. First, notice that the sets $\hat{\Pi}_i$ defined in (31) are centred at the origin, since so are the sets $\tilde{\Phi}_i$ and Δ_i defined, respectively, in (18) and (29). The sets $\hat{\Pi}_j^F$, defined in (32), depend on the sets Δ_j^F and $\hat{\Psi}_j^F$. The former are sets centred around the origin (see (30)). In the case of the sets $\hat{\Psi}_j^F$, defined in Definition 16, they are RPI sets corresponding to the *under-fault* estimator dynamics (23), containing the sets $\hat{\Phi}_j$ given by (22). Notice that the *under-fault* estimator dynamics (23) tend to ultimate bound sets centred around the offset point $\hat{x}_{j0}^F$ given in (26) [cf. (25)]. The sets $\hat{\Phi}_j$, given in (22), are centred around the offset point $\hat{x}_{j0}$ given in (21). Noting that the offsets y_{j0} in (21) are related to the offset u_0 (see Assumption 10 and Remark 11), we conclude from (21) and (26) that effective separation of the sets $\hat{\Pi}_i$ and $\hat{\Pi}_j^F$ can be achieved by proper selection of the value of the input offset u_0 . This makes the multisensor switching scheme particularly well suited for reference tracking problems, especially when the reference signals contain an offset component.

3.5 Fault Tolerant Properties of the Switching Scheme

The analysis of Section 3.4 motivates us to impose the following assumption that describes the failure scenario that we can consider in order to obtain fault tolerance guarantees within the proposed framework.

Assumption 19. (Fault scenario). At any time instant the following conditions are satisfied:

- (1) There exists at least one healthy sensor with associated state estimation error dynamics (7) inside the RPI set $\tilde{\Phi}_i$ defined in (18);
- (2) All faulty sensors have associated estimator dynamics (23) inside the RPI sets $\hat{\Psi}_j^F$ of Definition 16. $\circ$

The fault scenario of Assumption 19 allows for any sequence of persistent sensor faults, including simultaneous faults of several sensors, as long as at least one sensor remains operational and the first fault occurs after sufficiently long time of operation without fault (such that all variables have entered the corresponding invariant sets).

The following theorem states the main property of the switching estimation scheme; namely, its capability to keep—under sensor failure—the estimation errors bounded, with bounds determined by the healthy sensors.

Theorem 20. Assume that the conditions of Assumptions 1 and 3 on the system and estimator matrices are satisfied. Also, assume that the conditions of Assumptions 6 and 10 on the disturbance and noise bounds and operating conditions are satisfied. Further, assume that the set separation conditions of Assumption 17 are fulfilled. Then, the switching state estimation error $\tilde{x}_{sw}(k)$ defined by (11), corresponding to the switching scheme, remains bounded under any sensor failure that satisfies the fault scenario of Assumption 19. Moreover, $\tilde{x}_{sw}(k)$ converges to zero in the absence of disturbances and measurement noise in the healthy sensors. $\circ$

Proof. Since Assumptions 17 and 19 are satisfied, Proposition 18 guarantees that the switching mechanism based on criterion (9) selects exclusively healthy sensors. The result then follows from Proposition 9. $\square$

3.6 Implementation Aspects of the Switching Scheme

A notable feature of the proposed multisensor switching scheme is the simplicity of its implementation and of the conditions that guarantee fault tolerance of the scheme. The *online* implementation of the switching estimation scheme simply requires evaluation of N quadratic functions and selection of the sensor-estimator that provides the minimum according to the switching criterion (9). In contrast, alternative approaches (e.g., using conventional sensor fusion with an FDI module, as in the scheme used in the comparison example of Section 4 below) are considerably more involved and require substantially heavier online computations. The fault-tolerance capabilities of the switching scheme are guaranteed by a simply verifiable set of conditions. In effect, the conditions to check “set separation”, given by (33)–(35), can be tested *offline*. The evaluation of (33)–(35) is particularly simple if the eigenvalues of the matrices A_{L_i} are real (which implies that the sets (31) and (32) are polyhedral). In this case, the maximum in (33) is achieved at one of the vertices of the set $\hat{\Pi}_i$ and the minimisation in (34) is a quadratic programme. Thus, the optimisations required by the off-line tests can be solved using standard numerical algorithms.

4. SIMULATION EXAMPLE

Consider system (1) with matrices $B = [-0.58; 2.18]$, $E = [0.1; 0.1]$, and $A = [0.73, 0; -0.44, 0.174]$, with input $u(k)$ bounded as $|u(k) - 2.5| \leq 1.2$ and process disturbance $w(k)$ bounded as $|w(k)| \leq 0.5$. The state $x(k)$ is measured via a set of 3 scalar sensors of the form (2) with identical characteristics: $C_i = [0.76 \ 0.55]$, for $i = 1, 2, 3$, and bounded measurement noises $|v_i(k)| \leq 0.2$, $i = 1, 2, 3$. The measured output of a failed sensor is given by (12) where $v_j^F(k)$ is bounded as $|v_j^F(k)| \leq 0.5$, $j = 1, 2, 3$. The measurement offsets and bounds are computed as in Remark 11 with $y_{i0} = -9.28$ and $\bar{y}_i = 3.8$, for $i = 1, 2, 3$. The estimator dynamics and estimate update equations are given by (3)–(4) with values of $L_i = [0.252 \ -0.163]^T$ (computed by pole placement), and $M_i = [0.346 \ -0.063]^T$, for $i = 1, 2, 3$. (In order to obtain polyhedral sets, see

Fig. 2. Sets $\hat{\Pi}_i$ and $\hat{\Pi}_j^F$ satisfying (35) for $i, j = 1, 2, 3$.

Fig. 3. Multisensor switching scheme. a) Plant state $x(k)$ (dotted) and switching state estimate $\hat{x}_{sw}(k)$ (dashed). b) Switching sequence.

Remark 13, the eigenvalues of A_{L_i} were chosen to be real.) The weights Θ_i in (9) are taken as $\Theta_i = 1$ for $i = 1, 2, 3$. The set separation condition (35) is illustrated in Figure 2.

In the simulation, each sensor fails for a period of time and then recovers. Sensor 1 fails during $15s \leq t \leq 20s$ and $25s \leq t \leq 30s$, sensor 2 fails during $25s \leq t \leq 30s$ and $35s \leq t \leq 40s$, and sensor 3 fails during $35s \leq t \leq 40s$. Figure 3a) depicts the plant state $x(k)$ (dotted) together with the switching state estimate $\hat{x}_{sw}(k)$ (dashed) (both practically undistinguishable). The switching sequence, plotted in Figure 3b), shows that during the periods of sensor failure the switching scheme exclusively selects among healthy sensors. The shaded areas in Figure 3 indicate the periods of sensor failure. Notice that, during the period $25s \leq t \leq 30s$, both sensors 1 and 2 fail and during $35s \leq t \leq 40s$, both sensors 2 and 3 fail. This simulation illustrates that the proposed switching scheme is able to handle multiple simultaneous sensor failures with the provision that at least one healthy sensor is available.

Finally, we present a comparison between the multisensor switching estimator proposed in this paper and an alternative multisensor estimation scheme based on *sensor fusion* (see Zhuo et al. [2006] for full details). In the fusion scheme, the states of the plant are estimated using individual Kalman filters, based on measurements obtained from each sensor. In addition, a fault detection and isolation (FDI) unit is used to perform fault detection based on hypothesis testing. For our comparison we executed 100 runs under identical conditions but with different noise realisations and computed the *average* of the resulting state estimation errors. To satisfy the assumptions required by the sensor fusion scheme, the disturbance $w(k)$ and measurement noises $v_i(k)$ were selected as i.i.d. zero mean Gaussian white noise sequences $w(k) \sim N(0, 0.02)$ and $v_i(k) \sim N(0, 0.002)$, $i = 1, 2, 3$, and independent from each other. The initialisation values for the Kalman filters were taken as $\hat{x}_i(0|0) = 0$ and the error covariance $P_i(0|0) = 0$, for $i = 1, 2, 3$. To satisfy Assumption 6 of

Fig. 4. Comparison of estimation errors for the switching scheme (solid) and fusion-based scheme (dashed) (average over 100 realisations). a) First, and b) Second, element of the state estimation error vector.

the multisensor switching scheme, all 100 realisations of the Gaussian noises for the simulation were selected such that they were bounded as $|w(k)| \leq 0.5$, $|v_i(k)| \leq 0.2$, $i = 1, 2, 3$. Figure 4 depicts the average (over the 100 runs) of the switching state estimation error $\hat{x}_{sw}(k)$ [see (11)] (solid) and the fusion-based scheme state estimation error (dashed). Both schemes exhibit similar levels of error during the periods when all sensors are healthy, with a slightly better performance for the fusion-based scheme. When one or more sensors fail the average performance of the fusion-based scheme is degraded noticeably with respect to the switching scheme. The switching scheme exhibits superior performance under these circumstances.

5. CONCLUSIONS

A novel multisensor switching state estimation scheme was presented. It was shown that, if a set of conditions on the system parameters is satisfied then the switching estimation scheme is able to guarantee fault tolerance capabilities under multiple sensor failures. The scheme was compared with an alternative approach based on sensor fusion, equipped with fault detection and isolation. The proposed switching scheme evidenced superior performance, in the presence of sensor failures, in all the examples studied. Current work includes more general sensor fault models.

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