Inverse Minimax Optimality of Model Predictive Control Policies

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Abstract

We present novel results linking model predictive control (MPC) and minimax optimal control theory. Specifically, we show that the closed-loop optimal solutions of a particular class of minimax optimal control problems are a class of typical MPC policies, for linear discrete-time systems with constraints and disturbance inputs. We also present conditions which ensure that the inverse optimal MPC policies achieve a prescribed (regional) $\ell_2$-gain from the disturbance input.

Key words: Predictive Control, Constrained Control, Inverse Optimality

1 Introduction

In the absence of model errors and disturbances, model predictive control (MPC), with a suitably chosen terminal weighting, can be seen as an exact implementation of an optimal feedback policy (Chmielewski and Manousiouthakis, 1996; Mayne, Rawlings, Rao, and Scokaert, 2000). In this case, the MPC methodology circumvents the need to solve a functional dynamic programming equation for the explicit control policy. On the other hand, when model errors and disturbances are to be taken into account, certain advantages provided by a dynamic programming formulation cannot be achieved within the usual MPC framework. For example, using an open-loop problem formulation to predict the effects of uncertainty over a long prediction horizon can lead to feasibility issues and conservative results (Scokaert and Mayne, 1998; Mayne, 2001).

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To address these problems, one should, in principle, optimize over a set of control policies as opposed to control sequences. This optimization, however, can be difficult since it usually requires (minimax) dynamic programming, (Lee and Yu, 1997; Diehl and Björnberg, 2004). Various simplifications, where the admissible control policies are restricted, a priori, to a parameterized form (e.g., \( u_k = -K_p x_k + \bar{u}_k \)) have therefore been suggested (Lee and Kouvaritakis, 2000a; Lee, 2003; Mayne, Seron, and Raković, 2005). In these algorithms, the parameters (e.g., \( \bar{u}_k \)) are determined as usual, in an open-loop fashion, but the presence of a fixed stabilizing feedback law (e.g., \(-K_p x_k\)), enables robust feasibility to be established. The optimization criteria used in these simplified schemes are not explicitly of the (closed-loop) minimax type. Nevertheless, certain associated robustness properties have been established by, for example, Lee and Kouvaritakis (2000a); Lee (2003); Mayne et al. (2005), Chisci, Rossiter, and Zappa (2001); Kouvaritakis, Rossiter, and Schuurmans (2000), and in Jadbabaie and Morse (2002); Kerrigan and Maciejowski (2003), input-to-state stability of the closed-loop system with respect to the disturbance input was proven.

In this paper, we show, using minimax dynamic programming, that some of the above mentioned approaches, which are based on open-loop criteria, can, in fact, be seen as the solutions to closed-loop minimax optimal control problems albeit with different cost functions. Furthermore, the associated minimax cost function is piecewise quadratic and “meaningful” in the sense that it puts positive definite weight on both the system state and the control input. The class of MPC policies described here contains both well-known control policies and policies with novel features. For example, we consider an MPC policy which uses linear constraints of the type proposed by Richards and How (2006); Kuwata, Richards, and How (2007) and MPC cost function parameters satisfying an algebraic Riccati equation of the type that appears in the bounded real lemma (de Souza and Xie, 1992).

A preliminary version of parts of the present paper has appeared in Løvaas, Seron, and Goodwin (2006). Our initial investigations were, in part, inspired by the inverse optimality results of Krstić and Li (1998) for continuous-time nonlinear control design. Other related results on inverse minimax optimality for the case of linear feedback gain control of discrete-time systems have been established by Kogan (1997).

The remainder of this paper is organized as follows: The following section presents a class of MPC policies and a brief review of minimax optimal control theory. Section 3 establishes inverse minimax optimality of the class of MPC policies. Section 4 concludes the paper. Throughout we use the following notation: \( \|x\|_P^2 \) denotes \( x^T P x \), [\( a, \cdots, c \)] denotes \([a^T \cdots c^T]^T\) and \( I_q \) denotes the \( q \times q \) identity matrix.
2 Preliminaries

2.1 System Description

We consider a linear constrained system with state space representation

$$x_{k+1} = f(x_k, u_k, w_k) = Ax_k + Bu_k + B_w w_k,$$

(1)

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ the control input and $w_k \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ an unmeasured disturbance input. The system (1) is subject to mixed state and control input constraints, $[x_k, u_k] \in \mathbb{C}_0 \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$. We assume that the sets, $\mathbb{C}_0$ and $\mathbb{W}$, contain in their interior the origin of their respective spaces. Moreover, the pair $(A, B)$ is stabilizable.

2.2 Model Predictive Control

We consider a general class of MPC policies based on a quadratic programme of the form

$$K(x) = \arg \min_{\mu \in \mathbb{R}^{n_\mu}} \left\{ V(x, \mu) \text{ s.t. } [x, \mu] \in S \right\}.$$

(2)

Here, the set $S$ is polyhedral (i.e., $S = \{[x, \mu] \mid T[x, \mu] \leq t\}$ for some matrix $T$ and vector $t$) and the cost function

$$V(x, \mu) = \|[x, \mu]\|_P^2,$$

(3)

where $P$ is some positive definite matrix. We denote the set of feasible states $x$ by

$$S_x = \{x \mid \exists \mu \text{ such that } [x, \mu] \in S\},$$

(4)

and define the MPC value function $V^* : S_x \to \mathbb{R}$ as

$$V^*(x) = V(x, K(x)).$$

(5)

We assume that the dimension $n_\mu$ of the decision variable $\mu$ satisfies $n_\mu \geq n_u$ and that the control input $u_k$ is calculated as

$$u_k = -K_p x_k + D_1 K(x_k),$$

(6)

where $D_1 \triangleq [I_{n_u}, 0 \cdots 0]$, and where $K_p$ is a “pre-stabilizing” feedback gain (see, e.g., Chisci et al. (2001)). Hence, the MPC policy for the system (1) is as follows:
Algorithm 1 For the current state \( x_k \in \mathbb{S} \), compute (6) and apply \( u_k \) to the system (1).

We require the following assumption on the constraint set \( \mathbb{S} \) and the feedback gain \( K_p \).

Assumption 2.1 There exist matrices, \( F_1, F_2, F_3 \), such that

\[
\begin{bmatrix}
A_p & BD_1 \\
F_1 & F_2 \\
\hat{A} & B_w
\end{bmatrix}
\begin{bmatrix}
x \\
\mu
\end{bmatrix}
+ 
\begin{bmatrix}
B_w \\
F_3
\end{bmatrix}w 
\in \mathbb{S}, \forall 
\begin{bmatrix}
x \\
\mu
\end{bmatrix} \in \mathbb{S}, \forall w \in \mathbb{W},
\]

(7)

where \( A_p \triangleq A - BK_p \), and where the matrix \( \hat{A} \) is stable. Moreover, \( 0 \in \mathbb{S} \) and

\[
\mathbb{S} \subseteq \left\{ [x, \mu] \mid [x, -K_p x + D_1 \mu] \in \mathbb{C}_0 \right\}.
\]

(8)

Finally, if \( \mathbb{W} \neq \mathbb{R}^{n_w} \), then \( \mathbb{S} \) is bounded.

Remark 2.1 For a broad class of MPC policies with \( n_\mu = N n_u \), where \( N \geq 1 \) is the control prediction horizon, Assumption 2.1 holds using \( F_1 = 0, F_3 = 0 \) and

\[
F_2 = \begin{bmatrix}
0 & I_{n_u} & 0 & \cdots & 0 \\
\vdots & 0 & I_{n_u} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & I_{n_u} \\
0 & \cdots & 0 & 0 & 0
\end{bmatrix}.
\]

(9)

As a specific example, consider the case when \( A \) is stable, \( \mathbb{W} = \mathbb{R}^{n_w} \) and \( \mathbb{C}_0 = \mathbb{R}^{n_x} \times \mathbb{U} \), with \( 0 \in \mathbb{U} \), and an MPC policy with \( K_p = 0 \) and \( \mathbb{S} = \mathbb{R}^{n_x} \times \mathbb{U}^N \), where \( \mathbb{U}^N = \mathbb{U} \times \cdots \times \mathbb{U} \) \( N \) times. Further examples can be found in Chisci et al. (2001) and Kerrigan and Maciejowski (2003).

Remark 2.2 Recently in Richards and How (2006); Kuwata et al. (2007), the “constraint tightening” approach of Gossner et al. (1997); Chisci et al. (2001) has been generalized. The techniques for MPC constraint set computations proposed by Richards and How (2006); Kuwata et al. (2007) lead to MPC policies that satisfy Assumption 2.1 with a non-zero value for \( F_3 \). For example, by using the pre-stabilized dynamics \( x_{k+1} = A_p x_k + B u_k + B_w w_k \) and the techniques in Richards and How (2006), one can satisfy Assumption 2.1 using
\(F_1 = 0, F_2\) as in (9) and \(F_3\) of the form
\[
\begin{bmatrix}
  K_w \\
  K_w(A_p - BK_w) \\
  K_w(A_p - BK_w)^2 \\
  \vdots \\
  K_w(A_p - BK_w)^{N-1}
\end{bmatrix},
\]
where \(K_w\) is a dead-beat feedback gain such that \((A_p - BK_w)^N = 0\).

We require the following assumption on the cost function matrix \(P\).

**Assumption 2.2** For some invertible matrix \(C \in \mathbb{R}^{q \times q}\), \(q = n_x + n_u\), we have
\[
P - \bar{A}^T P \bar{A} - D_2^T C^T C D_2 \geq 0,
\]
where \(\bar{A}\) is as in (7),
\[
D_2 \triangleq \begin{bmatrix}
  I & 0 \\
  -K_p & D_1
\end{bmatrix},
\]
and where \(D_1\) is as in (6).

**Remark 2.3** Suppose that Assumption 2.1 holds with \(F_1, F_2, \) and \(F_3\) as described in Remark 2.1 and that (11) holds with equality using \(C = \text{diag}[Q^{1/2}, R^{1/2}]\), \(Q \in \mathbb{R}^{n_x \times n_x}\), \(Q > 0\), \(R \in \mathbb{R}^{n_u \times n_u}\), \(R > 0\). Then, the cost function (3) takes the following conventional form:
\[
V(x_{0|k}, \mu) = \|x_{N|k}\|_S^2 + \sum_{n=0}^{N-1} \left( \|x_{n|k}\|_Q^2 + \|u_{n|k}\|_R^2 \right),
\]
subject to
\[
x_{n+1|k} = A_p x_{n|k} + B \bar{u}_{n|k}, \\
\begin{bmatrix}
  x_{n|k} \\
  u_{n|k}
\end{bmatrix} = \begin{bmatrix}
  I & 0 \\
  -K_p & I
\end{bmatrix} \begin{bmatrix}
  x_{n|k} \\
  \bar{u}_{n|k}
\end{bmatrix},
\]
where \(\mu = [\bar{u}_{0|k}^T, \ldots, \bar{u}_{N-1|k}^T]\) and where \(S \in \mathbb{R}^{n_x \times n_x}\) satisfies
\[
S - A_p^T S A_p - Q - K_p^T R K_p = 0.
\]
In (13), \(\left( \|x_{n|k}\|_Q^2 + \|u_{n|k}\|_R^2 \right)\) is the “stage cost”.

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In Section 3, we will show that Assumptions 2.1 and 2.2 are sufficient for Algorithm 1 to be the optimal solution to a closed-loop minimax optimal control problem, which is “meaningful” in the sense that its stage cost puts positive definite weight on both $x_k$ and $u_k$.

2.3 Minimax Optimal Control

Next we give a brief review of minimax optimal control; further details may be found in Mayne (2001); Basar and Bernhard (1995). Consider the system (1) and suppose we wish to determine the best state feedback policy, with respect to a minimax criterion, among all admissible policies $\pi$ of length $L$

$$\pi = [\pi_0(\cdot), \pi_1(\cdot), \cdots, \pi_{L-1}(\cdot)].$$

A policy $\pi$ is said to be admissible for the initial state $x_0 = x$ if the following constraints

$$[x_k, u_k] = [x_k, \pi_k(x_k)] \in C, \forall k \in \{0, \ldots, L-1\},$$

$$x_L \in X_f,$$  \hspace{1cm} (15) \hspace{1cm} (16)

are satisfied by the closed-loop system, with dynamics

$$x_0 = x,$$

$$x_{k+1} = f(x_k, \pi_k(x_k), w_k), \forall k \in \{0, \ldots, L-1\},$$  \hspace{1cm} (17) \hspace{1cm} (18)

for all admissible disturbance input sequences, that is, for all

$$w = [w_0, w_1, \cdots, w_{L-1}] \in W^L \triangleq W \times \cdots \times W.$$ \hspace{1cm} (19)

The constraint sets, $C \subseteq C_0$ in (15), $X_f$ in (16), are assumed to be given.

Let $\Pi(x)$ denote the set of such admissible control policies, and consider the following cost function:

$$\theta_L(x, \pi, w) = \sum_{k=0}^{L-1} \ell(x_k, u_k, w_k) + \phi(x_L),$$ \hspace{1cm} (20)

subject to (17)-(18) and $u_k = \pi_k(x_k)$, for some policy $\pi \in \Pi(x)$. Here, the stage cost $\ell(x_k, u_k, w_k)$ and the terminal penalty $\phi(x_L)$ are given. The minimax optimal control problem is then expressed as

$$P_L(x) : \inf_{\pi} \{ J_L(x, \pi) | \pi \in \Pi(x) \},$$ \hspace{1cm} (21)
where we have defined the upper value function \( J_L(x, \pi) \) as
\[
J_L(x, \pi) = \max_{w \in W^L} \theta_L(x, \pi, w).
\] (22)

The solution to the minimax problem \( P_L(x) \), when it exists, is denoted
\[
\pi^*(x) = [\pi_0^*(\cdot), \pi_1^*(\cdot), \ldots, \pi_{L-1}^*(\cdot)],
\] (23)
and the corresponding value function is
\[
J^*_L(x) = J_L(x, \pi^*(x)).
\] (24)

If the solution to \( P_L(x) \) exists, it may be obtained recursively, using \( i = 1, \ldots, L \), by solving the following minimax dynamic programming equation (see, e.g., Mayne (2001); Diehl and Björnberg (2004)):
\[
J^*_i(x) = \min_{u \in \mathbb{R}^n_u} \max_{w \in W} \ell(x, u, w) + J^*_{i-1}(f(x, u, w))
\]
subject to \([x, u] \in X^C_i\},
\] (25)
\[
\pi^*_{L-i}(x) = \arg \min_{u \in \mathbb{R}^n_u} \max_{w \in W} \ell(x, u, w) + J^*_{i-1}(f(x, u, w))
\]
subject to \([x, u] \in X^C_i\},
\] (26)
where
\[
X^C_i = \{[x, u] \in \mathbb{C} | f(x, u, W) \subseteq X_{i-1}\},
\] (27)
and where the domain \( X_i \) of \( J^*_i(x) \) is computed according to
\[
X_i = \{x | \exists u \in \mathbb{R}^n_u \text{ such that } [x, u] \in X^C_i\}.
\] (28)

The boundary conditions of the dynamic programming recursions are
\[
J^*_0(x) = \phi(x), \quad X_0 = \mathbb{X}_f.
\] (29)

3 Inverse Optimality of MPC Policies

3.1 Inverse Optimality Result

Here we present the main result of the paper, namely, the inverse minimax optimality of Algorithm 1 for the system described in Section 2.1. Hence, our goal is to find the constraint sets, \( \mathbb{C} \subseteq \mathbb{C}_0 \) in (15) and \( \mathbb{X}_f \) in (16), and the cost function, \( \theta_L(x, \pi, w) \) in (20), such that the resulting minimax optimal control problem (21) has Algorithm 1 as its solution.
Our solution starts by defining the terminal penalty of the minimax cost function (20) to be equal to the value function of the MPC scheme (5), that is, 
\[ \phi(x) = V^*(x), \]
and accordingly, the terminal constraint set will be taken as its domain
\[ X_f = S_x, \]
as defined in (4).

To prepare for the remaining definitions, that is, our definitions of the constraint set \( C \) and the stage cost \( \ell(x, u, w) \), we consider the following parameterization for \( \mu \in \mathbb{R}^{n_u} \)
\[ F(x, u) \triangleq \left[ K_p x + u, \mu^+(x, u) \right]. \] (30)
Here, \( K_p \) is the “pre-stabilizing” feedback gain in (6), \( x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u} \) and the function \( \mu^+(x, u) \) is defined by the following quadratic program:
\[ \mu^+(x, u) = \arg \min_{\mu^+ \in \mathbb{R}^{n_u-n_x}} \left\{ V \left( x, \left[ K_p x + u, \mu^+ \right] \right) \right\} \text{ s.t. } [x, K_p x + u, \mu^+] \in S. \] (31)

The set \( \mathbb{C} \) is defined as the domain of \( \mu^+(x, u) \) and \( F(x, u) \), which is
\[ \mathbb{C} \triangleq \{ [x, u] \mid \exists \mu^+ \in \mathbb{R}^{n_u-n_x} \text{ such that } [x, K_p x + u, \mu^+] \in S \}, \] (32)
where the set \( S \) is as given in (2). Note that, subject to Assumption 2.1, we then have \( \mathbb{C} \subseteq \mathbb{C}_0 \).

**Remark 3.1** The above definition of \( F : \mathbb{C} \to \mathbb{R}^{n_u} \) implies the following optimal sub-structure:
\[ V^*(x) = \min_{u \in \mathbb{R}^{n_u}} \left\{ V \left( x, F(x, u) \right) \right\} \text{ s.t. } [x, u] \in \mathbb{C}. \] (33)

Furthermore, the function \( \mu^+(x, u) \) in (31) is piecewise affine and Lipschitz continuous in the parameters \( [x, u] \in \mathbb{C} \) (see, e.g., Bemporad et al. (2002)). Hence, since \( F(0, 0) = 0 \) (c.f. \( P > 0 \) and \( 0 \in S \)), we have
\[ \| F(x, u) \| \leq c \| [x, u] \|, \quad \forall [x, u] \in \mathbb{C}, \] (34)
for some scalar \( c \).

The stage cost of the minimax optimal control problem is defined as \( \ell : \mathbb{C} \times \mathbb{W} \to \mathbb{R} \), where
\[ \ell(x, u, w) \triangleq \left\| [x, F(x, u)] \right\|_{L^2(P)}^2 + \Delta(x, u, w) - \gamma^2 \| w \|^2, \] (35)
and where $L_\gamma(P)$ and $\Delta : \mathbb{C} \times \mathbb{W} \to \mathbb{R}$ are defined by,

$$L_\gamma(P) \triangleq P - \bar{A}^T P \bar{A} - \bar{A}^T P \bar{B}_w \left(\gamma^2 I - \bar{B}_w^T P \bar{B}_w\right)^{-1} \bar{B}_w^T P \bar{A}, \quad (36)$$

$$\Delta (x, u, w) \triangleq \|\bar{A}[x, F(x, u)] + \bar{B}_w w\|_P^2 - V^* (f(x, u, w)). \quad (37)$$

Here, we have made use of the matrices $\bar{A}$ and $\bar{B}_w$, as defined in (7), and the function $F(\cdot)$, as defined in (30). (Conditions on the scalar $\gamma$ will be given below.) Finally, the cost function for the minimax optimal control problem is defined by

$$\theta_L (x, \pi, w) \triangleq \sum_{k=0}^{L-1} \ell(x_k, u_k, w_k) + V^* (x_L), \quad (38)$$

where $V^* (x)$ denotes the MPC value function (5) and the stage cost $\ell(x, u, w)$ is as given in (35)-(37).

**Remark 3.2** Note that, subject to Assumption 2.1, the piecewise quadratic function $\Delta (x, u, w)$ in (37) is non-negative, that is,

$$\Delta (x, u, w) \geq 0, \forall [x, u, w] \in \mathbb{C} \times \mathbb{W}. \quad (39)$$

This follows by optimality of the MPC solution $\mu = K(f(x, u, w))$ and feasibility of the “candidate solution” $\mu = F_1 x + F_2 F(x, u) + F_3 w$ (see (7)).

The scalar $\gamma$ in (35)-(36) is a parameter of the minimax optimal control problem. It can be chosen freely subject to the following assumption:

**Assumption 3.1**

**G1** The following set inclusion holds

$$\mathcal{K}_\gamma(P)[x, \mu] \in \mathbb{W}, \forall [x, \mu] \in \mathbb{S},$$

where

$$\mathcal{K}_\gamma(P) \triangleq \left(\gamma^2 I - \bar{B}_w^T P \bar{B}_w\right)^{-1} \bar{B}_w^T P \bar{A}. \quad (40)$$

**G2** $\gamma^2 I - \bar{B}_w^T P \bar{B}_w > 0$.

**G3** The stage cost (35) satisfies:

$$\ell(x, u, w) \geq \|z\|^2 - \gamma^2 \|w\|^2, \forall [x, u, w] \in \mathbb{C} \times \mathbb{W}, \quad (41)$$

where $z = H[x, u]$, and where $H$ is some invertible matrix.

We note that this assumption is not restrictive. It can be satisfied by choosing $\gamma$ as we next show.
Lemma 1 If Assumptions 2.1 and 2.2 hold, then Assumption 3.1 can be satisfied by choice of a sufficiently large scalar $\gamma$.

Proof Clearly, whenever Assumption 2.1 holds, conditions $G1-G2$ are satisfied for a range of sufficiently large $\gamma$, since either: (i) $W = \mathbb{R}^{nw}$; or (ii) the set $S$ is bounded and $0 \in \text{int}(W)$. The fact that $G3$ can be satisfied for a sufficiently large $\gamma$ follows from (35), (11), (34), (39) and the property that $(\gamma^2 I - \bar{B}_w^T \bar{P} \bar{B}_w)^{-1} \to 0$ as $\gamma \to \infty$. \hfill \Box

The following theorem is the main result of this paper; it establishes inverse minimax optimality of the complete class of MPC policies described in Section 2.2.

Theorem 2 Consider the following minimax optimal control problem:

$$P_L(x) : \inf_{\pi} \{ \max_{w \in W_L} \theta_L(x, \pi, w) \mid \pi \in \Pi(x) \},$$

(42)

where the dynamics are given by (1) and $x_0 = x$. Let $\Pi(x)$ in (42) be the set of admissible control policies associated with the following constraints:

$$[x_k, u_k] \in C, \quad w_k \in W, \quad \forall \; k \in \{0, 1, \cdots, L - 1\},$$

$$x_L \in S_x,$$

where the sets $S_x$ and $C$ are as defined in (4) and (32), respectively. Suppose that Assumptions 2.1 and 2.2 hold, and let the cost function $\theta_L(x, \pi, w)$ be as in (38) with the parameter $\gamma > 0$ sufficiently large so that Assumption 3.1 holds. Then, for any integer $L \geq 1$ and any initial state $x \in S_x$, the solution to $P_L(x)$ is time-invariant and uniquely given by Algorithm 1. That is,

$$\pi^*_k(x_k) = -K_p x_k + D_1 K(x_k), \quad \forall \; k \in \{0, 1, \cdots, L - 1\},$$

where, as in (2),

$$K(x) = \arg \min_{\mu \in \mathbb{R}^{nw}} \left\{ V(x, \mu) \right\} \text{ s. t. } [x, \mu] \in S,$$

Furthermore, the associated value functions, as defined in (24)-(25), are

$$J^*_{i}(x) = V^*(x), \quad \forall \; i \in \{1, \cdots, L\}.$$

Proof We begin solving $P_L(x)$ using (25) with $i = 1$ and the boundary condi-
tions, \( J_0^*(x) = V^*(x) \), \( X_0 = S_x \), yielding:

\[
J_1^*(x) = \min_{u \in \mathbb{R}^n} \max_{w \in \mathbb{W}} \left\{ \ell(x, u, w) + V^*(f(x, u, w)) \right\}
\]

subject to \([x, u] \in X_1^C \}, \tag{43}

where, from (27), we have

\[
X_1^C = \{ [x, u] \in C | f(x, u, \mathbb{W}) \subseteq S \}
= \mathbb{C}. \tag{44}
\]

This follows since, by Assumption 2.1 and the definitions of \( C \) and \( S_x \), the following implications hold:

\[
[x, u] \in C \quad \implies \quad [f(x, u, \mathbb{W}), F_1x + F_2F(x, u) + F_3\mathbb{W}] \subseteq S \quad \implies \quad f(x, u, \mathbb{W}) \subseteq S_x,
\]

where \( F_1, F_2, F_3 \) are as in (7). Let us proceed by denoting the objective function in (43) by

\[
\mathcal{H}(x, u, w) \triangleq \ell(x, u, w) + V^*(f(x, u, w)), \tag{45}
\]

and letting \( \Gamma = \Gamma^T > 0 \) denote the unique positive definite square root of \( \gamma^2I - \bar{B}_w^T \bar{P} \bar{B}_w > 0, \)

\[
\gamma^2I - \bar{B}_w^T \bar{P} \bar{B}_w = \Gamma \Gamma = \Gamma^2. \tag{46}
\]

We may then substitute (35)-(37) into (45), and use (40) and (46) to obtain

\[
\mathcal{H}(x, u, w) = \left\| [x, F(x, u)] \right\|_{\mathcal{L}_f(P)}^2 + \Delta(x, u, w) - \gamma^2 \| w \|^2 + V^*(f(x, u, w))
= \left\| [x, F(x, u)] \right\|_{P - A^TPA}^2 - \left\| \mathcal{K}_\gamma(P) x, F(x, u) \right\|^2
+ \| A [x, F(x, u)] + \bar{B}_w w \|_P^2 - w^T \left( \Gamma^2 + \bar{B}_w^T \bar{P} \bar{B}_w \right) w. \tag{47}
\]

Further, by adding and subtracting \( \| \mathcal{K}_\gamma(P) x, F(x, u) \| - \Gamma w \| \|^2 \), we obtain

\[
\mathcal{H}(x, u, w) = \left\| [x, F(x, u)] \right\|_{P - A^TPA}^2
- \| \mathcal{K}_\gamma(P) x, F(x, u) \|_P^2 + \| \mathcal{K}_\gamma(P) x, F(x, u) \| - \Gamma w \|^2 - w^T \Gamma^2 w
- \bar{B}_w^T \bar{P} \bar{B}_w w + \| A [x, F(x, u)] + \bar{B}_w w \|_P^2
- \| \mathcal{K}_\gamma(P) x, F(x, u) \| - \Gamma w \|^2
= V(x, F(x, u)) - \| \mathcal{K}_\gamma(P) x, F(x, u) \| - \Gamma w \|^2, \tag{48}
\]
where we have made use of (3). We then substitute (48) and (44) into (43) to obtain

\[ J_1^*(x) = \min_{u \in \mathbb{R}^n_u} \max_{w \in \mathcal{W}} \left\{ V(x, F(x, u)) - \|G\mathcal{K}_\gamma(P)[x, F(x, u)] - \Gamma w\|^2 \right\} \mathrm{s.~t.} [x, u] \in \mathcal{C} \}.

Since we have \([x, F(x, u)] \in \mathcal{S}\), for all \([x, u] \in \mathbb{C}\) [see (30)-(32)], it follows from \textbf{G1} and (46) that the unconstrained maximum \(\mathcal{K}_\gamma(P)[x, F(x, u)] \in \mathcal{W}\) is feasible, for all \([x, u] \in \mathbb{C}\). Hence, using (33),

\[ J_1^*(x) = \min_{u \in \mathbb{R}^n_u} \{ V(x, F(x, u)) \} \mathrm{s.~t.} [x, u] \in \mathbb{C} \} = V^*(x) = J_0^*(x). \tag{49} \]

Also, since using (2) and (26) we have that \(F(x, \pi(x, w)) = K(x)\) at the minimum, we have from (30) that the unique minimizer is

\[ \pi_{L-1}^*(x) = -Kp + D_1K(x) = \arg\min_{u \in \mathbb{R}^n_u} \{ V(x, F(x, u)) \} \mathrm{s.~t.} [x, u] \in \mathbb{C} \}.

To complete the first step of the dynamic programming recursion, we use (28) and (44) to compute the domain of \(J_1^*(x)\), yielding:

\[ X_1 = \{ x \mid \exists u \in \mathbb{R}^n_u \text{ such that } [x, u] \in \mathbb{C} \} = \mathcal{S}_x = X_0. \tag{50} \]

It follows, by induction, that all remaining steps of the dynamic programming problem will be identical to the above. In particular, we get \(X_L = \mathcal{S}_x\) and since \(x_0 \in \mathcal{S}_x\), by assumption, the results follow. \hfill \Box

\textbf{Remark 3.3} Note that the solution to the minimax optimal control problem of \textbf{Theorem 2} is invariant with respect to the choice of \(\gamma\) in the cost function \(\theta_L(x, \pi, w)\). Hence, since a range of values for \(\gamma\) will satisfy \textbf{Assumption 3.1}, equation (38) defines a class of cost functions, parameterized by \(\gamma\), for which a particular instance of \textbf{Algorithm 1} is closed-loop minimax optimal.

The statement of \textbf{Theorem 2} does not actually require \textbf{G3} of \textbf{Assumption 3.1} to hold. However, whenever \textbf{G3} holds, we have the following interesting corollary:

\textbf{Corollary 3} Suppose that \textbf{Assumptions 2.1, 2.2 and 3.1} hold. Then, for any integer \(L \geq 1\), the following \(\ell_2\)-gain property holds for system (1) in closed-loop under \textbf{Algorithm 1}:

\[ \sum_{k=0}^{L-1} \|z_k\|^2 \leq V^*(x_0) + \gamma^2 \sum_{k=0}^{L-1} \|w_k\|^2, \forall w = [w_0, w_1, \ldots, w_{L-1}] \in \mathcal{W}^L, \forall x_0 \in \mathcal{S}_x, \tag{51} \]
where \( z_k = H[x_k, u_k] \) and \( u_k = \pi_k^*(x_k) = -K_p x_k + D_1 K(x_k) \).

**Proof** Theorem 2 establishes that for all \( w \in \mathbb{W}^L \) and all \( x_0 \in S_x \):

\[
\theta_L(x, \pi^*(x), w) \leq \max_{w \in \mathbb{W}^L} \theta_L(x, \pi^*(x), w) = V^*(x_0)
\]

The claim follows by substituting (38) into the left hand side of (52) and then using (41) and the fact that \( V^*(x_L) \geq 0 \).

Theorem 2 and Corollary 3 provide a novel closed-loop characterization which captures both optimality and disturbance attenuation properties of a broad class of MPC policies (that are based on open-loop predictions). Moreover, since the considered class contains policies that, to the authors’ knowledge, have not been studied before, these results motivate new MPC policies that achieve a closed-loop \( \ell_2 \)-gain less than \( \gamma \) from the disturbance \( w_k \) to a system output \( z_k = H[x_k, u_k] \).

### 3.2 Prescribed Closed-Loop \( \ell_2 \)-Gain

To achieve an \( \ell_2 \)-gain less than a prescribed value \( \gamma \) from the disturbance input \( w_k \) to a system output \( z_k = H[x_k, u_k] \), it suffices (under Assumptions 2.1-2.2 by Corollary 3) to choose the matrix \( P \) in the MPC cost function (3) so that Assumption 3.1 holds using \( \gamma \) and \( H \). Clearly, under Assumption 2.1, \( G_1-G_2 \) of Assumption holds provided \( \gamma \) is relatively large as compared to \( P \) and \( H \). Moreover, a sufficient condition for \( G_3 \) of Assumption 3.1 to hold is as follows:

\[
\mathcal{L}_\gamma (P) - D_2^T H^T H D_2 = 0,
\]

where \( D_2 \) is as in (12) and \( \mathcal{L}_\gamma \) is as in (36). To verify this, note that subject to (53) and using (12), (30), the stage cost (35) satisfies

\[
\ell(x, u, w) = ||H[x, u]||^2 + \Delta (x, u, w) - \gamma^2 ||w||^2.
\]

Condition \( G_3 \) then holds since the function \( \Delta (x, u, w) \) is non-negative, as shown in Remark 3.2.

Note that (53) is an algebraic Riccati equation of the type that appears in the bounded real lemma (de Souza and Xie, 1992). Hence, we have the following result regarding the existence of a matrix \( P \) satisfying (53).

**Lemma 4 (Bounded Real Lemma (de Souza and Xie, 1992))** Consider the Riccati equation (53), where \( \mathcal{L}_\gamma (P) \) is defined in (36). The following two statements are equivalent:
(i) The matrix $\bar{A}$ defined in (7) is stable and $^1$

$$\|HD_2(zI - \bar{A})^{-1}\bar{B}_w\|_\infty < \gamma.$$  

(ii) There exists a solution $P = P^T$ to (53) which is such that the matrix $(I - \gamma^{-2}\bar{B}_w\bar{B}_w^TP)^{-1}\bar{A}$ is stable and $\gamma^2 I - \bar{B}_w^TP\bar{B}_w > 0$.

In the sequel, we use $P = P_\gamma$ to denote a (so-called “stabilizing”) solution satisfying statement (ii) of Lemma 4, that is, $P = P_\gamma$ denotes a solution to (53) which is such that the matrix $(I - \gamma^{-2}\bar{B}_w\bar{B}_w^TP)^{-1}\bar{A}$ is stable and $\gamma^2 I - \bar{B}_w^TP\bar{B}_w > 0$. Note that, as $\gamma \to \infty$, the matrix $K_\gamma(P_\gamma)$ in (40) goes to zero. Hence, provided that Assumption 2.1 holds, it follows by Lemma 4 that, for any given matrix $H$, there exists a suitably large $\gamma > \gamma^*$ such that $P_\gamma$ exists and Assumption 3.1 (i.e., G1-G3) holds using $P = P_\gamma$, $\gamma$ and $H$. In particular, we may determine such a matrix $P = P_\gamma$ in the MPC cost function (3) by performing a search of the scalar variable $\gamma$. These steps result in a new approach to cost function selection that leads to a class of MPC policies which contains, as special cases, some existing (open-loop) min-max MPC designs.

To clarify, the following result shows that, under certain conditions, the cost function obtained using $P = P_\gamma$ is a cost function of the form employed by Lee and Kouvaritakis (2000b); Lee (2003).

**Lemma 5** Suppose that Assumption 2.1 holds with $F_1$, $F_2$, and $F_3$ as described in Remark 2.1 and consider $H = \text{diag}[Q^{1/2}, R^{1/2}]$, $Q \in \mathbb{R}^{n_x \times n_x}$, $Q > 0$, $R \in \mathbb{R}^{n_u \times n_u}$, $R > 0$ in (53). Then, the cost function (3) obtained using $P = P_\gamma$ can be expressed as follows:

$$V(x_0[k], \mu) = \max_{W \in \mathbb{R}^{N_x \times n_w}} \left\{ \|x_N[k]\|_S^2 + \sum_{n=0}^{N-1} \left( \|x_n[k]\|_Q^2 + \|u_n[k]\|_R^2 - \gamma^2\|w_n[k]\|^2 \right) \right\},$$

subject to

$$x_{n+1}[k] = A_p x_n[k] + B\bar{u}_n[k] + B_w w_n[k],$$

$$\begin{bmatrix} x_n[k] \\ u_n[k] \end{bmatrix} = \begin{bmatrix} I & 0 \\ -K_p & I \end{bmatrix} \begin{bmatrix} x_{n+1}[k] \\ \bar{u}_n[k] \end{bmatrix},$$

where $\mu = [\bar{u}_0[k], \ldots, \bar{u}_{N-1}[k]]$ and $W = [w_0[k], w_1[k], \ldots, w_{N-1}[k]]$. Here, $S \in \mathbb{R}^{n_x \times n_x}$ satisfies:

$$S - A_p^T S A_p - A_p^T S B_w \left( \gamma^2 I - B_w^T S B_w \right)^{-1} B_w^T S A_p - Q - K_p^T R K_p = 0,$$

$^1\|A(z)\|_\infty$ denotes the usual “$H_\infty$-norm”, i.e., $\sup_{w \in [0,2\pi]} \sigma_{\max}[A(e^{jw})]$, where $\sigma_{\max}[M]$ denotes the largest singular value of $M$.
where $\gamma^2 I - B_w^T S B_w > 0$ and $(I - \gamma^{-2} B_w B_w^T S)^{-1} A_\gamma$ is stable.

**Proof** The result is proven by using (53) and Theorem 2 in Kogan (1997) to express $V(x_{0\mid k}, \mu)$ in (3) as the maximum of an infinite horizon cost function, yielding:

$$V(x_{0\mid k}, \mu) = \|x_{0\mid k}, \mu\|_{p_\gamma}^2$$

$$= \max_{\{w_{n\mid k}\}_{n=0}^\infty} \left\{ \sum_{n=0}^{\infty} \left( \|x_{n\mid k}, \mu_n\|_{D_2^T H^T H D_2}^2 - \gamma^2 \|w_{n\mid k}\|_2^2 \right) \right\}$$

$$\text{s.t. } [x_{n+1\mid k}, \mu_{n+1\mid k}] = A[x_{n\mid k}, \mu_n] + B w_{n\mid k}, \mu_0 = \mu$$

$$= \max_{\{w_{n\mid k}\}_{n=0}^\infty} \left\{ \sum_{n=0}^{\infty} \left( \|x_{n\mid k}\|_Q^2 + \|u_{n\mid k}\|_R^2 - \gamma^2 \|w_{n\mid k}\|_2^2 \right) \right\}$$

$$\text{s.t. } (56) - (57) \text{ and } [\bar{u}_{0\mid k}, \cdots, \bar{u}_{N-1\mid k}] = \mu, \{\bar{u}_{n\mid k}\}_{n=N}^\infty = 0$$

$$= \max_{\{w_{n\mid k}\}_{n=0}^\infty} \left\{ \sum_{n=0}^{N-1} \left( \|x_{n\mid k}\|_Q^2 + \|u_{n\mid k}\|_R^2 - \gamma^2 \|w_{n\mid k}\|_2^2 \right) \right\} + \sum_{n=N}^{\infty} \left( \|x_{n\mid k}\|_Q^2 + \|u_{n\mid k}\|_R^2 - \gamma^2 \|w_{n\mid k}\|_2^2 \right)$$

$$\text{s.t. } (56) - (57) \text{ and } [\bar{u}_{0\mid k}, \cdots, \bar{u}_{N-1\mid k}] = \mu, \{\bar{u}_{n\mid k}\}_{n=N}^\infty = 0$$

where we have also made use of (7). The result then follows, since (again by Theorem 2 in Kogan (1997) and by the “principle of optimality”) the contribution of the last summation in the last maximization in (59) can be expressed as $\|x_{N\mid k}\|_S^2$, leading to the maximization problem (55). [Also note from Lemma 4 that a solution $S$ to (58) exists whenever $P_\gamma$ exists.] \hfill \Box

**Remark 3.4** Cost functions of the type (55) have been proposed and used in, for example, Lee and Kouvaritakis (2000b); Lee (2003). A common variation is to require equality (58) to hold as an inequality.

It is an interesting topic for future work to further explore MPC policies using cost functions satisfying (55) in the general case when $F_1 \neq 0$, $F_3 \neq 0$ in (7). For example, by employing the results in Richards and How (2006) (see Remark 2.2). Another possible extension, which is captured by the stability test proposed by Løvaas, Seron, and Goodwin (2007), is to require (53) to hold as an inequality. That is,

$$L_\gamma(P) - D_2^T H^T H D_2 \geq 0.$$  

(60)
Note that the latter inequality also is a sufficient condition for G3 to hold using $\gamma$ and $H$.

### 3.3 Connections to Results on “One-Player” Optimality

Here we connect Theorem 2 to a well-known result on inverse optimality of conventional MPC policies (Mayne et al., 2000; Mayne, 2001).

Suppose the conditions of Remark 2.3 hold and that the set $\mathbb{W}$ is bounded and convex. Clearly, if we reduce the size of the convex disturbance set as $\mathbb{W} \to \rho \mathbb{W}$, for some $0 < \rho \leq 1$, but keep the constraint set $\mathbb{S}$, Assumption 2.1 continues to hold true. Furthermore, for each $0 < \rho \leq 1$, there exists a sufficiently large $\gamma = \gamma(\rho)$ such that Assumption 3.1 is satisfied (c.f. Lemma 1). Next consider $\rho \to 0$ and $\gamma(\rho) \to \infty$ and the associated sequence of minimax problems covered by Theorem 2. In the limit of this sequence of problems we obtain a problem having disturbance set $\mathbb{W} = \{0\}$ and $\gamma = \infty$, which corresponds to a one-player problem with dynamics $f(x, u, w) = f(x, u, 0)$. This problem, however, is not covered by Theorem 2. At the same time, its solution should be arbitrarily close to Algorithm 1, which, by Theorem 2, is the solution to each problem in the sequence of problems.

To verify this, let $w_k = 0$ and $\gamma \to \infty$ in the minimax stage cost (35)-(37) to yield the following stage cost [see also (12), the equality in (11), Remark 2.3, and the definitions of $\bar{A}$ and $\bar{B}_w$ in (7)]:

$$
\ell(x, u) \triangleq \ell(x, u, 0) = \|C[x, u]\|^2 + \Delta(x, u, 0) \\
= \|x\|_Q^2 + \|u\|^2_R + V(f(x_k, u_k, 0), F_2F(x_k, u_k)) - V^*(f(x_k, u_k, 0))
$$

(61)

where $F_2$ is as in (9) and $V(\cdot)$ is the MPC cost function in (3), which we have assumed to be of the form (13) in Remark 2.3. We may then write (33) as

$$
V^*(x_k) = \min_{u \in \mathbb{R}^n_u} \{ \ell(x_k, u_k) + V^*(f(x_k, u_k, 0)) \} \text{ s. t. } [x, u] \in \mathcal{C},
$$

(62)

which is, in fact, the Hamilton-Jacobi equation associated with conventional MPC policies (Mayne et al., 2000; Mayne, 2001). Hence, Algorithm 1 is the solution to the problem obtained with $\gamma = \infty$ and $w_k = 0$.

To further connect equation (62) to the dynamic programming recursions associated with the open-loop optimal control problem over $N \geq 2$ stages, let us consider the unconstrained case and take $K_p = 0$. Then use (9), (14) and (31) to obtain

$$
V(f(x, u, 0), F_2F(x, u)) = V_{N-1}^*(f(x, u, 0)),
$$

(63)
where $V^*_N(x)$ denotes the value function in (5) obtained with horizon $N-1$. By combining (61)-(63), we obtain

$$V^*(x_k) = \min_{u_k \in \mathbb{R}^nu} \{\ell(x_k, u_k) + V^*(f(x_k, u_k, 0))\},$$

$$= \min_{u_k \in \mathbb{R}^nu} \{\|x_k\|^2_Q + \|u_k\|^2_R + V^*_{N-1}(f(x, u, 0))\}.$$ 

4 Conclusions

We have identified a parameterized class of constrained closed-loop minimax optimal control problems having instances of Algorithm 1 as their optimal solution. Hence we have established inverse minimax optimality of a broad class of MPC policies. The result provides a novel characterization of both conventional and robust MPC for linear systems with disturbance inputs. Moreover, it motivates the use of non-standard quadratic MPC cost functions. In particular, we have shown that, by choosing the MPC cost function parameters so as to satisfy an algebraic Riccati equation (or an LMI condition), one can achieve a prescribed upper bound on the closed-loop (regional) $\ell_2$-gain from the disturbance input.

References

A. Jadababaie and A. S. Morse. On the ISS property for receding horizon


