AN ITERATIVE METHOD FOR APPROXIMATING LTI SYSTEMS USING SUBBANDS

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ABSTRACT

A linear system can be approximated in the time-frequency domain by the composition of an analysis filterbank, a transfer matrix (subband model) and a synthesis filterbank, a method known as subband technique. In this paper we propose an iterative method to jointly optimize the subband model, analysis and synthesis filterbanks. To this end we propose a minimization criterion which we solve using the so-called alternating least-squares method. As a possible application we consider the implementation of the so-called head related transfer functions which are used in virtual acoustics. Simulation results suggest that the subband technique, optimized using the proposed method proposed method, is a promising approach.

keywords: Subband filters, Time-frequency analysis, Head-related transfer functions.

1. INTRODUCTION

The subband technique represents a linear system in the time-frequency domain. More precisely, the system is replaced by the composition of an analysis filterbank, followed by a (usually diagonal) transfer matrix (called the subband model) and a synthesis filterbank. This approach can be used for system approximation [1], system identification [2], adaptive filtering [3], channel equalization [4], etc., with the advantage of having a higher numerical efficiency. However, the analysis of this technique and optimal setup are not trivial.

The approximation of Hilbert-Schmidt operators (i.e., a kind of linear time-variant (LTV) system) by the so-called Gabor multipliers (i.e., a diagonal subband model without memory) has been studied in [5, 6], where, for given choices of analysis and synthesis filterbanks, the subband model is chosen to minimize the Hilbert-Schmidt norm. The case of linear time-invariant (LTI) systems, was studied in [1], where the optimal subband model was chosen to minimize the power of the output error signal, assuming a white input signal.

In this paper we proceed a step further from [1]. More precisely, we propose an iterative method to jointly optimize the choices of the subband model, analysis and synthesis filterbanks, when the quality of the approximation is measured by the power of the output error signal, assuming that the input signal has an arbitrary but known power spectrum. To this end, we consider the so-called alternating least-squares (ALS) algorithm [7], which consists of cyclic iterations where, at each step, two of the three elements to be optimized are fixed, and the third is optimized using linear least-squares (LLS).

The time-domain implementation of an LTI system is computationally inefficient, and it is often prohibitive for some applications, e.g., real-time audio applications where impulse responses are in the order of several hundreds. A computationally efficient alternative implements the system in the frequency domain; however, this approach is not suitable for real-time applications, because it requires the block processing of the “whole history” of the involved signals. To address this issue, and for the case of systems of finite impulse response (FIR) type, the so-called overlap-save and overlap-add methods (OS/A) have been proposed [8]. Both methods permit accommodating a trade-off between computational complexity and latency (i.e., time-delay). Simulation results show that a subband scheme, optimized using the algorithm described above, offers a better trade-off than that of the OS/A methods, provided a tolerance on the implementation error is allowed. (Notice that an implementation error is unavoidably introduced when using OS/A methods to implement a system of infinite impulse response (IIR) type.)

In order to illustrate this point, we consider the implementation of the so-called head-related transfer functions (HRTFs), which find applications in the so-called binaural virtual acoustics synthesis [9]. This technique consists of modifying a sound source signal, so as to give the listener the sensation that it is located in space. This is done by filtering the signal from the source using an HRTF, which describes the filtering effects of the listener’s morphology (i.e. pinna, head, torso, etc.) The HRTFs are given separately for the left and right ears, and their associated head related impulse responses (HRIR) are functions of the source location relative to the listener head’s position. In a virtual acoustics application, a pair of filters needs to be computed for every sound source location, which is often computationally unaffordable in real-time applications.

Throughout the paper we use the following notation: com-
plex sequences (indexed by the integers $\mathbb{Z}$) are denoted us-
ing non-bold lowercase letters (e.g., $a$). Vectors and matrices whose elements are complex sequences are denoted by low-
ercase bold letters (e.g., $\mathbf{a}$) and uppercase bold letters (e.g., $\mathbf{A}$), respectively. Complex-valued vectors and matrices are denoted using underlined letters (e.g., $\mathbf{a}$ and $\mathbf{A}$). An LTI system (or filter) $g(q)$ is denoted as a function of the forward shift operator $q$ (i.e., $qx(t) = x(t + 1)$), and its impulse response is denoted by $g(t)$. Finally, we say that an LTI system $g(q)$ is FIR, with tap size $l_g$ and $d_g$ non-causal taps, if $g(t) = 0$ for all $t \not\in \{-d_g, \ldots, l_g - d_g - 1\}$.

The proofs of results are not included in the paper and will be included in a journal version.

2. SYSTEM APPROXIMATION USING SUBBANDS

The subband approximation scheme is shown in Fig. 1. The linear system $g(q)$ is approximated by splitting the input signal $x(t)$ into $M$ subbands using the array of filters $h(q) = [h_1(q), \ldots, h_M(q)]^T$, followed by a downsampling operation of factor $D$ (i.e., one out of $D$ samples is kept). In this way, the subband vector signal $\xi(t)$ is generated. The subband model is an $M \times M$ transfer matrix $\Gamma(q)$ whose output is denoted by $\hat{\theta}(t)$. Finally, the output signal $\hat{y}(t)$ is generated by upsampling $\hat{\theta}(t)$ by a factor of $D$ (i.e., $D - 1$ zero valued samples are added between every two samples), then filtering each component using the array of filters $f^*(q) = [f_1^*(q), \ldots, f_M^*(q)]$, and adding together all the resulting signals.

![Fig. 1. System approximation in the time-frequency domain.](image)

We assume that the system $g(q)$ is an IIR system. Also, the filters in the arrays $h(q)$ and $f(q)$ are causal and FIR with tap sizes $l_h$ and $l_f$, respectively. Finally, the subband model $\Gamma(q)$ is diagonal with FIR filters of tap size $l_\Gamma$ and $d_\Gamma$ non-causal taps, on each diagonal entry.

**Comparison with OS/A methods:** These methods permit the implementation of an FIR system, whose tap size we denote by $l_g$. They consist of an iterative procedure in which, at each iteration, a block of $l_g + L - 1$ samples is processed in the frequency domain. Then, $L$ samples are skipped, and the next iteration is processed [8]. The computational cost and latency of the OS/A and subband methods are given in Table 1, where we assume that the filterbanks are of Gabor type (i.e., there exists $h_0(t)$ such that, for all $m \in \{1, \ldots, M\}$ and all $t \in \mathbb{Z}$, $h_m(t) = e^{j \frac{2\pi}{M}(m-1)t}h_0(t)$, and a similar property for $f(q)$), for which a numerically efficient algorithm exists [10]. We also assume that a $k$-point FFT can be implemented with $\frac{k}{2} \log_2 k$ multiplications.

### Table 1. Computational cost (CC) and latency for the OS/A and subband (SB) methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>CC [mult./sample]</th>
<th>Latency [samples]</th>
</tr>
</thead>
<tbody>
<tr>
<td>OS/A</td>
<td>$\frac{2}{L}(l_g + 2 \log_2 M)$</td>
<td>$L - 1$</td>
</tr>
<tr>
<td>SB</td>
<td>$\frac{2}{l_g D}(l_g + \log_2 M)$</td>
<td>$d_g D + l_g - 1$</td>
</tr>
</tbody>
</table>

3. PROPOSED APPROXIMATION CRITERION

The goal is to find the filters $h(q), f(q)$ and the subband model $\Gamma(q)$, for given values of $l_h, l_f, l_\Gamma, d_\Gamma, M$ and $D$, that minimize the power of the error signal $\hat{y}(t) = y(t) - \hat{y}(t)$, when $x(t)$ has a given auto-correlation function $r_x(t)$. In this section we express this problem as a minimization problem. To this end, we transform the setting in Fig. 1 using the so-called polyphase representation [11].

#### 3.1. Polyphase Representation

Let $x(t)$ be a scalar random process. The polyphase representation of $x(t)$ is the vector random process $x(t)$ satisfying

$$ [x]_d = \phi_d^{\Gamma} x \text{ for each } d \in \{1, \ldots, D\}, $$

where $[x]_d$ denotes the $d$-th entry of $x(t)$ and $(\phi_d^{\Gamma} x)(t) = x(tD + e)$. Also, let $g(t)$ be the impulse response of an LTI system $g(q)$. The polyphase representation of $g(q)$ is the $D \times D$ transfer matrix $G(q)$ whose impulse response satisfies

$$ [G]_{d,e} = \phi_d^{G} g \text{ for each } d, e \in \{1, \ldots, D\}. $$

Also, the polyphase representation of an analysis filterbank with filters $h(q)$ and downsampling factor $D$ is the $M \times D$ transfer matrix $H(q)$ whose impulse response satisfies

$$ [H]_{m,d} = \phi_d^{H} h_m \text{ for each } m \in \{1, \ldots, M\}, d \in \{1, \ldots, D\}. $$

The polyphase representation of the synthesis filterbank $f(q)$ with upsampling factor $D$ is the $D \times M$ transfer matrix $F(q)$, where $F(q)$ is defined as in (1), and $F^*(q) = F(q^{-1})^*$, with $F(q^{-1})^*$ being the transpose conjugate of $F(q^{-1})$. If $h(q)$ is of Gabor type, and $h_0(q)$ is a causal FIR filter with tap size $l_h$, then its polyphase representation is given by [10]

$$ H = W_M L_2 A_h L_1 $$

where $W_M \in \mathbb{C}^{M \times M}$ is the DFT matrix, i.e., $[W_M]_{k,l} = \frac{1}{2\pi} e^{-j \frac{2\pi}{M} (k-1)(l-1)}$, and

$$ L_1 = [I_D, q^{-1} I_D, \ldots, q^{-(nD + 1)} I_D]_{:1:l_h}, $$

$$ L_2 = [I_M, I_M, \ldots, I_M]_{:1:l_h}^{n_M \text{ times}}, $$

$$ A_h = \text{diag}\{h_0(0), \ldots, h_0(l_h - 1)\}. $$
with \( n_D = \left\lceil \frac{b}{b} \right\rceil \), \( n_M = \left\lceil \frac{b}{a} \right\rceil \) and \( |A|_{k:2} \) denoting the matrix formed with the columns from \( k \) to \( l \) of \( A \). Also, for a vector \( \text{diag}(x) \) denotes the diagonal matrix with elements \( |x| \) in its main diagonal.

By using the polyphase representation, the scheme in Fig. 1 can be represented by the LTI system shown in Fig. 2.

![Fig. 2. Polyphase representation of the time-frequency system approximation scheme.](image)

**3.2. Approximation as a Minimization Problem**

Let \( G(t) \) be the impulse response of the polyphase representation of an LTI system, let \( x(t) \) and \( y(t) \) denote its input and output, respectively, and let \( \hat{x}(t) \) and \( \hat{y}(t) \) denote their polyphase representations. Then, it is straightforward to verify that

\[
R_y = G * R_x * G^*
\]

where \( * \) denotes convolution of matrix sequences and \( R_v \) is the auto-correlation function of the vector random process \( v(t) \). It follows that

\[
S_y = \text{Tr}\{G * R_x * G^*\}(0)S_x
\]

where \( S_v \) is the power of the scalar random process \( v(t) \), and for a \( B \times B \) matrix sequence \( X(t) \), \( \text{Tr}\{X\}(t) = \sum_{i=1}^{B} [X]_{i,i}(t) \). Hence, we define the following norm

\[
\|G\|^2 = S_x^{-1} \text{Tr}\{G * R_x * G^*\}(0)
\]

which measures the power gain of the system when the input has auto-correlation \( r_z(t) \).

Let \( (\mathbb{C}^B)^A \) be the space of sequences of complex \( B \times A \) matrices indexed by \( Z \). Let \( G \subset (\mathbb{C}^Z)^M \), \( H \subset (\mathbb{C}^Z)^X \) and \( F \subset (\mathbb{C}^Z)^D \) be the subspaces of allowed subband models, analysis and synthesis filterbank polyphase matrices, respectively. Then, the system approximation problem can be written as

\[
(\Gamma, H, F) = \arg\min_{\Gamma \in G, H \in H, F \in F} \|G - \hat{F}^* \hat{\Gamma} - \hat{H}\|_{r_x}
\]  

**4. OPTIMIZATION ALGORITHM**

The problem (4) is a non-linear least-squares optimization problem, which we solve using the ALS method. More precisely, if we fix two of the elements in \( (\Gamma, H, F) \), the optimization of the third element is a linear optimization problem which can be solved using linear least-squares (LLS). The ALS algorithm cyclically repeats these three steps. Below we define the following transformations:

If \( X \) is a vector in \( (\mathbb{C}^Z)^B \), and \( m, n \in Z \), then \( X = \text{seq2col}_{m,n}(X) \) is given by \( X = [X^T, \cdots, X^T]^T \) with \( X = [X_{i,j}(m), \cdots, X_{i,j}(n)]^T \).

If \( X \) is a matrix in \( (\mathbb{C}^Z)^B \times A \), and \( n \in Z \), then \( X = \text{convmat}_{n}(X) \) denotes the matrix \( \tilde{X} = [\tilde{X}_{i,a}(0), \cdots, \tilde{X}_{i,a}(n)] \) with each submatrix \( \tilde{X}_{i,a}(n) \), where \( \tilde{X}_{i,a}(n) = X(i-j) \) for all \( i, j \in \{1, \cdots, n\} \).

If \( X, Y \in (\mathbb{C}^Z)^B \times A \), then \( X \otimes Y \in (\mathbb{C}^Z)^B \times A \) denotes the entry-wise convolution, i.e., \( [X \otimes Y]_{b,a} = [X]_{b,a} \otimes [Y]_{b,a} \).

**Theorem 1 (Optimization of \( \Gamma \))** Let \( G \subset (\mathbb{C}^Z)^M \) be the subspace of diagonal matrix sequences with FIR filters of tap size \( l_F \) and \( d_F \) non-causal taps, on each diagonal entry, and let \( m_F = -d_F \) and \( n_F = l_F - d_F - 1 \). Then, the solution of (4), for fixed \( H \) and \( F \), is given by

\[
\Gamma = \text{diag}\{\text{seq2col}_{m_F,n_F}(M^1v)\}
\]

where

\[
M = \text{convmat}_{m_F,n_F-1}\{\phi_D^0(h * r_x * h^*) \otimes \phi_D^0(f * f^*)\}
\]

\[
v = \text{seq2col}_{m_F,n_F}\{\text{diag}^{-1}\{\phi_D^0(f * g * r_x * h^*)\}\}
\]

**Theorem 2 (Optimization of \( H \))** Let \( H \subset (\mathbb{C}^Z)^M \) be the subspace of polyphase representations of Gabor filterbanks whose filters are causal and FIR with tap sizes \( l_H \). The solution of (4), for fixed \( \Gamma \) and \( F \), is given by

\[
H = W_M L_2 \text{diag}^{-1}\{M^1v\} L_1
\]

where

\[
M = (A \circ C^T)(0)
\]

\[
A = L_2^* W_M^* B W_M L_2
\]

\[
B = \text{diag}^{-1}\{\Gamma^* \} * \text{diag}^{-1}\{\Gamma^*\} \circ \phi_D^0(h * r_x * h^*)
\]

\[
C = L_1 * R_x * L_1^*
\]

\[
v = \text{diag}^{-1}\{L_2^* W_M^* \Gamma^* \phi_D^0(f * r_x * g) * L_1^*\}(0)
\]
<table>
<thead>
<tr>
<th></th>
<th>Azy=0, Ele=0</th>
<th>Azy=45, Ele=40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>-37.27dB</td>
<td>-40.47dB</td>
</tr>
<tr>
<td>Right</td>
<td>-39.41dB</td>
<td>-40.47dB</td>
</tr>
<tr>
<td>SB</td>
<td>-38.67dB</td>
<td>-39.41dB</td>
</tr>
<tr>
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<td>-40.47dB</td>
</tr>
<tr>
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<td>-40.47dB</td>
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