APPROXIMATE FRÉCHET SUBDIFFERENTIABILITY OF CONVEX FUNCTIONS

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Abstract. A generalisation of strong subdifferentiability and its characterisations are given along with implications for a Banach space to be Asplund or reflexive.

1. Introduction


Given a continuous convex function $\phi$ on a nonempty open convex subset $A$ of a Banach space $X$, we say that $\phi$ is Fréchet differentiable at $a \in A$ if there exists a continuous linear functional $\phi'(a)$ and given $0 < \varepsilon < 1$ there exists $0 < \delta(a, \varepsilon) < 1$ such that

$$0 \leq \phi(a + y) - \phi(a) - \phi'(a)(y) < \varepsilon \|y\|$$

for all $\|y\| < \delta$.

We say that $\phi$ is Fréchet subdifferentiable at $a \in A$ if the continuous sublinear functional $\phi'_+(a)$, given $0 < \varepsilon < 1$ there exists $0 < \delta(a, \varepsilon) < 1$ such that

$$0 \leq \phi(a + y) - \phi(a) - \phi'_+(a)(y) < \varepsilon \|y\|$$

for all $\|y\| < \delta$.

As with Fréchet differentiability, this property has a characterisation by a continuity property of the subdifferential mapping, [6, p.28] and when $\phi$ is a gauge a characterisation in terms of weak*—exposed subsets of the appropriate polar set in the dual, [8, p.400].

Here we are interested in a further generalisation of this concept. Given $0 < d < 1$, we say $\phi$ is d-Fréchet subdifferentiable at $a \in A$ if there exists $0 < \delta(a, \varepsilon) < 1$ such that

$$0 \leq \phi(a + y) - \phi(a) - \phi'_+(a)(y) < d \|y\|$$

for all $\|y\| < \delta$.

Now this generalisation is of interest because d-Fréchet subdifferentiability has similar characterisations to Fréchet subdifferentiability and proofs are obtained by slight modification of the earlier proofs. But significantly,

(i) if for some $0 < d < 1$, the norm of the space $X$ is d-Fréchet subdifferentiable on the unit sphere $S(X)$ then $X$ is an Asplund space, and

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Theorem 2.1. Consider a continuous convex function $\phi$ on a nonempty open convex subset $A$ of a Banach space $X$ and $a \in A$, the weak$^*-$compact convex subset of the dual space $X^*$. $\partial \phi(a) \equiv \{ f \in X^*: f(x) \leq \phi'_+(a)(x) \text{ for all } x \in X \}$ is the subdifferential of $\phi$ at $a$ and the set-valued mapping $x \mapsto \partial \phi(x)$ on $A$ is called the subdifferential mapping of $\phi$. It is well known that the Fréchet differentiability of $\phi$ at $a$ can be characterised by single-valuedness and norm upper semicontinuity of the subdifferential mapping of $\phi$ at $a$, [10, p.19]. Further, Fréchet subdifferentiability of $\phi$ at $a$ can be characterised by Hausdorff norm upper semicontinuity of the subdifferential mapping of $\phi$ at $a$, [6, p.28]. A modification of this proof provides us with a similar characterisation of approximate Fréchet subdifferentiability.

Theorem 2.1. Consider a continuous convex function $\phi$ on a nonempty open convex subset $A$ of a Banach space $X$ and $0 < d < 1$. $\phi$ is Fréchet differentiable at $a$ if and only if

\begin{align*}
\text{(i)} & \quad \text{if for some } 0 < d < 1, \text{ the norm of the dual space } X^* \text{ is } d-\text{Fréchet subdifferentiable on the dual unit sphere } S(X^*) \text{ then } X \text{ is reflexive.}
\end{align*}

2. The Continuity Characterisation

Given a continuous convex function $\phi$ on a nonempty open convex subset $A$ of a Banach space $X$ and $a \in A$, the weak$^*-$compact convex subset of the dual space $X^*$

\begin{align*}
\partial \phi(a) \equiv \{ f \in X^*: f(x) \leq \phi'_+(a)(x) \text{ for all } x \in X \}
\end{align*}

is the subdifferential of $\phi$ at $a$ and the set-valued mapping $x \mapsto \partial \phi(x)$ on $A$ is called the subdifferential mapping of $\phi$.

Then $\partial \phi(a)$ is the closed unit ball of $\partial \phi(a)$.

Proof. (i) Suppose that there exists a sequence $\{y_n\}$ in $A$, $y_n \to a$ and $f_n \in \partial \phi(y_n)$ such that $f_n \notin \partial \phi(a) + dB[X^*]$ for all $n \in N$. Now $\partial \phi(a) + dB[X^*]$ is weak$^*-$closed and convex so for each $n \in N$, $f_n$ can be strongly separated from $\partial \phi(a) + dB[X^*]$ by some $x_n \in X$, $\|x_n\| = 1$. Since for any $y \in X$, $\phi'_+(a)(y) = \sup \{ f(y) : f \in \partial \phi(a) \}$ we have that $(f_n - \phi'_+(a))(x_n) > d$.

Put $z_n = \delta x_n$. Then $0 \leq \phi(a + z_n) - \phi(a) - \phi'_+(a)(z_n) < d\delta$.

Now $f_n((a + z_n) - y_n) \leq \phi(a + z_n) - \phi(y_n)$ so

\begin{align*}
f_n(z_n) & \leq \phi(a + z_n) - \phi(a) + f_n(y_n - a) + \phi(a) - \phi(y_n).
\end{align*}

Then

\begin{align*}
\|z_n\| & \leq (f_n - \phi'_+(a))(z_n) \\
& \leq \phi(a + z_n) - \phi(a) - \phi'_+(a)(z_n) + M\|y_n - a\| + \phi(a) - \phi(y_n)
\end{align*}

for some $M > 0$. As $y_n \to a$ and $\phi$ is continuous we have

\begin{align*}
\|z_n\| & \leq \phi(a + z_n) - \phi(a) - \phi'_+(a)(z_n)
\end{align*}
but this contradicts the given $d$-Fréchet subdifferentiability condition of $\phi$ at $a$.

(ii) The continuity condition implies that, given $a + y \in A$, $\|y\| < \delta$ for each $f_{a+y} \in \partial \phi(a + y)$ there exists $f_a \in \partial \phi(a)$ such that $\|f_{a+y} - f_a\| \leq d$. Now
\[
\phi'_+(a)(y) \leq \phi(a + y) - \phi(a) \leq f_{a+y}(y) \leq f_a(y) + d\|y\| \leq \phi'_+(a)(y) + d\|y\|
\]
so we conclude that
\[
0 \leq \phi(a + y) - \phi(a) - \phi'_+(a)(y) \leq d\|y\| \quad \text{for all } \|y\| < \delta.
\]
\[\square\]

3. The Dual Characterisation for Gauges

A nonnegative continuous sublinear functional $p$ on a Banach space $X$ is the gauge of the closed bounded convex set $C \equiv \{ x \in X : p(x) \leq 1 \}$. The polar of $C$ is the set $C^0 \equiv \{ f \in X^* : f(x) \leq p(x) \text{ for all } x \in X \}$. It is well known that $f \in \partial p(x)$ if and only if $f \in C^0$ and $f(x) = p(x)$, [10, p.84].

It is said that $\partial p(x)$ is a weak*—exposed subset of $C^0$ and $C^0$ is weak*—exposed by $x$. Further $C^0$ is weak*—strongly exposed by $x$ if for all $\{f_n\} \subset C^0$ such that $f_n(x) \to \partial p(x)(x)$ then $\text{dist}(f_n, \partial p(x)) \to 0$ as $n \to \infty$. It is known that $C^0$ is weak*—strongly exposed by $x$ if and only if $p$ is Fréchet subdifferentiable at $x$, [p.400]8. We generalise this result modifying the standard arguments.

Given $0 < d < 1$, we say that $C^0$ is $d$-weak*—strongly exposed by $x$ if for all $\{f_n\} \subset C^0$ such that $f_n(x) \to \partial p(x)(x)$ then $\text{dist}(f_n, \partial p(x)) \to d$ as $n \to \infty$; equivalently, there exists $\delta > 0$ such that
\[
\mathcal{S}\ell(C^0, \tilde{x}, \delta) \subset \partial p(x) + dB[X^*]
\]
where $\mathcal{S}\ell(C^0, \tilde{x}, \delta) \equiv \{ f \in C^0 : f(x) > \sup \{ f(x) : f \in C^0 \} - \delta \}$, a weak*—slice of $C^0$.

**Theorem 3.1.** Consider $p$ the gauge of a closed bounded convex set $C$ in a Banach space $X$.

(i) If at $x \in X$ there exists $0 < \delta(x) < 1$ such that
\[
0 \leq p(x + y) - p(x) - p'_+(x)(y) < d\|y\| \quad \text{for all } \|y\| \leq \delta
\]
then the polar $C^0$ is $d$-weak*—strongly exposed by $x$.

(ii) If the polar $C^0$ is $d$-weak*—strongly exposed by $x$ then there exists $0 < \delta(x) < 1$ such that
\[
0 \leq p(x + y) - p(x) - p'_+(x)(y) \leq d\|y\| \quad \text{for all } \|y\| < \delta.
\]

**Proof.** (i) Suppose there exists a sequence $\{f_n\} \subset C^0$ where $f_n(x) \to \partial p(x)(x)$ but $\text{dist}(f_n, \partial p(x)) > d$ for all $n \in N$. Now for each $n \in N$ we can strongly separate $f_n$ from $\partial p(x) + dB[X^*]$ by some $z_n \in X$, $\|z_n\| = 1$. Then for any $f \in \partial p(x)$,
\[
(f_n - f)(z_n) > d \quad \text{for all } n \in N.
\]
Put $y_n \equiv \delta z_n$. Then
\[
d\|y_n\| \leq (f_n - f)(y_n)
\]
\[
= (f_n - f)(x + y_n) - (f_n - f)(x)
\]
\[
\leq p(x + y_n) - p(x) - f(y_n) - (f_n - f)(x)
\]
since \( f_n \in C^0 \) and \( f \in \partial p(x) \). As \( f_n(x) \to f(x) = \partial p(x)(x) \) and this holds for all \( f \in \partial p(x) \) we have that
\[
d\|y_n\| \leq p(x + y_n) - p(x) - p^*(x)(y_n)
\]
and this contradicts the given \( d \)-Fréchet subdifferentiability condition of \( p \) at \( x \).

(ii) We show that for all sequences \( \{x_n\} \), \( \partial p(x_n) \subset \partial p(x) + dB[X^*] \) when \( x_n \to x \). Now \( \partial p(x_n) \subset C^0 \) and \( p(x_n) = \partial p(x_n)(x_n) \). Given any \( f_n \in \partial p(x_n) \) and \( f \in \partial p(x) \),
\[
|f_n - f(x)| = |(f_n - f)(x - x_n) + f_n(x_n) - f(x_n)| \\
\leq \|f_n - f\| \|x_n - x\| + |p(x_n) - f(x_n)|.
\]
Now as \( x_n \to x \) we have \( p(x_n) \to p(x) \), \( f(x_n) \to f(x) = p(x) \) and \( \|f_n - f\| \) is bounded. So \( f_n(x) \to \partial p(x)(x) \). Since \( \partial p(x) \) is a \( d \)-weak*-strongly exposed subset of \( C^0 \), then \( \text{dist}(f_n, \partial p(x)) \leq d \); but this is true for all \( f_n \in \partial p(x_n) \) so \( \partial p(x_n) \subset \partial p(x) + dB[X^*] \) as \( x_n \to x \). Our conclusion follows from Theorem 2.1(ii).

4. The Condition Implying that the Space is Asplund

A Banach space \( X \) is an Asplund space if every continuous convex function \( \phi \) on a nonempty open convex subset \( A \) of \( X \) is Fréchet differentiable at the points of a dense \( G_a \) subset of \( A \). We will use the significant characterisation that \( X \) is Asplund if and only if every closed separable subspace of \( Y \) of \( X \) has separable dual \( Y^* \), \([10, p.32]\). It was proved by Godefroy that a Banach space \( X \) with norm Fréchet subdifferentiable on \( S(X) \) is an Asplund space \([5, p.68]\, [7, p.64]\). We use our approximate Fréchet subdifferentiability to generalise this.

The following crucial result was devised by Godefroy as a consequence of Simon's inequality. The statement is as given by Franchetti and Paya, \([5, p.68]\).

**Lemma 4.1.** Consider a Banach space \( X \). If there exists a countable set \( C \) in \( S(X^*) \) and \( 0 < \alpha < 1 \) such that for each \( x \in S(X) \) we have \( \|f_x - f\| \leq \alpha \) for some \( f_x \in \partial \|x\| \) and some \( f \in C \), then \( X^* = \text{sp} \) \( C \) and so \( X \) and \( X^* \) are separable.

**Proof.** Denote by \( \sigma \) the selection on \( S(X) \) where \( \sigma(x) \in \partial \|x\| \) and satisfying the given inequality. Suppose \( X^* \neq \text{sp} \) \( C \). Then there exists an \( F \in S(X^{**}) \) such that
\[
F(\text{sp} C) = 0.
\]
Choose \( 0 < \alpha < \beta < 1 \) and \( g \in S(X^*) \) such that
\[
F(g) > \beta.
\]
Since \( \hat{X} \) is weak*-dense in \( X^{**} \) and \( C \) is countable there exists a sequence \( \{x_n\} \) in \( B[X] \) such that
\[
(a) \ f(x_n) \to F(f) \text{ for all } f \in C \text{ and } (b) \ g(x_n) \to F(g).
\]
We may assume that \( g(x_n) \geq \beta \) for all \( n \in N \). Now the function \( \Phi \) on \( X^* \) where
\[
\Phi(f) = \limsup_{n \to \infty} |f(x_n)|
\]
is convex and norm lower semi continuous. Now from (a) we have
\[
\sup \{\Phi(f) : f \in C\} = 0
\]
but our given inequality gives
\[ \sup \{ \Phi(f) : f \in \sigma(S(X)) \} \leq \alpha. \]
Also from (b) we have
\[ \sup \{ \Phi(f) : f \in B[X^*] \} \geq \beta. \]
But this contradicts Simon’s Theorem, \([4, p.81]\).

It is important to note that in this result the constant \(0 < \alpha < 1\) is to hold generally for all \(x \in S(X)\). Godefroy \([7, p.62]\) has given a renorming of the separable space \(\ell_1\) where, under this renorming there is a sequence \(\{f_n\}\) in \(B(\ell_1^*)\) and a sequence of real numbers \(\{\epsilon_n\}\) where \(\epsilon_n \to 0\) and
\[ B(\ell_1^*) \subset \bigcup_{n=1}^{\infty} B[f_n; 1-\epsilon_n] \]
but of course \(\ell_1^*\) is non-separable.

**Theorem 4.2.** A Banach space \(X\) is an Asplund space if for some \(0 < d < 1\) the norm is \(d\)-Fréchet subdifferentiable on \(S(X)\).

**Proof.** Since the restriction to a closed linear subspace \(Y\) has norm \(d\)-Fréchet subdifferentiable on \(S(Y)\) and the Asplund property is separably determined, it is sufficient to consider \(X\) separable and prove that \(X^*\) is separable. By Mazur’s Theorem \([10, p.12]\) there exists a dense countable subset \(E\) of \(S(X)\) where the norm is Gâteaux differentiable so \(\partial \|E\|\) is countable. By Theorem 2.1(i) the subdifferential mapping \(x \mapsto \partial \|x\|\) has the property that, given \(x \in S(X)\) there exists an open neighbourhood \(U\) of \(x\) such that
\[ \partial \|y\| \subset \partial \|x\| + dB[X^*] \quad \text{for all} \quad y \in U \]
so
\[ \partial \|E\| \cap (\partial \|x\| + dB[X^*]) \neq \emptyset. \]
Then, by Lemma 4.1, we have that \(X^*\) is separable. \(\square\)

**5. The Condition Implying that the Space is Reflexive**

A Banach space \(X\) whose dual \(X^*\) has norm Fréchet subdifferentiable on \(S(X^*)\) is reflexive. This was proved by Aparicio et al. \([1, p.29]\) and by Godefroy \([7, p.69]\). An attractive proof was given by Franchetti and Paya \([5, p.65]\). We follow Godefroy’s proof in establishing our more general result. He takes his proof through several steps, \([7, p.65-69]\).

Godefroy’s proof is based on ‘ball topology’ techniques: given a Banach space \(X\), the ball topology \(b_X\) on \(X\) is the weakest topology for which the norm closed balls are closed. He proves:
(i) that a Banach space \(X\) where every \(f \in X^*\) is \(b_X\)-continuous on \(B[X]\) has \(X^*\) with no proper norming subspace, \([7, p.68]\);
(ii) that a Banach space \(X\) where every closed separable subspace \(Y\) of \(X\) has \(Y^*\) with no proper norming subspace has every \(f \in X^*\) \(b_X\)-continuous on \(B[X]\), \([7, p.67]\);
(iii) that a Banach space where the norm is Fréchet subdifferentiable on $S(X)$ has every closed separable subspace $Y$ of $X$ with $Y^*$ having no proper norming subspace, [7, p.68].

So his proof that a Banach space $X$ where the norm of $X^*$ is Fréchet subdifferentiable is reflexive, follows from (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

We present Godefroy’s proof of (i) because our statement is more general than that given in his paper. We recall that a typical $b_X$-open neighbourhood of 0 is of the form

$$X \setminus \bigcup_{k=1}^{n} B[x_k; r_k] \quad \text{where} \quad 0 < r_k < \|x_k\| \quad \text{for} \quad k \in \{1, 2, \ldots, n\}.$$

**Proof.** (i) It is sufficient to show that for any $F \in S(X^{**})$, $\ker F$ is not a norming subspace of $X^*$. If it were norming then

$$\|F - \widehat{x}\| \geq \|x\| \quad \text{for all} \quad x \in X, \ [4, p.134].$$

Choose an $f \in B[X^*]$ such that $F(f) > 0$ and consider a net $\{\widehat{x}_\alpha\}$ in $B[\widehat{X}]$ weak$^*$-convergent to $F$. Since the norm is weak$^*$-lower semi continuous then

$$\liminf_{\alpha} \|x_\alpha - x\| \geq \|x\| \quad \text{for all} \quad x \in X \quad \text{and so} \quad b_X \lim_{\alpha} x_\alpha = 0.$$

But $\lim_{\alpha} f(x_\alpha) = F(f) > 0$ so we conclude that $f$ is not $b_X$-continuous on $B[X]$. □

We achieve our result by generalising (iii) for approximate Fréchet subdifferentiability.

**Lemma 5.1.** Consider a Banach space $X$ and $0 < d < 1$. If the norm on $X$ is $d$-Fréchet subdifferentiable on $S(X)$ then for every closed separable subspace $Y$ of $X$, $Y^*$ contains no proper norming subspace.

**Proof.** If $N \subset Y^*$ is norming then $N \cap B[Y^*]$ is weak$^*$-dense in $B[Y^*]$. Since $B[Y^*]$ is weak$^*$-compact and metrisable there exists a sequence $\{f_n\}$ in $N \cap B[Y^*]$ which is weak$^*$-dense in $B[Y^*]$. Then since the norm on $Y$ is also $d$-Fréchet subdifferentiable on $S(Y)$ we have from Theorem 3.1(i) that

$$\partial \|y\| \cap \bigcup_{n=1}^{\infty} B[f_n; d] \neq \emptyset \quad \text{for all} \quad y \in S(Y).$$

So, by Lemma 4.1, $Y^* = \overline{\mathcal{P}\{f_n : n \in \mathcal{N}\}}$ and so $N = Y^*$. □

**Theorem 5.2.** A Banach space $X$ where, for some $0 < d < 1$, the dual $X^*$ has norm $d$-Fréchet subdifferentiable on $S(X^*)$ is reflexive.

**Proof.** Lemma 5.1 $\Rightarrow$ (ii) $\Rightarrow$ (i) and a nonreflexive space always has $X^{**}$ with proper norming subspace $\widehat{X}$. □

**6. Postscript**

Prof. Cascales pointed out that the characterisations in §2 and §3 suggest a generalisation of the standard result that a Banach space $X$ is Asplund if every nonempty bounded subset of $X^*$ has weak$^*$-slices of arbitrarily small diameter, [10, Th 2.32, p.31].
Theorem 6.1. A Banach space $X$ is an Asplund space if for some $0 < d < 1$, every nonempty subset of $B[X^*]$ has weak*--slices of diameter less than or equal to $d$.

Proof. Consider a nonempty open subset $E$ of $B[X^*]$ with a weak*--slice $S\ell_1$ of diameter less than or equal to $d$. Then there exists a closed ball $B_1$ with radius $r_1 \leq d$ such that $S\ell_1 \subset B_1$. Consider this ball $B_1$ magnified from its centre by $\frac{1}{r_1}$. Then by our hypothesis, this magnified slice $\frac{1}{r_1}S\ell_1$ has a weak*--slice of diameter less than or equal to $d$. So the original slice $S\ell_1$ has a weak*--slice $S\ell_2$ of diameter less than or equal to $r_1d$. Then there exists a weak*--open set $W_2$ such that $E \cap W_2 = S\ell_2$ with diameter less than or equal to $d$.

Repeating this process, there exists a closed ball $B_2$ with radius $r_2 \leq r_1d$ such that $S\ell_2 \subset B_2$. Again consider this ball $B_2$ magnified from its centre by $\frac{1}{r_2}$. Then by our hypothesis, this magnified slice $\frac{1}{r_2}S\ell_2$ has a weak*--slice of diameter less than or equal to $d$. So the original slice $S\ell_2$ has a weak*--slice $S\ell_3$ of diameter less than or equal to $r_2r_1d$. Then there exists a weak*--open set $W_3$ such that $E \cap W_3 = S\ell_3$ with diameter less than or equal to $d$.

So given $0 < \varepsilon < 1$ there exists $n \in \mathbb{N}$ such that after $n$ steps in this process we have that there exists a weak*--open set $W_n$ such that $E \cap W_n = S\ell_n$ with diameter less than or equal to $r_n \cdots r_1d < \varepsilon$. That is, every nonempty subset of $B[X^*]$ with the weak*--topology is fragmentable by the norm of the space, and this implies that $X$ is an Asplund space, [3, Th 5.2.3, p90].

We can go on to generalise the standard result that a Banach space $X$ is Asplund if every equivalent norm has a point of Fréchet differentiability, [10, Corol 2.35, p31].

Corollary 6.2. A Banach space $X$ is an Asplund space if for some $0 < d < 1$ and every equivalent norm $p$ on $X$ there exists a point $x \in X$ where $\text{diam} \partial p(x) < d$ and $p$ is Fréchet subdifferentiable at $x$.

Proof. Consider a bounded subset $E$ of $B[X^*]$. Let $K = \text{co}(E \cup -E) + B[X^*]$. Then $K$ is a bounded symmetric convex subset of $X^*$ with nonempty interior and the functional $p$ on $X$ where $p(x) = \sup\{f(x) : f \in K\}$ is an equivalent norm for $X$.

If for $x \in X$, $\text{diam} \partial p(x) < d$ and $p$ is Fréchet subdifferentiable at $x$ then for $0 < 2\varepsilon < d - \text{diam} \partial p(x)$ there exists $0 < \delta(\varepsilon, x) < 1$ such that

$$0 \leq p(x + y) - p(x) - p_+(x)(y) < \varepsilon \|y\| \quad \text{for all} \quad \|y\| < \delta.$$

Now

$$0 \leq p_+(x)(y) + p_+(x)(-y) = \sup\{f(y) : f \in \partial p(x)\} - \inf\{f(y) : f \in \partial p(x)\} \leq \text{diam} \partial p(x) \cdot \|y\|.$$

Then

$$0 \leq p(x + y) + p(x - y) - 2p(x) < d\|y\| \quad \text{for all} \quad \|y\| < \delta \quad (\ast)$$

But if every weak*--slice of $E$ has diameter greater than or equal to $d$ then $K$ has the same property. Then given $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\text{diam}\{f \in K : f(x) > p(x) - \frac{\varepsilon}{n}\} \geq d.$$
Now there exist $f_{1n}, f_{2n} \in K$ such that
\[
f_{1n}(x) > p(x) - \frac{\varepsilon}{n}, \quad f_{2n}(x) > p(x) - \frac{\varepsilon}{n} \quad \text{and} \quad \|f_{1n} - f_{2n}\| > d - \frac{1}{n},
\]
So there exists $\{y_n\}, \|y_n\| = 1$ such that $(f_{1n} - f_{2n})(y_n) > d - \frac{1}{n}$.
Therefore
\[
p \left( x + \frac{y_n}{n} \right) + p \left( x - \frac{y_n}{n} \right) - 2p(x) \geq f_{1n} \left( x + \frac{y_n}{n} \right) + f_{2n} \left( x - \frac{y_n}{n} \right) - (f_{1n} + f_{2n})(x) - \frac{2\varepsilon}{n}
\]
\[
= \frac{1}{n}(f_{1n} - f_{2n})(y_n) - \frac{2\varepsilon}{n} > \frac{d}{n} - \frac{1}{n^2} - \frac{2\varepsilon}{n}
\]
which contradicts $(\ast)$. So we conclude that $E$ has a weak$^\ast$−slice with diameter less than or equal to $d$ and the result follows from Theorem 6.1.
\[\square\]

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