A FAMILY OF HIGHER-RANK GRAPHS ARISING
FROM SUBSHIFTS

NATASHA WEAVER, B. MATH. (HONS)

THE UNIVERSITY OF NEWCASTLE

Thesis submitted for the degree of
DOCTOR OF PHILOSOPHY IN MATHEMATICS

MARCH 2009
Declaration

The thesis contains no material which has been accepted for the award of any other
degree or diploma in any university or other tertiary institution and, to the best of my
knowledge and belief, contains no material previously published or written by another
person, except where due reference has been made in the text. I give consent to this copy
of my thesis, when deposited in the University Library, being made available for loan and
photocopying subject to the provisions of the Copyright Act 1968.

I hereby certify that the work embodied in this thesis contains published papers of
which I am a joint author. I have included as part of the thesis a written statement,
endorsed by my supervisors, attesting to my contribution to the joint publications:

During the writing of this thesis I have received advice, guidance, and mathematical
assistance from my supervisors. Their assistance has been within the scope of normal
supervisor-student relations. Apart from their help, this thesis has been all my own work.
Some of the results in this thesis are contained in the paper \[46\], written jointly with my
supervisors. The names on the paper are in alphabetical order, to conform with the usual
convention in Pure Mathematics.

______________________________
Natasha Weaver

______________________________
Professor Iain Raeburn

______________________________
Doctor David Pask
Acknowledgements

To my supervisors Professor Iain Raeburn and Doctor David Pask I extend my sincerest gratitude: for their guidance and support; for sharing their knowledge and enthusiasm; for improving my writing; and for their patience while I turned drawings of socks and dominos into proofs.

I thank my fellow postgrad students and friends, Daniel, Matt, Steve, and Russell, for interesting conversations — mathematical and otherwise — and for programming advice.

Finally, I dedicate this thesis to my family: Mum, Dad, and Danielle.
Contents

Abstract ix

Chapter 1. Introduction 1
  1.1. Outline of the thesis 4

Chapter 2. Directed graphs and their $C^*$-algebras 7
  2.1. Directed graphs 7
  2.2. Representations of shifts of finite type 8
  2.3. Cuntz-Krieger algebras 10
  2.4. $C^*$-algebras of row-finite directed graphs 11
  2.5. Properties of graph algebras 12
  2.6. $K$-theory of graph algebras 13

Chapter 3. Higher-rank graphs and their $C^*$-algebras 15
  3.1. Robertson-Steger algebras 15
  3.2. Higher-rank graphs 15
  3.3. 2-graphs 17
  3.4. The infinite path space 22
  3.5. $C^*$-algebras of row-finite $k$-graphs with no sources 23
  3.6. Properties of higher-rank graph $C^*$-algebras 24
  3.7. $K$-theory and classifiability 25
  3.8. Higher-dimensional dynamical systems 26

Chapter 4. A family of 2-graphs arising from two-dimensional subshifts 29
  4.1. Tiles and 2-graphs 29
  4.2. Connections with shift spaces 37
  4.3. Strong connectivity 39
  4.4. Aperiodicity 41
  4.5. The $C^*$-algebras 44
  4.6. $K$-theory methods 45
  4.7. $K$-theory results 48

Chapter 5. Crossed products of $C^*$-dynamical systems 55
  5.1. The Pontryagin Duality Theorem 55
  5.2. Crossed products 56
  5.3. Twisted crossed products 58
  5.4. Restricted crossed products 61
Abstract

There is a strong connection between directed graphs and the shifts of finite type which are an important family of dynamical systems. Higher-rank graphs (or $k$-graphs) and their $C^*$-algebras were introduced by Kumjian and Pask to generalise directed graphs and their $C^*$-algebras. Kumjian and Pask showed how higher-dimensional shifts of finite type can be associated to $k$-graphs, but did not discuss how one might associate $k$-graphs to $k$-dimensional shifts of finite type. In this thesis we construct a family of 2-graphs $\Lambda$ arising from a certain type of algebraic two-dimensional shift of finite type studied by Schmidt, and analyse the structure of their $C^*$-algebras.

Graph algebras and $k$-graph algebras provide a rich source of examples for the classification of simple, purely infinite, nuclear $C^*$-algebras. We give criteria which ensure that the $C^*$-algebra $C^*(\Lambda)$ is simple, purely infinite, nuclear, and satisfies the hypotheses of the Kirchberg-Phillips Classification Theorem. We perform $K$-theory calculations for a wide range of our 2-graphs $\Lambda$ using the Magma computational algebra system. The results of our calculations lead us to conjecture that the $K$-groups of $C^*(\Lambda)$ are finite cyclic groups of the same order. We are able to prove under mild hypotheses that the $K$-groups have the same order, but we have only numerical evidence to suggest that they are cyclic. In particular, we find several examples for which $K_1(C^*(\Lambda))$ is nonzero and has torsion, hence these are examples of 2-graph $C^*$-algebras which do not arise as the $C^*$-algebras of directed graphs.

Finally, we consider a subfamily of 2-graphs with interesting combinatorial connections. We identify the nonsimple $C^*$-algebras of these 2-graphs and calculate their $K$-theory.
CHAPTER 1

Introduction

Directed graphs are combinatorial objects used to model networks. The action of the shift map on the space of two-sided infinite paths in a finite directed graph is known as a shift of finite type or topological Markov shift. Shifts of finite type form an important class of examples in dynamical systems as their theory has been well-understood using graph theory techniques. Directed graphs have a natural representation as a nonnegative integer matrix and vice-versa [40]. In [8], Cuntz and Krieger introduced a class of \( C^* \)-algebras associated to \( \{0,1\} \)-matrices arising from shifts of finite type. If \( A \) is an \( n \times n \) \( \{0,1\} \)-matrix, the Cuntz-Krieger algebra \( O_A \) is a \( C^* \)-algebra generated by a family of partial isometries \( \{S_i\}_{i=1}^n \) with certain properties. Cuntz and Krieger showed that under certain conditions the isomorphism class of \( O_A \) is independent of the choice of the \( S_i \) (this result is known as the Cuntz-Krieger Uniqueness Theorem) and gave criteria for simplicity of \( O_A \). By realising \( A \) as the vertex matrix of a directed graph, Enomoto, Fujii, and Watatani described properties of the graph which correspond to properties of \( O_A \) [16, 25, 15].

Analogues of the Cuntz-Krieger algebras for infinite directed graphs and matrices were considered by various authors [38, 37, 18, 21, 2, 14, 58]. In [38] and [37], Kumjian, Pask, Raeburn, and Renault defined the \( C^* \)-algebra \( C^*(E) \) of a row-finite directed graph \( E \), in which every vertex sends and receives only finitely many edges. They proved a Cuntz-Krieger Uniqueness Theorem for \( C^*(E) \) if every cycle in \( E \) has an entry. The structure theory results of [37] also relate the properties of cycles in the graph to properties of the \( C^* \)-algebra such as simplicity and pure infiniteness. The \( C^* \)-algebras of directed graphs, also known as graph algebras, are always nuclear [63]. Graph algebras have since been associated to all infinite graphs and the general theory is well-understood.

In [61], Robertson and Steger introduced higher-rank analogues of Cuntz-Krieger algebras associated to higher-dimensional shifts of finite type with special properties. Each Robertson-Steger algebra \( A \) is the unique \( C^* \)-algebra generated by partial isometries whose properties are determined by a family of \( \{0,1\} \)-matrices satisfying certain conditions. Robertson and Steger showed that \( A \) is simple, purely infinite, and nuclear.

In 2000, Kumjian and Pask, motivated by [61], introduced combinatorial objects called higher-rank graphs (or \( k \)-graphs) as higher-dimensional analogues of directed graphs [36]. The definition of a \( k \)-graph, given abstractly in category theoretical terms, is modelled on the path space of a directed graph. Kumjian and Pask showed how to construct a \( C^* \)-algebra associated to a higher-rank graph in direct generalisation of the Cuntz-Krieger algebras of directed graphs. Although all \( k \)-graphs studied in [36] were assumed to be row-finite with no sources, the definition was modified in [56] to include \( k \)-graphs which may have sources.
Every $k$-graph has a **skeleton** which consists of $k$ directed graphs on the same vertices; we think of each of these graphs as being one of $k$ colours. Thus a $k$-graph gives rise to a family of $k$ integer matrices (called the **coordinate matrices**) which describe connectivity in the $k$ coloured graphs comprising the skeleton. Conversely, a $k$-graph can be constructed from certain families of commuting integer matrices (or $k$ directed graphs with the same vertex set [22, Remark 2.3]). There can be more than one $k$-graph associated to the same family of matrices; however, if the matrices satisfy the conditions of Robertson and Steger then the $C^*$-algebra of the associated higher-rank graph (defined in [36, Examples 1.7(iv)]) is the same as the Robertson-Steger algebra [36, Corollary 3.5(ii)]. Thus Robertson-Steger algebras are included in the class of $C^*$-algebras in [36].

Aperiodicity is a property of higher-rank graphs which appears as hypothesis for many theorems about the associated $C^*$-algebras. Kumjian and Pask proved a Cuntz-Krieger Uniqueness Theorem for the $C^*$-algebras of aperiodic $k$-graphs. Aperiodicity is also among their criteria for determining whether the $C^*$-algebra is simple and purely infinite. The definition of aperiodicity in [36] is phrased in terms of the shift map on the infinite path space of the $k$-graph. Since this definition is hard to use in practice, there had not been many examples of $k$-graphs for which aperiodicity had been established. In 2007, D. I. Robertson and Sims provided a reformulation of aperiodicity which was inspired by the original aperiodicity condition of [61]. The new condition is much easier to apply as it only involves finite paths.

The theory of $k$-graphs is especially complicated for $k \geq 3$ [22]; working in the rank 2 case is much easier since 2-graphs can be constructed from pairs of “commuting” directed graphs [36]. A 2-graph is specified by a pair of directed graphs (with the same vertex set) called the **blue graph** and the **red graph** together with a set of factorisation relations which pair up the blue-red paths of length 2 with the red-blue paths of length 2. In general there is a choice of suitable factorisation relations and this affects the structure of the $C^*$-algebra [56, 54]. For example, the 2-graphs in [45] are constructed from a Bratteli diagram as the blue graph and a red graph consisting of disjoint cycles. The accompanying $C^*$-algebras are $AT$ algebras that are not purely infinite and are simple under certain conditions on the factorisation relations. As another example, the 2-graphs studied by Davidson, Power, and Yang [9, 10, 11] are associated to different sets of factorisation relations on a given blue graph and red graph with a single vertex. The characterisation of periodicity for these 2-graphs indicates that simplicity of the $C^*$-algebras depends on the choice of factorisation relations [11].

Since their introduction, higher-rank graph $C^*$-algebras have become a popular area of study, and many of the fundamental results about the $C^*$-algebras of directed graphs have analogues in higher-dimensions. As with one dimension, we have good criteria to determine when a higher-rank graph algebra is simple or purely infinite. It is not known which $C^*$-algebras arise as $C^*$-algebras of $k$-graphs and recent work has involved studying examples. Examples in noncommutative geometry have appeared [48, 47, 52] and some important operator algebras have been identified as higher-rank graph algebras [9, 34, 45].
A Kirchberg algebra is a simple, purely infinite, nuclear, separable $C^*$-algebra [63]. The Kirchberg-Phillips Classification Theorem [29, 50] says that Kirchberg algebras satisfying the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet [65] are classified by their $K$-theory. Szymański showed that every Kirchberg algebra which satisfies the UCT and has $K_1$ free is stably isomorphic to a graph algebra [72]. It is possible that the $K_1$-group of a tensor product of graph algebras has torsion, and every tensor product of graph algebras is a higher-rank graph algebra, so the class of higher-rank graph $C^*$-algebras includes more Kirchberg algebras than the graph algebras do. (Although we shall not use these constructions here, we mention that all Kirchberg algebras have been modeled by other variations on the graph algebra construction [71, 28].)

Associated to a $k$-graph is a dynamical system generated by $k$ commuting homeomorphisms on the higher-dimensional analogue of the two-sided infinite path space of a directed graph [35]. These dynamical systems generalise the topological Markov shifts associated to nonnegative integer matrices of [8]. Associating a $k$-graph to a dynamical system is a much harder problem due to the complex nature of higher-rank graphs and because the general theory of higher-dimensional shifts of finite type is not well-understood. Many difficulties arise when determining dynamical properties of higher-dimensional shifts of finite type; for example, it may be undecidable whether a given shift space is nonempty (in one dimension this is easy to ascertain) [40]. To avoid these difficulties study has concentrated on understanding tractable examples such as the subclass of higher-dimensional Markov group shifts in [31, 32]. The algebraic structure of Markov group shifts admits an easy visual description and can be used to determine dynamical properties such as mixing [68].

The purpose of this thesis is to introduce a class of 2-graphs associated to two-dimensional algebraic subshifts as studied by Schmidt [68, 32] and analyse the associated $C^*$-algebras. Our motivating example was introduced by Ledrappier [39] to highlight a key difference between higher-dimensional shifts of finite type and one-dimensional shifts of finite type. For a two-dimensional Markov group shift satisfying certain hypotheses we construct a 2-graph $\Lambda$ which is finite and has no sources. The basic object in the construction is a hereditary subset of the quarter plane $\mathbb{N}^2$, which we call a tile, and paths in the associated 2-graph are obtained by writing elements of an alphabet in overlapping translates of the tile. The resulting 2-graph comes from a blue graph (consisting of paths obtained by translating the tile in the horizontal direction) and a red graph (translating in the vertical direction) for which there is exactly one choice of factorisation relations. For a two-dimensional Markov group shift satisfying certain hypotheses we construct a 2-graph $\Lambda$ which is finite and has no sources. The basic object in the construction is a hereditary subset of the quarter plane $\mathbb{N}^2$, which we call a tile, and paths in the associated 2-graph are obtained by writing elements of an alphabet in overlapping translates of the tile. The resulting 2-graph comes from a blue graph (consisting of paths obtained by translating the tile in the horizontal direction) and a red graph (translating in the vertical direction) for which there is exactly one choice of factorisation relations. Our construction is designed to produce good models for the chosen dynamical systems; we demonstrate this by identifying the two-sided infinite path space of $\Lambda$ with the original shift space. This correspondence breaks down if we relax the hypotheses on the shift space and then have to make a choice for the factorisation relations. (Similar problems arise when we try to generalise our construction to dimensions higher than 2.)
Next we analyse the $C^*$-algebras of our 2-graphs. We give conditions under which $\Lambda$ satisfies the aperiodicity condition of [60] and justify their imposition with counterexamples. Under the same conditions, $C^*(\Lambda)$ is simple, purely infinite, and satisfies the hypotheses of the Kirchberg-Phillips Theorem, which says that these $C^*$-algebras are classified by their $K$-theory.

We compute the $K$-theory of $C^*(\Lambda)$ using techniques developed by Robertson and Steger [62] and Evans [17] which identify the $K$-groups of a 2-graph in terms of kernels and cokernels of the coordinate matrices. These matrices get quite large in our family of 2-graphs and so we use the computational algebra system Magma [4] to perform calculations for a wide range of examples. Numerical evidence suggests that the $K_0$-group and the $K_1$-group of $C^*(\Lambda)$ are cyclic groups of the same finite cardinality which depends on the shape of the tile. Under mild hypotheses we are able to prove that $|K_0| = |K_1|$, but in general we do not know whether $K_0$ and $K_1$ are actually cyclic, or whether they are isomorphic. For many of our examples $|K_1|$ is finite and greater than 1, hence our 2-graph construction provides models of $C^*$-algebras which are Kirchberg algebras but do not arise as $C^*$-algebras of ordinary directed graphs.

Finally, we consider a subfamily of our 2-graphs consisting of those based on a one-dimensional tile called a domino which are very different from the ones above which give Kirchberg algebras. Connections to combinatorial concepts such as necklaces and Lyndon words arise in our description of the structure of these 2-graphs, which we call domino graphs. The associated $C^*$-algebras are not simple as domino graphs do not satisfy the aperiodicity condition of [60]. Apart from a few special cases, every domino graph $\Lambda$ is a crossed product of a certain directed graph by an action of $\mathbb{Z}$ as studied in [19]. Results of [19] imply that $C^*(\Lambda)$ is a crossed product of an ordinary graph algebra by an action of $\mathbb{Z}$. We analyse the structure of this crossed product using results of Olesen and Pedersen [44] and Kishimoto [30], and our main result is a realisation of $C^*(\Lambda)$ as an induced $C^*$-algebra with simple fibres. We compute the $K$-theory of $C^*(\Lambda)$ explicitly, finding, as the Magma experiments suggest, that $K_0$ and $K_1$ are isomorphic to the same cyclic group whose order is determined by the length of the domino.

1.1. Outline of the thesis

Chapter 2. In this chapter we review relevant material about the $C^*$-algebras of directed graphs and their role in representing dynamical systems. We begin in Section 2.1 with basic facts about directed graphs. In Section 2.2, we give an overview of terminology and notation used in dynamical systems, recalling the relationship between shifts of finite type and directed graphs and their matrix representations. Section 2.3 contains a brief description of the Cuntz-Krieger algebras associated to finite matrices arising in dynamical systems. In Section 2.4, we give the definition of the $C^*$-algebra of a row-finite directed graph. We discuss important theorems about properties of graph algebras in Section 2.5 and discuss the $K$-theory of graph algebras in Section 2.6.
Chapter 3. In this chapter we survey higher-rank graph $C^*$-algebras and their connections to higher-dimensional dynamical systems. In Section 3.1, we briefly recall Robertson and Steger’s construction of higher-rank Cuntz-Kreiger algebras. Section 3.2 is a collection of facts about $k$-graphs. In Section 3.3, we describe the structure of 2-graphs with emphasis on how they are visualised. We explain in detail how 2-graphs are constructed from directed graphs (this procedure will be used in Chapter 4), illustrating with examples. In Section 3.4, we state the definition of the two-sided infinite path space of a $k$-graph and recall its topological properties. In Section 3.5, we define the $C^*$-algebra of a row-finite higher-rank graph with no sources. In Section 3.6, we review the development of the major theorems about properties of higher-rank graph algebras. In Section 3.7, we present formulas for calculating the $K$-theory of $C^*$-algebras of 2-graphs and discuss the hypotheses of the Kirchberg-Phillips Theorem in relation to our work. In Section 3.8, we describe the sort of higher-dimensional shifts of finite type which form the basis for our study.

Chapter 4. This chapter, our main body of work, comprises a construction of a family of 2-graphs arising from two-dimensional dynamical systems and an analysis of their $C^*$-algebras. With reference to our motivating example (the Ledrappier example) we start with a visual description of our construction in Section 4.1. We then prove that under certain hypotheses our construction yields a 2-graph $\Lambda$ (Theorem 4.1.7) with the properties listed in Proposition 4.1.8. We provide examples to justify these hypotheses. In Section 4.2, we prove that the two-sided infinite path space of $\Lambda$ is homeomorphic to a two-dimensional Markov shift (Theorem 4.2.1). As a step towards aperiodicity, we show in Section 4.3 that our 2-graphs are always strongly connected. In Section 4.4, we investigate the aperiodicity of our 2-graphs and prove in Theorem 4.4.3 that if $\Lambda$ satisfies certain conditions then $\Lambda$ is aperiodic in the sense of [60]. We justify these conditions with a periodic example. In Section 4.5, we state our main theorem, Theorem 4.5.1, which says that $C^*(\Lambda)$ is simple, nuclear, and purely infinite under the same conditions that ensure aperiodicity of $\Lambda$.

In Section 4.6, we discuss the procedures used for $K$-theory calculations and we give details for the Ledrappier example. Magma code for these procedures, which we used in our computer experiments, is included in Appendix C. We comment on the results of the Magma calculations which are listed in tables in Appendix D. (The notation used in the tables is borrowed from that of integer partitions in combinatorics. See Appendix A for background on integer partitions and connections with our work.) Based on our results, we make three conjectures about the $K$-theory of $C^*(\Lambda)$ and discuss the implications of these conjectures for the classification of $C^*(\Lambda)$. In Section 4.7, we prove two of our conjectures given mild assumptions. We resolve our first conjecture in Theorem 4.7.1. Of most interest is our second conjecture, that $|K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))|$, which we prove in Theorem 4.7.10.

Chapter 5. In this chapter we define the crossed product of a $C^*$-dynamical system $(A, G, \alpha)$ in which $G$ is a discrete abelian group. These crossed products arise in our analysis of the $C^*$-algebras of domino graphs later in Chapter 6. Following an overview of Pontryagin duality in Section 5.1, we summarise the construction of the crossed product
in Section 5.2. Section 5.3 contains a similar outline of twisted crossed products. We also state a result, Theorem 5.3.1, which follows from the work of Olesen and Pedersen and identifies the $C^*$-algebra of a dynamical system with certain properties as an induced algebra. The corollary to this theorem, Corollary 5.3.3, will be important later in Chapter 6 in identifying the $C^*$-algebras of domino graphs. We provide a proof of Theorem 5.3.1 in Section 5.4. To prove this result we review the restricted crossed products studied by Olesen and Pedersen.

**Chapter 6.** Here we consider a subfamily of the 2-graphs introduced in Chapter 4 which have connections to combinatorial concepts such as necklaces and Lyndon words. (Background information on necklaces and Lyndon words is compiled in Appendix B.) We begin in Section 6.1 with an example to visually describe our construction in the particular case of domino graphs. In Section 6.2, we describe the structure of domino graphs. We use the combinatorial connections to list the special properties of domino graphs in Proposition 6.2.1. We then show that domino graphs are of two types: they either arise as a product graph or can be realised as the crossed product of the underlying red graph by a $\mathbb{Z}$-action (Proposition 6.2.5).

We identify the nonsimple $C^*$-algebras of domino graphs $\Lambda$ in Section 6.3. The first part of our major result, Theorem 6.3.3, says that when the domino graph is a product graph, $C^*(\Lambda)$ is a tensor product of ordinary graph algebras. The second part of Theorem 6.3.3 says that when the domino graph is a crossed product graph, $C^*(\Lambda)$ can be realised as a $C^*$-algebraic crossed product and is induced from a simple algebra. To prove this we use results of Chapter 5.

We then consider the $K$-theory of $C^*(\Lambda)$ in Section 6.4. The form of $C^*(\Lambda)$ lends itself to analysis via either the Künneth formula or the Pimsner-Voiculescu sequence for crossed products by $\mathbb{Z}$. We give the $K$-theory of $C^*(\Lambda)$ in Proposition 6.4.1. For domino graphs, this result agrees with our conjectures about the $K$-theory of $C^*(\Lambda)$ in Section 4.6 of Chapter 4.
CHAPTER 2

Directed graphs and their $C^*$-algebras

2.1. Directed graphs

A directed graph $E$ consists of a countable set $E^0$ of vertices, a countable set $E^1$ of edges between them, and maps $r, s : E^1 \to E^0$ which indicate the direction of the edges: an edge $e \in E^1$ points from its source $s(e)$ to its range $r(e)$. If each vertex of $E$ receives only finitely many edges, that is, if $r^{-1}(v)$ is a finite set for every $v \in E^0$, then $E$ is called row-finite. A graph $E$ is called finite if $E^0$ and $E^1$ are finite sets. A sink is a vertex $v \in E^0$ that does not emit any edges, that is, $s^{-1}(v) = \emptyset$, and a source is a vertex $v \in E^0$ that does not receive any edges, $r^{-1}(v) = \emptyset$. We say that $E$ has no sources if none of its vertices are sources and $E$ has no sinks if none of its vertices are sinks.

A path of length $n$ in $E$ is a sequence $\lambda = e_1 e_2 \cdots e_n$ of edges such that $s(e_i) = r(e_{i+1})$ for $1 \leq i < n$, and we define $|\lambda| = n$. We regard the vertices as paths of length 0. We denote by $E^n$ the paths of length $n$ in $E$ and write $E^* = \bigcup_{n \geq 0} E^n$. Extend the maps $r$ and $s$ to $E^*$ in the following way: define the range and source of $\lambda \in E^*$ by $r(\lambda) = r(e_1)$ and $s(\lambda) = s(e_{|\lambda|})$. If $\lambda = e_1 e_2 \cdots e_{|\lambda|}$ and $\mu = f_1 f_2 \cdots f_{|\mu|}$ are paths in $E^*$ with $s(\lambda) = r(\mu)$ then there is a path $\lambda\mu = e_1 e_2 \cdots e_{|\lambda|} f_1 f_2 \cdots f_{|\mu|} \in E^*$. (This definition may seem “backwards” but we want to view the collection $E^*$ as a category and so path composition corresponds to composition of morphisms which is right-to-left. We have adopted this convention — although it is opposite to that of [38, 37] and much of the graph theory literature — in order to be consistent with the definition of higher-rank graphs. In particular, a directed graph with “no sinks” in [38, 37] corresponds to “no sources” here.) The category with objects $E^0$, morphisms $E^*$ and composition of paths defined above is called the path category of $E$ and is denoted by $E^*$.

A path $\mu \in E^*$ is called a loop if $r(\mu) = s(\mu)$. A cycle is a loop with no repeated edges and a cycle of length 1 is called a self-loop. Denote by $E^\infty$ the set of infinite paths $\lambda = \lambda_1 \lambda_2 \lambda_3 \cdots$. An isomorphism of directed graphs $E$ and $F$ is a pair of bijections $\phi^0 : E^0 \to F^0$ and $\phi^1 : E^1 \to F^1$ such that $\phi^0 \circ s_E = s_F \circ \phi^1$ and $\phi^0 \circ r_E = r_F \circ \phi^1$.

A category $\mathcal{C}$ consists of a set of objects $\text{Obj}(\mathcal{C})$, a set $\text{Mor}(\mathcal{C})$ consisting of morphisms $\text{hom}(A, B)$ from $A$ to $B$ for every $(A, B) \in \mathcal{C} \times \mathcal{C}$, and a composition function $\circ : \text{hom}(A, B) \times \text{hom}(B, C) \to \text{hom}(A, C)$ (we write $g \circ f = g \circ f$). We also require that $\circ$ is associative and that for every $A \in \mathcal{C}$ there exists an identity morphism $\text{id}_A \in \text{hom}(A, A)$ such that $f \circ \text{id}_A = f$ for every $f \in \text{hom}(A, B)$ and $\text{id}_A \circ g = g$ for every $g \in \text{hom}(B, A)$. 

---

1A category $\mathcal{C}$ consists of a set of objects $\text{Obj}(\mathcal{C})$, a set $\text{Mor}(\mathcal{C})$ consisting of morphisms $\text{hom}(A, B)$ from $A$ to $B$ for every $(A, B) \in \mathcal{C} \times \mathcal{C}$, and a composition function $\circ : \text{hom}(A, B) \times \text{hom}(B, C) \to \text{hom}(A, C)$ (we write $g \circ f = g \circ f$). We also require that $\circ$ is associative and that for every $A \in \mathcal{C}$ there exists an identity morphism $\text{id}_A \in \text{hom}(A, A)$ such that $f \circ \text{id}_A = f$ for every $f \in \text{hom}(A, B)$ and $\text{id}_A \circ g = g$ for every $g \in \text{hom}(B, A)$. 

7
2. DIRECTED GRAPHS AND THEIR C*-ALGEBRAS

2.2. Representations of shifts of finite type

Let \( \mathcal{A} \) be a finite set called an alphabet consisting of elements called symbols. The full shift \( \mathcal{A}^\mathbb{Z} \),
\[
\mathcal{A}^\mathbb{Z} = \{ x = (x_i)_{i \in \mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z} \},
\]
is the collection of all bi-infinite sequences of symbols from \( \mathcal{A} \). A word over \( \mathcal{A} \) is a finite sequence of symbols from \( \mathcal{A} \). A shift space \( X \) over \( \mathcal{A} \) is a closed subset of \( \mathcal{A}^\mathbb{Z} \) that can be described by a collection \( \mathcal{F} \) of forbidden words over \( \mathcal{A} \), that is, \( X \) consists of elements of \( \mathcal{A}^\mathbb{Z} \) which do not contain any word of \( \mathcal{F} \). Note that many collections can describe a given shift space. The shift map \( \sigma \) on \( \mathcal{A}^\mathbb{Z} \) is given by \( \sigma(x)_i = x_{i+1} \) and is a bijection. Shift spaces are subsets of the full shift which are invariant under \( \sigma \), that is, \( \sigma(X) = X \), and we write \( \sigma_X \) for the restriction to \( X \) of the shift map on \( \mathcal{A}^\mathbb{Z} \).

Example 2.2.1. Let \( \mathcal{A} = \{0, 1\} \). The shift space \( X \) with \( \mathcal{F} = \{11\} \), called the Golden Mean Shift, is the collection of binary sequences in which no two 1s appear together.

A shift of finite type is a shift space for which there is a finite set of forbidden words. We say a shift of finite type has memory \( M \) if there is a set of forbidden words in which every word has length \( M + 1 \). Every shift of finite type has the “Markov” property of finite memory [40, Proposition 2.1.7]. Every finite directed graph \( E \) gives rise to an edge shift \( X_E \) which is the shift space over the alphabet \( \mathcal{A} = E^1 \) consisting of of all bi-infinite sequences of edges:
\[
X_E = \{ (x_i)_{i \in \mathbb{Z}} \in (E^1)^\mathbb{Z} : s(x_i) = r(x_{i+1}) \}.
\]
The edge shift \( X_E \) can be described by the forbidden set \( \mathcal{F} \) consisting of pairs \((e, f)\) of edges \( e, f \in E^1 \) such that \( s(e) \neq r(f) \). Hence \( X_E \) has memory 1 since every word in \( \mathcal{F} \) has length 2.

Example 2.2.2. Let \( E \) be the graph with one vertex and \( n \) self-loops at that vertex. Then the edge shift \( X_E \) is the full shift on \( \mathcal{A} = \{1, \ldots, n\} \).

Every edge shift is a shift of finite type [40, Proposition 2.2.6] and the shift map defines a homeomorphism on the space of two-sided infinite paths of the corresponding directed graph.

Directed graphs are excellent models for representing shifts of finite type because many properties of the shift can be determined by analysing the vertex matrix of the associated directed graph. The vertex matrix of a directed graph \( E \) is the matrix \( A_E \) indexed by \( E^0 \) defined by
\[
A_E(v, w) = \# \{ \lambda \in E^1 : r(\lambda) = v \text{ and } s(\lambda) = w \}.
\]
Conversely, every square matrix \( A \) with nonnegative integer entries has representation as a directed graph \( E_A \): the vertex set of \( E_A \) is the index set \( I \) of \( A \) and edge connectivity is given by drawing \( A(i, j) \) edges from vertex \( j \in I \) to vertex \( i \in I \). Then we have \( A = A_{E_A} \) and \( E \cong E_{A_E} \) (up to naming of vertices) and we use this correspondence to go from a directed graph to a matrix and vice versa.
Example 2.2.3. The graph $E$ below

has vertex matrix

$$A_E = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

One of the reasons why shifts of finite type are so nice to study is because we can use linear algebraic methods on the vertex matrices of their corresponding edge shifts. For example, $A^n_E(v, w)$ is the number of paths of length $n$ in $E$ from $w$ to $v$ [40, Proposition 2.2.12]. If $E$ is row-finite then the sum of entries in each row of $A_E$ is finite hence the name.

A directed graph $E$ is irreducible (also called transitive or strongly connected) if for every $v, w \in E^0$ there exists a path $\lambda \in E^*$ with $r(\lambda) = v$ and $s(\lambda) = w$. Equivalently, $E$ is irreducible if and only if its vertex matrix $A_E$ is an irreducible matrix in the sense that for every ordered pair $v, w \in E^0$ there exists $n \in \mathbb{N}$ such that $A^n_E(v, w) > 0$. The language $L_X$ of a shift space $X$ is the set of all words which do not contain any forbidden word as a subword. A shift of finite type $X \subset A^Z$ is called irreducible if for every ordered pair of words $u, w \in L_X$ there exists $v \in L_X$ such that the concatenation $uvw$ is in $L_X$. These three notions of irreducibility coincide: the edge shift of an irreducible directed graph (with no isolated vertices) is an irreducible shift of finite type [40, Proposition 2.2.14].

Two shift spaces $X$ and $Y$ are conjugate if there is a bijection $\phi : X \to Y$ which commutes with the shift maps ($\phi \circ \sigma_X = \sigma_Y \circ \phi$) and there exists $n \geq 0$ such that $\phi(x)_0$ is a function of $x_{-n}x_{-n+1} \cdots x_n$. Conjugacy preserves properties of the shift space (such as being a shift of finite type [40, Theorem 2.1.10]) and in this way we view conjugate shift spaces as being “the same”. In particular, isomorphic directed graphs yield conjugate edge shifts.

We have seen that the collection of all sequences of edges which form the infinite paths in a directed graph is a shift of finite type. Other shifts of finite type can be specified as collections of bi-infinite sequences of adjacent vertices. If $B$ is a finite $\{0, 1\}$-matrix with index set $A$ (e.g. the vertex matrix of a directed graph in which there is at most one edge between every pair of vertices), then the vertex shift $\hat{X}_B$ is

$$\hat{X}_B = \{(v_i)_{i \in \mathbb{Z}} \in A : B(v_i, v_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\}.$$ 

For example, the vertex shift of the matrix in Example 2.2.3 is the Golden Mean Shift of Example 2.2.1. The vertex shift $\hat{X}_B$ can be described by the forbidden set $F$ consisting of pairs $(u, v) \in A \times A$ such that $B(u, v) = 0$. Hence vertex shifts are shifts of finite type with memory 1.

In probability theory, a Markov chain on a finite set $S$ of states specifies the probability of a state $I$ given a sequence of states from $S$ that have already occurred. Associated to a given Markov chain is a directed graph in which there is at most one edge between every pair of vertices. Thus Markov chains correspond to shifts of finite type having memory 1 which specify allowed sequences of states (vertices) but not their probabilities. Such shift
spaces are the same as vertex shifts, which is why vertex shifts were originally known as topological Markov chains. Every edge shift can be realised as a vertex shift (on a different graph) but not conversely. For example, the Golden Mean Shift of Example 2.2.1 can be recoded as the vertex shift of the graph in Example 2.2.3 but it is not an edge shift \[40\], Example 2.3.8.

There are two matrices associated to every directed graph \( E \): the vertex matrix \( A_E \) (defined in (2.2.1)) and the edge matrix \( B_E \) which is the matrix indexed by \( E^1 \) and defined by

\[
B_E(e, f) = \begin{cases} 
1 & \text{if } s(e) = r(f) \\
0 & \text{otherwise.}
\end{cases}
\]

For example, the edge matrix of the graph in Example 2.2.3 is the matrix

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

In general, \( A_E \) is not a \( \{0,1\} \)-matrix but \( B_E \) is always a \( \{0,1\} \)-matrix by definition. Every \( \{0,1\} \)-matrix arises as the vertex matrix of some directed graph but not all \( \{0,1\} \)-matrices are edge matrices (for example, the matrix \( A_E \) in Example 2.2.3 is not the edge matrix of any directed graph).

Recall from \[54\], page 17, for example, that the dual graph of a directed graph \( E \) is the directed graph \( \hat{E} \) with \( \hat{E}^0 := E^1 \), \( \hat{E}^1 := \{(e, f) \in E^1 \times E^1 : r(f) = s(e)\} \), and range and source maps given by \( r(e, f) = e \) and \( s(e, f) = f \) (so that the vertices of \( \hat{E} \) represent the edges of \( E \)).\(^2\) If \( E \) is a directed graph in which there is at most one edge between every pair of vertices, then the edge shift \( X_{\hat{E}} \) of the dual graph \( \hat{E} \) can be viewed as the vertex shift \( \hat{X}_E \) of \( E \) \[40\], Proposition 2.3.9].

### 2.3. Cuntz-Krieger algebras

In 1980, Cuntz and Krieger introduced a class of \( C^* \)-algebras associated to finite \( \{0,1\} \)-matrices arising from shifts of finite type \[8\]. Let \( A \) be an \( n \times n \) \{0,1\}-matrix. A Cuntz-Krieger \( A \)-family consists of \( n \) partial isometries \( S_i \) on Hilbert space satisfying

\[
1 = \sum_{j=1}^{n} S_j S_j^* \quad \text{and} \quad S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^*.
\]

All such families \( \{S_i\}_{i=1}^{n} \) generate isomorphic \( C^* \)-algebras provided that the matrix \( A \) satisfies a certain Condition (I) introduced by Cuntz and Krieger in \[8\] (this result is now known as the Cuntz-Krieger Uniqueness Theorem). The universal \( C^* \)-algebra generated by a family \( \{S_i\}_{i=1}^{n} \) satisfying these conditions is called the Cuntz-Krieger algebra \( O_A \). Under a stronger condition on \( A \), Cuntz classified the ideals of \( O_A \) in \[6\].

As in Section 2.2, the matrix \( A \) has a natural representation as a directed graph and so \( O_A \) can be described in graph-theoretical terms. Enomoto and Watatani reformulated

\(^2\)Note that \( \hat{E} \) is isomorphic to the directed graph with vertex matrix \( B_E \) and so up to a permutation of the index sets \( B_E \) and \( A_{\hat{E}} \) are the same matrix.
Cuntz and Krieger’s matrix Condition (I) in terms of the associated directed graph in [15]. The directed graph $G$ with vertex matrix $A$ satisfies Condition (I) if $G$ has no sources and if every vertex in $G$ lies on two different cycles [15]. In particular, Condition (I) is satisfied if the directed graph $G$ is irreducible.

### 2.4. $C^*$-algebras of row-finite directed graphs

If $A$ is an infinite matrix the relations in (2.3.1) do not make sense since in a $C^*$-algebra infinite sums of projections cannot converge in norm. This problem is avoided by dropping the first condition and assuming that $A$ is row-finite (if there are finitely many 1s in each row then the sum in (2.3.1) has only finitely many nonzero terms). Cuntz-Krieger algebras associated to row-finite matrices $A$ were studied in [38, 37, 2] and a different generalisation of Cuntz-Krieger algebras to infinite $\{0,1\}$-matrices with no zero rows appeared in [18].

In [38, 37], the matrix $A$ arises as the edge matrix of a row-finite directed graph $E$. The uniqueness theorem of [8] and Cuntz’s ideal theory of [6] were generalised in [38] by realising $O_A$ as the $C^*$-algebra of a groupoid arising from $E$. A Cuntz-Krieger $E$-Family $\{S, P\}$ on a Hilbert space $\mathcal{H}$ consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$ which satisfy the following Cuntz-Krieger relations:

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$,

(CK2) $P_v = \sum_{r(e) = v} S_e S_e^*$ whenever $r(v) \neq \emptyset$.

The $C^*$-algebra of $E$ is the $C^*$-algebra $C^*(E)$ which is generated by the family $\{S, P\}$. The $C^*$-algebra $C^*(E)$ is unital if and only if $E^0$ is finite [37, Proposition 1.4], and the identity is $\sum_{v \in E^0} P_v$. (Note that $E$ is allowed to have sources in the above definition of $C^*(E)$, which is from [37]; the definition in [38] only applies to graphs with no sources.)

If every cycle in $E$ has an entry,\(^3\) then we have the Cuntz-Krieger Uniqueness Theorem which says that all Cuntz-Krieger $E$-families generate isomorphic $C^*$-algebras: if $\{S, P\}$ and $\{T, Q\}$ are Cuntz-Krieger $E$-families such that $P_v, Q_v \neq 0$ for all $v \in E^0$, then there is an isomorphism $\phi$ from $C^*(S, P)$ onto $C^*(T, Q)$ satisfying $\phi(P_v) = Q_v$ for all $v \in E^0$ and $\phi(S_e) = T_e$ for all $e \in E^1$ [2, Theorem 3.1]. (This theorem was proved for graphs with no sources previously in [37, Theorem 3.7]. The proofs of the uniqueness theorem and other major results in [2] are more direct than those in [37] and they apply to graphs with sources. The extra generality is important because the Cuntz-Krieger algebras of arbitrary infinite graphs can be approximated by the $C^*$-algebras of finite graphs with sources [58].)

If $E$ is row-finite then the dual graph $\hat{E}$ is row-finite and $C^*(\hat{E}) \cong C^*(E)$ [54, Corollary 2.6]. Then $C^*(\hat{E}) \cong O_{B_\partial}$ implies that the Cuntz-Krieger algebras are the $C^*$-algebras of finite directed graphs with no sinks or sources. These facts were first proved by Enotomo and Watatani in [16, 25]. The Cuntz-Krieger families of a vertex matrix correspond

\(^3\)Recall that by our convention the term “sources” and the condition that “every cycle has an entry” correspond to “sinks” and “every cycle has an exit” in [37].
to the Cuntz-Krieger families of the accompanying edge matrix: if $A_E$ is a $\{0, 1\}$-matrix satisfying Condition (I), then $B_E$ satisfies Condition (I) and $C^*(E) \cong \mathcal{O}_{A_E} \cong \mathcal{O}_{B_E}$ [43, Proposition 4.1].

In [18], Exel and Laca defined the Cuntz-Krieger algebra $\mathcal{O}_A$ of an infinite $\{0, 1\}$-matrix $A$ with no zero rows and generalised the uniqueness theorem and the simplicity criteria of [8] to the infinite case. Exel-Laca algebras include graph algebras but there are matrices $A$ for which $\mathcal{O}_A$ is not a graph algebra [58, Remark 4.4]. In [21], Fowler, Laca, and Raeburn used the results of [18] to define the $C^*$-algebras of infinite directed graphs which are not necessarily row-finite. Drinen and Tomforde then proved the uniqueness theorem and results about simplicity and pure infiniteness for the $C^*$-algebras of arbitrary directed graphs in [14].

2.5. Properties of graph algebras

There are four equivalent definitions of what it means for a $C^*$-algebra to be nuclear - all of which are quite complicated (see [3, Theorem 5.8.1] and [63, Definition 2.1.1]). The class of nuclear $C^*$-algebras includes many familiar $C^*$-algebras (for example, all commutative $C^*$-algebras, finite-dimensional $C^*$-algebras, the group algebras of locally compact amenable\(^4\) groups) and is closed under taking operations such as stable isomorphism, quotients, tensor products and crossed products by amenable groups (see [3, Theorem 5.8.2] or [63, Proposition 2.1.2]). Graph algebras are always nuclear and they fall into the bootstrap class $\mathcal{N}$ of Rosenberg and Schochet [65, page 439] for which the Universal Coefficient Theorem (UCT) applies [63, Theorem 2.4.6].

A graph algebra is simple if its only ideals are the trivial ones. Simplicity of $C^*(E)$ can be easily determined from the graph $E$. A row-finite directed graph $E$ with no sources is cofinal if for every $\lambda \in E^\infty$ and every $v \in E^0$ there exists a path $\mu \in E^*$ with $r(\mu) = v$ and $s(\mu) = w$ for some $w$ on $\lambda$. Then $C^*(E)$ is simple if and only if $E$ is cofinal and satisfies Condition (L): every cycle in $E$ has an entry [37, Corollary 3.11]. Condition (L) was introduced in [37] as the graph-theoretic version of Condition (I) for infinite matrices and graphs: when $E^0$ is finite Condition (L) is equivalent to Condition (I) [37, Lemma 3.3].

The property of a $C^*$-algebra $A$ being purely infinite can be described in terms of projections in $A$. If $A$ is a simple algebra, then there are six equivalent conditions for $A$ to be purely infinite given in [63, Proposition 4.1.1]. We can easily establish if a graph algebra is purely infinite from the graph: $C^*(E)$ is purely infinite if and only if $E$ satisfies Condition (L) and for all $v \in E^0$ there exists a cycle $\mu$ and a path $\lambda \in E^*$ with $s(\lambda) = r(\mu)$ and $r(\lambda) = v$ [37, Theorem 3.9]. Indeed, since $C^*(E)$ is AF (Approximately-finite dimensional) if and only if $E$ has no cycles [37, Theorem 2.4], there is a dichotomy for simple graph algebras: they are either AF or purely infinite [37, Corollary 3.11].

A Kirchberg algebra is a purely infinite, simple, nuclear, separable $C^*$-algebra. From results in [38, 37], a graph algebra $C^*(E)$ is a Kirchberg algebra if the following conditions hold: $s^{-1}(v)$ and $r^{-1}(v)$ are finite for all $v \in E^0$, $E$ is cofinal and has no sources, and $E$ is not a graph algebra [18, Remark 4.4].

\(^4\)A discrete group $G$ is amenable if there is a left-invariant mean on $l^\infty(G)$ [74, Appendix A.2].
2.6. K-theory of graph algebras

has at least one cycle and every cycle in \( E \) has an entry. For example, the algebras \( \mathcal{O}_n \) are Kirchberg algebras.

Example 2.5.1. (The Cuntz algebras \( \mathcal{O}_n \)) In 1977, Cuntz introduced a class of simple \( C^* \)-algebras generated by isometries which are now known as Cuntz algebras \([7]\). Let \( n \geq 2 \) be an integer and let \( \{S_i\}_{i=1}^n \) be a sequence of isometries (that is, \( S_i^*S_i = 1 \)) on a Hilbert space \( \mathcal{H} \) such that \( \sum_{i=1}^n S_iS_i^* = 1 \). Consider the \( C^* \)-algebra \( C^*(S_1, \ldots, S_n) \) generated by the family of isometries \( \{S_i\}_{i=1}^n \). If \( \{T_i\}_{i=1}^n \) is a second family of isometries satisfying \( \sum_{i=1}^n T_iT_i^* = 1 \), then \( C^*(T_1, \ldots, T_n) \) is canonically isomorphic to \( C^*(S_1, \ldots, S_n) \) \([7\), Theorem 1.12\]. Hence we write \( \mathcal{O}_n \) for \( C^*(S_1, \ldots, S_n) \) since the isomorphism class of \( \mathcal{O}_n \) does not depend on the choice of \( \{S_i\}_{i=1}^n \). Cuntz showed that \( \mathcal{O}_n \) is nuclear by representing it as a crossed product and simplicity of \( \mathcal{O}_n \) follows from \([7\), Theorem 1.12\]. (See \([63\), Theorem 4.2.2\] for a proof that \( \mathcal{O}_n \) is purely infinite.) The Cuntz-Krieger algebras \( \mathcal{O}_A \) of \([8]\) are generalised versions of the Cuntz algebras \( \mathcal{O}_n \).

As shown in \([54\), Example 2.16\], \( \mathcal{O}_n \) is a graph algebra. Let \( n \geq 2 \) and suppose that \( E \) is the directed graph consisting of one vertex \( v \) and \( n \) self-loops at \( v \) labelled 1, \ldots, \( n \) (see also Example 2.2.2). Let \( S = \{S_1, \ldots, S_n\} \) be a family of partial isometries on a Hilbert space \( \mathcal{H} \) and let \( P = \{P_v\} \) consist of an orthogonal projection on \( \mathcal{H} \) so that \( \{S, P\} \) is a Cuntz-Krieger \( E \)-family. Since \( P_v = 1 \) is the identity for \( C^*(E) \), the relation (CK1) implies that each \( S_i \) is an isometry and the relation (CK2) implies that \( \sum_{i=1}^n S_iS_i^* = 1 \). Each self-loop is a cycle with an entry (any other self-loop) and so the Cuntz-Krieger Uniqueness Theorem applies. Since the family \( S \) of isometries generates the Cuntz algebra \( \mathcal{O}_n \), we have \( C^*(E) \cong \mathcal{O}_n \).

2.6. K-theory of graph algebras

The \( K \)-theory of a \( C^* \)-algebra \( A \) is a pair of abelian groups containing information about \( A \). The \( K_0 \)-group of a unital \( C^* \)-algebra \( A \) is defined in terms of equivalence classes of projections in \( A \) as follows \([64\), §2-3\]. Let \( P_n(A) = P(M_n(A)) \) be the projections in the algebra of \( n \times n \) matrices having entries in \( A \) and let \( P_\infty = \bigcup_{n=1}^\infty P_n(A) \). There is an equivalence relation \( \sim_0 \) on \( P_\infty(A) \) defined for \( p \in P_n(A) \) and \( q \in P_m(A) \) by

\[
p \sim_0 q \text{ if there exists } v \in M_{m,n}(A) \text{ with } p = v^*v \text{ and } q = vv^*,
\]

and we define a binary operation \( \oplus \) on \( P_\infty(A) \) by

\[
(2.6.1) \quad p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.
\]

Then \( D(A) := P_\infty(A)/\sim_0 \) is an abelian semigroup with addition

\[
[p] + [q] = [p \oplus q],
\]

and \( K_0(A) \) is defined to be the Grothendieck group of \( D(A) \) (the Grothendieck construction produces an abelian group from an abelian semigroup in the same way as \( \mathbb{Z} \) is constructed from \( \mathbb{N} \)).
2. DIRECTED GRAPHS AND THEIR $C^*$-ALGEBRAS

The $K_1$-group is defined in terms of equivalence classes of unitary elements in $A$ in the following way [64, §8.1]. Let $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ be the unitary elements in the matrix algebra $M_n(A)$ and let $\mathcal{U}_\infty = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$. Then $K_1(A)$ is defined to be the set of homotopy equivalence classes of $\mathcal{U}_\infty(A)$, and $K_1(A)$ is an abelian group with addition defined by

$$[u] + [v] = [u \oplus v],$$

where $\oplus$ is a binary operation defined on $\mathcal{U}_\infty(A)$ similarly to (2.6.1).

The $K$-theory of graph algebras is described in [54, Chapter 7]; for example if $E$ is a row-finite directed graph with no sources then $K_0(C^*(E))$ is generated by $\{[p_v] : v \in E^0\}$ subject to the relation

$$[p_v] = \sum_{r(e)=v} [p_{s(e)}]$$

imposed by the Cuntz-Krieger relations. Then [54, Theorem 7.16] implies that $K_0(C^*(E))$ and $K_1(C^*(E))$ are the kernel and cokernel of the map $1 - A_E^t : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$.

The Cuntz algebras $\mathcal{O}_n$ were the first examples of $C^*$-algebras whose $K$-theory contains torsion. Cuntz showed in [5] that $K_0(\mathcal{O}_n)$ is generated by the class of the identity $p_v = 1$. The $K$-theory of $\mathcal{O}_n$ is given in [64, page 234] by

$$K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z} \text{ and } K_1(\mathcal{O}_n) = 0.$$

In [6, Proposition 3.1], Cuntz calculated that the $K$-theory of the Cuntz-Krieger algebra $\mathcal{O}_A$ is

$$K_0(\mathcal{O}_A) = \text{coker}(1 - A_E^t) \text{ and } K_1(\mathcal{O}_A) = \ker(1 - A^t).$$

When $A = A_E$ arises as the vertex matrix of a directed graph $E$, which has accompanying edge matrix $B_E$, then the maps $1 - A_E^t$ and $1 - B_E^t$ have the same kernel and cokernel [43, Proposition 4.1] and so either of the matrices $A_E$ or $B_E$ may be used for $K$-theory calculations of $C^*(E)$. (This observation will be useful in our $K$-theory calculations in Section 4.7 in Chapter 4.)

---

5A group $G$ is said to contain torsion if it has an nonzero element of finite order. Otherwise $G$ is called torsion-free.
CHAPTER 3

Higher-rank graphs and their $C^*$-algebras

3.1. Robertson-Steger algebras

In [61], Robertson and Steger constructed as the higher-dimensional analogue of a Cuntz-Krieger algebra a $C^*$-algebra $\mathcal{A}$ associated to an $r$-dimensional shift of finite type. The construction starts with a set of $r$-dimensional words over an alphabet and defines a family $\{M_1, \ldots, M_r\}$ of commuting $\{0,1\}$-matrices satisfying certain conditions. The unique $C^*$-algebra $\mathcal{A}$ is generated by a family of partial isometries indexed by compatible $r$-dimensional words satisfying some relations. The $C^*$-algebra $\mathcal{A}$ is a direct generalisation of a Cuntz-Krieger algebra (if $r = 1$, then $\mathcal{A}$ is a Cuntz-Krieger algebra). Under conditions which they label (H0)–(H3), the authors showed that the $C^*$-algebra $\mathcal{A}$ is simple [61, Theorem 5.9] and purely infinite [61, Proposition 5.11]. By [61, Corollary 6.3], $\mathcal{A}$ is stably isomorphic to the crossed product of an AF-algebra by a $\mathbb{Z}^r$-action, and it follows that $\mathcal{A}$ belongs to the bootstrap class $\mathcal{N}$ [61, Remark 6.5] and so $\mathcal{A}$ is nuclear [61, Corollary 6.4]. Thus the $C^*$-algebras of Robertson and Steger are classified by the theorem of Kirchberg and Phillips [29, 50].

3.2. Higher-rank graphs

In [36], Kumjian and Pask introduced the concept of a higher-rank graph and showed how to associate a $C^*$-algebra to it. There are two motivations for the form of the construction in [36]: firstly, to build upon the higher-dimensional Cuntz-Krieger algebras of Robertson and Steger, and secondly, to generalise the construction of the $C^*$-algebra of a directed graph [8, 38, 37] to the higher-dimensional setting. Sufficient conditions on the higher-rank graph are given in [36] for the associated $C^*$-algebra to be simple and purely infinite (for further discussion of these results see Section 3.6). The definition of a higher-rank graph is modelled on the path category of a directed graph. The following definitions and notations are mostly from [36].

Notation 3.2.1. Let $\mathbb{N} = \{0,1,2,3,\ldots\}$ denote the monoid of natural numbers under addition and let $\mathbb{Z}$ be the group of integers. For $k \geq 1$, we view $\mathbb{N}^k$ as the set of morphisms in a category with one object and composition given by addition. We write $n_i$ for the $i$th coordinate of $n \in \mathbb{Z}^k$, and $\{e_i\}$ for the usual basis of $\mathbb{Z}^k$. For $m, n \in \mathbb{Z}^k$ we say $m \leq n$ if $m_i \leq n_i$ for each $i$, and write $m \vee n$ and $m \wedge n$ for the coordinate-wise maximum and minimum.

Definition 3.2.2. A $k$-graph is a pair $(\Lambda, d)$ consisting of a countable category $\Lambda$ and a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, satisfying the factorisation property: for

\footnote{We discuss conditions (H0)–(H3) in the context of our construction later in Remark 4.4.5.}
every \( \lambda \in \Lambda \) and \( m, n \in \mathbb{N}^k \) with \( d(\lambda) = m + n \), there exist unique elements \( \mu, \nu \in \Lambda \) such that \( d(\mu) = m \), \( d(\nu) = n \) and \( \lambda = \mu \nu \). A \( k \)-graph is also called a higher-rank graph or a graph of rank \( k \).

In practice, we drop the degree map from the notation and write \( \Lambda \) for \((\Lambda, d)\). When \( \Lambda \) is a \( k \)-graph, the morphisms in the category are called paths, the objects are called vertices, and the codomain and domain maps are called the range map \( r \) and source map \( s \). Recall that a category has an associative composition which is written right-to-left, and so \( \lambda \) and \( \mu \) in \( \Lambda \) are composable if \( s(\lambda) = r(\mu) \) and we write \( \lambda \mu := \lambda \circ \mu \) for the composition. It follows from the category axioms that \( r(\lambda \mu) = r(\lambda) \) and \( s(\lambda \mu) = s(\mu) \).

If \( \lambda \in \Lambda \) has \( d(\lambda) = n \) we say \( \lambda \) has degree \( n \); define \( \Lambda^n \) to be the set \( \{\lambda \in \Lambda : d(\lambda) = n\} \) and let \( \Lambda^* := \bigcup_{n \geq 0} \Lambda^n \). If \( E \subset \Lambda \) and \( \lambda, \mu \in \Lambda \) then we write \( \lambda E \) for the set \( \{\lambda \mu : \mu \in E : s(\lambda) = r(\mu)\} \), and similarly \( E \lambda = \{\mu \lambda : \mu \in E : s(\mu) = r(\lambda)\} \). For example, \( v\Lambda \) is the set of paths with range \( v \), \( \Lambda v \) is the set of paths with source \( v \), and \( v\Lambda w := v\Lambda \cap \Lambda w \) is the set of paths from \( w \) to \( v \). We say \( \Lambda \) is row-finite if each \( v\Lambda^n \) is a finite set; \( \Lambda \) is finite if \( \Lambda \) is a finite set. We say \( \Lambda \) has no sources if each \( v\Lambda^n \) is a nonempty set. Note that all \( k \)-graphs in [36] are row-finite and have no sources, whereas the \( k \)-graphs studied in [56] are allowed to have sources.

**Remark 3.2.3.** The factorisation property has several consequences. First, it implies that for each vertex \( v \) there is only one path of degree 0 from \( v \) to \( v \). To see this, let \( v \) be a vertex in a 2-graph \( \Lambda \). Since \( v \) is an object in the category \( \Lambda \), there is an identity morphism \( \iota_v \in \Lambda^* \) such that \( r(\iota_v) = v = s(\iota_v) \) and \( \iota_v \lambda = \lambda \) for \( \lambda \in v\Lambda \) and \( \mu \iota_v = \mu \) for \( \mu \in \Lambda v \). Then \( d(\lambda) = d(\iota_v \lambda) = d(\iota_v) + d(\lambda) \) implies \( d(\iota_v) = 0 \). Suppose that \( \kappa_v \) is another morphism with these properties. Then \( \iota_v \lambda = \kappa_v \lambda \) implies \( \iota_v = \kappa_v \) since the factorisation property implies cancellation [36, Remark 2.1]. Thus there is only one identity morphism for each object and this allows us to identify the objects of the category \( \Lambda \) with the morphisms of degree 0. Under the identification of \( \text{Obj}(\Lambda) \) with \( \Lambda^0 \) we view \( r \) and \( s \) as maps from \( \Lambda \) to \( \Lambda^0 \), and then \( r(v) = v = s(v) \) for \( v \in \Lambda^0 \).

Second, the factorisation property implies that for every triple \( m, n, p \in \mathbb{N}^k \) satisfying \( 0 \leq m \leq n \leq p \) and \( \lambda \in \Lambda^p \), there are unique segments \( \lambda(0, m) \in \Lambda^m \), \( \lambda(m, n) \in \Lambda^{n-m} \), \( \lambda(n, p) \in \Lambda^{p-n} \) such that \( \lambda = \lambda(0, m) \lambda(m, n) \lambda(n, p) \). The paths \( \lambda(m, m) \) have degree 0, and hence are vertices; in the literature it is common to write \( \lambda(m) := \lambda(m, m) \), but we will refrain from doing this as in our 2-graphs \( \lambda(m) \) will have another more natural meaning.

**Examples 3.2.4.** ([36, Examples 1.7]) (1) Let \( E^* \) be the path category of a directed graph \( E \) and let \( d \) be the length function \( d : E^* \to \mathbb{N} \) defined by \( d(\lambda) = n \) if \( \lambda \in E^n \). Then \( (E^*, d) \) is a 1-graph, and by identifying a directed graph with its path category the definition of higher-rank graphs reduces to that of directed graphs when \( k = 1 \).

(2) Consider the category \( \Omega_k \) with object set \( \mathbb{N}^k \), morphism set \( \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\} \), and maps \( r(m, n) = m \), \( s(m, n) = n \) (note that the composition of \( (m, n) \) and \( (n, p) \) is \( (m, p) \)). Then \( (\Omega_k, d) \) is a \( k \)-graph with \( d : \Omega_k \to \mathbb{N}^k \) defined by \( d(m, n) = n - m \).
3.3. 2-graphs

Since we are primarily interested in 2-graphs, we will describe the implications of the factorisation property specifically for the $k=2$ case and illustrate how 2-graphs and their paths may be visualised. The construction of [36, §6] shows how a 2-graph can be viewed as a bicoloured directed graph together with a set of relations specifying the factorisation property, and certain conditions ensure that there is exactly one way this association can be made. We will use this procedure in constructing our 2-graphs in Chapter 4. Our construction yields only one choice for the factorisations, as compared to [9, 10, 11] in which there is a choice of factorisations.

By a bicoloured directed graph we mean a directed graph in which the edges are assigned one of two colours (usually blue and red). Such a graph $S$ can be formed from two directed graphs $E$ and $F$ with $S^0 = E^0 = F^0$ by setting $S^1 = E^1 \sqcup F^1$ and colouring the edges from $E$ blue and the colouring the edges from $F$ red (and $S$ inherits range and source maps from $E$ and $F$). The vertex matrices of $S$ are the vertex matrices of $E$ and $F$.

3.3.1. Skeletons and factorisation. We visualise a 2-graph $\Lambda$ by drawing its skeleton, which is the bicoloured directed graph with vertex set $\Lambda^0$ in which the elements of $\Lambda^e_1$ are represented by blue edges from $s(\beta) \in \Lambda^0$ to $r(\beta) \in \Lambda^0$, and the elements of $\Lambda^e_2$ by red edges. (In print we use black curves to represent blue edges and dashed curves for red edges.) The skeleton of a 2-graph $\Lambda$ yields two component graphs with vertices $\Lambda^0$ called the blue graph $B\Lambda$ (consisting of edges $\Lambda^e_1$) and the red graph $R\Lambda$ (consisting of edges $\Lambda^e_2$). The vertex matrices of $\Lambda$ are the vertex matrices of the component graphs of the skeleton.

Example 3.3.1. The skeleton of $\Omega_2$ is

A higher-rank graph $\Lambda$ is row-finite if and only if its skeleton is row-finite and $\Lambda$ has no sources if and only if its skeleton has no sources. In general, a 2-graph is not uniquely determined by its skeleton (see [56, §2] and [54, Example 10.10]), and so different 2-graphs can have the same skeleton. The extra information needed to determine a 2-graph from a given skeleton is the factorisation property.

A path $\lambda$ of degree $(1,1)$ in a 2-graph $\Lambda$ is called a commuting square. The factorisation property implies that since $(1,1) = e_1 + e_2 = e_2 + e_1$, there exist blue edges $\beta, \beta' \in \Lambda^e_1$ and red edges $\rho, \rho' \in \Lambda^e_2$ such that $s(\beta) = r(\rho)$ and $s(\rho') = r(\beta')$ and $\lambda$ factorises as
\( \lambda = \beta \rho = \rho' \beta' \). We visualise \( \lambda \) as the square

\[
\begin{array}{c|c|c}
\beta' & \downarrow & \rho' \\
\hline
\rho & \downarrow & \rho \\
\hline
\beta & \downarrow & \beta \\
\end{array}
\]

in which blue edges are solid lines and red edges are dashed lines. It follows that for every pair consisting of \( \beta \in \Lambda^{e_1} \) and \( \rho \in \Lambda^{e_2} \) with \( s(\beta) = r(\rho) \) there exists a pair consisting of \( \beta' \in \Lambda^{e_1} \) and \( \rho' \in \Lambda^{e_2} \) such that \( s(\rho') = r(\beta') \), \( r(\beta) = r(\rho') \) and \( s(\rho) = s(\beta') \).

In other words, there is a bijection between the blue-red paths of length 2 and the red-blue paths of length 2 in the skeleton of \( \Lambda \) which preserves the range and the source. In fact, a 2-graph is completely determined by a collection \( C \) of such commuting squares (3.3.1) in which each blue-red path and each red-blue path occurs exactly once [36, §6]. Therefore, an isomorphism of 2-graphs is induced by a bijection between their skeletons which preserves colour and the commuting squares.

A 2-graph can be constructed from a pair of coloured graphs \( E \) and \( F \) with common vertex set if their vertex matrices commute [36, §6]. (This assertion is equivalent to that in [36, §6] because directed graphs are in one-to-one correspondence with 1-graphs, see Example 3.2.4). Let \( E \) and \( F \) be directed graphs with \( E^0 = F^0 \) whose vertex matrices commute. We will define a 2-graph \( \Lambda \) with \( \Lambda^0 = E^0 = F^0, \Lambda^{e_1} = E^1 \) and \( \Lambda^{e_2} = F^1 \). Let \( u, v \) be vertices and consider the sets

\[
\Lambda_{EF}(u,v) := \{ (\beta, \rho) \in E^1 \times F^1 : s(\beta) = r(\rho), r(\beta) = u, s(\rho) = v \}
\]

\[
\Lambda_{FE}(u,v) := \{ (\rho, \beta) \in F^1 \times E^1 : s(\rho) = r(\beta), r(\rho) = u, s(\beta) = v \}.
\]

Since the vertex matrices commute, the sets \( \Lambda_{EF}(u,v) \) and \( \Lambda_{FE}(u,v) \) have the same cardinality for every pair of vertices. So there is a bijection which takes each element of \( \Lambda_{EF} \) to an element of \( \Lambda_{FE} \), and we extend this to a bijection \( \theta \)

\[
\theta : \{ (\beta, \rho) \in E^1 \times F^1 : s(\beta) = r(\rho) \} \to \{ (\rho, \beta) \in F^1 \times E^1 : s(\rho) = r(\beta) \}
\]

which takes \( (\beta, \rho) \) to \( \theta(\beta, \rho) = (\rho', \beta') \) with \( r(\beta) = r(\rho') \) and \( s(\rho) = s(\beta') \). Thus \( \theta \) relates the blue-red paths of length 2 with the red-blue paths of length 2. Because of the relations \( (\beta, \rho) = \theta(\beta, \rho) \) we identify \( \Lambda_{EF}(u,v) \) with \( \Lambda_{FE}(u,v) \) and define

\[
\Lambda^{(1,1)} := \bigcup_{u,v \in \Lambda^0} \Lambda_{EF}(u,v) = \bigcup_{u,v \in \Lambda^0} \Lambda_{FE}(u,v).
\]

(Here \( C = \Lambda^{(1,1)} \) is the collection of commuting squares in which \( \beta \rho \) factorises as \( \rho' \beta' \) where \( \theta(\beta, \rho) = (\rho', \beta') \). It is shown in [36, pages 17-18] that there is a unique 2-graph \( \Lambda \) such that \( \Lambda^{(1,1)} = \bigcup \Lambda_{EF} = \bigcup \Lambda_{FE} \) and with components \( B\Lambda = E \) and \( R\Lambda = F \). So \( \Lambda \) is the 2-graph determined by \( E, F \) and \( \theta \).

Moreover, every 2-graph can be viewed in this way: if \( \Lambda \) is a 2-graph then the vertex matrices of the component graphs \( B\Lambda \) and \( R\Lambda \) commute, and \( \Lambda \) can be recovered (up to a choice of isomorphism \( \theta \)) via the above construction in terms of \( B\Lambda \) and \( R\Lambda \). For example,
the rank-2 Bratteli diagrams of [45] are constructed from a Bratteli diagram as the blue graph and a red graph consisting of disjoint cycles.

3.3.2. Construction of a 2-graph from a bicoloured directed graph. Suppose that we have a bicoloured directed graph $S$ for which the vertex matrices $B$ and $R$ commute ($BR = RB$). As shown in [36, §6], there exists a collection $C$ of commuting squares in which each blue-red path and each red-blue path occurs exactly once. Here we will describe visually how $C$ determines a set of paths which has the factorisation property. We also give the condition on $S$ which ensures there is only one choice for $C$ which gives a 2-graph.

Just as $P_n$ is the prototype for a path of length $n \in \mathbb{N}$ in a directed graph, a path of degree $m \in \mathbb{N}_2$ in a 2-graph $\Lambda$ is modelled upon the path $m = (m_1, m_2)$ in $\Omega_2$ by labelling the skeleton of $\Omega_2$ with compatible vertices and edges from $\Lambda$ so that colour is preserved. For example, a path of degree $(3, 2)$ from $u$ to $v$ in which each square (such as $ef = gh$) is in the collection $C$ and all the squares fit together compatibly may be drawn as

So the collection $C$ determines a set of paths. Composition of paths involves taking the complex hull, and the set of paths has the factorisation property if there is a unique path composition. For example, if $\lambda$ is a path of degree $(1, 1)$ with factorisation $ef = gh$ and $\mu$ is a path of degree $(1, 2)$ with $s(\lambda) = r(\mu)$, then $\lambda\mu$ is obtained by filling the missing areas in the diagram

with squares in $C$. There is only one square in $C$ in which $fi$ appears which completes the top left-hand area. Similarly, the bottom right-hand area is completed uniquely by filling in the squares in the appropriate order. So there is a unique path $\lambda\mu$ which is the composition of $\lambda$ and $\mu$. An inductive argument can be used to show that the composition of arbitrary composable paths is unique. Thus the factorisation property holds for the set of paths determined by the collection $C$, and so $C$ determines a 2-graph whose skeleton is $S$.

For the analogous construction to work for $k$-graphs when $k \geq 3$, there is an extra associativity condition which has to be imposed on the collection $C$ [22, Remark 2.3].

In general, when $S$ is a bicoloured directed graph whose vertex matrices $B$ and $R$ commute, there are many choices for the collection $C$. Each of these choices determines a
(possibly different) 2-graph with skeleton $S$. However, if $B$ and $R$ are \{0, 1\}-matrices and $BR = RB$ is also a \{0, 1\}-matrix, then there is exactly one collection $C$ determining a set of paths with the factorisation property and so there is a unique 2-graph $\Lambda_S$ with skeleton $S$ (cf. Lemma 1.1 and Lemma 1.4 of [61] which together say that these matrix conditions are equivalent to the existence of a unique path composition).

The above process of associating a 2-graph to a bicoloured graph and a collection of paths is exactly what we will use to construct our family of 2-graphs in Chapter 4. In our situation, the bicoloured directed graph is formed from a given 2-dimensional shift space satisfying certain hypotheses and there is exactly one choice of a collection of paths for which the factorisation property holds. Hence there is exactly one 2-graph which is associated in this way to each shift space. (All our graphs have at most one edge of each colour between any two vertices, and so their vertex matrices are \{0, 1\}-matrices; see Proposition 4.1.8. It is a consequence of the hypotheses on the shift space that the vertex matrices commute and the product is a \{0, 1\}-matrix; see Remark 4.4.5.)

3.3.3. Choices for the factorisation property. It is intrinsic in our construction that there is exactly one 2-graph associated to a given skeleton. Different 2-graphs having the same skeleton were studied by Davidson, Power, and Yang [9, 10, 11].

In [33], Kribs and Power introduced a class of operator algebras, called free semigroupoid algebras $\mathfrak{L}_G$, generated by families of partial isometries and projections arising from directed graphs. In particular the class includes the analytic Toeplitz algebra $H^\infty$ which is realised as the algebra $\mathfrak{L}_{C_1}$ of the directed graph $C_1$ with a single vertex and one cycle. Further examples include the noncommutative analytic Toeplitz algebras $\mathfrak{L}_n$ which are realised as the algebras $\mathfrak{L}_{F_n}$, where $F_n$ is the directed graph with a single vertex and $n$ cycles ($n \geq 2$). The main result is that $\mathfrak{L}_G$ is unitarily equivalent to $\mathfrak{L}_{G'}$ if and only if the directed graphs $G$ and $G'$ are isomorphic [33, Theorem 9.1].

Kribs and Power generalised their algebras $\mathfrak{L}_G$ to the setting of higher-rank graphs in [34] and defined the semigroupoid algebra $\mathfrak{L}_\Lambda$, which is generated by the family $\{S_\lambda : \lambda \in \Lambda\}$. Their elementary example is the 2-graph $\Lambda$ with $\Lambda^0 = \{v\}$, $\Lambda^e_1 = \{\beta\}$ and $\Lambda^e_2 = \{\rho\}$ for which $\mathfrak{L}_\Lambda$ is unitarily equivalent to $H^\infty \otimes H^\infty$ [34, Example 4.1]. The skeleton of $\Lambda$ is the bicoloured directed graph with a single vertex, one blue cycle and one red cycle, and the factorisation property imposes a single relation, $\beta \rho = \rho \beta$, which generates all the paths. For more general 2-graphs, the relations on a bicoloured directed graph are encoded by a permutation $\theta$ which is built in to ensure the factorisation property. In [53], Power focused on $k$-graphs with a single vertex and classifies their algebras $\mathfrak{L}_\Lambda$, which are a higher-rank version of the analytic Toeplitz algebras $\mathfrak{L}_n$. The representation theory and a characterisation of periodicity for single vertex algebras $\mathfrak{L}_\Lambda$ was investigated in the rank 2 case in [9, 10, 11], and the results were extended to the general case in [12].

Examples show that in general the factorisation property affects the structure of the $C^\ast$-algebra, for example, in [11, Example 4.3] and [45, Example 6.5] we need to choose the right factorisation property to ensure the $C^\ast$-algebra is simple. This is not an issue in our work because there is only one choice for the factorisation property (see Proposition 4.1.4).
3.3.4. Examples.

Example 3.3.2. In this example we describe a bicoloured directed graph for which there is exactly one associated 2-graph. Let $E$ and $F$ be the directed graphs with $E^0 = F^0 = \{u, v\}$, $E^1 = \{\beta_1, \beta_2\}$, $F^1 = \{\rho_1, \rho_2, \rho_3, \rho_4\}$ and connectivity given by the matrices

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$ 

The vertex matrices commute and the bicoloured directed graph with components $E$ and $F$ is

![Diagram](image)

The collection $C$ of commuting squares

![Diagram](image)

which encodes the factorisation relations

$$\beta_1 \rho_1 = \rho_1 \beta_1, \quad \beta_1 \rho_3 = \rho_3 \beta_2, \quad \beta_2 \rho_2 = \rho_2 \beta_2, \quad \beta_2 \rho_4 = \rho_4 \beta_1,$$

determines a 2-graph with skeleton $S$. The collection $C$ above is the only one determining a 2-graph associated to $S$ since $BR = RB = R$ is a $\{0,1\}$-matrix. (This graph is an example of the periodic graphs we consider in Chapter 6.)

Example 3.3.3. There is more than one way to associate a 2-graph to the bicoloured directed graph below.

![Diagram](image)

We will find all collections of commuting squares which determine a 2-graph. The two commuting squares with range $v$

![Diagram](image)
must be included in any collection since they are the only squares containing the pairs \( \beta_2\rho_2, \beta_2\rho_4 \) and the pairs \( \rho_2\beta_2, \rho_4\beta_1 \). The nine other commuting squares with range \( u \) are

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\rho_3 & \rho_3 & \rho_3 \\
\beta_1 & \beta_1 & \beta_1
\end{array}
\quad
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\rho_3 & \rho_3 & \rho_3 \\
\beta_1 & \beta_1 & \beta_1
\end{array}
\quad
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\rho_3 & \rho_3 & \rho_3 \\
\beta_1 & \beta_1 & \beta_1
\end{array}
\quad
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\rho_3 & \rho_3 & \rho_3 \\
\beta_1 & \beta_1 & \beta_1
\end{array}
\]

and only one square from each row and column can appear in a collection. Clearly, every collection must include the square in the last row with factorisation \( \beta_1\rho_1 = \rho_1\beta_1 \), since it is the only one containing the pair \( \rho_1\beta_1 \). Then there is a choice for the remaining two squares, either

\[
\beta_1\rho_3 = \rho_3\beta_2 \quad \text{and} \quad \beta_1\rho_5 = \rho_5\beta_2
\]

or

\[
\beta_1\rho_3 = \rho_5\beta_2 \quad \text{and} \quad \beta_1\rho_5 = \rho_3\beta_2.
\]

So there are two collections which determine 2-graphs. (Note that the vertex matrices commute since the blue matrix is the identity, but their product is not a \([0,1]\)-matrix since the red matrix is not a \([0,1]\)-matrix.)

### 3.4. The infinite path space

A morphism between two \( k \)-graphs \((\Lambda_1, d_1)\) and \((\Lambda_2, d_2)\) is a functor \( f : \Lambda_1 \rightarrow \Lambda_2 \) which respects the degree maps (that is, \( d_2(f(\lambda)) = d_1(\lambda) \) for \( \lambda \in \Lambda_1 \)). The infinite path space of a row-finite \( k \)-graph \( \Lambda \) with no sources is defined in [36, Definition 2.1] by

\[
\Lambda^\infty = \{ x : \Omega_k \rightarrow \Lambda \mid x \text{ is a } k\text{-graph morphism} \}.
\]

We endow \( \Lambda^\infty \) with the topology generated by the collection \( \{ Z(\lambda) : \lambda \in \Lambda \} \) of sets

\[
Z(\lambda) = \{ x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda \}.
\]

Since each \( Z(\lambda) \) is compact [36, Lemma 2.6] and the sets \( \{ Z(v) : v \in \Lambda^0 \} \) partition \( \Lambda \), the infinite path space \( \Lambda^\infty \) is compact if and only if \( \Lambda^0 \) is a finite set. For \( p \in \mathbb{N}^k \) define \( \sigma^p : \Lambda^\infty \rightarrow \Lambda^\infty \) by

\[
\sigma^p(x)(m, n) = x(m + p, n + p)
\]

for \( x \in \Lambda^\infty \) and \( (m, n) \in \mathbb{N}^{2k} \). Each \( \sigma^p \) is a continuous surjection of \( \Lambda^\infty \).
3.5. $C^*$-algebras of row-finite $k$-graphs with no sources

We can also consider the space of two-sided infinite paths of the $k$-graph $\Lambda$. First we define (as in [35, §3])

$$\Delta_k = \{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k : m \leq n\}$$

with similar structure maps to $\Omega_k$ and then $\Delta_k$ is a $k$-graph. The prototype of a two-sided infinite path in a $k$-graph is $\Delta_k$, in analogy with infinite paths being modelled on $\Omega_k$. The two-sided infinite path space of $\Lambda$ is

$$\Lambda^\Delta = \{x : \Delta_k \to \Lambda \mid x \text{ is a } k\text{-graph morphism}\}.$$ 

Whenever $\Lambda$ has no sources or sinks, $\Lambda^\Delta \neq \emptyset$. For $\lambda \in \Lambda$ and $n \in \mathbb{Z}^2$ define

$$Z(\lambda, n) = \{x \in \Lambda^\Delta : x(n, n + d(\lambda)) = \lambda\}.$$ 

The collection of all such sets forms a basis for topology on $\Lambda^\Delta$ and each set $Z(\lambda, n)$ is compact. One can also define a metric on $\Lambda^\Delta$ which induces the same topology as this one. If $\Lambda^0$ is finite then $\Lambda^\Delta$ is compact since $\Lambda^\Delta = \bigcup_{v \in \Lambda^0} Z(v, 0)$. By extending the definition in (3.4.1) to $\mathbb{Z}^k$, define for each $p \in \mathbb{Z}^k$ a map $\sigma^p : \Lambda^\Delta \to \Lambda^\Delta$. Observe that for all $n, p \in \mathbb{Z}^k$ and $\lambda \in \Lambda$ we have $\sigma^p(Z(\lambda, n)) = Z(\lambda, n + p)$, so $\sigma^p$ is a homeomorphism for each $p \in \mathbb{Z}^k$. Since $\sigma^{p+q} = \sigma^p \sigma^q$ for $p, q \in \mathbb{Z}^k$ and $\sigma^0$ is the identity, $\Lambda^\Delta$ is a $\mathbb{Z}^k$-space.

3.5. $C^*$-algebras of row-finite $k$-graphs with no sources

If $\Lambda$ is a row-finite $k$-graph with no sources, then a Cuntz-Krieger $\Lambda$-family is a collection of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ (either operators in a Hilbert space or elements of an abstract $C^*$-algebra) satisfying the following Cuntz-Krieger relations:

1. $\{S_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
2. $S_{\lambda \mu} = S_\lambda S_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$,
3. $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$, and
4. $S_v = \sum_{\lambda \in v \Lambda^0} S_\lambda S_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

There is a universal generating family $\{S_\lambda : \lambda \in \Lambda\}$ for $C^*(\Lambda)$ in the sense that if $\{T_\lambda : \lambda \in \Lambda\}$ is another family of partial isometries in a $C^*$-algebra $B$ which satisfy the above relations then there exists a $*$-homomorphism $\pi$ from $C^*(\Lambda)$ to $B$ which takes $S_\lambda$ to $T_\lambda$ for all $\lambda \in \Lambda$. The $C^*$-algebra of $\Lambda$ is the $C^*$-algebra $C^*(\Lambda)$ generated by $\{s_\lambda : \lambda \in \Lambda\}$.

The basic facts about these $C^*$-algebras are discussed in [36] and [54, Chapter 10], for example. Define $p_\lambda = s_\lambda s_\lambda^*$, then $p_v = s_v$ for all $v \in \Lambda^0$. If $\Lambda^0$ is a finite set then $\sum_{v \in \Lambda^0} p_v$ is the identity for $C^*(\Lambda)$ [36, Remark 1.6(v)].

Row-finite directed graphs which can have sources were analysed in [2], but there is difficulty in extending these results to higher-dimensions because in a $k$-graph a vertex can receive edges of some degrees and not others. In [56], Raeburn, Sims, and Yeend modified the Cuntz-Krieger relations so that they apply to $k$-graphs which have sources and they prove the uniqueness theorems and classifications of ideals for the $C^*$-algebras in that situation. The definitions in [36] are sufficient for our work however, since all of our 2-graphs are finite and have no sinks or sources (see Proposition 4.1.8).
3.6. Properties of higher-rank graph C*-algebras

The C*-algebras of row-finite k-graphs are always nuclear (it is shown in [36, Theorem 5.5] using Takai duality that C*(Λ) is stably isomorphic to the crossed product of an AF algebra by the dual action of Z^k and hence C*(Λ) belongs to the bootstrap class N of [65] for which the UCT applies) and we have good criteria for deciding whether the C*-algebra of a higher-rank graph is simple and purely infinite. The major theorems depend upon an aperiodicity hypothesis on the k-graph.

Aperiodicity is the property of a k-graph Λ which ensures that Λ has a Cuntz-Krieger uniqueness theorem which says that all Cuntz-Krieger Λ-families generate isomorphic C*-algebras. Kumjian and Pask defined aperiodicity for row-finite higher-rank graphs with no sources in [36, Definition 4.3]: a k-graph Λ satisfies Condition (A) if for every v ∈ Λ^0 there exists x in the set vΛ^∞ := \{x ∈ Λ^∞ : x(0) = v\} such that for m, n ∈ N^k with m ≠ n, we have σ^m(x) ≠ σ^n(x). Condition (A) is the higher-dimensional analogue of Condition (L) for directed graphs [37]. The problem with Condition (A) is that it is hard to check in practice as it is phrased in terms of infinite paths. Robertson and Steger’s original aperiodicity hypothesis (H3) is similarly phrased to Condition (A) and since (H3) is difficult to verify, the authors introduced condition (H3*) which is sufficient for (H3) in the presence of conditions (H0)–(H2) [61, Lemma 2.1].

In [60], D. I. Robertson and Sims resolved many of the shortcomings of Condition (A) by formulating a version of the aperiodicity condition (H3) of [61]. A row-finite k-graph Λ with no sources is aperiodic if it satisfies the condition in [60, Lemma 3.2(4)]: for every v ∈ Λ^0 and every m, n ∈ N^k with m ≠ n there exists a path λ ∈ vΛ such that d(λ) ≥ m ∨ n and

\[(3.6.1) \quad \lambda(m, m + d(\lambda) - (m ∨ n)) ≠ \lambda(n, n + d(\lambda) - (m ∨ n)).\]

This is equivalent to the aperiodicity hypotheses used in [36] and [56] which phrase aperiodicity as properties of the shifts on the one-sided path space Λ^∞ [60, Lemma 3.2]. (When the the row-finite or no sources hypotheses on the k-graph are removed, there are replacement conditions for Condition (A) given as Condition (B) and Condition (C) in [56] and [57] respectively. All three conditions are equivalent for row-finite k-graphs with no sources [60].) In our work it is not practical to verify either Condition (A) or (H3), and (H3*) is not even satisfied in some cases (see Remark 4.4.5). Hence we will use the results of [60] to establish aperiodicity of our 2-graphs.

Once the aperiodicity hypothesis has been verified simplicity of the C*-algebra can be readily determined. A higher-rank graph Λ is cofinal if for every x ∈ Λ^∞ and v ∈ Λ^0 there exists λ ∈ Λ and n ∈ N^k such that s(λ) = x(n, n) and r(λ) = v. Kumjian and Pask showed that if Λ is a row-finite k-graph with no sources satisfying Condition (A), then C*(Λ) is simple if and only if Λ is cofinal [36, Proposition 4.8]. Or, using [60, Theorem 3.2], C*(Λ) is simple if and only if Λ is cofinal and satisfies the aperiodicity condition in [60, Lemma 3.2(4)].
Kumjian and Pask gave sufficient conditions on the $k$-graph $\Lambda$ for $C^*(\Lambda)$ to be purely infinite. In [36, Proposition 4.9], $C^*(\Lambda)$ is purely infinite (in the sense of [63, Proposition 4.1.1(iv)]) that every hereditary subalgebra contains an infinite projection) if $\Lambda$ satisfies Condition (A) and satisfies the condition that every vertex can be reached from a non-trivial cycle: for every $v \in \Lambda^0$ there exists $\lambda, \mu \in \Lambda$ with $d(\mu) \neq 0$ such that $r(\lambda) = v$ and $s(\lambda) = r(\mu) = s(\mu)$. This result is not correct as it stands: for example, the 2-graphs in [45, Figures 3 and 4] satisfy the hypothesis of [36, Proposition 4.9], but their $C^*$-algebras are AT-algebras and hence are not purely infinite. In [69, Proposition 8.8], Sims gives the corrected result which uses a stronger version of the cycle condition under which the argument in the proof of [36, Proposition 4.9] stands up (although Sims’s proof is based on [2, Proposition 5.3]). Sims’s result says that $C^*(\Lambda)$ is purely infinite if $\Lambda$ is aperiodic, cofinal, and satisfies the condition that every vertex can be reached from a cycle with an entrance: for every $v \in \Lambda^0$ there exists a cycle $\mu \in \Lambda$ with $s(\mu) = r(\mu)$ and $v\Lambda s(\mu) \neq \emptyset$ and there exists an entrance $\alpha \in s(\mu)\Lambda$ such that $d(\mu) \geq d(\alpha)$. Thus we will use [69, Proposition 8.8] rather than [36, Proposition 4.9] to establish that a $C^*$-algebra is purely infinite.

The dichotomy for graph algebras of [37] does not hold in higher-dimensions; this was confirmed in [45, §6] where the $C^*$-algebras are simple AT algebras and are neither purely infinite nor AF.

### 3.7. $K$-theory and classifiability

The first $K$-theoretic computations for higher-rank graphs were performed by Robertson and Steger for their rank-2 Cuntz-Krieger algebras in [62]. Their methods were generalised by Evans [17] to produce explicit formulas for the $K$-groups of the $C^*$-algebras of 2-graphs. The vertex matrices $B$ and $R$ of a 2-graph $\Lambda$ are the vertex matrices of the blue and red graphs, defined for $u, v \in \Lambda^0$ by

$$B(u,v) = \# \{ \lambda \in \Lambda^1 : r(\lambda) = u, s(\lambda) = v \}$$

$$R(u,v) = \# \{ \lambda \in \Lambda^2 : r(\lambda) = u, s(\lambda) = v \}.$$ 

The entries $BR(u,v)$ in the product $BR$ are the numbers of blue-red paths from $v$ to $u$, which the factorisation property implies are the same as the entries $RB(u,v)$ in $RB$; in other words, $BR = RB$. Consider the maps $\delta_1 : \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 \to \mathbb{Z}\Lambda^0$ and $\delta_2 : \mathbb{Z}\Lambda^0 \to \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0$ with matrices

$$\delta_1 = \begin{pmatrix} 1 - B^t & 1 - R^t \\ 1 - B^t & 1 - R^t \end{pmatrix}$$

and

$$\delta_2 = \begin{pmatrix} R^t - 1 & 0 \\ 0 & 1 - B^t \end{pmatrix}.$$ 

If $\Lambda$ is a row-finite 2-graph with no sources then [17, Proposition 3.16] gives isomorphisms

$$K_0(C^*(\Lambda)) \cong \operatorname{coker} \delta_1 \oplus \ker \delta_2$$

$$K_1(C^*(\Lambda)) \cong \ker \delta_1 / \img \delta_2.$$ 

Recall that it is possible for several 2-graphs to have the same skeleton and thus have the same pair of vertex matrices (possibly up to a permutation of index sets). Such 2-graphs have the same $K$-groups by [17, Proposition 3.16]. So Evans’s result implies that
K-theory of a 2-graph depends only on the skeleton (and not on the choice of C, see Section 3.3.2).

Calculating K-theory for k-graphs with k > 2 is much harder as there are no such nice formulas, although some information about the K-groups can be determined in certain circumstances (see [17, Proposition 3.17]).

The Kirchberg-Phillips Classification Theorem [29, 50] implies that a separable C*-algebra which belongs to the bootstrap class \( \mathcal{N} \) and is simple and purely infinite is classified by its K-theory. Zhang’s Dichotomy implies that such a C*-algebra is either unital or stable [63, Proposition 4.1.3], and so there are two versions of the Kirchberg-Phillips Theorem: one for unital C*-algebras and one for stable C*-algebras. As stated in [63, Theorem 8.4.1(iv)] for example, the former version of the theorem says that if \( A \) and \( B \) are separable, unital, purely infinite, simple C*-algebras in \( \mathcal{N} \), then \( A \) and \( B \) are isomorphic if and only if \( K_1(A) \cong K_1(B) \) and there is an isomorphism of \( K_0(A) \) onto \( K_0(B) \) which takes the class \([1_A]\) of the identity to \([1_B]\). In other words, the triple \((K_0(A), [1_A], K_1(A))\) is a complete invariant for unital Kirchberg algebras in \( \mathcal{N} \) [63, §8.4].

The results of Section 3.6 tell us when graph algebras of row-finite higher-rank graphs satisfy the hypotheses of the Kirchberg-Phillips Theorem. As we will only be dealing with finite 2-graphs the unital version of the theorem applies. Graph algebras always belong to the bootstrap class \( \mathcal{N} \) [36, Theorem 5.5] and are always separable since we insist as part of the definition that the graphs are countable. We have theorems which tell us when higher-rank graph algebras are simple [36, Proposition 4.8] and purely infinite [69, Proposition 8.8], and so whenever these properties hold the C*-algebra satisfies the hypotheses of the Kirchberg-Phillips Theorem. In particular, if \( \Lambda \) is a finite 2-graph with no sources (such as those we will study in Chapters 4 and 6), then [17, Corollary 5.1] says the isomorphism \( \Phi \) of coker \( \delta_1 \oplus \ker \delta_2 \) onto \( K_0(C^*(\Lambda)) \) satisfies \( \Phi(e + \text{im} \delta_1) = [1] = \sum_{v \in \Lambda^0}[p_v] \) where \( e(v) = 1 \) for all \( v \in \Lambda^0 \). Thus if \( C^*(\Lambda) \) is simple and purely infinite, it is classified by its K-theory.

### 3.8. Higher-dimensional dynamical systems

In Section 2.2, we explored the close relationship between shifts of finite type and directed graphs. In the higher-dimensional setting, Kumjian and Pask associated higher-dimensional shift spaces to row-finite k-graphs with no sources [35]. Associating a higher-rank graph to a given higher-dimensional shift space is a technically harder problem. In this section we describe the sort of two-dimensional shift spaces to which we will associate a family of 2-graphs in the next chapter.

Fix \( d \geq 1 \). Let \( A \) be a finite set called the alphabet and let \( A^{\mathbb{Z}^d} \) be the set of all maps \( x : \mathbb{Z}^d \to A \). The space \( A^{\mathbb{Z}^d} \) is compact in the product topology. For a typical element \( x \in A^{\mathbb{Z}^d} \) we write \( x = (x_m) \) where \( x_m \in A \) is the value of \( x \) at \( m \in \mathbb{Z}^d \). Define \( (\sigma^n(x))_m = x_{m+n} \) for \( x \in A^{\mathbb{Z}^d} \), then \( \sigma : n \to \sigma^n \) is the shift-action of \( \mathbb{Z}^d \) on \( A^{\mathbb{Z}^d} \). A shift space is a closed subset \( X \subset A^{\mathbb{Z}^d} \) that is shift-invariant (\( \sigma^n(X) = X \)). The shift space \( X \) is a \( d \)-dimensional Markov shift or shift of finite type if there exists a finite subset \( T \subset \mathbb{Z}^d \) and a set \( V \subset A^T \) such that \( X \) consists of the points \( x \in A^{\mathbb{Z}^d} \) for which the coordinate
projection of $x$ onto every translate of $T$ is an element of $V$. For example, the shift space $X_L$ with $A = \mathbb{Z}/2\mathbb{Z}$ and $T = \{0, e_1, e_2\}$ defined by

\begin{equation}
X_L = \{ x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = 0 \text{ for all } (m_1,m_2) \in \mathbb{Z}^2 \}
\end{equation}

is a 2-dimensional shift of finite type known as Ledrappier’s example (see [41, Examples 1.8, 2.4], [68, Example 5.3(5)] and [39]). There are difficulties analysing general higher-dimensional Markov shifts, for example, given $T$ and $V$ it may be undecidable whether the corresponding Markov shift is nonempty [40, §13.10].

It is an open problem in ergodic theory as to whether, for $\mathbb{Z}$-actions, the topological property known as mixing implies the property $r$-mixing of all orders $r \geq 2$ [68, page 261].\footnote{See [68, page 2] for the definition of mixing and [68, page 169] for the definition of $r$-mixing. Note that the definitions are equivalent when $r = 2$.} For $\mathbb{Z}^d$-actions with $d \geq 2$, mixing does not imply mixing of all orders. This was shown by Ledrappier who introduced the shift space $X_L$ in (3.8.1) as the first example of a $\mathbb{Z}^2$-action which is mixing but not 3-mixing. Ledrappier’s work opened up a new area of ergodic theory which studies $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups, such as those studied in [68, 31, 32].

The dynamical systems studied by Kitchens and Schmidt in [32] are a special class of $d$-dimensional shifts of finite type called Markov group shifts which are much better understood than general higher-dimensional shifts of finite type. These shifts are always nonempty (since they always contain the point of all 0s) and by using the algebraic structure one can determine dynamical properties such as irreducibility and mixing [31]. What follows is a description of the notation used in [32] and [68, Chapter II] (which we will use in Section 4.2 of Chapter 4) in the case where the alphabet is the finite abelian ring $\mathbb{Z}/q\mathbb{Z}$. Let $R_d^q = (\mathbb{Z}/q\mathbb{Z})[u_1^\pm, \ldots , u_d^\pm]$ be the ring of Laurent polynomials in the commuting variables $u_1, \ldots , u_d$ with coefficients in $\mathbb{Z}/q\mathbb{Z}$. A typical element of $f \in R_d^q$ is written $f = \sum_{n \in \mathbb{Z}^d} c_f(n) u^n$ where $c_f(n) \in \mathbb{Z}/q\mathbb{Z}$ for all $n \in \mathbb{Z}^d$ and $u^n = u_1^{n_1} \cdots u_d^{n_d}$ for $n \in \mathbb{Z}^d$, and we suppose that $c_f(n) \neq 0$ for only finitely many $n \in \mathbb{Z}^d$. Since $R_d^q$ is isomorphic as a discrete group to $\sum_{\mathbb{Z}^d} \mathbb{Z}/q\mathbb{Z}$, it follows from Pontryagin duality (see Section 5.1) that the dual group $\hat{R}_d^q$ may be identified with the compact group $\prod_{\mathbb{Z}^d} \mathbb{Z}/q\mathbb{Z}$ which is a Hausdorff space with the product topology. We further identify $\prod_{\mathbb{Z}^d} \mathbb{Z}/q\mathbb{Z}$ with $(\mathbb{Z}/q\mathbb{Z})^{2d}$ which has the topology of uniform convergence on compact sets. Under these identifications the multiplicative action $\sigma$ of $\mathbb{Z}^d$ on $R_d^q$ given by $\sigma_m(g) = u^m g$ becomes an additive action $\hat{\sigma}$ of $\mathbb{Z}^d$ on $(\mathbb{Z}/q\mathbb{Z})^{2d}$ given by $\hat{\sigma}_m(f)(n) = f(n + m)$.

Now fix a polynomial $g \in R_d^q$ and let (g) be the principal ideal generated by $g$. Since $R_d^q/(g)$ is a quotient of the additive group $R_d^q$ by a $\sigma$-invariant subgroup, the dual group $\hat{R}_d^q/(g)$ may be identified with the closed $\hat{\sigma}$-invariant subgroup $X$ of $(\mathbb{Z}/q\mathbb{Z})^{2d}$ given in [32, page 719] as

$$X = \{ x = (x_n) \in (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2} : \sum_{m \in \mathbb{Z}^d} c_g(m)x_{m+n} = 0 \pmod{q} \text{ for all } n \in \mathbb{Z}^d \}.$$
With this identification, the additive action $\hat{\sigma}$ of $\mathbb{Z}^d$ on $\hat{R}_q^d$ restricts to an action $\alpha$ on $\hat{R}_q^d/(g)$ defined by $(\alpha_m(x))_n = x_{m+n}$ for $x = (x_n) \in X$ and $m, n \in \mathbb{Z}^d$. The space $X$ is a compact subspace of $(\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^d}$ in the product topology and also inherits the topology of uniform convergence from $(\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^d}$. Results about the dynamical properties of the action $\alpha$ are summarised in Figure 1 of [68, page xiii]. In particular, whether $\alpha$ is mixing is determined by the polynomial $g$ [68, Theorem 6.5].

Ledrappier’s system $X_L$ is a Markov group shift having the form $\hat{R}_2^2/(g)$ where $g = 1 + u_1 + u_2$. Part of our work in Chapter 4 involves realising $X_L$ and similar algebraically-defined shifts as the edge shifts of certain higher-rank graphs. (These Markov group shifts involve both mixing and nonmixing actions; see Remark 6.1.1).
A family of 2-graphs arising from two-dimensional subshifts

Every shift of finite type is equivalent to the backward shift $\sigma$ on the two-sided infinite path space of a finite directed graph [40, Theorem 2.3.2]. The two-sided infinite path space $\Lambda^\Delta$ of a finite $k$-graph $\Lambda$ introduced in [35] carries a set of $k$ commuting shifts $\sigma_i$, and these are examples of the higher-dimensional shifts of finite type studied by dynamicists.

In this chapter we consider a family of finite 2-graphs $\Lambda$ for which the path spaces $(\Lambda^\Delta, \sigma_i)$ are dynamical systems of algebraic origin, as studied by Schmidt and others [68]. (A particular motivating example for us was the system introduced by Ledrappier in [39].)

We analyse the $C^*$-algebras of these 2-graphs, find criteria under which they are simple and purely infinite, and compute their $K$-theory.

4.1. Tiles and 2-graphs

We begin with a visual description of our 2-graph construction with reference to a key example, and then give details of the general construction. Our key example, which we call the Ledrappier graph, is the 2-graph associated to Ledrappier’s system (3.8.1).

4.1.1. The Ledrappier graph. Each of our graphs $\Lambda$ is associated to a tile, which is a finite hereditary subset $T$ of $\mathbb{N}^2$ containing the origin. We picture $T$ as a collection of boxes into which we can put elements of the commutative ring $\mathbb{Z}/q\mathbb{Z}$, which we think of as an alphabet: for example, the tile in Ledrappier’s system is $T := \{0, e_1, e_2\}$, which we call the sock tile and picture as

$$
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
$$

The vertices in $\Lambda$ are copies of $T$ in which each box is filled with elements of $\mathbb{Z}/q\mathbb{Z}$ which together satisfy a fixed equation in $\mathbb{Z}/q\mathbb{Z}$; for example, the vertices in the Ledrappier graph are copies of the sock filled with 0s and 1s such that sum of the entries is 0 (mod 2); there are four vertices $a, b, c, d$ visualised as

$$
\begin{array}{c}
\begin{array}{c}
0 \ 0 & 1 \ 0 & 1 \ 1 \ 0 & 0 \ 1
\end{array}
\end{array}
\right)
$$

(Conventionally, we visualise a subset $S$ of $\mathbb{N}^2$ as the union of the unit squares whose bottom left-hand corners belong to $S$, and a function $f : S \to \mathbb{Z}$ as a diagram in which the number $f(i)$ is placed in the square with bottom left-hand corner $i$. Thus if $T = \{0, e_1, e_2\}$, the function $f : T \to \mathbb{Z}$ defined by $f(0) = 0$, $f(e_1) = 1$ and $f(e_2) = 1$ represents the vertex $b$ in (4.1.1)).

Paths in $\Lambda^*$ are diagrams covered by translates of $T$, filled in so that each
translate of \( T \) is a valid vertex. Thus for example,

\[
\begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
\end{array}
\] (4.1.2)

represents a path \( \lambda \) of degree \((3, 2)\) in the Ledrappier graph

from \( s(\lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) (the top right-hand one) to \( r(\lambda) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) (the bottom left-hand one).

So we can easily visualise the vertices and paths in our 2-graphs.

4.1.2. The construction. To generate more examples like the Ledrappier graph we change variables such as the tile, the alphabet and the equation, which together determine the vertices and paths in the graph. There are four variables in our main construction of 2-graphs. Recall that a subset \( T \) of \( \mathbb{N}^2 \) is hereditary if \( j \in T \) and \( 0 \leq i \leq j \) imply \( i \in T \).

**Definition 4.1.1.** Let basic data \((T, q, t, w)\) consist of the variables:

- a tile \( T \), which is a hereditary subset of \( \mathbb{N}^2 \) with finite cardinality \(|T|\);
- an alphabet \( \{0, 1, \ldots, q - 1\} \), where \( q \geq 2 \) is an integer; we view the alphabet as a commutative ring by identifying it with \( \mathbb{Z}/q\mathbb{Z} \) in the obvious way;
- an element \( t \) of the alphabet, called the trace; and
- a function \( w : T \to \{0, 1, \ldots, q - 1\} \) called the rule.

**Example 4.1.2.** The Ledrappier graph is associated to basic data consisting of the sock tile \( T = \{0, e_1, e_2\} \), \( q = 2 \), \( t = 0 \), and the constant function \( w \equiv 1 \).

For the rest of the section, we fix the basic data \((T, q, t, w)\). We now describe the vertices and paths in our 2-graph \( \Lambda = \Lambda(T, q, t, w) \) as functions from \( T \) and translates of \( T \) into \( \mathbb{Z}/q\mathbb{Z} \), and then we have to prove that they form the objects and morphisms in a category satisfying the axioms of a 2-graph (see Theorem 4.1.7).

The vertex set in our 2-graph will be

\[
\Lambda^0 = \left\{ v : T \to \mathbb{Z}/q\mathbb{Z} : \sum_{i \in T} w(i)v(i) = t \pmod{q} \right\}.
\]

To describe the paths, we need some notation. Let \( (c_1, c_2) := \bigvee \{i : i \in T\} \), so that the longest row in \( T \) (the bottom one) has \( c_1 + 1 \) boxes and the highest column in \( T \) (the left-hand one) has \( c_2 + 1 \) boxes. For \( S \subset \mathbb{Z}^2 \) and \( n \in \mathbb{Z}^2 \), we let \( S + n = \{i + n : i \in S\} \) denote the translate of \( S \) by \( n \), and we set \( T(n) := \bigcup_{0 \leq m \leq n} T + m \). When we visualise \( T(n) \) using our convention, it looks like a \( (c_1 + 1 + n_1) \times (c_2 + 1 + n_2) \) rectangle of boxes with a bite taken out of the top right-hand corner. If \( f : S \to \mathbb{Z}/q\mathbb{Z} \) is a function defined on a subset \( S \) of \( \mathbb{N}^2 \) containing \( T + n \), then we define \( f|_{T+n} : T \to \mathbb{Z}/q\mathbb{Z} \) by

\[
f|_{T+n}(i) = f(i + n) \text{ for } i \in T.
\]

A path of degree \( n \) is a function \( \lambda : T(n) \to \mathbb{Z}/q\mathbb{Z} \) such that \( \lambda|_{T+m} \) is a vertex for \( 0 \leq m \leq n \); then \( \lambda \) has source \( s(\lambda) = \lambda|_{T+n} \) and range \( r(\lambda) = \lambda|_{T} \).
The rule
Suppose that we have basic data
Suppose that
If
∈
Case
+l
appear in the factorisations of paths. For \( \lambda \in \Lambda^p \) and \( 0 \leq m \leq n \leq p \), the segment \( \lambda(m, n) \) is the path of degree \( n - m \) defined by

\[
\lambda(m, n)(i) = \lambda(m + i) \text{ for } i \in T(n - m).
\]

In particular, \( \lambda(m, m) \) is the vertex \( \lambda|_{T+m} \).

We want \( \Lambda^* := \bigcup_{n \in \mathbb{N}^2} \Lambda^n \) to be the morphisms in a category, and so we have to define composition. To make this work, we need to make an assumption on the rule \( w \).

**Definition 4.1.3.** The rule \( w \) has invertible corners if \( w(c_1e_1) \) and \( w(c_2e_2) \) are invertible elements of the ring \( \mathbb{Z}/q\mathbb{Z} \).

The next proposition tells us that when \( w \) has invertible corners there is exactly one candidate for the composition of two paths.

**Proposition 4.1.4.** Suppose that we have basic data \( (T, q, t, w) \) and the rule \( w \) has invertible corners. Suppose that \( \mu \in \Lambda^m \) and \( \nu \in \Lambda^n \) satisfy \( s(\mu) = r(\nu) \). Then there is a unique path \( \lambda \in \Lambda^{m+n} \) such that

\[
(4.1.4) \quad \lambda(0, m) = \mu \text{ and } \lambda(m, m + n) = \nu.
\]

Notice that Equation (4.1.4) defines \( \lambda \) uniquely on \( T(m) \cup (T(n) + m) \), so our problem is to show that there is a unique function \( \lambda' : T(m+n) \to \mathbb{Z}/q\mathbb{Z} \) such that \( \lambda'|_{T(m)} = \lambda \) and \( \lambda'|_{T+k} \) belongs to \( \Lambda^0 \) for every \( k \) such that \( 0 \leq k \leq m + n \); since \( \mu \) and \( \nu \) are paths and \( \lambda' \) extends \( \lambda \), we already know this for \( k \) such that \( T + k \subset T(m) \cup (T(n) + m) \).

Our strategy is to extend \( \lambda \) from \( T(m) \cup (T(n) + m) \) to \( T(m+n) \) by adding one point at a time in such a way that there is only one possible value for \( \lambda \) at the new point. The next lemma tells us how to do this. It depends crucially on the assumption that the rule has invertible corners (see §4.1.3), and it fails spectacularly for tiles in \( \mathbb{N}^k \) when \( k > 2 \) (see §4.1.4).

**Lemma 4.1.5.** Suppose that \( l \in \mathbb{N}^2 \) and \( S \) is a subset of \( \mathbb{N}^2 \) containing \( T + l - e_2 \) and \( T + l + e_1 \), and \( \lambda : S \to \mathbb{Z}/q\mathbb{Z} \) is a function such that \( \lambda|_{T+k} \) belongs to \( \Lambda^0 \) for every \( k \in \mathbb{N}^2 \) such that \( T + k \subset S \). Suppose also that \( l + e_1 - e_2 + c_1 e_1 + a_1 e_1 - a_2 e_2 \notin S \) for all \( a \in \mathbb{N}^2 \setminus \{0\} \). Then there is a unique function

\[
\lambda' : S' := S \cup \{l + e_1 - e_2 + c_1 e_1\} \to \mathbb{Z}/q\mathbb{Z}
\]

such that \( \lambda'|_S = \lambda \) and \( \lambda'|_{T+k} \) belongs to \( \Lambda^0 \) for every \( k \in \mathbb{N}^2 \) such that \( T + k \subset S' \).

**Proof.** If \( l + e_1 - e_2 + c_1 e_1 \) belongs to \( S \), there is nothing to do. So we suppose that \( l + e_1 - e_2 + c_1 e_1 \) is not in \( S \). Let \( i \in T \setminus \{c_1 e_1\} \). Then either \( i_2 = 0 \) and \( i_1 < c_1 \), in which case \( i + e_1 \) belongs to \( T \) and \( l + e_1 - e_2 + i \) belongs to \( T + l - e_2 \), or \( i_2 > 0 \), in which case \( i - e_2 \in T \) and \( l + e_1 - e_2 + i \) belongs to \( T + l + e_1 \). So \( (T \setminus \{c_1 e_1\}) + l + e_1 - e_2 \) is contained
in the domain $S$ of $\lambda$, and $l + e_1 - e_2 + c_1 e_1$ is the only point of $T + l + e_1 - e_2$ which is not in $S$. Thus we can define $\lambda'|S = \lambda$ and
\[(4.1.5) \quad \lambda'(l + e_1 - e_2 + c_1 e_1) := w(c_1 e_1)^{-1} \left( l - \sum_{i \in T \setminus \{c_1 e_1\}} w(i) \lambda(l + e_1 - e_2 + i) \right),\]

If $T + k \subset S'$, then either $T + k \subset S$, in which case $\lambda'|_{T+k} = \lambda|_{T+k} \in \Lambda^0$, or $T + k \not\subset S$, in which case we claim that $k = l + e_1 - e_2$ and (4.1.5) implies $\lambda'|_{T+k} \in \Lambda^0$. No other value of $\lambda'(l + e_1 - e_2 + c_1 e_1)$ would give $\lambda'|_{T+l+e_1-e_2} \in \Lambda^0$, so this function $\lambda'$ is the only one with the required property. To justify the claim that $k = 1 + e_1 - e_2$, we have
\[l + e_1 - e_2 + c_1 e_1 \in T + k,\]
so there is $\alpha \in T$ with
\[l + e_1 - e_2 + c_1 e_1 = \alpha + k.\]
Then
\[(4.1.6) \quad T + k = T + l + e_1 - e_2 + c_1 e_1 - \alpha.\]
Since $\alpha \in T$, (4.1.6) gives
\[l + e_1 - e_2 + c_1 e_1 \in T + k.\]
If $\alpha_2 > 0$ then $\alpha_1 e_1 \in T$, hence (4.1.6) gives
\[l + e_1 - e_2 + c_1 e_1 - \alpha_2 e_2 \in T + k,\]
which gives a second point in $S' \setminus S$, a contradiction. Thus $\alpha_2 = 0$. Now, if $\alpha_1 < c_1$, then $\alpha + e_1 \in T$, hence
\[l + e_1 - e_2 + c_1 e_1 + \alpha = (\alpha + e_1) + l + e_1 - e_2 + c_1 e_1 - \alpha \in T + k,\]
by (4.1.6), again giving a second point in $S' \setminus S$. Therefore $\alpha = c_1 e_1$, and thus $k = 1 + e_1 - e_2$. \qed

**Proof of Proposition 4.1.4.** The region $T(m+n)$ is obtained from $T(m) \cup \{T(n) + m\}$ by adding two rectangles
\[BR := \{ j \in \mathbb{N}^2 : c_1 + m_1 < j_1 \leq c_1 + m_1 + n_1, \ 0 \leq j_2 < m_2 \}, \text{ and} \]
\[UL := \{ j \in \mathbb{N}^2 : 0 \leq j_1 < m_1, \ c_2 + m_2 < j_2 \leq c_2 + m_2 + n_2 \}.\]
We order the bottom right rectangle $BR$ lexicographically, first going down the column $j_1 = c_1 + m_1 + 1$, then down the column $j_1 = c_1 + m_1 + 2$, and so on. We then apply Lemma 4.1.5 to each $j$ in order: when we come to define $\lambda(j)$, we have already defined $\lambda(i)$ for every $i$ above and to the left of $j$, and with $l := j - e_1 + e_2 - c_1 e_1$, $\lambda$ is defined on both $T + l - e_2$ and $T + l + e_1$. Since there is only one possible value of $\lambda(j)$ at each stage, there is only one way to extend $\lambda$ to $T(m) \cup \{T(n) + m\} \cup BR$.

To see that $\lambda$ extends uniquely to $UL$, we can either run the mirror image of this argument in the rectangle $UL$, or reflect everything in the line $n_1 = n_2$ and apply what we have just proved. \qed
Example 4.1.6. We illustrate the procedure of Proposition 4.1.4 for the sock tile of Example 4.1.2 which has basic data consisting of $T = \{0, e_1, e_2\}$, $q = 2$, $t = 0$, and the constant function $w \equiv 1$. (Note that the sock tile has $c_1 = 1$ and $c_2 = 1$.) Lemma 4.1.5 implies that the diagram below

(4.1.7)

determines a unique path of degree $(1,1)$ by filling in the missing box with $c + g \pmod 2$.

The idea of the proof of Proposition 4.1.4, shown in the diagram below,

is to start by filling in the entry for $*$ and then to proceed down the column. The rest of the area $BR$ is completed one column at a time left-to-right, starting at the top of each column. The area $UL$ is completed symmetrically. The resulting path is the unique path which factorises as $\lambda \mu$ by applying uniqueness in Lemma 4.1.5 at each stage of the process.

Theorem 4.1.7. Suppose that we have basic data $(T, q, t, w)$ and the rule $w$ has invertible corners. Say that $\mu \in \Lambda^m$ and $\nu \in \Lambda^n$ are composable if $s(\mu) = r(\nu)$, and define the composition $\mu \nu$ to be the unique path $\lambda$ satisfying (4.1.4). Define $d : \Lambda \to \mathbb{N}^2$ by $d(\lambda) = n$ for $\lambda \in \Lambda^n$. Then, with $\Lambda^0$, $\Lambda^*$, $r$ and $s$ defined at the beginning of the section, $\Lambda(T, q, t, w) := ((\Lambda^0, \Lambda^*, r, s), d)$ is a 2-graph.

Proof. We can view a vertex $v \in \Lambda^0$ as a path of degree 0; then $\lambda$ has the property (4.1.4) which characterises $r(\lambda)\lambda$ and $\lambda s(\lambda)$, so $v$ has the properties required of the identity morphism at $v$. For $\mu \in \Lambda^m$, $\nu \in \Lambda^n$ with $s(\mu) = r(\nu)$, (4.1.4) implies that $r(\mu \nu) = (\mu \nu)|_T = \mu|_T = r(\mu)$ and

$s(\mu \nu) = (\mu \nu)|_{T+m+n} = (\mu \nu)(m, m+n)|_{T+n} = \nu|_{T+n} = s(\nu)$.

To prove that $\Lambda$ is a category, it remains to show that composition is associative.

Suppose that $\mu \in \Lambda^m$, $\nu \in \Lambda^n$, and $\rho \in \Lambda^p$ satisfy $s(\mu) = r(\nu)$ and $s(\nu) = r(\rho)$. For $i \in T(n)$, we have

$((\mu \nu)\rho)(m, m+n+p)(i) = ((\mu \nu)\rho)(i+m) = ((\mu \nu)\rho)(0, m+n)(i+m)$

$= (\mu \nu)(i+m) = (\mu \nu)(m, m+n)(i) = \nu(i),$

and for $i \in T(p)$ we have

$((\mu \nu)\rho)(m, m+n+p)(i+n) = ((\mu \nu)\rho)(i+m+n) = ((\mu \nu)\rho)(m+n, m+n+p)(i) = \rho(i)$. 

4.1. TILES AND 2-GRAPHS 33
Thus \(((\mu\nu)\rho)(m, m + n + p)(0, n) = \nu\) and \(((\mu\nu)\rho)(m, m + n + p)(n, n + p) = \rho\), and hence\(((\mu\nu)\rho)(m, m + n + p) = \nu\rho\). On the other hand, for \(i \in T(m)\), we have
\[
((\mu\nu)\rho)(0, m)(i) = ((\mu\nu)\rho)(i) = ((\mu\nu)\rho)(0, m + n)(i)
= (\mu\nu)(i) = (\mu\nu)(0, m)(i) = \mu(i),
\]
so \(((\mu\nu)\rho)(0, m) = \mu\). Thus \((\mu\nu)\rho\) has the property which characterises \(\mu(\nu\rho)\), and we have \((\mu\nu)\rho = \mu(\nu\rho)\).

We have now shown that \(\Lambda\) is a category, and it is countable because each \(\Lambda^n\) is finite. The map \(d : \Lambda \rightarrow \mathbb{N}^2\) satisfies \(d(\mu\nu) = d(\mu) + d(\nu)\) and hence is a functor, so it remains to verify that \(d\) has the factorisation property. But this is easy: given \(\lambda \in \Lambda^{m+n}\), the paths \(\mu := \lambda(0, m)\) and \(\nu := \lambda(m, m + n)\) are the only ones which can satisfy \(\lambda = \mu\nu\). \(\square\)

To visualise the 2-graph \(\Lambda(T, q, t, w)\), we draw its skeleton. This skeleton has a few special properties.

**Proposition 4.1.8.** Suppose that we have basic data \((T, q, t, w)\) and the rule \(w\) has invertible corners. Then \(\Lambda = \Lambda(T, q, t, w)\) satisfies

(a) \(|\Lambda^0| = q^{|T|}-1;\)

(b) for \(v, u \in \Lambda^0\), \(v\Lambda^{e_i}u\) is nonempty if and only if
\[
v(m) = u(m - e_i) \text{ for every } m \in T \cap (T + e_i),
\]
in which case \(|v\Lambda^{e_i}u| = 1;\)

(c) \(|v\Lambda^{e_1}| = |\Lambda^{e_1}v| = q^{c_2} \text{ and } |v\Lambda^{e_2}| = |\Lambda^{e_2}v| = q^{c_1} \text{ for every } v \in \Lambda^0.\)

**Proof.** There are \(q^{|T|}-1\) functions \(v : T\setminus\{c_1e_1\} \rightarrow \mathbb{Z}/q\mathbb{Z}\), and each defines a unique vertex \(v\) by setting
\[
v(c_1e_1) = w(c_1e_1)^{-1}\left(t - \sum_{i \in T\setminus\{c_1e_1\}} w(i)v(i)\right).
\]
This gives (a). For (b), note that if \(\beta \in v\Lambda^{e_i}u\) and \(m \in T \cap (T + e_i)\), then
\[
v(m) = \beta|_T(m) = \beta|_{T+e_i}(m - e_i) = u(m - e_i).
\]
Conversely, if \(u, v\) satisfy (4.1.8), then we can define \(\beta : T(e_i) \rightarrow \mathbb{Z}/q\mathbb{Z}\) by
\[
\beta(m) = \begin{cases} v(m) & \text{for } m \in T, \\
u(m - e_i) & \text{for } m \in (T + e_i)\setminus T,
\end{cases}
\]
and (4.1.8) says that \(\beta|_{T+e_i} = u\). The constraints \(\beta|_T = v\) and \(\beta|_{T+e_i} = u\) completely determine \(\beta\), so \(|v\Lambda^{e_i}u| = 1.\)

To see (c), note that an edge \(\beta \in \Lambda^{e_1}v\) has \(\beta|_{T+e_1}\) determined by \(v\). The remainder \(T(e_1) \setminus (T + e_1)\) is the first column of \(T(e_1)\), which has \(c_2 + 1\) entries. There are \(q^{c_2}\) ways of filling in the bottom \(c_2\) squares, and then the top entry is determined by
\[
\beta(c_2e_2) = w(c_2e_2)^{-1}\left(t - \sum_{i \in T\setminus\{c_2e_2\}} w(i)\beta(i)\right).
\]
Thus $|\Lambda^e_v| = q^2$. On the other hand, an edge $\beta \in v\Lambda^e_1$ has $\beta|_T = v$, and $T(e_1) \setminus T$ also has $c_2 + 1$ squares. We can fill in all the squares except $c_1 e_1 + e_1$ arbitrarily in $q^{c_2}$ ways, and then $\beta(c_1 e_1 + e_1)$ is determined by
\[
\beta(c_1 e_1 + e_1) = w(c_1 e_1)^{-1} \left( t - \sum_{i \in T \setminus \{c_1 e_1\}} w(i) \beta(i + e_1) \right).
\]
The facts about the red edges follow by symmetry. \qed

Example 4.1.9. The Ledrappier graph $L(\Xi)$ is the 2-graph constructed from the basic data consisting of the sock tile $T$, $q = 2$, $t = 0$, and $w \equiv 1$. It has four vertices $a, b, c, d$ listed in (4.1.1). Examples of a blue edge (with range $b$ and source $d$) and a red edge (with range $a$ and source $b$) are visualised by
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]
The skeleton of $L(\Xi)$ is the 2-coloured graph in Figure 4.1.

4.1.3. The invertible corners hypothesis. It follows from Proposition 4.1.4 that when $w$ has invertible corners there is exactly one way to associate a 2-graph to the bicoloured directed graph of Proposition 4.1.8. Recall from Section 3.3.3 that this happens if there is only one possible bijection between the blue-red paths of length 2 and the red-blue paths of length 2.

If we start with a rule which does not have invertible corners, then we can still draw a bicoloured graph, which may or may not be the skeleton of a 2-graph. For example, suppose that $T$ is the sock, $w(0) = w(e_1) = 1$, and $w(e_2) = 0$ (with $q = 2$ and $t = 0$). Then there is exactly one blue-red path between each pair of vertices, but there are sometimes two and sometimes no red-blue paths, so the bicoloured graph cannot be the skeleton of a 2-graph. In pictures, every diagram like (4.1.7) determines a unique path of degree $(1, 1)$ since $w(e_1) = 1$ is invertible. But since $w(e_2) = 0$ is not invertible, a diagram such as the one below
\[
\begin{bmatrix}
1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
does not determine a path of degree \((1, 1)\) since there is no value for the missing square which makes the translate \(T + e_2\) a vertex (existence in Lemma 4.1.5 fails here). Diagrams such as

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
\]

have two possible entries for the missing square \((0 \text{ or } 1)\) and so uniqueness in Lemma 4.1.5 fails.

Choosing a rule which does not have invertible corners may also result in the bicoloured directed graph having sources. For example, if \(T\) is the sock tile, \(w(0) = 1\), and \(w(e_1) = w(e_2) = 0\) (with \(q = 2\) and \(t = 0\)), then every vertex \(v\) has \(v(0) = 0\). The vertex \(f = \begin{array}{c}1 \\
0 \\
0 \end{array}\) is a red source, \(g = \begin{array}{c}0 \\
0 \\
1 \end{array}\) is a blue source and the vertex \(\begin{array}{c}1 \\
0 \\
1 \end{array}\) does not receive edges of either colour. So there are no possible paths of degree \((1, 1)\) which contain a red-blue path with range \(f\) or a blue-red path with range \(g\).

On the other hand, if we use the zero rule \(w \equiv 0\) on the sock, then there are two blue-red paths and two red-blue paths between each pair of vertices, so there are many bijections between the sets of blue-red and red-blue paths, each of which determines a potentially different 2-graph. We will observe in Remark 4.2.4 that when we have to make choices to define a factorisation property, the correspondence between 2-graphs and shifts breaks down.

4.1.4. The construction in higher-dimensions. In two dimensions, the bicoloured directed graph determined by the basic data has the property that every two-coloured path of length 2 determines a unique commuting square in the 2-graph. If our construction generalises to dimensions \(k > 2\), we want a \(k\)-coloured directed graph to yield one choice for the factorisations of commuting \(k\)-cubes in a \(k\)-graph. For example, when \(k = 3\) the basic data should determine a tricoloured directed graph in which a three-coloured path of length 3 determines the other five three-coloured factorisations of the unit cube.

When we start with a tile \(T\) which is a finite hereditary subset of \(\mathbb{N}^3\), we can construct a tricoloured graph, but this will not completely determine a 3-graph because Lemma 4.1.5 fails. The crux of the proof of Lemma 4.1.5 (when \(l = e_2\)) is that the set \(T(e_1 + e_2) \setminus (T(e_2) \cup (T(e_1) + e_2))\), consists of the single point \((e_1 + 1) e_1\). When we consider the tile \(T = \{0, e_1, e_2, e_3\}\), which is a natural 3-dimensional analogue of the sock, we have

\[
T(e_1 + e_2) \setminus (T(e_2) \cup (T(e_1) + e_2)) = \{2e_1, e_1 + e_3\}.
\]

If we use a single rule \(w\) with invertible corners to define our vertices and paths, then there is still more than one way to fill in the two empty cubes. So one would have to impose more than one rule to get a uniquely defined red-blue factorisation of a blue-red path. However, the number of empty cubes to be filled depends on the dimensions of the original tile, so just one extra rule is not enough in general.
4.2. Connections with shift spaces

The infinite path spaces of our 2-graphs consist of diagrams like (4.1.2) covering the entire plane, and the shifts \( \sigma_1 \) and \( \sigma_2 \) simply move the diagram one row left and one column down respectively. In this section we show that the two-sided infinite path space \( \Lambda^\Delta \) of \( \Lambda(T, q, 0, w) \) is the underlying space for a higher-dimensional shift of the sort studied by Schmidt.

Assume the notation of [32] discussed in Section 3.8. Let \( R_2^q = \mathbb{Z}/q\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}] \) denote the commutative ring of Laurent polynomials in \( u_1, u_2 \) over the ring \( \mathbb{Z}/q\mathbb{Z} \), and define \( g = g_{T, w} \in R_2^q \) by
\[
g_{T, w} = \sum_{m \in T} w(m)u^m.
\]
The shift space \( \Omega = \Omega^{R_2^q/(g)} \), defined in [32, page 719] as
\[
\Omega = \left\{ f = (f(n)) \in (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2} : \sum_{i \in T} w(i)f(i + n) = 0 \pmod q \text{ for } n \in \mathbb{Z}^2 \right\}
\]
(4.2.1) is a compact subspace of \( (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2} \) in the product topology, and carries an action of \( \mathbb{Z}^2 \) defined by \((\alpha_p f)(n) = f(n + p)\). Recall from Section 3.4 that the two-sided infinite path space \( \Lambda^\Delta \) of a 2-graph \( \Lambda(T, q, 0, w) \) is compact and carries a \( \mathbb{Z}^2 \)-action \( \sigma \) defined for \( p \in \mathbb{Z}^2 \) by
\[
\sigma_p(x)(m, n) = x(m + p, n + p).
\]
In the next result we show that there is a homeomorphism from \( \Omega \) to the two-sided infinite path space \( \Lambda^\Delta \) of \( \Lambda(T, q, 0, w) \) which is compatible with the shift maps.

**Theorem 4.2.1.** Suppose that we have basic data \((T, q, 0, w)\) and \( w \) has invertible corners, let \( \Lambda := \Lambda(T, q, 0, w) \) be the associated 2-graph, and define \( \Omega \) as in (4.2.1). Then there is a homeomorphism \( h : \Lambda^\Delta \to \Omega \) such that \( \alpha_p \circ h = h \circ \sigma_p \).

**Proof.** Define \( h : \Lambda^\Delta \to (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}^2} \) by
\[
h(x)(i) = x(i, i)(0) \text{ for } x \in \Lambda^\Delta, i \in \mathbb{Z}^2.
\]
Let \( j \in T \). Since \( x(i - j, i) \) is a path in \( \Lambda \), it is a well-defined function on \( T(j) \) and
\[
x(i, i)(0) = x(i - j, i)|_{T+j}(0) = x(i - j, i)(j + 0) = x(i - j, i - j)(j).
\]
Then \( h(x) \in \Omega \) since for all \( j \in \mathbb{Z}^2 \) the formula (4.2.2) gives
\[
\sum_{i \in T} w(i)h(x)|_{T+j}(i) = \sum_{i \in T} w(i)h(x)(i + j)
= \sum_{i \in T} w(i)x(i + j, i + j)(0)
= \sum_{i \in T} w(i)x(j, j)(i),
\]
which is \( 0 \pmod q \) since \( x \in \Lambda^\Delta \).
To see that \( h : \Lambda^\Delta \rightarrow \Omega \) is a homeomorphism it suffices to show \( h \) is a continuous bijection (since every continuous bijection from a compact space to a Hausdorff space is a homeomorphism). For \( f \in \Omega \) define a function \( k(f) : \Delta \rightarrow \Lambda \) by

\[
k(f)(m,n) = f|_{T(n-m)+m} \text{ for } m \leq n.
\]

Condition (4.2.1) implies that \( k(f)(m,n) \) is a path in \( \Lambda \) of degree \( n - m \), so \( k \) is degree-preserving. An application of Proposition 4.1.4 says that \( k(f)(m,p) = f|_{T(p-m)+m} \) factors as

\[
f|_{T(p-m)+m} = f|_{T(n-m)+m} f|_{T(p-n)+n},
\]

which gives

\[
k(f)(m,p) = k(f)(m,n)k(f)(n,p),
\]

so \( k(f) \) is a functor.

We claim that \( k : \Omega \rightarrow \Lambda^\Delta \) is the inverse of \( h \). We have \( h(k(f)) = f \) since for \( i \in \mathbb{Z}^2 \)

\[
h(k(f))(i) = k(f)(i, i)(0) = f|_{T(0)+i}(0) = f(i + 0) = f(i).
\]

Now we must show \( k(h(x)) = x \). Let \( i \in T(n-m) \). Then \( i = j + l \) for some \( j \in T \) and \( 0 \leq l \leq n - m \). We have

\[
(k(h(x))(m,n))(i) = h(x)|_{T(n-m)+m}(i)
\]

\[
= h(x)(i + m)
\]

\[
= h(x)(j + l + m)
\]

\[
= x(j + l + m, j + l + m)(0)
\]

\[
= x(l + m, l + m)(j) \text{ by (4.2.2)}
\]

\[
= x(m,n)|_{T+l}(j)
\]

\[
= x(m,n)(j + l)
\]

\[
= x(m,n)(i),
\]

so \( k(h(x)) = x \).

To see that \( h \) is continuous, suppose that \( x_\gamma \rightarrow x \) in \( \Lambda^\Delta \). Since \( \Omega \) has the product topology, it suffices to prove that \( h(x_\gamma)(i) \rightarrow h(x)(i) \) for all \( i \in \mathbb{Z}^2 \). Let \( i \in \mathbb{Z}^2 \). Then \( Z(x(i,i), 0) \) is an open neighbourhood of \( x \) in \( \Lambda^\Delta \), so for large \( \gamma, x_\gamma \in Z(x(i,i), 0) \). But then for large \( \gamma \) we have \( h(x_\gamma)(i) = x_\gamma(i,i)(0) = x(i,i)(0) = h(x)(i) \), so certainly \( h(x_\gamma(i)) \rightarrow h(x)(i) \).

For the last part we have \( h(\sigma_p(x)) = \alpha_p(h(x)) \) since

\[
h(\sigma_p(x))(i) = (\sigma_p x)(i,i)(0) = x(i + p, i + p)(0) = h(x)(i + p) = \alpha_p(h(x))(i).
\]

\[\square\]

Remark 4.2.2. Theorem 4.2.1 implies in particular that the shift space \( \Omega \) associated to the Ledrappier graph \( L(\mathbb{Z}) \) is the 2-dimensional Markov system of (3.8.1) known as Ledrappier’s example.
Remark 4.2.3. There is a one-sided version of Theorem 4.2.1. The space
\[ \Omega^+ := \left\{ f : \mathbb{N}^2 \to \mathbb{Z} / q\mathbb{Z} : \sum_{i \in T} w(i)f|_{T+n(i)} = 0 \pmod q \text{ for all } n \in \mathbb{N}^2 \right\}. \]
has a natural action of \( \mathbb{N}^2 \), and the \( \mathbb{Z}^2 \) action on \( \Lambda^\Delta \) restricts to an \( \mathbb{N}^2 \) action on the one-sided infinite path space \( \Lambda^\infty \). Then the argument of Theorem 4.2.1 gives a homeomorphism of \( \Lambda^\infty \) onto \( \Omega^+ \) which commutes with the actions of \( \mathbb{N}^2 \).

Remark 4.2.4. We saw in §4.1.3 and §4.1.4 that relaxing our hypotheses on the rule or using higher-dimensional tiles would lead to situations where we have to nominate blue-red to red-blue factorisations to define a \( k \)-graph \( \Lambda \). In the two-dimensional case, this would mean that if \( d(\lambda) = e_1 + e_2 \), then \( \lambda((c_2+1)e_2) \) will depend on the choice of \( \lambda((c_1+1)e_1) \) as well as the values of \( \lambda \) on \( T \cup (T + e_1 + e_2) \). So the homeomorphism of Remark 4.2.3 will carry the infinite path space of \( \Lambda \) onto a proper subspace of the shift space \( \Omega^+ \).

### 4.3. Strong connectivity

Here we show that each of the 2-graphs \( \Lambda = \Lambda(T, q, t, w) \) of Theorem 4.1.7 is strongly connected in the sense that every \( v \Lambda^* u \) is nonempty. (We will use this result in our proof of aperiodicity in Section 4.4.)

Proposition 4.3.1. Suppose that \( k \in \mathbb{N} \) satisfies \( (k-1)(e_1+e_2) \in T \) and \( k(e_1+e_2) \notin T \). Then for every \( v, u \in \Lambda^0 \) there exists \( \lambda \in \Lambda^{(e_1+e_2)} \) such that \( r(\lambda) = v \) and \( s(\lambda) = u \).

The proof of Proposition 4.3.1 depends on the following variant of Proposition 4.1.8(b).

Lemma 4.3.2. If \( v, u \in \Lambda^0 \) satisfy
\[(4.3.1) \quad v(j) = u(j - e_1 - e_2) \text{ for } j \in T \cap (T + e_1 + e_2),\]
then there is a unique path \( \mu \in \Lambda^{e_1+e_2} \) such that \( r(\mu) = v \) and \( s(\mu) = u \).

Proof. We define
\[ \mu(j) = \begin{cases} v(j) & \text{if } j \in T \\ u(j - (e_1 + e_2)) & \text{if } j \in (T + e_1 + e_2) \setminus T. \end{cases} \]
then we obviously have \( \mu|_T = v \), and (4.3.1) says that \( \mu|_{T + e_1 + e_2} = u \). Now we observe that because the corners \( w(e_i e_i) \) are invertible, there are unique values of \( \mu((e_i + 1)e_i) \) such that \( \mu|_{T + e_i} \) belongs to \( \Lambda^0 \). So there is exactly one path \( \mu \) with the required property. \( \square \)

Proof of Proposition 4.3.1. We will prove by induction on \( p \) that for \( 0 \leq p \leq k \), there exists \( \mu^p \in \Lambda^{p(e_1+e_2)} \) such that \( r(\mu^p) = v \) and
\[(4.3.2) \quad s(\mu^p)(j) = u(j - (k-p)(e_1 + e_2)) \text{ for } j \in T \cap (T + (k-p)(e_1 + e_2)).\]
Then \( \mu := \mu^k \) is the required path.

For \( p = 0 \), we take \( \mu^0 := v \). Suppose that \( 0 \leq p < k \) and we have \( \mu^p \) with the required properties. Now we define
\[(4.3.3) \quad \nu^{p+1}(i) = \begin{cases} s(\mu^p)(i + e_1 + e_2) & \text{for } i \in T \cap (T - e_1 - e_2) \\ u(i - (k-p-1)(e_1 + e_2)) & \text{for } i \in T \cap (T + (k-p-1)(e_1 + e_2)). \end{cases} \]
if \( i \) belongs to both sets on the right-hand side, then we can apply (4.3.2) with \( j = i + e_1 + e_2 \) and deduce that the two possible values for \( v^{p+1}(i) \) coincide. We now define \( v^{p+1}(i) \) arbitrarily for other points \( i \) in \( T \setminus \{ e_1 e_1 \} \), and set

\[
v^{p+1}(c_1 e_1) := w(c_1 e_1)^{-1} \left( \sum_{i \in T \setminus \{ c_1 e_1 \}} w(i) v^{p+1}(i) \right),
\]

so that \( v^{p+1} \in \Lambda^0 \). The first option in (4.3.3) implies that the pair \( s(\mu^p) \) and \( v^{p+1} \) satisfy (4.3.1), and hence by Lemma 4.3.2 there exists a path \( \nu \in \Lambda^{e_1 + e_2} \) with \( r(\nu) = s(\mu^p) \) and \( s(\nu) = v^{p+1} \). Now we take \( \mu^{p+1} \) to be the composition \( \mu^p \nu \), and the second option in (4.3.3) implies that \( s(\mu^{p+1}) \) satisfies (4.3.2). \( \square \)

**Example 4.3.3.** (Outline of proof of Proposition 4.3.1) Suppose that \( q = 2 \), \( t = 0 \), \( T = \{ 0, e_1, 2e_1, e_2, e_1 + e_2, 2e_2 \} \), and \( w \equiv 1 \). Let \( v, u \in \Lambda^0 \) be the vertices defined by \( v(i) = 0 \), \( u(i) = 1 \) for all \( i \in T \). We claim that there is a path \( \lambda \) of degree \( 2(e_1 + e_2) \) from \( u \) to \( v \), which looks like the diagram below with the missing areas filled in.

![Diagram](image)

The first step is to complete the translate at \( T + e_1 + e_2 \) so that it is a vertex which overlaps appropriately with \( v \) and \( u \). In the diagram (a) we choose any values for the bullets • and then there is a unique value for * which completes a vertex (call it \( v^1 \)) on \( T + e_1 + e_2 \). For example, let the bullets be 0s, then * = 1 as in diagram (b).

![Diagram](image)

Lemma 4.3.2 says there is a path \( \nu \in \Lambda^{e_1 + e_2} \) from \( v^1 \) to \( v \), so there are values for the * symbols in diagram (c). So we get diagram (d). Let \( \mu^1 = v \nu = \nu \).

![Diagram](image)

Repeating the last step, there is a path \( \nu^1 \in \Lambda^{e_1 + e_2} \) from \( u \) to \( v^1 \), so we get diagram (e). Let \( \mu^2 = \mu^1 \nu^1 \), which is the path in diagram (f). Then \( \lambda := \mu^2 \) is a path of degree \( 2(e_1 + e_2) \).
4.4. APERIODICITY

from $u$ to $v$.

\begin{align*}
\begin{array}{cccc}
1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots
\end{array}
\end{align*}

\begin{align*}
\begin{array}{cccc}
1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
\end{align*}

Remark 4.3.4. Strong connectivity of $\Lambda = \Lambda(T, q, t, w)$ implies that the skeleton of $\Lambda$ is irreducible as a directed graph (see Section 2.2). In general, the blue and red component graphs of the skeleton of $\Lambda$ may not be irreducible. For example, neither the blue graph $B\Lambda = (\Lambda^0, \Lambda^e, r, s)$ or the red graph $R\Lambda = (\Lambda^0, \Lambda^{e2}, r, s)$ of the tile $\square$, with $q = 2$, $t = 0$, and $w \equiv 1$, is connected.

4.4. Aperiodicity

Here we investigate the aperiodicity of our 2-graphs and, as discussed in Section 3.6, we will use the formulation of aperiodicity which is due to Robertson and Sims: $\Lambda$ is aperiodic if for every $v \in \Lambda^0$ and $m, n \in \mathbb{N}^2$ with $m \neq n$, there is a path $\lambda \in \Lambda$ satisfying

$$r(\lambda) = v, \ d(\lambda) \geq m \lor n$$

Recall that $(c_1, c_2) = \bigvee \{i : i \in T\}$. To prove aperiodicity we assume that the tile $T$ has $c_1 \geq 0$ and $c_2 \geq 0$. (In Chapter 6 we will see examples with $c_1 = 1$ or $c_2 = 1$ for which aperiodicity fails.) We also need to make another restriction on the rule $w$.

Definition 4.4.1. The rule $w$ has three invertible corners if $w(0), w(c_1e_1)$, and $w(c_2e_2)$ are all invertible in $\mathbb{Z}/q\mathbb{Z}$.

We show in Example 4.4.2 that aperiodicity may fail if $w(0)$ is not invertible. We will also simplify things by assuming that the trace $t$ is zero, and we will discuss this hypothesis after the proof of Theorem 4.4.3.

Example 4.4.2. Consider the data consisting of the sock tile $T = \{0, e_1, e_2\}$, $q = 2$, $t = 0$, and rule defined by $w(0) = 0, w(e_1) = w(e_2) = 1$. The vertices are:

\begin{align*}
0 & \quad 0 \quad 0 \quad 0 \\
0 & \quad 0 \quad 1 \quad 0 \quad 1 \\
1 & \quad 0 \quad 1 \quad 1 \quad 1
\end{align*}

Since every vertex $v$ has $v(e_1) = v(e_2)$, every path is constant along the short diagonals $n_1 + n_2 = c$. In other words, for every path $\lambda$ and every $n$, we have $\lambda(n) = \lambda(n + e_1 - e_2)$ whenever $n$ and $n + e_1 - e_2$ lie in the domain of $\lambda$. This implies in particular that for every path $\lambda$ with $d(\lambda) \geq (1, 1)$, we have

$$\lambda(e_2, d(\lambda) - e_1) = \lambda(e_1, d(\lambda) - e_2),$$

so (4.4.1) fails for every $v$ with $m = e_2$ and $n = e_1$.

Theorem 4.4.3. If the rule $w$ in the basic data $(T, q, 0, w)$ has $c_1 \geq 1$, $c_2 \geq 1$, and three invertible corners, then the associated 2-graph $\Lambda$ is aperiodic.
We deal separately with the cases where \( m \leq v \) identically zero function \( A \) labelled satisfies (4.4.1); indeed, we claim that the two paths in (4.4.1) have different sources \( \mu \). Since \( c \not= 4.2.3 \), we choose a path \( \mu \) with \( r(\mu) = v \) and \( d(\mu) = m \lor n \). We aim to extend \( \mu \) to a path \( \lambda \) satisfying (4.4.1). If the vertices \( \mu|_{T+m} \) and \( \mu|_{T+n} \) are different, then \( \lambda := \mu \) will do. So we suppose that \( \mu|_{T+m} = \mu|_{T+n} \).

We deal separately with the cases where \( m \leq n \) or \( n \leq m \), and where they are not comparable.

Suppose first that \( m \) and \( n \) are comparable, say \( m \leq n \). Since \( m \not= n \), there exists \( i \) such that \( m + e_i \leq n \), and then we have

\[
\lambda(m, m + d(\lambda) - (m \lor n)) = \lambda(m, m + e_i) = \mu(m, m + e_i)
\]

is not equal to

\[
\lambda(n, n + d(\lambda) - (m \land n)) = \lambda(n, n + e_i) = \nu.
\]

Now suppose that \( m \) and \( n \) are not comparable, say \( m_1 > n_1 \) and \( m_2 < n_2 \). This is where we use the extra hypotheses on the rule \( w \) and the trace \( t \). Since \( t = 0 \), the identically zero function \( v_0 : T \to \mathbb{Z}/q\mathbb{Z} \) defines a vertex \( v_0 \), and the identically zero function \( x : \mathbb{N}^2 \to \mathbb{Z}/q\mathbb{Z} \) defines an infinite path \( x \in \Lambda^\infty \) (via the homeomorphism of Remark 4.2.3). Since \( \Lambda^{e_1}v_0 \) has more than one element, and there is just one blue edge from \( v_0 \) to \( v_0 \) (see Proposition 4.1.8), there must be a blue edge \( \beta \) with \( s(\beta) = v_0 \) and \( r(\beta) \not= v_0 \). By Proposition 4.3.1, there is a path \( \alpha \) with \( r(\alpha) = s(\mu) \) and \( s(\alpha) = r(\beta) \). We claim that

\[
\lambda := \mu \alpha \beta x(0, (m \lor n) - (m \land n) - e_1) = \mu \alpha \nu, \quad \text{say},
\]

satisfies (4.4.1); indeed, we claim that the two paths in (4.4.1) have different sources (labelled \( A \) and \( B \) in the diagram below).
Since $d(\lambda) = d(\mu\alpha) + (m \lor n) - (m \land n)$ and
\[
\lambda|_{T+m+d(\lambda)-(m\lor n)} = \lambda|_{T+m+d(\mu\alpha)-(m\land n)} \\
= \nu|_{T+m-(m\land n)} \\
= x|_{T+(m_1-n_1-1)e_1} \\
= \nu_0,
\]
(which is the vertex labelled $B$), it suffices to show that
\[
\lambda|_{T+n+d(\lambda)-(m\lor n)} = \nu|_{T+n-(m\land n)} = \nu|_{T+(n_2-m_2)e_2}
\]
(which is the vertex labelled $A$) is not equal to $\nu_0$.

We suppose that there exists $p \in \mathbb{N}$ such that $\nu|_{T+pe_2} = \nu_0$, and look for a contradiction. Then there is a smallest such $p$, and since $\nu|_{T} = r(\beta) \neq \nu_0$, we then have $p > 0$. Now $\nu|_{T+(p-1)e_2} \in \Lambda^0$ implies
\[
(4.4.2) \quad w(0)\nu((p-1)e_2) = - \sum_{i \in T \setminus \{0\}} w(i)\nu(i + (p-1)e_2);
\]
since we have $\nu(l) = x(l) = 0$ whenever $l_1 > 0$, (4.4.2) implies that
\[
w(0)\nu((p-1)e_2) = - \sum_{j=1}^{c_2} w(je_2)\nu((j + p - 1)e_2),
\]
which is 0 because $\nu((k + p)e_2) = \nu|_{T+pe_2}(ke_2) = \nu_0(ke_2) = 0$ for $k \geq 0$. Since $w(0)$ is invertible, this implies that $\nu((p - 1)e_2) = 0$. Thus we have $\nu|_{T+(p-1)e_2} = \nu_0$, and this contradicts the choice of $p$. Thus for every $p$, $\nu|_{T+pe_2}$ is not equal to $\nu_0$, and in particular $\nu|_{T+(n_2-m_2)e_2}$ is not equal to $\nu_0$, as required. \hfill \Box

Remark 4.4.4. The preceding proof also works when $t \neq 0$ provided there is a vertex $\nu_0$ which is constant, say $\nu_0(m) = c$ for all $m \in T$. There is such a vertex if and only if there exists $c \in \mathbb{Z}/q\mathbb{Z}$ such that
\[
(4.4.3) \quad c\left(\sum_{i \in T} w(i)\right) = t \pmod{q}.
\]
However, we do not obtain any new 2-graphs this way: if there is such a $c$, then $\Lambda(T, q, t, w)$ is isomorphic to $\Lambda(T, q, 0, w)$. To see this, note that for every path $\lambda$ in $\Lambda(T, q, 0, w)$, $\lambda_t : i \mapsto \lambda(i) + c \pmod{q}$ is a path in $\Lambda(T, q, t, w)$, and the map $\lambda \mapsto \lambda_t$ is an isomorphism of $\Lambda(T, q, 0, w)$ onto $\Lambda(T, q, t, w)$.

It is easy to find examples where (4.4.3) has no solution $c$. For example, if $|T|$ is even, $q = 2$, $t = 1$, and $w \equiv 1$, we have $\sum w(i) = |T|$ and $c|T| = 1 \pmod{2}$ has no solutions. We do not have general criteria for aperiodicity when (4.4.3) has no solution.

Remark 4.4.5. Recall from Section 3.6 that the aperiodicity condition formulated by D. I. Robertson and Sims is based on Robertson and Steger’s original aperiodicity condition (H3). Since it is hard to check (H3) using only the matrices comprising a Robertson-Steger family, another condition, (H3*), was given in [61, §2] which implies (H3) and is easier to check. Here we discuss Robertson and Steger’s conditions as they apply to our 2-graphs.
Condition (H3*) is sufficient for (H3) provided that the matrices satisfy conditions (H0)–(H2) [61, Lemma 2.1]. We claim that the vertex matrices $B$ and $R$ of a 2-graph $\Lambda = \Lambda(T, q, t, w)$ satisfy (H0)–(H2). Condition (H0) requires that $B$ and $R$ be $\{0, 1\}$-matrices, which we know is true from Proposition 4.1.8. Condition (H1) is essentially existence of a unique path composition, which we have proved in Proposition 4.1.4. In the rank 2 case, (H1) is also equivalent to the condition that $B$ and $R$ commute and $BR = RB$ is a $\{0, 1\}$-matrix ([61, Lemma 1.1, Lemma 1.4]). Condition (H2) is satisfied if the skeleton of $\Lambda$ is irreducible as a directed graph or, equivalently, $|e^T| = 1$. To see this, let $j \in T$ tile with $m \in \mathbb{N}$ and $m_j = 0$ and let $\lambda \in \Lambda^m$. Then (H3*) is satisfied if there exist $\mu_1, \mu_2 \in \Lambda^{m+e_j}$ such that $\mu_1(0, m) = \mu_2(0, m) = \lambda$ but $\mu_1(0, e_j) \neq \mu_2(0, e_j)$. We claim that if $T$ is a tile with $|T(e_1) \setminus T(e_2)| = 1$ (for example, the sock tile) then (H3*) is not satisfied when $j = 1$. To see this, let $m \in \mathbb{N}$ with $m_1 = 0$ and $m_2 > 0$, and suppose that $\lambda \in \Lambda^m$. Since $|T(e_1) \setminus T(e_2)| = 1$, there exists a unique edge $\beta \in \Lambda^{e_1}$ such that $\beta(i) = \lambda(0, e_2)(i)$ for $i \in (T(e_1) \cap T(e_2))$. Hence every $\mu \in \Lambda^{m+e_1}$ with $\mu(0, m) = \lambda$, must have $\mu(0, e_1) = \beta$, which contradicts (H3*). (By symmetry, tiles with $|T(e_2) \setminus T(e_1)| = 1$ do not satisfy (H3*) when $j = 2$.)

### 4.5. The $C^*$-algebras

Under the same hypotheses which ensure aperiodicity of a 2-graph $\Lambda(T, q, 0, w) —$ that $T$ has $c_1, c_2 \geq 1$ where $(c_1, c_2) = \bigvee \{i : i \in T\}$ and $w$ has three invertible corners — the $C^*$-algebra $C^*(\Lambda(T, q, 0, w))$ is nuclear, simple, and purely infinite. This follows quickly from general results in [60] and [69] about the structure of $k$-graph algebras.

**Theorem 4.5.1.** Suppose that $(T, q, 0, w)$ is basic data with $c_1 \geq 1$ and $c_2 \geq 1$, and the rule $w$ has three invertible corners. Then $C^*(\Lambda(T, q, 0, w))$ is unital, nuclear, simple and purely infinite, and belongs to the bootstrap class $\mathcal{N}$.

**Proof.** We write $\Lambda$ for $\Lambda(T, q, 0, w)$. We begin by observing that $C^*(\Lambda)$ is unital because $\Lambda^0$ is finite, and is nuclear and belongs to the bootstrap class by [36, Theorem 5.5]. It follows easily from Proposition 4.3.1 that $\Lambda$ is cofinal: if $x \in \Lambda^\infty$ and $v \in \Lambda^0$, then there is a path from $r(x)$ to $v$. Since we know from Theorem 4.4.3 that $\Lambda$ is aperiodic (that is, satisfies property (iv) of [60, Lemma 3.2]), it follows from Theorem 3.1 and Lemma 3.2 of [60] that $C^*(\Lambda)$ is simple.

To see that $\Lambda$ is purely infinite, we need to check that every vertex $v$ can be reached from a “cycle with an entrance” (see [69, Proposition 8.8]). But we know that $v$ receives at least two blue edges $\alpha, \beta$, and then Proposition 4.3.1 implies that there is a path $\nu$ from $v$ to $s(\alpha)$, so there is a path $\mu = \alpha \nu$ with $d(\mu) \neq 0$ such that $r(\mu) = s(\mu) = v$. Since
\( \beta \) is an entrance to \( \mu \), we have verified the hypothesis of \cite[Proposition 8.8]{69}, and can deduce that \( C^*(\Lambda) \) is purely infinite. \( \square \)

4.6. \textit{K-theory methods}

The main theorem of the previous section (Theorem 4.5.1) implies that, when the rule has three invertible corners, the \( C^* \)-algebra falls into the class which is classified by the celebrated theorem of Kirchberg and Phillips, which says that \( C^*(\Lambda) \) is determined up to isomorphism by its \textit{K}-theory \cite{29, 50, 63}. So we want to compute the \( K \)-groups of \( C^*(\Lambda(T, q, t, w)) \). We do this using the techniques developed by Robertson-Steger \cite{62} and Evans \cite{17} which identify \( K_0(C^*(\Lambda(T, q, t, w))) \) and \( K_1(C^*(\Lambda(T, q, t, w))) \) in terms of the kernels and cokernels of the vertex matrices of \( \Lambda \). Our 2-graphs are finite but large, so we have used the computational algebra system \textit{Magma} \cite{4} to compute these kernels and cokernels. In this section we outline the procedure used in our calculations and make some general conjectures about the \textit{K}-theory of our 2-graphs based on the results.

4.6.1. \textbf{Procedure for calculating \textit{K}-theory.} Suppose that we have basic data satisfying the hypotheses of Proposition 4.1.8, so that in particular the associated 2-graph \( \Lambda \) is finite with no sources, and the methods of \cite{17} apply. As in Section 3.7, if \( B \) and \( R \) are the vertex matrices of \( \Lambda \) and \( \delta_1 : \mathbb{Z}^0 \oplus \mathbb{Z}^0 \to \mathbb{Z}^0 \) and \( \delta_2 : \mathbb{Z}^0 \to \mathbb{Z}^0 \oplus \mathbb{Z}^0 \) are the maps with matrices

\[
\delta_1 = \begin{pmatrix} 1 - B^t & 1 - R^t \end{pmatrix} \quad \text{and} \quad \delta_2 = \begin{pmatrix} R^t - 1 & 1 - B^t \end{pmatrix},
\]

then Proposition 3.16 of \cite{17} says that the \( K \)-groups are given by

\[
K_0(C^*(\Lambda)) \cong \text{coker} \ \delta_1 \oplus \text{ker} \ \delta_2 \quad \text{and} \quad K_1(C^*(\Lambda)) \cong \text{ker} \ \delta_1 / \text{img} \ \delta_2.
\]

We will see later in Lemma 4.7.9 that both \( K \)-groups are finite. We were able to calculate the size of the \( K \)-groups for a large number of examples by implementing the following procedure in the \textit{Magma} computational algebra system. The \textit{Magma} system recognises that we are dealing with integer matrices and so it performs calculations over the integers; for example, on being asked to find a basis for the columnspace of an integer matrix it returns an integer basis. When calculating \( |K_0(C^*(\Lambda))| \), we noticed that \( \ker \delta_2 = 0 \) in every example. To calculate \( |\text{coker} \ \delta_1| \), we find a basis matrix \( M \) whose columns are an integer basis for the columnspace of the matrix of \( \delta_1 \). Then

\[
|K_0(C^*(\Lambda))| = |\text{coker} \ \delta_1| = |\det M|.
\]

To calculate \( |K_1(C^*(\Lambda))| \), first we find a basis matrix \( H \) for \( \ker \delta_1 \). Since the columns of \( H \) are linearly independent, for each column vector \( z \) of the matrix of \( \delta_2 \) the equation \( Hw = z \) has a unique solution \( w \). Form the matrix whose columns are the solutions \( w \); then the \( i \)-th column of \( W \) contains the coordinates of the basis vector \( z \) with respect to the basis for \( \ker \delta_1 \). Thus

\[
|K_1(C^*(\Lambda))| = |\ker \delta_1 / \text{img} \ \delta_2| = |\det W|.
\]
We give details of these calculations for the Ledrappier graph.

**Example 4.6.1.** Consider the sock tile \(\begin{array}{c|c|c} \end{array}\) with \(q = 2\), \(t = 0\), and rule \(w \equiv 1\). Vertex matrices for the 2-graph \(\Lambda\) are

\[
(4.6.1) \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.
\]

The matrices \(\delta_1 : \mathbb{Z}^8 \to \mathbb{Z}^4\) and \(\delta_2 : \mathbb{Z}^4 \to \mathbb{Z}^8\) are

\[
\delta_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \delta_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

Here, \(\delta_1\) is onto so we can choose a basis for \(\text{img} \delta_1\) such that \(M\) is the \(4 \times 4\) identity matrix; hence \(|\det M| = 1\), and \(|K_0(C^\ast(\Lambda))| = 1\): Magma gives us the matrices \(H\) and \(W\) below, which satisfy \(HW = \delta_2\).

\[
H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.
\]

Then since \(|\det W| = 1\), we have \(|K_1(C^\ast(\Lambda))| = 1\).

We have included Magma code for these procedures in \(\S C.2.1\) in Appendix C. The results of our calculations are presented in tables in Appendix D.

### 4.6.2. Conjectures.

We begin with the following comments.

- A tile \(T\) can be uniquely described as an integer partition by listing the lengths of its rows from longest to shortest (see Appendix A); for example, we write \([2, 1]\) for the sock. The tile obtained by reflecting \(T\) about the line \(y = x\) is called the *conjugate tile* of \(T\); for example, the conjugate of the tile \([3, 1]\) is \([2, 1, 1]\) and the sock tile is its own conjugate. A tile and its conjugate give \(C^\ast\)-algebras with the same \(K\)-theory since this amounts to swapping the roles of \(B\) and \(R\) in the \(K\)-theory formulas. So in the tables we list conjugate tiles in pairs.
4.6. K-THEORY METHODS

- The results in the tables refer to basic data with \( t = 0 \) and \( w \equiv 1 \). We also performed calculations for other rules and traces, but we obtained the same values of \( |K_0| \) and \( |K_1| \). A partial explanation for this is in Remark 4.4.4.

A more detailed look at the results of our calculations suggests the following conjectures.

**Conjectures.**

1. \( \ker \delta_2 = 0 \).
2. \( |K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))| \).
3. \( |K_i(C^*(\Lambda))| \) always has the form \( q^n - 1 \). Our calculations are consistent with the formula \( |K_i(C^*(\Lambda))| = \gcd(q^{c_2} - 1, q^{c_1} - 1) \).

We prove Conjectures (1) and (2) in Theorems 4.7.1 and 4.7.10 in Section 4.7. We do not know whether \( K_0(C^*(\Lambda)) \) is isomorphic to \( K_1(C^*(\Lambda)) \) in general, though calculations in Magma confirm that \( K_i(C^*(\Lambda)) \) is cyclic in all the examples listed in Table D.1, and hence \( K_0 \) is isomorphic to \( K_1 \) for all these examples (see §C.2.2 in Appendix C). However, this is automatically true in most cases because there is only one group of the given order, so the number of examples we have considered where there is something to prove (that is, the ones where \( |K_i| = 4 \) or \( 8 \)) is fairly small. Certainly we have not yet identified a potential reason for the existence of such an isomorphism, and our proof of Conjecture (2) does not help. We have only numerical evidence for Conjecture (3).

4.6.3. Implications for the classification. The graphs whose \( K \)-theory is computed in Table D.1 are of two types. For tiles with \( c_2 = 0 \), which we consider in Chapter 6, the aperiodicity condition of [60] fails, so the \( C^* \)-algebras of these graphs are not simple (see Lemma 6.3.1). For all other graphs, the basic data satisfies the hypotheses of Theorem 4.5.1, and hence the \( C^* \)-algebras are simple and satisfy the hypotheses of the Kirchberg-Phillips Theorem. The Kirchberg-Phillips Theorem (as stated in [63, Theorem 8.4.1(iv)], for example) says that two suitable unital \( C^* \)-algebras \( A \) and \( B \) are isomorphic if and only if \( K_1(A) \cong K_1(B) \) and there is an isomorphism of \( K_0(A) \) onto \( K_0(B) \) which takes the class \( [1_A] \) of the identity to \( [1_B] \). When \( |K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))| = 1 \), the last condition is trivially satisfied and \( C^*(\Lambda) \) is isomorphic to the Cuntz algebra \( O_2 \).

(Somewhat disappointingly, the Ledrappier graph is one of these graphs.) When \( |K_i| > 1 \), we computed the class of \( [1] = \sum_{v \in \Lambda^0} [p_v] \) in \( K_0(C^*(\Lambda)) = \ker \delta_1 \) (see [17, Corollory 5.1]), and found that it is always a generator for \( K_0(C^*(\Lambda)) \) (see §C.2.2 in Appendix C). So in all our examples, \( K_0(C^*(\Lambda)) \) is cyclic. We do not know whether this is always true. To sum up: if \( \Lambda_1 \) and \( \Lambda_2 \) are any graphs in the table with \( c_2 \geq 1 \), and if \( K_0(C^*(\Lambda_1)) = K_0(C^*(\Lambda_2)) \), then \( C^*(\Lambda_1) \cong C^*(\Lambda_2) \).

None of the \( C^* \)-algebras of graphs in Table D.1 with nonzero \( K \)-theory can be isomorphic to the \( C^* \)-algebra of an ordinary directed graph \( E \), because \( K_1(C^*(E)) \) is always free (being a subgroup of the free group \( \mathbb{Z}E^0 \)).
4.7. K-theory results

In this section we prove some results about $K_i(C^*(\Lambda(T,q,t,w)))$, under mild hypotheses on the shape of the tile. Perhaps the most surprising result is that, under mild hypotheses, $K_0$ and $K_1$ have the same finite cardinality — though our proof of this is indirect, and gives us no hint of whether the groups are actually isomorphic.

Let $T$ be a tile, and again write $(c_1,c_2) = \sqrt{\{j : j \in T\}}$. For $0 \leq i \leq c_1$, we let $h_i$ denote the second coordinate of the top box $(i,h_i)$ in each column of $T$; for $0 \leq i \leq c_2$, $w_i$ is the first coordinate of the right-hand box $(w_i,i)$ in each row.

**Theorem 4.7.1.** Suppose that we have basic data $(T,q,t,w)$ in which $w$ has invertible corners and $c_1,c_2 \geq 1$. Suppose further that either $h_0 > h_1$ or $w_0 > w_1$. If $B$ and $R$ are the vertex matrices associated to $\Lambda = \Lambda(T,q,t,w)$, then the map

$$\delta_2 = \begin{pmatrix} R^t - 1 \\ 1 - B^t \end{pmatrix} : \mathbb{Z}^{\Lambda^0} \to \mathbb{Z}^{\Lambda^0} \oplus \mathbb{Z}^{\Lambda^0}$$

has trivial kernel and $K_0(C^*(\Lambda)) = \text{coker} \, \delta_1$.

Proposition 3.16 of [17] says that the bijection of $\text{coker} \, \delta_1$ onto $K_0(C^*(\Lambda))$ carries the generator $\delta_e$ of $\mathbb{Z}^{\Lambda^0}$ to $[p_e]$, and therefore Theorem 4.7.1 says that these generate $K_0(C^*(\Lambda))$. The image of $\delta_1$ is then generated by the images of the elements $(1 - B^t)\delta_e$ and $(1 - R^t)\delta_e$. Thus Theorem 4.7.1 says that, for one of our 2-graphs $\Lambda$, $K_0(C^*(\Lambda))$ is generated by $\{[p_e] : v \in \Lambda^0\}$ modulo the relations

$$[p_e] = \sum_{r(e)=v,d(e)=e_1} [p_{s(e)}], \quad [p_e] = \sum_{r(e)=v,d(e)=e_2} [p_{s(e)}]$$

imposed by the blue and red Cuntz-Krieger relations.

Theorem 4.7.1 will follow immediately from the following proposition. For the rest of the section, we fix a set of basic data $(T,q,t,w)$ in which $w$ has invertible corners and $c_1,c_2 \geq 1$.

**Proposition 4.7.2.** Let $B$ and $R$ be the vertex matrices of $\Lambda = \Lambda(T,q,t,w)$.

1. If $h_0 > h_1$ then the map $1 - B^t : \mathbb{Z}^{\Lambda^0} \to \mathbb{Z}^{\Lambda^0}$ has trivial kernel.
2. If $w_0 > w_1$ then the map $1 - R^t : \mathbb{Z}^{\Lambda^0} \to \mathbb{Z}^{\Lambda^0}$ has trivial kernel.

To prove this we use the special structure of the vertex matrices of $\Lambda$. By symmetry it suffices to prove part (1). From Proposition 4.1.8 we know that $B$ is a $\{0,1\}$-matrix; that the number of 1s in each row/column is $q^2$; and that any two rows/columns are either equal or orthogonal. The crucial observation is that the matrices with these properties are the ones which arise as the vertex matrices of dual graphs (see Section 2.2).

To describe the graphs whose duals arise we need some notation. Let $S$ be the tile $S := T \cap (T - e_1)$ and let $S^+$ be the tile $S^+ := S \cup \{(h_1+1)e_2\}$. In the visual model, $S$ is the tile obtained from $T$ by deleting the first column and shifting one unit to the left, and $S^+$ is obtained from $S$ by adding one box to the top of its first column. For a directed graph $F$ and an integer $n \geq 1$, the directed graph $nF$ has vertex set $(nF)^0 = F^0$, edge set

$$(nF)^1 = F^1 \times \{1, \ldots , n\} = \{(f,i) : f \in F^1, 1 \leq i \leq n\}$$
and range and source maps given by \( r(f, i) = r(f) \) and \( s(f, i) = s(f) \). Then the vertex matrix of \( nF \) is \( n \) times the vertex matrix of \( F \). (Note if \( n = 1 \) then \( 1F \cong F \).

**Proposition 4.7.3.** Suppose that \( h_0 > h_1 \). Set \( r_B = q^{h_0 - h_1 - 1} \), and let \( \Lambda(S^+, q, 0, 1) \) be the blue graph of the tile \( S^+ \) with alphabet \( q \), trace 0 and rule which is identically 1. Then the blue graph \( \Lambda \) of \( \Lambda(T, q, t, w) \) is isomorphic to the dual of \( r_B \Lambda(S^+, q, 0, 1) \).

To prove this we need the following lemma.

**Lemma 4.7.4.** Let \( v_1, v_2 \in \Lambda^0 \). Then the set

\[
(4.7.1) \quad \{ u \in \Lambda^0 : u|_{S+e_1} = v_2|_S \text{ and } u|_S = v_1|_S \}
\]

contains \( r_B = q^{h_0 - h_1 - 1} \) vertices if \( v_1, v_2 \) satisfy

\[
(4.7.2) \quad v_1(i) = v_2(i - e_1) \text{ for } i \in S \cap (S + e_1),
\]

and is empty otherwise.

**Proof.** If \((4.7.2)\) holds, then define a function \( u : T \to \mathbb{Z}/q\mathbb{Z} \) by \( u|_{S+e_1} = v_2|_S \) and \( u|_S = v_1|_S \). Since \(|T \setminus (S \cup (S + e_1))| = h_0 - h_1 > 0\), there are \( r_B \) such functions \( u \) which define vertices in \( \Lambda^0 \).

Define a relation \( \sim \) on \( \Lambda^0 \) by

\[
v_1 \sim v_2 \iff v_1|_S = v_2|_S.
\]

It is straightforward to check that \( \sim \) is an equivalence relation. Let \([v]\) denote the equivalence class of \( v \in \Lambda^0 \) under \( \sim \). By definition of \( \sim \) the set in Lemma 4.7.4 does not change if we replace \( v_1 \) and \( v_2 \) by other elements of \([v_1]\) and \([v_2]\). So for \( v_1, v_2 \) satisfying \((4.7.2)\), we can list the vertices in the set \((4.7.1)\) as \( u_i([v_1], [v_2]) \) for \( 1 \leq i \leq r_B \).

**Proof of Proposition 4.7.3.** Let \( F \) be the directed graph with vertices \( F^0 = \Lambda^0/\sim \) and edges

\[
F^1 = \{ ([v_1], [v_2]) \in F^0 \times F^0 : v_1, v_2 \text{ satisfy } (4.7.2) \}.
\]

We prove first that \( \Lambda \) is isomorphic to the dual of the directed graph \( r_B F \), and then that \( F \) is isomorphic to \( \Lambda \Lambda(S^+, q, 0, 1) \).

By definition, \( r_B F \) has vertices \( F^0 \) and edges

\[
(r_B F)^1 = \{ ([v_1], [v_2], i) : ([v_1], [v_2]) \in F^1, 1 \leq i \leq r_B \}.
\]

So the dual \( \widehat{r_B F} \) has vertices \( (\widehat{r_B F})^0 = (r_B F)^1 \) and there is an edge from \(([v_3], [v_4], j)\) to \(([v_1], [v_2], i)\) if and only if \([v_2] = [v_3]\).

Define a map \( \phi^0 : (r_B F)^0 \to \Lambda^0 \) by \( \phi^0((v_1), [v_2], i) = u_i([v_1], [v_2]) \). Then \( \phi^0 \) is a bijection since \( u_i([v_1], [v_2]) \) is uniquely determined by \( ([v_1], [v_2]) \) and \( i \), and every \( v \in \Lambda^0 \) belongs to a set in \((4.7.1)\) for some \( v_1 \) and \( v_2 \) (for example, take \( v_1 = v \) and \( v_2 \) to be any vertex adjacent to \( v \)).
Suppose that there is an edge in $r_BF$ from $([v_3], [v_4], j)$ to $([v_1], [v_2], i)$ — that is, suppose that $[v_2] = [v_3]$. Then Proposition 4.1.8 says there is a unique edge in $BA$ from $\phi([v_3], [v_4], j) = u_j([v_3], [v_4])$ to $\phi([v_1], [v_2], i) = u_i([v_1], [v_2])$ since

$$u_i([v_1], [v_2])[S + e_1] = v_2|S = v_3|S = u_j([v_3], [v_4])|S.$$ 

Define the map $\phi^1 : (r_BF)^+ \rightarrow \Lambda^{e_1}$ by taking $\phi^1(([v_1], [v_2], i), ([v_3], [v_4], j))$ to be the unique edge in $BA$ with source $u_j([v_3], [v_4])$ and range $u_i([v_1], [v_2])$. Then $\phi^1$ is a bijection since $\phi^0$ is a bijection. We have $r \circ \phi^1 = \phi^0 \circ r$ since

$$r(\phi(([v_1], [v_2], i), ([v_3], [v_4], j))) = u_i([v_1], [v_2])$$

$$= \phi^0([v_1], [v_2], i)$$

$$= \phi^0(r(([v_1], [v_2], i), ([v_3], [v_4], j))),$$

and similarly $s \circ \phi^1 = \phi^0 \circ s$. Thus $\phi = (\phi^0, \phi^1)$ is a graph isomorphism from $r_BF$ to $BA$.

It remains to show that $F$ is isomorphic to $BA(S^+, q, 0, 1)$. Let $[v] \in F^0$. Define $v^+ : S^+ \rightarrow \mathbb{Z}/q\mathbb{Z}$ by

$$v^+|S = v|S \text{ and } v^+(0, h_1 + 1) = -\sum_{j \in S} v(j) \pmod{q}.$$ 

This is well-defined since $S^+ \setminus S = \{(0, h_1 + 1)\}$ and each element in $[v]$ takes the same values on $S$. We also have that $v^+$ is uniquely determined by $[v]$, and $v^+$ is clearly a vertex in $BA(S^+, q, 0, 1)$. Define $\psi^0 : F^0 \rightarrow BA(S^+, q, 0, 1)^0$ by $\psi^0([v]) = v^+$. Then we claim that $\psi^0$ is a bijection. It is one-to-one because $v^+$ is uniquely determined by $[v]$. To see that $\psi^0$ is onto, let $u$ be a vertex in $BA(S^+, q, 0, 1)$ and suppose that $u^- \in \Lambda^0$ with $u^-|S^+ = u$. Then $(u^-)^+|S = u^-|S = u|S$ which implies $(u^-)^+|S^+ = u|S^+$; this says that $\psi^0([u^-]) = (u^-)^+ = u$, so $\psi$ is onto.

Suppose that $([v_1], [v_2]) \in F^1$. Then (4.7.2) and $S^+ \cap (S^+ + e_1) = S \cap (S + e_1)$ imply

$$v^+_1(i) = v^+_2(i - e_1) \text{ for } i \in S^+ \cap (S^+ + e_1).$$

Now Proposition 4.1.8 implies that there is a unique edge in $BA(S^+, q, 0, 1)$ from $\psi^0([v_2]) = v^+_2$ to $\psi^0([v_1]) = v^+_1$. Define a map $\psi^1$ from $F^1$ to the edge set of $BA(S^+, q, 0, 1)$ by taking $\psi^1(([v_1], [v_2]))$ to be the unique edge in $BA(S^+, q, 0, 1)$ with source $v^+_2$ and range $v^+_1$. Then $\psi^1$ is a bijection since $\psi^0$ is a bijection. We have $r \circ \psi^1 = \psi^0 \circ r$ since

$$r(\psi(([v_1], [v_2]))) = \psi^0([v_1]) = \psi^0(r([v_1], [v_2]))$$

and similarly $s \circ \psi^1 = \psi^0 \circ s$. Thus $\psi = (\psi^0, \psi^1)$ is a graph isomorphism from $F$ to $BA(S^+, q, 0, 1)$. \qed

So the blue graph of $T$ is related to the blue graph of $S^+$, which is a tile with one fewer column than $T$. In fact we can repeatedly apply Proposition 4.7.3 since the new tile $S^+$ satisfies the hypotheses of that proposition. The tile $S^+_i$ in the next proposition is obtained from $T$ by deleting the first $i$ columns, shifting to the origin and adding one box to the new first column.
**Corollary 4.7.5.** Suppose that $h_0 > h_1$. Let $S^+_0 = T$ and for $1 \leq i \leq c_1$, let $S^+_i$ be the tile

$$S^+_i = (S^+_i \cap (S^+_i - e_1)) \cup \{(h_i + 1)e_2\},$$

define $r_{B_i}$ by

$$r_{B_i} = \begin{cases} q^{h_0 - h_1 - 1} & \text{if } i = 1 \\ q^{h_1 - h_i} & \text{if } i > 1, \end{cases}$$

and let $H_i$ be the directed graph $B\Lambda(S^+_i, q, 0, 1)$. Then $B\Lambda(T, q, t, w) \cong (r_{B_i}H_1)$ and $H_i \cong (r_{B_i}H_{i+1})$ for $1 \leq i \leq c_1 - 1$.

**Proof.** Applying Proposition 4.7.3 to $B\Lambda(T, q, t, w)$ gives the result for $i = 1$. Let $1 \leq i \leq c_1 - 1$. Each tile $S^+_i$ has columns $h_i + 1, h_i + 2, \ldots, h_{c_1}$. Since $T$ is hereditary, $h_i \geq h_{i+1}$. Then $S^+_i$ satisfies $(h_i + 1) - h_{i+1} > 0$ and so we can apply Proposition 4.7.3 to $H_i$ to get the result for $i > 1$.

**Examples 4.7.6.** (1) Suppose that $T$ is the tile with $c_1 = c_2 = 3$ and columns $h_0 = 3$, $h_1 = h_2 = 1$, $h_3 = 0$. Let $q = 2$, $t \in \mathbb{Z}/2\mathbb{Z}$, and $w$ is a rule with invertible corners. The tiles in Corollary 4.7.5 are

$$T = \begin{array}{ccc} \square & \square \end{array} \quad S^+_1 = \begin{array}{ccc} \square & \square & \square \end{array} \quad S^+_2 = \begin{array}{ccc} \square & \square \end{array} \quad S^+_3 = \begin{array}{ccc} \square \end{array}$$

and the constants are $r_{B_1} = 2$, $r_{B_2} = 1$, $r_{B_3} = 2$.

(2) Suppose that $T$ has columns $h_0, \ldots, h_{c_1}$ satisfying $h_0 = h_1 + 1$ and $h_1 = h_2 = \cdots = h_{c_1}$. Since $S^+_1$ is the tile with one column with $h_{c_1} + 1$ boxes, we have $S_1 \cap (S_1 + e_1) = \emptyset$. So in $H_{c_1}$ there is a directed edge between every pair of vertices, that is, $H_{c_1}$ is the complete graph $K_{q^{h_{c_1}}}$ with $q^{h_{c_1}}$ vertices. Corollary 4.7.5 implies that $r_{B_1} = r_{B_2} = \cdots = r_{B_{c_1}} = 1$ and so $B\Lambda(T, q, t, w)$ is obtained by $c_1$ times taking the dual of $K_{q^{h_{c_1}}}$. For example, the sock tile has $c_1 = 1$ and columns $h_0 = 1$ and $h_1 = 0$, and so the blue graph of the Ledrappier graph is isomorphic to $\widehat{K}_2$.

We need two more lemmas for the proof of Proposition 4.7.2.

**Lemma 4.7.7.** If $n \in \mathbb{Z}$ with $n > 1$ and $B$ is an integer matrix, then

$$\ker(1 - nB) = \{0\}.$$

**Proof.** Suppose that $v \in \ker(1 - nB)$, that is, $nBv = v$. We claim that $n^p|v$ for all $p \geq 1$. To see this, in the $p = 1$ case we have $v = nBv$ and so $v$ is $n$ times some vector $Bu \in \mathbb{Z}^B$. For the inductive step suppose that $n^p|v$. Then there exists $u \in \mathbb{Z}^B$ such that $v = n^pu$. Then

$$v = nBv = nB(n^pu) = n^{p+1}Bu$$

and so $n^{p+1}|v$. Hence $n^p|v$ for all $p \geq 1$, which is only possible if $v = 0$.

**Lemma 4.7.8.** Let $n > 1$ be an integer. If $K$ is the $n \times n$ matrix of all 1s, then

$$\ker(1 - K) = \{0\}.$$
The matrix $1 - K$ is the circulant matrix\(^1\) $\text{Circ}(v)$ with $v = (0, -1, \ldots, -1) \in \mathbb{Z}^n$. If $\omega$ is a primitive $n$th root of unity then using the formula for determinant of a circulant given in \([26]\) we have
\[
\det(1 - K) = \det \text{Circ}(v) = \prod_{j=0}^{n-1} \sum_{i=0}^{n-1} \omega^{ij} = \prod_{j=0}^{n-1} \sum_{i=1}^{n-1} \omega^{ij} = \prod_{j=1}^{n-1} \sum_{i=1}^{n-1} \omega^{ij} = \prod_{j=1}^{n-1} \sum_{i=1}^{n-1} \omega^{0j} = \prod_{j=1}^{n-1} (-1)^j \prod_{i=1}^{n-1} 1 = (-1)^{n-1}(n - 1).
\]
In particular $\det(1 - K) \neq 0$, so we have $\ker(1 - K) = \{0\}$.

\[\text{Proof of Proposition 4.7.2.}\] We can deduce (2) by applying part (1) to the conjugate tile, so it suffices to prove (1). Choose $r_{B_1}, \ldots, r_{B_{c_1}}$ and $B\Lambda, H_1, \ldots, H_{c_1}$ as in Corollary 4.7.5. Let $B, B_1, \ldots, B_{c_1}$ be the vertex matrices of $BA, H_1, \ldots, H_{c_1}$. Since the vertex matrix of a dual graph $\hat{E}$ is the edge matrix of $E$, Proposition 4.1 of \([43]\) gives isomorphisms
\[(4.7.3) \quad \ker(1 - B^i) \cong \ker(1 - r_{B_i}B_i^1) \text{ and } \ker(1 - B^i) \cong \ker(1 - r_{B_{i+1}}B_{i+1}^i),\]
for $1 \leq i \leq c_1 - 1$. If $r_{B_i} = 1$ for all $i$, then since the tile $S_{c_1}^+$ has only one column, Proposition 4.1.8(b) implies that every entry in the matrix $B_{c_1}$ is 1, $\ker(1 - B_{c_1}^i) = \{0\}$ by Lemma 4.7.8, and all the kernels in (4.7.3) are trivial. If there exists $r_{B_j}$ which is bigger than 1, then there is a first such $j$; then Lemma 4.7.7 implies $\ker(1 - r_{B_j}B_j^1) = \{0\}$, and $\ker(1 - B_j^i) = \{0\}$ for $i < j$. Hence $\ker(1 - B^i) = \{0\}$.

This completes the proof of Theorem 4.7.1, and hence settles Conjecture (1).

\[\text{Lemma 4.7.9.}\] Suppose that $\Lambda = \Lambda(T, q, t, w)$ is the 2-graph with basic data $(T, q, t, w)$ in which $w$ has invertible corners and $c_1, c_2 \geq 1$. If either $h_0 > h_1$ or $w_0 > w_1$, then both $K_0(C^*(\Lambda))$ and $K_1(C^*(\Lambda))$ are finite.

\[\text{Proof.}\] Let $C := 1 - B^t$ and $D := 1 - R^t$. First suppose that $h_0 > h_1$, so that $\ker C$ is trivial by Proposition 4.7.2. If $(x y)^t \in \ker \delta_1$, then $Cx + Dy = 0$. Then
\[
(C \oplus C)(x y)^t = (Cx - Cz)^t = (-Dy C y)^t \in \text{img} \delta_2.
\]
Thus $(C \oplus C) \ker \delta_1 \subseteq \text{img} \delta_2$, and hence
\[
|\ker \delta_1/\text{img} \delta_2| \leq |\ker \delta_1/(C \oplus C) \ker \delta_1|,
\]
which is finite since $C$ is invertible over $\mathbb{Q}$. Hence $K_1(C^*(\Lambda))$ is finite. Further, Theorem 4.7.1 says $K_0(C^*(\Lambda)) = \text{coker} \delta_1$. We have $\text{img} \delta_1 = \{(C \ D)(x y)^t : (x y)^t \in \mathbb{Z}^{A_0} \oplus \mathbb{Z}^{A_0}\} = \{Cx + Dy: x, y \in \mathbb{Z}^{A_0}\}$, which contains $\text{img} C$. Then $K_0(C^*(\Lambda)) = \mathbb{Z}^{A_0}/(CZ^{A_0} + DZ^{A_0})$ is finite since $C$ is invertible over $\mathbb{Q}$. Hence $K_0(C^*(\Lambda))$ is finite.

---

\(^1\)A circulant matrix is a square matrix in which each row vector is obtained from the previous row vector by rotating one element to the right. Then an $n \times n$ circulant matrix $\text{Circ}(v)$ can be fully specified by the first row vector $v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}^n$. (See \([13]\).)
If we suppose \( w_0 > w_1 \) instead, then \( D \) is invertible over \( \mathbb{Q} \) and similar arguments apply.

□

The next theorem settles Conjecture (2).

**Theorem 4.7.10.** Suppose that we have basic data \((T, q, t, w)\) in which \( w \) has invertible corners and \( c_1, c_2 \geq 1 \). Suppose further that \( h_0 > h_1 \) or \( w_0 > w_1 \). Then the \( C^* \)-algebra of the 2-graph \( \Lambda = \Lambda(T, q, t, w) \) has

\[
|K_0(C^*(\Lambda))| = |K_1(C^*(\Lambda))|.
\]

**Proof.** By Lemma 4.7.9, both \( K_0(C^*(\Lambda)) \) and \( K_1(C^*(\Lambda)) \) are finite. Let \( C := 1 - B^t \) and \( D := 1 - R^t \). Then \( C \) and \( D \) commute because \( B \) and \( R \) do, and

\[
\delta_1 = \begin{pmatrix} C & D \end{pmatrix}: \mathbb{Z}^A_0 \oplus \mathbb{Z}^A_0 \to \mathbb{Z}^A_0 \text{ and } \delta_2 = \begin{pmatrix} -D \\ C \end{pmatrix}: \mathbb{Z}^A_0 \to \mathbb{Z}^A_0 \oplus \mathbb{Z}^A_0.
\]

By Theorem 4.7.1 we have \( \ker \delta_2 = \{0\} \) and \( K_0(C^*(\Lambda)) = \coker \delta_1 \), that is,

\[
K_0(C^*(\Lambda)) = \mathbb{Z}^A_0 / (C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0).
\]

On the other hand we have

\[
K_1(C^*(\Lambda)) = \ker \delta_1 / \im \delta_2
\]

\[
= \{(u, v): u, v \in \mathbb{Z}^A_0, Cu + Dv = 0\} / \{(-Dw, Cw): w \in \mathbb{Z}^A_0\}.
\]

The map \( C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0 \to \ker \delta_1 \) defined by \( w \mapsto (-C^{-1}w, D^{-1}w) \) carries \( C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0 \) onto \( \im \delta_2 \). This induces an isomorphism of \( (C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0) / C\mathbb{Z}^A_0 \) onto \( \ker \delta_1 / \im \delta_2 \), hence

\[
K_1(C^*(\Lambda)) = (C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0) / C\mathbb{Z}^A_0.
\]

We have \( C\mathbb{Z}^A_0 \leq (C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0) \leq \mathbb{Z}^A_0 \) and \( C(D\mathbb{Z}^A_0) \leq (C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0) \leq D\mathbb{Z}^A_0 \).

Since \( D \) is an isomorphism of \( \mathbb{Z}^A_0 \) onto \( D\mathbb{Z}^A_0 \) which carries \( C\mathbb{Z}^A_0 \) onto \( C\mathbb{Z}^A_0 \), we have \( |D\mathbb{Z}^A_0 : C\mathbb{Z}^A_0| = |\mathbb{Z}^A_0 : C\mathbb{Z}^A_0| \). Then

\[
|D\mathbb{Z}^A_0 : C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0| |C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0 : C\mathbb{Z}^A_0|
\]

\[
= |D\mathbb{Z}^A_0 : C\mathbb{Z}^A_0|
\]

\[
= |\mathbb{Z}^A_0 : C\mathbb{Z}^A_0|
\]

\[
= |\mathbb{Z}^A_0 : (C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0)| |(C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0) : C\mathbb{Z}^A_0|.
\]

The inclusion of \( D\mathbb{Z}^A_0 \) in \( C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0 \) induces an isomorphism of \( D\mathbb{Z}^A_0 / (C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0) \) onto \( (C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0) / C\mathbb{Z}^A_0 \), and hence

\[
|D\mathbb{Z}^A_0 : C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0| = |(C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0) : C\mathbb{Z}^A_0|.
\]

Equation (4.7.5) allows us to cancel in (4.7.4) and obtain

\[
|C\mathbb{Z}^A_0 \cap D\mathbb{Z}^A_0 : C\mathbb{Z}^A_0| = |\mathbb{Z}^A_0 : (C\mathbb{Z}^A_0 + D\mathbb{Z}^A_0)|,
\]

which gives the result.

□
Remark 4.7.11. Notice that our proof does not give an explicit isomorphism between $CZ^\Lambda_0 \cap DZ^\Lambda_0 / C DZ^\Lambda_0$ and $Z^\Lambda_0 / (CZ^\Lambda_0 + DZ^\Lambda_0)$, so we cannot deduce that $K_0 \cong K_1$, only that they have the same number of elements.
CHAPTER 5

Crossed products of $C^*$-dynamical systems

The automorphisms of a $C^*$-algebra $A$ form a group $\text{Aut} A$ under composition. An action of a group $G$ on $A$ is a homomorphism $\alpha : G \to \text{Aut} A$, and we usually write $\alpha_t(a)$ for $\alpha(t)(a)$. A $C^*$-dynamical system $(A, G, \alpha)$ consists of a $C^*$-algebra $A$, a locally compact group $G$ and a continuous action $\alpha : G \to \text{Aut} A$. In this chapter we will briefly outline the construction of the crossed product of $(A, G, \alpha)$ for the particular case where $G$ is a discrete abelian group and $A$ is a $C^*$-algebra with identity. Assuming that $G$ is discrete makes the construction remarkably simpler and we suggest [74, Chapter 2] or [59, §7.1] for the full generality. We make this restriction because the systems which arise in our work in Chapter 6 involve actions of the discrete abelian group $\mathbb{Z}$.

5.1. The Pontryagin Duality Theorem

Let $G$ be a locally compact abelian group. The set $\hat{G}$ of continuous homomorphisms of $G$ into the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is an abelian group under pointwise multiplication. We call $\hat{G}$ the dual group of $G$. The group operations are continuous in the topology of uniform convergence on compact sets and so $\hat{G}$ is a topological group. Then $\hat{G}$ is locally compact and Hausdorff [20, §4.1].

Now suppose that $G$ is a discrete abelian group. Then the topology of uniform convergence on compact sets on $\hat{G}$ is the topology of pointwise convergence in which

$\gamma_n \to \gamma \iff \gamma_n(s) \to \gamma(s)$ for all $s \in G$.

The map $\gamma \mapsto \{\gamma(s)\}_{s \in G}$ embeds $\hat{G}$ as a closed subset of the compact Hausdorff space $\mathbb{T}^G$, and so $\hat{G}$ is compact.

Example 5.1.1. In this example we show that the dual group $\hat{\mathbb{Z}}$ of the discrete abelian group $\mathbb{Z}$ is $\mathbb{T}$. Every homomorphism $\gamma : \mathbb{Z} \to \mathbb{T}$ is determined by $\gamma(1)$ since $\gamma(n) = \gamma(n.1) = \gamma(1)^n$. So every homomorphism $\gamma : \mathbb{Z} \to \mathbb{T}$ has the form $\gamma_z : n \mapsto z^n$ for $z = \gamma(1)$. The map $z \mapsto \gamma_z$ is a group homomorphism $\mathbb{T} \to \hat{\mathbb{Z}}$ since

$\gamma_{yz}(n) = (yz)^n = y^n z^n = \gamma_y(n)\gamma_z(n),

and is a bijection since it is one-to-one:

$\gamma_z = \gamma_y \implies \gamma_z(1) = \gamma_y(1) \implies z = y.$

Since $\gamma_1 \to \gamma$ in $\hat{\mathbb{Z}}$ implies $\gamma_1(1) \to \gamma(1)$ in $\mathbb{T}$, the map is a homeomorphism since it is a continuous bijection $\mathbb{T} \to \hat{\mathbb{Z}}$ from a compact space to a Hausdorff space. Hence $\hat{\mathbb{Z}}$ and $\mathbb{T}$ are topologically isomorphic.
The elements of $\hat{G}$ are often called characters on $G$. We can also regard the elements of $G$ as characters on $\hat{G}$, that is, each $s \in G$ defines a character $\hat{s}$ on $\hat{G}$ by $\hat{s}(\gamma) = \gamma(s)$. The Pontryagin Duality Theorem [20, Theorem 4.31] says that the map $G \to \hat{G}$ which takes $s \mapsto \hat{s}$ is an isomorphism of topological groups, and so we identify $G$ with $\hat{G}$.

The Pontryagin Duality Theorem gives a nice duality between subgroups and quotients of a locally compact abelian group. If $H$ is a closed subgroup of $G$, then

$$H^\perp := \{ \gamma \in \hat{G} : \gamma(t) = 1 \text{ for all } t \in H \}$$

is a closed subgroup of $\hat{G}$ and $(H^\perp)^\perp = H$ [20, Proposition 4.38]. Define $\Xi : \hat{G}/H \to H^\perp$ by $\Xi(\chi) = \chi \circ q$ where $q : G \to G/H$ is the quotient map. Then $\Xi$ is an isomorphism of topological groups [20, Theorem 4.39], and so we use this isomorphism to identify $\hat{G}/H$ with $H^\perp$.

**Example 5.1.2.** Let $G = \mathbb{Z}$ and $H = n\mathbb{Z}$. Using [20, Theorem 4.39], the dual group of $\mathbb{Z}/n\mathbb{Z}$ is identified with

$$\left(\mathbb{Z}/n\mathbb{Z}\right)^\wedge = (n\mathbb{Z})^\perp = \{ \gamma \in \hat{\mathbb{Z}} : \gamma|_{n\mathbb{Z}} = 1 \} = \{ z \in \mathbb{T} : z^n = 1 \} = C_n,$$

the group of complex $n$th roots of unity.

### 5.2. Crossed products

Suppose that $(A, G, \alpha)$ is a $C^*$-dynamical system in which $G$ is a discrete abelian group and $A$ is a $C^*$-algebra with identity. A covariant representation of $(A, G, \alpha)$ on a Hilbert space $H$ is a pair $(\pi, U)$ consisting of a representation $\pi : A \to \mathcal{B}(H)$ which is nondegenerate $(\pi(1_A) = 1_{\mathcal{B}(H)})$ and a unitary representation $U : G \to \mathcal{U}(H)$ such that $\pi(\alpha_s(a)) = U_s \pi(a) U^*_s$. The crossed product of $(A, G, \alpha)$ is a $C^*$-algebra $A \times_\alpha G$ whose representations are in one-to-one correspondence with the covariant representations of the system.

The standard way to construct $A \times_\alpha G$ is as follows. The vector space $C_c(G, A)$ of finitely supported functions $f : G \to A$ is a $*$-algebra with multiplication and involution given by

$$f * g(s) = \sum_{r \in G} f(r) \alpha_r(g(s - r))$$
$$f^*(s) = \alpha_s(f(-s)^*).$$

If $(\pi, U)$ is a covariant representation of $(A, G, \alpha)$ on a Hilbert space $H$ then there exists a nondegenerate $*$-representation $\pi \times U$ of $C_c(G, A)$ on $H$ such that

$$\pi \times U(f) = \sum_{s \in G} \pi(f(s)) U_s \text{ for } f \in C_c(G, A).$$

(5.2.1)

The maps $i_A : A \to C_c(G, A)$ and $i_G : G \to \mathcal{U}(C_c(G, A))$ defined by

$$i_A(a)(s) = \begin{cases} a & \text{if } s = e \\ 0 & \text{if } s \neq e \end{cases} \quad \text{and} \quad i_G(r)(s) = \begin{cases} 1_A & \text{if } s = r \\ 0 & \text{if } s \neq r \end{cases}$$

(5.2.2)

are homomorphisms such that

$$i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^* \text{ for } a \in A, s \in G,$$
and every \( f \in C_c(G, A) \) can be written as \( f = \sum_{s \in G} i_A(f(s))i_G(s) \). The map \( \pi, U \mapsto \pi \times U \) is a bijection from the set of covariant representations of \((A, G, \alpha)\) on a Hilbert space \( \mathcal{H} \) to the set of nondegenerate representations of \( C_c(G, A) \) on \( \mathcal{H} \) with inverse given by \( \rho \mapsto (\rho \circ i_A, \rho \circ i_G) \). So \( C_c(G, A) \) is an algebraic crossed product in that it has the right representation theory, and now we need to make it into a \( C^* \)-algebra.

The crossed product \( A \rtimes_\alpha G \) is defined to be the completion of \( C_c(G, A) \) in the \( C^* \)-norm

\[
\|f\| := \sup \{\|\rho(f)\| : \rho \text{ is a } *\text{-representation of } C_c(G, A)\}.
\]

So every representation of \( C_c(G, A) \) extends to a representation of \( A \rtimes_\alpha G \). The maps of (5.2.2) extend to a homomorphism \( i_A : A \to A \rtimes_\alpha G \) and a homomorphism \( i_G : G \to \mathcal{U}(A \rtimes_\alpha G) \) of \( G \) into the group of unitary elements of \( A \rtimes_\alpha G \) such that

1. \((i_A, i_G)\) is covariant: \( i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^* \) for all \( a \in A \), \( s \in G \),

2. for every covariant representation \( (\pi, U) \) of \((A, G, \alpha)\) on \( \mathcal{H} \) there exists a nondegenerate representation \( \pi \times U \) of \( A \rtimes_\alpha G \) on \( \mathcal{H} \) such that \( (\pi \times U) \circ i_A = \pi \) and \( (\pi \times U) \circ i_G = U \), and

3. the elements \( i_A(a)i_G(s) \) span a dense subspace of \( A \rtimes_\alpha G \).

The crossed product \((A \rtimes_\alpha G, i_A, i_G)\) is universal for covariant representations; this concept was formalised by Raeburn in [55].

The crossed product \( A \rtimes_\alpha G \) carries an action \( \hat{\alpha} : \hat{G} \to \text{Aut}(A \rtimes_\alpha G) \) called the dual action, and we call the dynamical system \((A \rtimes_\alpha G, \hat{G}, \hat{\alpha})\) the dual system. The dual action is characterised by its behaviour on the generators of the system

\[
(5.2.3) \quad \hat{\alpha}_\sigma(i_A(a)i_G(s)) = i_A(a)\sigma(s)i_G(s) \quad \text{for } \sigma \in \hat{G}, a \in A \text{, } s \in G,
\]

or by its behaviour on the dense subalgebra \( C_c(G, A) \)

\[
\hat{\alpha}_\sigma(x)(s) = \sigma(s)x(s),
\]

and \( \hat{\alpha} \) is continuous: if \( \sigma_n \to \sigma \) pointwise in \( \hat{G} \) then \( \hat{\alpha}_{\sigma_n}(x) \to \hat{\alpha}_\sigma(x) \) for \( x \in A \rtimes_\alpha G \).

The crossed product \( A \rtimes_\alpha G \) encodes information about the original dynamical system \((A, G, \alpha)\) which can be recovered using duality. An important result (which we will not need) is the Takai Duality Theorem ([74, §7.1], [55, Theorem 6], [73]) which generalises the Pontryagin Duality Theorem to crossed products by abelian groups and relates the crossed product \((A \rtimes_\alpha G) \times_\alpha \hat{G}\) of the dual system \((A \rtimes_\alpha G, \hat{G}, \hat{\alpha})\) to the original algebra \( A \). Note that \( \hat{G} \) is not necessarily a discrete group even if \( G \) is (for example, when \( G = \mathbb{Z} \) and \( \hat{G} = \mathbb{T} \)). So making sense of \((A \rtimes_\alpha G) \times_\alpha \hat{G}\) requires the construction of crossed products by arbitrary locally compact groups and we have only described the discrete case. However, we will get by without needing the extra generality because we will not need to apply this result.

One can also define the reduced crossed product \( A \rtimes_{\alpha, r} G \) using regular representations [74, §7.2]. If \( G \) is an amenable group, which is automatic if \( G \) is abelian, then the crossed product and the reduced crossed product coincide [74, Theorem 7.13].
5.3. Twisted crossed products

There are two twisted crossed product constructions. In this section we follow the approach due to Green [27] and construct the twisted crossed product as a quotient of the ordinary crossed product. As before we will simplify things by supposing that \((A, G, \alpha)\) is a \(C^*\)-dynamical system in which \(A\) is a \(C^*\)-algebra with identity and \(G\) is a discrete abelian group. For twisted crossed products involving arbitrary locally compact groups \(G\) see [74, Section 7.4].

A twisting map is a homomorphism \(\tau : N \to U(A)\) from a subgroup \(N\) of \(G\) into \(A\) satisfying the following for all \(t \in N, a \in A\) and \(s \in G\)

\[
\alpha_t(a) = \tau(a)t^*_\tau = \text{Ad}\, \tau(a)
\]

\[
\alpha_s(\tau_t) = \tau_t.
\]

Then \((A, G, N, \alpha, \tau)\) is called a twisted dynamical system. We say that a covariant representation \((\pi, U)\) of \((A, G, \alpha)\) preserves the twist \(\tau\) if

\[
\pi(\tau_t) = U_t \text{ for all } t \in N.
\]

The twisted crossed product \(A \times^\tau_\alpha G\) is defined to be the quotient of the ordinary crossed product \(A \times_\alpha G\) whose representations correspond bijectively to the covariant representations of \((A, G, \alpha)\) which preserve the twist \(\tau\). To see how the quotient is constructed, let \(I_\tau\) be the ideal of \(A \times_\alpha G\) generated by

\[
\{i_G(t) - i_A(\tau_t) : t \in N\}.
\]

Then by [74, Proposition 7.23] a representation \((\pi, U)\) preserves \(\tau\) if and only if \(I_\tau \subset \ker \pi \times U\), thus

\[
I_\tau = \bigcap \{\ker \pi \times U : (\pi, U) \text{ is covariant and } \pi(\tau_t) = U_t \text{ for all } t \in N\}.
\]

The twisted crossed product of \((A, G, \alpha, \tau)\) is the quotient

\[
A \times^\tau_\alpha G := (A \times_\alpha G)/I_\tau.
\]

Since \(A \times_\alpha G\) is generated by a universal covariant representation \((i_A, i_G)\), the quotient \(A \times^\tau_\alpha G\) is generated by a covariant representation \((j_A, j_G) := (Q \circ i_A, Q \circ i_G)\), where \(Q : A \times_\alpha G \to A \times^\tau_\alpha G\) is the quotient map. Since \(i_A(\tau_t) - i_G(t) \in I_\tau = \ker Q\) for \(t \in N\), we have \(j_A(\tau_t) = j_G(t)\), so \((j_A, j_G)\) preserves the twist. In fact, \((j_A, j_G)\) is universal for representations \((\pi, U)\) which preserve \(\tau\); given such a pair \((\pi, U)\) the representation \(\pi \times U\) vanishes on \(I_\tau\), so there exists a representation \(\pi \times^\tau U\) of \(A \times^\tau_\alpha G\) such that \(\pi \times^\tau U(Q(f)) = \pi \times U(f)\); then

\[
(\pi \times^\tau U) \circ j_A = \pi \times^\tau U(Q \circ i_A) = \pi \times U \circ i_A = \pi
\]

\[
(\pi \times^\tau U) \circ j_G = \pi \times^\tau U(Q \circ i_G) = \pi \times U \circ i_G = U.
\]

For \(\sigma \in N^\perp \subset \hat{G}\), the dual action \(\hat{\alpha}\) of \(\hat{G}\) on \(A \times_\alpha G\) satisfies

\[
\hat{\alpha}_\sigma(i_G(t) - i_A(\tau_t)) = \sigma(t)i_G(t) - i_A(\tau_t) = i_G(t) - i_A(\tau_t) \text{ for } t \in N,
\]
so $\hat{\alpha}_t : I_\tau \to I_\tau$. Thus there is an action $\beta$ of $N^{\perp}$ on $A \times^\gamma_{\alpha} G$ characterised by

$$\beta_\sigma(Q(x)) = Q(\hat{\alpha}_\sigma(x)) \text{ for } x \in A \times^\gamma_{\alpha} G.$$  

(5.3.4)

On generators we have

$$\beta_\sigma(j_A(a)j_G(s)) = j_A(a)\sigma(s)j_G(s) \text{ for } \sigma \in N^{\perp}, a \in A, s \in G.$$  

If $L$ is a closed subgroup of a compact abelian group $K$ and $\nu : L \to \text{Aut } B$ is an action of $L$ on a $C^*$-algebra $B$, then the induced algebra is

$$\text{Ind}_L^K(B, \nu) = \{ f \in C_c(K, B) : f(st) = \nu_t^{-1}(f(s)) \text{ for all } s \in G, t \in L \}.$$  

The induced algebra is a $C^*$-algebra with the supremum norm since it is a closed $*$-subalgebra of $C_c(K, B)$ [59, §6.3]. (The property that the function $s + L \mapsto \|f(s)\|$ vanishes at $\infty$ on $K/L$, which is part of the definition in [59], is automatic here because $K$ is compact.) We need the following variation of a theorem of Olesen and Pedersen [44].

**Theorem 5.3.1.** Suppose that $(A,G,N,\alpha,\tau)$ is a twisted dynamical system in which $G$ is a discrete abelian group. Let $Q : A \times^\alpha G \to A \times^\gamma_{\alpha} G$ be the quotient map and let $\beta$ be the action of $N^{\perp}$ on $A \times^\gamma_{\alpha} G$ characterised by (5.3.4). Then the map $\Psi$ defined by

$$\Psi(x)(\sigma) = Q(\hat{\alpha}_\sigma^{-1}(x)) \text{ for } x \in A \times^\alpha G, \sigma \in \hat{G}$$  

is an isomorphism of $A \times^\alpha G$ onto $\text{Ind}_{N^{\perp}}(A \times^\gamma_{\alpha} G, \beta)$ such that $\Psi \circ \hat{\alpha} = \text{lt} \circ \Psi$, where it is the action of $\hat{G}$ defined for $f \in C_c(G, A)$ by $\text{lt}_\xi(f)(\sigma) = f(\xi^{-1}\sigma)$.  

In the next section we will deduce this theorem from what Olesen and Pedersen actually proved, [44, Theorem 2.4]. Right now we will say why we are interested in Theorem 5.3.1.

Suppose that $\gamma$ is an action of $G/N$ on $A$, let $q : G \to G/N$ be the quotient map, and let $\alpha = \gamma \circ q$. We view the dual action $\hat{\gamma}$ as an action of the closed subgroup $N^{\perp} = G/\bar{N}$ of the compact group $\hat{G}$ on $A \times^1 G$, which we will see in Lemma 5.3.2 is isomorphic to $A \times^\gamma (G/N)$, and then we get the induced algebra $\text{Ind}_{N^{\perp}}(A \times^\gamma (G/N), \hat{\gamma})$. For $t \in N$ we have

$$\alpha_t = (\gamma \circ q)_t = \gamma(q(t)) = \gamma(t + N) = 1 = \text{Ad}_1 t.$$  

Taking $\tau : N \to U(A)$ to be the identity $1_t \equiv 1$, we have that $(A,G,N,\gamma \circ q,1)$ is a twisted dynamical system. The ideal of (5.3.3) becomes

$$I_1 = \bigcap \{ \ker \pi \times U : (\pi, U) \text{ is covariant and } U_t = 1 \text{ for all } t \in N \},$$  

and the twisted crossed product $A \times_{\gamma \circ q}^1 G$ is, by definition, the quotient $(A \times_{\gamma \circ q} G)/I_1$.

In the next lemma we identify the twisted crossed product $A \times_{\gamma \circ q}^1 G$ with $A \times^\gamma (G/N)$. Recall that $(i_A, i_G)$ is the universal covariant representation of $(A,G)$ in $A \times^\alpha G$, and since $i_G(t) = i_A(\tau_t) = 1$ for $t \in N$ there exists a unique unitary homomorphism $i_{G/N} : G/N \to U(A \times^\alpha G)$ such that $i_G = i_{G/N} \circ q$. Then $i_A \times i_G = i_A \times (i_{G/N} \circ q)$, defined in (5.2.1), is a representation of $A \times^\alpha G$.

**Lemma 5.3.2.** The map $\Theta : A \times^\alpha G \to A \times^\gamma (G/N)$ defined by $\Theta := i_A \times (i_{G/N} \circ q)$ induces an isomorphism $\tilde{\Theta}$ of $A \times^\gamma_{\alpha} G$ onto $A \times^\gamma (G/N)$ such that $\tilde{\Theta} \circ \beta_\chi = \tilde{\gamma}_\chi$ for $\chi \in N^{\perp}$.  


Proof. We claim that \( \ker \Theta = I_1 \). We have \( \ker \Theta \supset I_1 \) since \( (i_A, i_{G/N} \circ q) \) is a covariant representation of \( (A, G, \alpha) \) which preserves the twist:

\[
i_{G/N} \circ q(t) = i_{G/N}(q(t)) = i_{G/N}(t + N) = 1 \quad \text{for all} \quad t \in N.
\]

For the other containment, suppose that \( (\pi, U) \) is a covariant representation of \( (A, G, \alpha) \) with \( U_t = 1 \) for all \( t \in N \). We will show that \( \ker \Theta \subset \ker \pi \times U \), and then it will follow that \( \ker \Theta \subset I_1 \) since \( I_1 \) is the intersection of the kernels of all such representations \( \pi \times U \). There exists a unique representation \( \tilde{U} \) of \( G/N \) such that \( U = \tilde{U} \circ q \). Then \( (\pi, \tilde{U}) \) is a covariant representation of \( A \times_\gamma (G/N) \) since

\[
\tilde{U}_{s+N} \pi(a) \tilde{U}^*_s = \tilde{U}_q(s) \pi(a) \tilde{U}_q^*(s) = U_s \pi(a) U_s^* = \pi(\alpha_s(a)) = \pi((\gamma \circ q)_s(a)) = \pi(\gamma_{s+N}(a))
\]

for \( a \in A \) and \( s + N \in G/N \). Then there exists a representation \( \pi \times \tilde{U} \) of \( (A, G/N, \gamma) \) such that \( (\pi \times \tilde{U}) \circ i_A = \pi \) and \( (\pi \times \tilde{U}) \circ (i_{G/N} \circ q) = U \). Then

\[
\pi \times \tilde{U}(i_A(a)) = \pi(a) \quad \text{for} \quad a \in A,
\]

and

\[
\pi \times \tilde{U}(i_{G/N} \circ q(s)) = \pi \times \tilde{U}(i_{G/N}(s + N)) = \tilde{U}_{s+N} = \tilde{U}_{q(s)} = U_s \quad \text{for} \quad s \in G.
\]

So we have \( (\pi \times \tilde{U}) \circ \Theta = \pi \times U \), which implies that \( \ker \Theta \subset \ker \pi \times U \). Hence \( \ker \Theta = I_1 \).

The elements \( i_A(a) i_{G/N}(s + N) = \Theta(i_A(a) i_{G/N} \circ q(s)) \) span a dense subspace of \( A \times_\gamma (G/N) \) and are in the range of \( \Theta \). So \( \Theta \) is onto and since \( \ker \Theta = I_1 \), \( \Theta \) induces an isomorphism \( \Theta \) of \( A \times_\alpha G := (A \times_\alpha G)/I_1 \) onto \( A \times_\gamma (G/N) \), and \( \Theta = \Theta \circ Q \) where \( Q : A \times_\alpha G \to A \times_\gamma G \) is the quotient map. We have \( \Theta \circ \beta_\chi = \hat{\gamma}_\chi \) for \( \chi \in N^\perp \) since

\[
\Theta \circ \beta_\chi(j_A(a) j_G(s)) = \Theta(j_A(a) \chi(s) j_G(s))
\]

\[
= \Theta(Q(i_A(s)) \chi(s) Q(i_G(s)))
\]

\[
= \Theta(i_A(s) \chi(s) i_G(s))
\]

\[
i_A(a) \chi(s) i_{G/N} \circ q(s)
\]

\[
= i_A(a) \chi(s) i_{G/N}(s + N)
\]

\[
= \hat{\gamma}_\chi(i_A(a) i_{G/N}(s + N)), \text{ by (5.2.3)}
\]

\[
= \hat{\gamma}_\chi(i_A(a) i_{G/N} \circ q(s)). \quad \Box
\]

The following corollary to Theorem 5.3.1, which we will use later in Chapter 6, is essentially [44, Corollary 2.5].

Corollary 5.3.3. Suppose that \( \gamma : G/N \to \text{Aut} A \) is an action and let \( \alpha = \gamma \circ q \) where \( q : G \to G/N \) is the quotient map. Then the function

\[
\Psi' : A \times_\alpha G \to \text{Ind}_{N^\perp}^G(A \times_\gamma (G/N), \hat{\gamma})
\]

defined by

\[
\Psi'(x)(\sigma) = \Theta(\hat{\alpha}_\sigma^{-1}(x)) \quad \text{for} \quad x \in C_c(G, A), \sigma \in \hat{G},
\]

is an isomorphism.
5.4. Restricted crossed products

Proof. Theorem 5.3.1 gives an isomorphism $\Psi$ of $A \times G$ onto $\text{Ind}_{N}^{G}(A \times G, \hat{\gamma})$ such that $\Psi(x)(\sigma) = \beta_{\sigma}^{-1}(Q(x))$. By Lemma 5.3.2, there is an isomorphism $\Theta$ of $A \times G$ onto $A \times (G/N)$ which converts $\beta$ to $\hat{\gamma}$ and such that $\Theta = \Theta \circ Q$. So there exists an isomorphism $L : \text{Ind}_{N}^{G}(A \times G, \beta) \to \text{Ind}_{N}^{G}(A \times (G/N), \hat{\gamma})$ such that $L(f)(\sigma) = \Theta(\rho(f(\sigma)))$.

We then have

$$(L \circ \Psi)(x)(\sigma) = \Theta(\Psi(x)(\sigma))$$

$$= \Theta(Q(\hat{\alpha}_{\sigma}^{-1}(x)))$$

$$= \Theta(\hat{\alpha}_{\sigma}^{-1}(x))$$

$$= \Psi'(x)(\sigma).$$

So $\Psi' = L \circ \Psi$ is an isomorphism as it is a composition of isomorphisms.

5.4. Restricted crossed products

As in Section 5.3, suppose that $(A, G, N, \alpha, \tau)$ is a twisted dynamical system in which $A$ is a $C^*$-algebra with identity and $G$ is a discrete abelian group. These systems appear in [44]: Olesen and Pedersen call a dynamical system $(A, G, \alpha)$ $N$-inner for a closed subgroup $N$ of $G$ if there is a unitary representation $\tau : N \to \mathcal{U}(A)$ such that $\alpha_t = \text{Ad} \tau_t$ and $\alpha_s(\tau_t) = \tau_t$ for all $t \in N$ and $s \in G$. Saying that $(A, G, \alpha)$ is $N$-inner with respect to $\tau$ is equivalent to saying that $(A, G, N, \alpha, \tau)$ is a twisted dynamical system. We will review the construction of Olesen and Pedersen’s “restricted crossed product”, and then we will prove that it is isomorphic to the twisted crossed product. This is mentioned in [44, page 264] with reference to Green [27], but there are no details; since we need a concrete realisation of the isomorphism between the restricted crossed product and the twisted crossed product, we construct it here.

Consider the set $K(G, A, \tau)$ of functions $y : G \to A$ satisfying $y(s - t) = y(s)\tau_t$ for all $s \in G$ and $t \in N$, such that the function $s + N \mapsto \|y(s)\|$ has finite support in $G/N$. Then $K(G, A, \tau)$ is a $*$-algebra with operations

$$y \times z(s) = \sum_{r+N \in G/N} y(r)\alpha_r(z(s - r))$$

$$y^*(s) = \alpha_s(y(-s)^*)$$

and

$$\|y\|_1 = \sum_{s+N \in G/N} \|y(s)\|$$

is a norm on $K(G, A, \tau)$. (Note that $y \times z$ is well-defined since the function $r \mapsto y(r)\alpha_r(z(s-r))$ depends only on the coset $r + N$ in $G/N$.) The restricted crossed product $A \times_{\alpha|N} G$, which appears in [44, Theorem 2.4], is defined to be the completion of $K(G, A, \tau)$ in the norm

$$\|y\| := \sup\{\|\rho(y)\| : \rho \text{ is a } *\text{-representation of } K(G, A, \tau)\}.$$
Lemma 5.4.1. For $x \in C_c(G, A)$, the function $\Phi(x)$ defined by

$$\Phi(x)(s) = \sum_{t \in \mathbb{N}} x(s + t)\tau_t$$

belongs to $K(G, A, \tau)$ and is a $\ast$-homomorphism. The map $\Phi : C_c(G, A) \to K(G, A, \tau)$ extends to a homomorphism $\Phi : A \times_\alpha G \to A \times_{\alpha|\mathbb{N}} G$ which has kernel $I_\tau$.

Proof. The function $\Phi(x)$ has finite support in $G/\mathbb{N}$ because only finitely many cosets $s + \mathbb{N}$ can meet the finite set $\text{supp } x$. Then $\Phi(x) \in K(G, A, \tau)$ since, for $s \in G$ and $t \in \mathbb{N}$,

$$\Phi(x)(s - t) = \sum_{r \in \mathbb{N}} x(s - t + r)\tau_r$$

$$= \sum_{r \in \mathbb{N}} x(s + (r - t))\tau_{r-t}\tau_t$$

$$= \left( \sum_{r \in \mathbb{N}} x(s + (r - t))\tau_{r-t} \right) \tau_t$$

$$= \Phi(x)\tau_t.$$

We now show that $\Phi$ is a $\ast$-homomorphism. Using properties (5.3.1) and (5.3.2) of the twisting map, we have that $\Phi$ is multiplicative:

$$\Phi(y \times z)(s) = \sum_{t \in \mathbb{N}} (y \times z)(s + t)\tau_t$$

$$= \sum_{t \in \mathbb{N}} \sum_{r \in \mathbb{N}} y(r)\alpha_r(z(s + t - r))\tau_t$$

$$= \sum_{t \in \mathbb{N}} \sum_{r \in \mathbb{N}} \sum_{q \in \mathbb{N}} y(r + q)\alpha_{r+q}(z(s + t - r - q))\tau_t$$

$$= \sum_{t \in \mathbb{N}} \sum_{r \in \mathbb{N}} \sum_{q \in \mathbb{N}} y(r + q)\tau_q\alpha_r(z(s + t - r - q))\tau_{r-q}$$

$$= \sum_{p \in \mathbb{N}} \sum_{r \in \mathbb{N}} \sum_{q \in \mathbb{N}} y(r + q)\tau_q\alpha_r(z(s - r + p))\tau_p$$

$$= \sum_{r \in \mathbb{N}} \sum_{q \in \mathbb{N}} \left( \sum_{q \in \mathbb{N}} y(r + q)\tau_q \right) \alpha_r \left( \sum_{p \in \mathbb{N}} (z(s - r + p))\tau_p \right)$$

$$= \sum_{r \in \mathbb{N}} \Phi(y)(r)\alpha_r(\Phi(z)(s - r))$$

$$= \Phi(y) \times \Phi(z)(s).$$
and $\Phi$ is $\ast$-preserving:

$$
\Phi(y)^\ast(s) = \alpha_s(\Phi(y)(-s)^\ast)
$$

$$
= \alpha_s \left( \left( \sum_{t \in N} y(-s + t)\tau_t \right)^\ast \right)
$$

$$
= \alpha_s \left( \sum_{t \in N} \tau_{-t} y(-s + t)^\ast \right)
$$

$$
= \sum_{t \in N} \alpha_s(\tau_{-t})\alpha_s(y(-s + t)^\ast)
$$

$$
= \sum_{t \in N} \tau_{-t}\alpha_s(y(-s + t)^\ast)
$$

$$
= \sum_{t \in N} \tau_{-t}\alpha_s(y(-s + t)^\ast)\tau_t \tau_{-t}
$$

$$
= \sum_{t \in N} \alpha_{s-t}(y(-s + t)^\ast)\tau_{-t}
$$

$$
= \sum_{t \in N} y^\ast(s - t)\tau_{-t}
$$

$$
= \sum_{r \in N} y^\ast(s + r)\tau_r
$$

$$
= \Phi(y^\ast)(s).
$$

If $\pi$ is a $\ast$-representation of $K(G, A, \tau)$ then $\pi \circ \Phi$ is a $\ast$-representation of $C_c(G, A)$, and so

$$
\|\pi(\Phi(x))\| = \|\pi \circ \Phi(x)\| \leq \|x\|,
$$

which implies that

$$
\|\Phi(x)\| = \sup_{\pi} \|\pi(\Phi(x))\| \leq \|x\|.
$$

So $\Phi$ is norm-decreasing and hence extends to a linear map from $A \times \alpha G$ to $A \times \alpha |N G$.

To prove that $\Phi$ has kernel $I_\tau$ we show first that $I_\tau \subset \ker \Phi$. Since $I_\tau$ is generated by the elements $i_G(t) - i_A(\tau_t)$, we check that $\Phi(i_G(t) - i_A(\tau_t)) = 0$ for $t \in N$. By definition of $\Phi$ we have

$$
\Phi(i_A(a))(s) = \begin{cases} 0 & \text{if } s \notin N \\ a\tau_{-s} & \text{if } s \in N \end{cases}
$$

and

$$
\Phi(i_G(t))(s) = \begin{cases} 0 & \text{if } t - s \notin N \\ \tau_{t-s} & \text{if } t - s \in N. \end{cases}
$$

It follows that for $t \in N$

$$
\Phi(i_G(t))(s) = \begin{cases} 0 & \text{if } s \notin N \\ \tau_{t-s} & \text{if } s \in N \end{cases} = \Phi(i_A(\tau_t))(s),
$$

and so $I_\tau \subset \ker \Phi$.

Conversely, we prove that $\ker \Phi \subset I_\tau$. It follows from the Gelfand-Naimark Theorem that there exists a representation $\pi \times U$ of $A \times \alpha G$ on a Hilbert space $\mathcal{H}_{\pi,U}$ with $\ker \pi \times U = \{0\}$. If $x \in I_\tau$, then $x = \phi y$ for some $y \in C_c(G, A)$, and so $\pi(x) = \pi(\phi y) = 0$ for all $\pi$. Therefore, $\ker \Phi = I_\tau$. 

5.4. RESTRICTED CROSSED PRODUCTS 63
We will show that there exists a representation $\rho : A \times_{\alpha} G \to B(\mathcal{H})$ such that $\rho \circ \Phi = \pi \times U$, from which it will follow that

$$I_\tau = \ker \pi \times U = \ker \rho \circ \Phi \supset \ker \Phi.$$ 

Define $\rho$ for $y \in K(G, A, \tau)$ by

$$\rho(y) = \sum_{s+N \in G/N} \pi(y(s))U_s.$$ 

To see that this is well-defined we show that $s+N = r+N$ implies $\pi(y(r))U_r = \pi(y(s))U_s$. We may suppose that $r = s+t$ for some $t \in N$. Using $\pi(\tau_t) = U_t$ and $y(s+t) = y(s)\tau_{-t}$ we have

$$\pi(y(r))U_r = \pi(y(s+t))U_{s+t} = \pi(y(s)\tau_{-t})U_tU_s = \pi(y(s))\pi(\tau_t)^*U_tU_s = \pi(y(s))U_s^*U_tU_s = \pi(y(s))U_s.$$ 

Next we must verify that $\rho$ is a $\ast$-representation of $K(G, A, \tau)$. Using the fact that $(\pi, U)$ is covariant for $\alpha$, we have that $\rho$ is a homomorphism:

$$\rho(y \times z) = \sum_{s+N \in G/N} \pi(y \times z(s))U_s = \sum_{s+N \in G/N} \pi \left( \sum_{r+N \in G/N} y(r)\alpha_r(z(s-r)) \right)U_s = \sum_{s+N \in G/N} \left( \sum_{r+N \in G/N} \pi(y(r))\pi(\alpha_r(z(s-r))) \right)U_s = \sum_{s+N \in G/N} \left( \sum_{r+N \in G/N} \pi(y(r))U_r\pi(z(s-r))U_r^* \right)U_s = \sum_{s+N \in G/N} \sum_{r+N \in G/N} \pi(y(r))U_r\pi(z(s-r))U_{s-r} = \sum_{s+N \in G/N} \sum_{r+N \in G/N} \pi(y(r))U_r\pi(z(a))U_a = \left( \sum_{r+N \in G/N} \pi(y(r))U_r \right) \left( \sum_{a+N \in G/N} \pi(z(a))U_a \right) = \rho(y)\rho(z),$$

1The Gelfand-Naimark Theorem says every $C^\ast$-algebra has a faithful nondegenerate representation. If $J$ is a closed two-sided ideal in a $C^\ast$-algebra $B$, then $B/J$ is a $C^\ast$-algebra in the quotient norm and has a faithful representation $\pi_0$. Let $Q : B \to B/J$ be the quotient map. Then $\pi := \pi_0 \circ Q$ is a representation of $B$ and $\ker \pi = \ker Q = J$ since $\ker \pi_0 = \{0\}$. 
and $\rho$ is $\ast$-preserving since

$$
\rho(y^*) = \sum_{s+N \in G/N} \pi(y^*(s))U_s \\
= \sum_{s+N \in G/N} \pi(\alpha_s(y(-s)^*))U_s \\
= \sum_{s+N \in G/N} U_s\pi(y(-s)^*)U_s^*U_s \\
= \sum_{s+N \in G/N} U_s\pi(y(-s))^* \\
= \sum_{r+N \in G/N} U_{-r}\pi(y(r))^* \\
= \sum_{r+N \in G/N} U_r^*\pi(y(r))^* \\
= \rho(y)^*.
$$

Thus $\rho$ is a $\ast$-representation of $K(G, A, \tau)$ and is norm-decreasing for the norm on $K(G, A, \tau)$, and hence extends to a homomorphism $\rho : A \times_{\alpha|_N} G \to B(H_\pi, U)$. Finally, we have $\rho \circ \Phi = \pi \times U$ since $\rho \circ \Phi(x) = \sum_{s+N \in G/N} \pi(\Phi(x)(s))U_s \\
= \sum_{s+N \in G/N} \pi \left( \sum_{t \in N} x(s+t)\tau_t \right) U_s \\
= \sum_{s+N \in G/N} \sum_{t \in N} \pi(x(s+t))\pi(\tau_t)U_s \\
= \sum_{s+N \in G/N} \sum_{t \in N} \pi(x(s+t))U_tU_s \\
= \sum_{r \in G} \pi(x(r))U_r \\
= \pi \times U(x).

Then $\ker \Phi \subset \ker \pi \times U = I_\tau$. \hfill \Box

As in [44, Lemma 2.1], the action $\hat{\alpha}_\sigma^{OP}$ defined for each $x \in K(G, A, \tau)$ and $\sigma \in N^\perp \subset \hat{G}$ by

$$
(5.4.1) \quad \hat{\alpha}_\sigma^{OP}(x)(s) = \sigma(s)x(s)
$$

extends to an automorphism $\hat{\alpha}^{OP}$ of $A \times_{\alpha|_N} G$, and $(A \times_{\alpha|_N} G, N^\perp, \hat{\alpha}^{OP})$ is a $C^*$-dynamical system. In [44], this action is denoted, rather confusingly, by $\hat{\alpha}$: with our notation [44, Lemma 2.2] says that $\Phi(\hat{\alpha}_\sigma(x)) = \hat{\alpha}_\sigma^{OP}(\Phi(x))$.

Corollary 5.4.2. There is an isomorphism $\bar{\Phi} : A \times_{\alpha|_N} G \to A \times_{\alpha|_N} G$ such that $\Phi = \bar{\Phi} \circ Q$ and $\bar{\Phi}$ converts the action $\beta$ into the action $\hat{\alpha}^{OP} : N^\perp \to \text{Aut}(A \times_{\alpha|_N} G)$ described in (5.4.1).
Proof. Since \( \Phi \) is onto \([44, \text{Lemma 2.2}] \) and \( \ker \Phi = I_r \) by Lemma 5.4.1, \( \Phi \) induces an isomorphism \( \tilde{\Phi} \) of \( A \times_{\alpha} G := (A \times_\alpha G)/I_r \) onto \( A \times_{\alpha|N} G \) such that \( \Phi = \tilde{\Phi} \circ Q \). For \( \sigma \in N^\perp \) and \( x \in A \times_\alpha G \) we have

\[
\tilde{\Phi}(\beta_\sigma(Q(x)))(s) = \tilde{\Phi}(\hat{\alpha}_\sigma(x))(s) = \hat{\alpha}_\sigma^\op(\Phi(x))(s)
\]

since \( \Phi(\hat{\alpha}_\sigma(x)) = \hat{\alpha}_\sigma^\op(\Phi(x)) \) by \([44, \text{Lemma 2.2}] \). □

Proof of Theorem 5.3.1. By \([44, \text{Theorem 2.4}] \) there is an isomorphism \( \Upsilon \) of \( A \times_\alpha G \) onto \( \text{Ind}_{N^\perp}^\hat{G}(A \times_{\alpha|N} G, \hat{\alpha}^\op) \) defined by

\[
\Upsilon(x)(\sigma) = \Phi(\hat{\alpha}_\sigma^{-1}(x)) \quad \text{for} \quad x \in A \times_\alpha G, \sigma \in \hat{G}.
\]

Since the isomorphism \( \tilde{\Phi} : A \times^\tau_\alpha G \to A \times_{\alpha|N} G \) of Corollary 5.4.2 converts \( \beta \) into \( \hat{\alpha}^\op \), there is an isomorphism \( M : \text{Ind}_{N^\perp}^\hat{G}(A \times_{\alpha|N} G, \hat{\alpha}^\op) \to \text{Ind}_{N^\perp}^G(A \times^\tau_\alpha G, \beta) \) such that

\[
M(f)(\sigma) = \tilde{\Phi}^{-1}(f(\sigma)).
\]

For \( x \in C_c(G, A) \) and \( \sigma \in \hat{G} \) we have

\[
(M \circ \Upsilon)(x)(\sigma) = \tilde{\Phi}^{-1}(\Upsilon(x)(\sigma)) = \tilde{\Phi}^{-1}(\Phi(\hat{\alpha}_\sigma^{-1}(x))) = Q(\hat{\alpha}_\sigma^{-1}(x)) = \Psi(x)(\sigma),
\]

and so \( \Psi = M \circ \Upsilon \). Thus \( \Psi \), being a composition of isomorphisms, is an isomorphism. We have \( \Psi \circ \hat{\alpha} = \text{lt} \circ \Psi \) since

\[
\Psi(\hat{\alpha}_\xi(x))(\sigma) = Q(\hat{\alpha}_\xi^{-1}(\hat{\alpha}_\xi(x))) = Q(\hat{\alpha}_{\xi^{-1}}^{-1}(x)) = \Psi(x)(\xi^{-1}\sigma) = \text{lt}_\xi(\Psi(x))(\sigma).
\]

□
CHAPTER 6

Periodic 2-graphs arising from subshifts

In Chapter 4, we constructed 2-graphs whose path spaces are rank 2 subshifts of finite type, and showed that this construction yields aperiodic 2-graphs whose $\mathcal{C}^*$-algebras are simple and are not ordinary graph algebras. In this chapter we show that the construction also gives a family of periodic 2-graphs which we call *domino graphs*. The graphs we consider here are based on a one-dimensional tile which we call a *domino*; vertical translates of dominos do not overlap, and infinite paths in the resulting 2-graph are periodic in the horizontal direction. This gives the structure of the path space a combinatorial character. We investigate the connections between domino graphs and known combinatorial objects, such as necklaces and Lyndon words (see Appendix B). Apart from a few special cases, every domino graph $\Lambda$ is a crossed product $\mathbb{R}\Lambda \rtimes_\alpha \mathbb{Z}$ of the underlying red graph $\mathbb{R}\Lambda$ by an action of $\mathbb{Z}$, as studied in [19]. Then $\mathcal{C}^*(\Lambda)$ is a crossed product of an ordinary graph algebra by an action of $\mathbb{Z}$ which we realise as an induced $\mathcal{C}^*$-algebra with simple fibres. We compute the $K$-theory of $\mathcal{C}^*(\Lambda)$, finding, as the experiments in Section 4.6 suggested, that $K_0$ and $K_1$ are isomorphic to the same cyclic group whose order is determined by the length of the domino.

6.1. Domino graphs

A *domino* is a tile of the form $T = \{0, e_1, 2e_1, \ldots, (n-1)e_1\}$, which is determined by $|T| := n$. In this chapter, we write $(n, q, t)$ for the basic data consisting of the domino of cardinality $n$, alphabet $\mathbb{Z}/q\mathbb{Z}$ and trace $t$, and we always take the rule $w$ to be the function which is constantly 1. Then the hypothesis on $w$ in Theorem 4.1.7 is satisfied (since every value of $w$ is invertible), and the construction of Chapter 4 gives a 2-graph $\Lambda = \Lambda(n, q, t)$, which we call a *domino graph*. Domino graphs have the same properties as the 2-graphs of tiles in Chapter 4 (they are finite, have no sources, are strongly connected, and there is at most one edge of each colour between any pair of vertices), but their structure is even more special, as we will see in Section 6.2. (Note that the domino $T$ with $|T| = n$ is the tile with $c_1 = n - 1$ and $c_2 = 0$. By swapping the roles of $B$ and $R$ we can apply any of the results in this chapter to the conjugate of the domino which is the tile with $c_1 = 0$ and $c_2 = n - 1$.)

We begin with a visual description of the vertices and paths in a domino graph with reference to an example. A domino $T$ with $|T| = n$ is pictured as a row of $n$ boxes; for example, we draw the domino with $n = 6$ as ................................

We picture a vertex as a copy of $T$ in which each box is filled with an element of the alphabet $\mathbb{Z}/q\mathbb{Z}$ so that the sum of the entries is $t \pmod{q}$. The graph $\Lambda(6, 2, 0)$ has 32 vertices; for example, the vertex $v : T \to \mathbb{Z}/2\mathbb{Z}$ with $v(0) = v(3e_1) = 0$, $v(e_1) = v(2e_1) = v(4e_1) = v(5e_1) = 1$ is drawn as
In pictures, paths in \( \Lambda^* \) are block diagrams covered by translates of \( T \), filled in so that each translate is a valid vertex. For example, in \( \Lambda(6,2,0) \), the diagram
\[
\mu = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]
(6.1.1)
represents a path \( \mu \) of degree \((3,2)\) from \( s(\mu) = [0 0 0 0 0 0] \) (the upper RH translate of \( T \)) to \( r(\mu) = [0 1 1 0 1 1 0 1] \) (the lower LH translate).

**Remark 6.1.1.** Theorem 4.2.1 says the two-sided infinite path space \( \Lambda(n, q, 0)^{\Lambda} \) is conjugate to the 2-dimensional shift of finite type with underlying space
\[
\Omega = \left\{ f : \mathbb{Z}^2 \to \mathbb{Z}/q\mathbb{Z} : \sum_{i \in T} f(i) = 0 \pmod{q} \right\}.
\]
The shift \( \Omega \) corresponds to the model \( R_2^g/(g) \) of [32, §3] in which \( g \) is the cyclotomic polynomial \( g_T(u_1, u_2) = 1 + u_1 + \cdots + u_2^{n-1} \). However, Theorem 6.5(2) of [68] implies that \( \Omega \) is a nonmixing shift, whereas those corresponding to the graphs in Sections 4.4 and 4.5 of Chapter 4 are mixing.

### 6.2. The structure of domino graphs

We can explicitly describe the skeletons of domino graphs as follows.

**Proposition 6.2.1.** Suppose that \( \Lambda = \Lambda(n, q, t) \) is a domino 2-graph. Then:

1. The red graph \( R\Lambda \) is the complete directed graph \( K_{q^{n-1}} \), so that \( |v\Lambda^2 u| = 1 \) for all \( u, v \in \Lambda^0 \).

2. \( \Lambda^0 \) can be identified with the set \( A_t^n \) of words of length \( n \) with trace \( t \pmod{q} \) over the alphabet \( A = \mathbb{Z}/q\mathbb{Z} \).

3. The blue graph \( B\Lambda \) consists of disjoint cycles whose orders divide \( n \), and which correspond to the necklaces of \( A_t^n \) discussed in Appendix B. For each divisor \( d \) of \( n \), the number \( h_d \) of blue cycles of order \( d \) in \( \Lambda \) is
\[
h_d = \sum_{s \in \mathbb{Z}/q\mathbb{Z}} L_q(d, s),
\]
where \( L_q(d, s) \) is the number of Lyndon words of \( A_t^d \) described in (B.0.3).

**Proof.** Proposition 4.1.8 says that \( |\Lambda^0| = q^{n-1} \) and \( |v\Lambda^2| = |\Lambda^{\varepsilon_2}v| = q^{n-1} \) for all \( v \in \Lambda^0 \). Since \( |v\Lambda^2 u| \leq 1 \) for all \( u, v \in \Lambda^0 \), this forces \( |v\Lambda^2 u| = 1 \) for all \( u, v \in \Lambda^0 \). For (2), we identify each vertex \( v \in \Lambda^0 \) with its image written as the concatenation \( v(0)v(e_1)v(2e_1)\cdots v((n-1)e_1) \), which is an element of the set \( A_t^n \). For (3), \( B\Lambda \) must consist of disjoint cycles since \( |v\Lambda^1| = |\Lambda^{e_1}v| = 1 \) for all \( v \in \Lambda^0 \) by Proposition 4.1.8. The equivalence classes of \( A_t^n \) under \( \rho \) defined in (B.0.1) of Appendix B are known as necklaces. Under the identification \( \Lambda^0 = A_t^n \), the cycles in \( B\Lambda \) correspond to the necklaces of \( A_t^n \), and the length of each cycle equals the period of the corresponding necklace which must divide \( n \). Suppose that \( [a] \) is a necklace with period \( d \) and let \( b \) be its Lyndon subword. Then
since \( t = \text{trace}(a) = \text{trace}(b^n/d) \), we have \( n/d \times \text{trace}(b) = t \). So the number of \( d \)-cycles in \( BA \) equals the number of Lyndon words \( L_q(d, s) \) of length \( d \) whose trace \( s \in \mathbb{Z}/q\mathbb{Z} \) satisfies \( sn/d \equiv t \) (mod \( q \)).

\[
\begin{align*}
\text{Example 6.2.2.} & \quad \text{Suppose that } \Lambda = \Lambda(6,2,0). \quad \text{The divisors of } 6 \text{ are } d_1 = 1, \ d_2 = 2, \ d_3 = 3, \ d_4 = 6. \quad \text{The constants of Proposition 6.2.1(3) are } \\
& \quad h_1 = L_2(1,0) + L_2(1,1) = 1 + 1 = 2 \\
& \quad h_2 = L_2(2,0) = 0 \\
& \quad h_3 = L_2(3,0) + L_2(3,1) = 1 + 1 = 2 \\
& \quad h_4 = L_2(6,0) = 4,
\end{align*}
\]

and so \( BA \) consists of two 1-cycles, two 3-cycles and four 6-cycles. Identifying the vertex set of \( \Lambda(6,2,0) \) with the set \( A^6_0 \) of words of length 6 over \( A = \{0,1\} \) with trace 0 (mod 2), the cycles in \( BA \) correspond to the necklaces of \( A^6_0 \) which are listed in the first column of Table B.1 in Appendix B. For example, the 3-cycle in the diagram below corresponds to the necklace [011011] with period 3 of (B.0.2).

\[
\begin{array}{cccc}
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1
\end{array}
\]

\[
\text{Lemma 6.2.3. Suppose that } \Lambda = \Lambda(n,q,t) \text{ is a domino 2-graph. Define } \sigma : \Lambda^0 \to \Lambda^0 \text{ by}
\]

\[
\sigma(v) = s(e) \text{ where } e \in v\Lambda^1.
\]

Then \( \sigma \) is a bijection. Further, \( \sigma \) is the identity if \( n = 1 \) or \( (n,q,t) = (2,2,0) \), and otherwise \( \sigma \) has order \( n \).

\[
\text{Proof.} \quad \text{Under the identification } \Lambda^0 = A^6_0 \text{ of Proposition 6.2.1(2), } \sigma \text{ corresponds to } \rho \text{ defined in (B.0.1). There is a unique blue edge entering and leaving each vertex by Proposition 6.2.1(3), and so } \sigma \text{ is a bijection.}
\]

The order of a vertex \( v \in \Lambda^0 \) under \( \sigma \) is the length of the cycle on which \( v \) lies. The blue graph of \( \Lambda(1,q,t) \) has only one vertex \( \begin{array}{c} \downarrow \\
\end{array} \\
\begin{array}{c} \downarrow \\
\end{array} \) and one blue edge \( \begin{array}{c} \downarrow \\
\end{array} \begin{array}{c} \downarrow \\
\end{array} \), which is a 1-cycle, so \( \sigma = \text{id} \). The blue graph of \( \Lambda(2,2,0) \) has two vertices \( \begin{array}{c} 0 \\
\end{array} \begin{array}{c} 0 \\
\end{array} \) and \( \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \), and the only blue edges are \( \begin{array}{c} 0 \\
\end{array} \begin{array}{c} 0 \\
\end{array} \begin{array}{c} 0 \\
\end{array} \) and \( \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \) which are 1-cycles. So \( \sigma = \text{id} \).

Otherwise, note that \( \sigma \) has order \( n \) if and only if \( BA \) contains an \( n \)-cycle. If \( t \neq 0 \), then the vertex \( v \) with \( v(0) = t \) and \( v(ie_1) = 0 \) for \( 1 \leq i \leq n - 1 \) lies on an \( n \)-cycle. If \( t = 0 \), then the vertex \( u \) with \( u(0) = u(e_1) = 1 \), and \( u(i) = 0 \) for \( 2 \leq i \leq n - 1 \) lies on an \( n \)-cycle.

The permutation \( \sigma \) satisfies \( \sigma(r(e)) = s(e) \), and hence moves against the direction of the edges. For example, in \( \Lambda(6,2,0) \), \( \sigma \) takes \( \begin{array}{c} 0 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 0 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \) to \( \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 0 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 1 \\
\end{array} \begin{array}{c} 0 \\
\end{array}. \)
Let $E$ and $F$ be 1-graphs. The product graph $(E \times F, d)$ is the 2-graph consisting of the product category $E \times F$ with the degree map $d(\lambda_1, \lambda_2) = (d(\lambda_1), d(\lambda_2))$. As we show in the following lemma, the domino graphs for which $\sigma$ is the identity are the only ones which are product graphs.

**Lemma 6.2.4.** The domino graph $\Lambda(n, q, t)$ is a product graph if and only if $n = 1$, in which case $\Lambda \cong K_1 \times K_1$, or $(n, q, t) = (2, 2, 0)$, in which case $\Lambda \cong K_1 \times K_2$.

**Proof.** Suppose that $\Lambda$ is a product graph $E \times F$. Then

$$\Lambda^e = (E \times F)^e = \{(v, f) : v \in E^0, f \in F^1\}$$

with $s(v, f) = (v, s(f))$ and $r(v, f) = (v, r(f))$. Since $RA$ is connected (Proposition 6.2.1), we must have $|E^0| = 1$. Then $|F^0| = q^{n-1}$ since

$$|E^0| \times |F^0| = |E^0 \times F^0| = |(E \times F)^0| = |\Lambda^0| = q^{n-1},$$

and so $BA$ has $q^{n-1}$ copies of $E$.

Recall from Proposition 4.1.8(b) that there is at most one edge of each colour between any pair of vertices in the skeleton of $\Lambda$. So since $E$ has only one vertex, there is either no edge or exactly one edge (a self-loop). If $E$ has no edge, then $\Lambda = E \times F = F$ which cannot be true because then $BA$ would have no edges. Hence $E$ must have an edge, and $E = K_1$. Since $BA$ has $q^{n-1}$ copies of $E = K_1$, there is a cycle at each vertex. Thus $\sigma = id$ and Lemma 6.2.3 implies that either $n = 1$ or $(n, q, t) = (2, 2, 0)$. If $n = 1$, then Proposition 6.2.1 implies that $h_1 = 1$ and $F = K_1$; if $(n, q, t) = (2, 2, 0)$, then $h_1 = 2$ and $F = K_2$. \hfill $\Box$

Suppose that $\alpha$ is an action of $\mathbb{Z}$ on a 1-graph $\Lambda$. Farthing, Pask, and Sims [19] show that there is a 2-graph $\Lambda \times_\alpha \mathbb{Z}$ called the crossed product of $\Lambda$ by $\alpha$, which as a set is the cartesian product $\Lambda \times \mathbb{N}$, and which has degree map\(^1\) $d(\lambda, m) := (m, d(\lambda))$, range and source maps defined by $r(\lambda, m) := (r(\lambda), 0)$ and $s(\lambda, m) := (\alpha^{-m}(s(\lambda)), 0)$, and composition defined by $(\mu, m)(\nu, n) = (\mu a^m(\nu), m + n)$ when $s(\mu, m) = r(\nu, n)$.

We now show that a domino graph $\Lambda$ which is not a product graph is isomorphic to the crossed product $RA \times_\alpha \mathbb{Z}$ of the red graph $RA$ by a $\mathbb{Z}$-action $\alpha$ which is determined by the permutation $\sigma$ defined in (6.2.1) in Lemma 6.2.3.

**Proposition 6.2.5.** Suppose that $\Lambda = \Lambda(n, q, t)$ is the domino graph associated to basic data $(n, q, t)$ with $n \geq 2$. For each $e \in \Lambda^e$, there is a unique edge $\sigma_1(e)$ in $\Lambda^e$ from $\sigma(s(e))$ to $\sigma(r(e))$, and then $\sigma_1 : \Lambda^e \to \Lambda^e$ is a bijection. The pair $(\sigma, \sigma_1)$ is an automorphism of $RA$. Let $\alpha$ be the action of $\mathbb{Z}$ on RA generated by $(\sigma^{-1}, \sigma_1^{-1})$. Then $\Lambda$ is isomorphic to the crossed product $RA \times_\alpha \mathbb{Z}$.

**Proof.** Since $RA$ is complete (by Proposition 6.2.1), there is exactly one red edge between every pair of vertices in $\Lambda^0$, so $\sigma_1$ is well-defined. To see that $\sigma_1$ is surjective, let

---

\(^1\)This is slightly different from the definition in [19], where the degree map is defined by $d(\lambda, m) := (d(\lambda), m)$. The change has the effect of repainting the red edges blue and vice-versa; we have changed to ensure that the isomorphism of the Proposition 6.2.5 matches red edges with red edges.
To see that $f \in \Lambda^{c_2}$. Then there exist unique blue edges $f_1, f_2$ with $s(f_1) = r(f)$ and $s(f_2) = s(f)$; take $e$ to be the unique edge from $r(f_2)$ to $r(f_1)$, and then we have $\sigma(r(e)) = s(f_1) = r(f)$, $\sigma(s(e)) = s(f_2) = s(f)$ and $\sigma_1(e) = f$. To see that $\sigma_1$ is injective, suppose that $\sigma_1(e) = \sigma_1(h)$. Then $s(e) = \sigma^{-1}(s(\sigma_1(e))) = \sigma^{-1}(s(\sigma_1(h))) = s(h)$ and $r(e) = r(h)$, and since there is exactly one red edge between two given vertices, we must have $e = h$.

The pair $(\sigma, \sigma_1)$ is an automorphism of $RA$ since $\sigma$ and $\sigma_1$ are bijections and we have $s(\sigma_1(e)) = \sigma(s(e))$ and $r(\sigma_1(e)) = \sigma(r(e))$ by definition.

We build a coloured graph isomorphism $\phi$ from the skeleton of $\Lambda$ to the skeleton of $RA \times_{\alpha} \mathbb{Z}$ and prove that it preserves commuting squares. Then by [36, §6], $\phi$ extends uniquely to a 2-graph isomorphism, and the result follows.

We define

$$
\phi_0 : \Lambda^0 \to (RA \times_{\alpha} \mathbb{Z})^0 \text{ by } \phi_0(v) = (v, 0) \text{ for } v \in \Lambda^0,
\phi_1 : \Lambda^{\epsilon_1} \to (RA \times_{\alpha} \mathbb{Z})^{\epsilon_1} \text{ by } \phi_1(\beta) = (r(\beta), 1) \text{ for } \beta \in \Lambda^{\epsilon_1}, \text{ and }
\phi_2 : \Lambda^{\epsilon_2} \to (RA \times_{\alpha} \mathbb{Z})^{\epsilon_2} \text{ by } \phi_2(\rho) = (\rho, 0) \text{ for } \rho \in \Lambda^{\epsilon_2}.
$$

Then $\phi_0$ and $\phi_2$ are bijections because $(RA \times_{\alpha} \mathbb{Z})^0 = \Lambda^0 \times \{0\}$ and $(RA \times_{\alpha} \mathbb{Z})^{\epsilon_2} = \Lambda^{\epsilon_2} \times \{0\}$.

To see that $\phi_1$ is a bijection, note that

$$(RA \times_{\alpha} \mathbb{Z})^{\epsilon_1} = \{(r(\beta), 1) : \beta \in \Lambda^{\epsilon_1}\} = \{(v, 1) : v \in \Lambda^0\}$$

since $|v\Lambda^{\epsilon_1}| = 1$ for all $v \in \Lambda^0$, and let $\beta \in \Lambda^{\epsilon_1}$. Then $\phi_1(\beta) = (r(\beta), 1)$ is the unique edge with range $r(r(\beta), 1) = (r(\beta), 0)$ and source $s(r(\beta), 1) = (\alpha^{-1}(r(\beta)), 0) = (s(\beta), 0)$. Hence $\phi_1$ is a bijection. So $\phi = (\phi_0, \phi_1, \phi_2)$ is an isomorphism and it remains to show that it preserves commuting squares.

Every commuting square $\lambda \in \Lambda^{(1,1)}$ is uniquely determined by the red edge $\lambda(0,e_2) = \lambda_{|T(e_2)}$. To see this, let $\rho \in \Lambda^{\epsilon_2}$. There are unique blue edges $\beta_1$ and $\beta_2$ with $r(\beta_1) = r(\rho)$ and $s(\beta_2) = s(\rho)$, and we then have $s(\beta_1) = \alpha^{-1}(r(\beta_1))$ and $s(\beta_2) = \alpha^{-1}(r(\beta_2))$. There is a unique red edge from $s(\beta_2)$ to $s(\beta_1)$, and it must be $\alpha^{-1}(\rho)$ since

$$r(\alpha^{-1}(\rho)) = \alpha^{-1}(r(\rho)) = \alpha^{-1}(r(\beta_1)) = s(\beta_1), \text{ and }$$
$$s(\alpha^{-1}(\rho)) = \alpha^{-1}(s(\rho)) = \alpha^{-1}(r(\beta_2)) = s(\beta_2).$$

So each $\rho \in \Lambda^{\epsilon_2}$ determines a commuting square $\rho \beta_2 = \beta_1 \alpha^{-1}(\rho)$ and every commuting square $\lambda$ in $\Lambda$ has this form with $\rho := \lambda_{|T(e_2)}$.

Every commuting square in $(RA \times_{\alpha} \mathbb{Z})^{(1,1)}$ has the form $(\rho, 1)$ for some $\rho \in \Lambda^{\epsilon_2}$ and has factorisations

$$(\rho, 0)(s(\rho), 1) = (r(\rho), 1)(\alpha^{-1}(\rho), 0).$$

We will show that $\phi$ maps the commuting square $\lambda \in \Lambda^{(1,1)}$ to the commuting square $(\lambda_{|T(e_2)}, 1)$ in $RA \times_{\alpha} \mathbb{Z}$. Let $\rho := \lambda_{|T(e_2)}$. In pictures,
We have $\phi_2(\rho) = (\rho, 0)$ and $\phi_2(\alpha^{-1}(\rho)) = (\alpha^{-1}(\rho), 0)$ by definition, and $\phi_1(\beta_2) = (r(\beta_2), 1) = (s(\rho), 1)$ and $\phi_1(\beta_1) = (r(\beta_1), 1) = (r(\rho), 1)$. So $\phi(\lambda)$ has factorisations $(\rho, 0)(s(\rho), 1) = (r(\rho), 1)(\alpha^{-1}(\rho), 0)$ which gives $\phi(\lambda) = (\rho, 1) = (\lambda|_{T(e_2)}, 1)$. □

6.3. The $C^*$-algebras of domino graphs

The $C^*$-algebras of dominoes are quite different from the simple $C^*$-algebras studied in Sections 4.4 and 4.5 in Chapter 4. In this section we verify that domino graphs $\Lambda$ are periodic and prove a structure theorem which identifies $C^*(\Lambda)$.

**Lemma 6.3.1.** The domino graph $\Lambda = \Lambda(n,q,t)$ does not satisfy the aperiodicity condition (iv) of [60, Lemma 3.2]. Hence $C^*(\Lambda)$ is not simple.

**Proof.** The infinite path space of $\Lambda$ consists of block diagrams like (6.1.1) covering the quarter plane. Since $\Lambda$ is complete, there is no vertical restriction on which vertices can follow others in a path. In the horizontal direction one vertex determines all the others in its row since the blue graph consists of disjoint cycles, and so each row in a path diagram like (6.1.1) contains the vertices from only one blue cycle and exhibits periodicity equal to the length of that cycle. Since all blue cycle lengths divide $n$, every path of $\Lambda$ has a horizontal repeating pattern of a block of width $n$.

Formally: for $\lambda \in \Lambda$ and $l \in \mathbb{Z}^2$, Proposition 6.2.1(2) implies that $\lambda|_{T+l} = \lambda|_{T+l+ne_1}$ whenever $T+l$ and $T+l+ne_1$ are in the domain of $\lambda$. Fix $v \in \Lambda^0$ and $m,p \in \mathbb{N}$ satisfying $p_1 - m_1 = n$ and $p_2 = m_2 = 0$. Then every $\lambda \in v\Lambda$ with $d(\lambda) \geq p$ has

$$\lambda(m,m+d(\lambda)-p) = \lambda(p,d(\lambda)),$$

which contradicts [60, Lemma 3.2(iv)]. Thus [60, Theorem 3.2] implies that $C^*(\Lambda)$ is not simple. □

**Remark 6.3.2.** As mentioned in Section 3.3.3, aperiodicity of the single vertex 2-graphs in [11], and hence simplicity of the $C^*$-algebras, is determined by choice of the factorisation property. Periodicity is the atypical situation and only occurs when the factorisations are given by a “flip” operation (such as in a domino graph with $n = 1$, there is one vertex and one edge of each colour and $\alpha \beta$ factorises as $\beta \alpha$). The resulting nonsimple $C^*$-algebra is exhibited as a tensor product of $C(T)$ by a simple algebra in [11, §5.3]. Similarly in our case, the periodic structure of domino graphs explored in Section 6.2 will be used in identifying the $C^*$-algebras.

Since the $C^*$-algebras of domino graphs are not simple, the Kirchberg-Phillips classification theorem does not apply, and we must use other methods to analyse $C^*(\Lambda)$.

**Theorem 6.3.3.** Suppose that $\Lambda = \Lambda(n,q,t)$ is a domino graph. Then

$$C^*(\Lambda) \cong \begin{cases} C(T) \otimes C(T) & \text{if } n = 1 \\ C(T) \otimes \mathcal{O}_2 & \text{if } (n,q,t) = (2,2,0). \end{cases}$$

Otherwise, let $\sigma$ be the permutation of $\Lambda^0$ defined by (6.2.1) and let $\{T_v : v \in \Lambda^0\}$ be a Cuntz family parametrised by $\Lambda^0$. Define $\gamma : \mathbb{Z}/n\mathbb{Z} \rightarrow \text{Aut} C^*(T_v)$ by $\gamma_{k+n\mathbb{Z}}(T_v) = T_{\sigma^{-k}(v)}$. 

Let $C_n$ be the group of $n$th roots of unity, and identify $C_n$ with $(\mathbb{Z}/n\mathbb{Z})^\wedge$ by sending $z \in C_n$ to $\chi_z: k + n\mathbb{Z} \mapsto z^k$. Then

\begin{equation}
C^*(\Lambda) \cong \text{Ind}_{C_n}^{\mathbb{Z}}(C^*(T_v) \times_{\gamma} (\mathbb{Z}/n\mathbb{Z}), \hat{\gamma}).
\end{equation}

and $C^*(T_v) \times_{\gamma} (\mathbb{Z}/n\mathbb{Z})$ is simple.

The interesting case is the general one, when $n \geq 2$ and $(n, q, t) \neq (2, 2, 0)$, so that $\sigma$ has order $n$. Then the graph $\Lambda(n, q, t)$ is isomorphic to a crossed product $RA \rtimes_{\alpha} \mathbb{Z}$, and the action $\alpha$ of $\mathbb{Z}$ on $RA$ induces an action $\tilde{\alpha}$ of $\mathbb{Z}$ on the $C^*$-algebra $C^*(RA)$ such that $\tilde{\alpha}(s_{\lambda}) = s_{\alpha(\lambda)}$ for all $\lambda \in \Lambda [19$, Proposition 3.1]. Next we identify $\tilde{\alpha}$.

**Lemma 6.3.4.** Suppose that $\Lambda = \Lambda(n, q, t)$ is a domino graph with $n \geq 2$. The $C^*$-algebra $C^*(RA)$ is generated by a Cuntz family $\{T_v : v \in \Lambda^0\}$ such that $\tilde{\alpha}(T_v) = T_{\sigma^{-1}(v)}$, where $\sigma$ is the permutation of $\Lambda^0$ defined in Lemma 6.2.3.

**Proof.** Suppose that $\{s_{\lambda} : \lambda \in RA\}$ is the universal Cuntz-Krieger family which generates $C^*(RA)$. For each $v \in \Lambda^0$ define $T_v := \sum_{r(e)=v} S_e$ and, as in the proof of [54, Corollary 2.6], $C^*(RA)$ is generated by $\{T_v : v \in \Lambda^0\}$. We now check that $\{T_v : v \in \Lambda^0\}$ is a Cuntz family. We have

\[ T_v T_v^* = \left( \sum_{r(e)=v} S_e \right) \left( \sum_{r(f)=v} S_f \right)^* = \sum_{r(e)=v} S_e \sum_{r(f)=v} S_f^* = \sum_{r(e)=v} S_e S_e^*, \]

since $S_e S_e^* \neq 0$ implies $r(e) = r(f)$. The projections $P_v = \sum_{r(e)=v} S_e S_e^*$ are mutually orthogonal in $C^*(RA)$, and since $|\Lambda^0|$ is finite, $\sum_{v \in \Lambda^0} T_v T_v^* = \sum_{v \in \Lambda^0} P_v$ is the identity 1. Each $T_v$ is an isometry since

\[ T_v T_v^* = \left( \sum_{r(e)=v} S_e \right) \left( \sum_{r(f)=v} S_f \right)^* = \sum_{r(e)=v} S_e^* S_e = \sum_{r(e)=v} P_{s(e)} = \sum_{v \in \Lambda^0} P_v = 1, \]

using the Cuntz-Krieger relation $S_e^* S_e = P_{s(e)}$ and the fact that $RA$ is complete. On the other hand, recall that $\alpha$ is generated by the automorphism $(\sigma^{-1}, \sigma_1^{-1})$ of $RA$, and then

\[ \tilde{\alpha}(T_v) = \sum_{r(e)=v} \tilde{\alpha}(S_e) = \sum_{r(e)=v} S_{\sigma_1(e)} = \sum_{r(f)=\sigma^{-1}(v)} S_f = T_{\sigma^{-1}(v)}. \]

Since $\sigma$ has order $n$, $\sigma$ induces an action $\gamma$ of $\mathbb{Z}/n\mathbb{Z}$ on $C^*(T_v)$ as in Theorem 6.3.3, and then Lemma 6.3.4 says that $\tilde{\alpha}$ is the composition $\gamma \circ \pi$ of $\gamma$ with the quotient map $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Now Theorem 3.5 of [19] implies $C^*(\Lambda) \cong C^*(T_v) \times_{\gamma \circ \pi} \mathbb{Z}$, and we realise this crossed product as an induced $C^*$-algebra using Corollary 5.3.3. To prove Theorem 6.3.3 we need one more lemma, the proof of which is a modification of the argument in [1, Theorem 1].

**Lemma 6.3.5.** Suppose that $n \geq 2$ and $\sigma$ is a permutation of $\{1, \ldots, n\}$ which is not the identity. If $\alpha$ is the automorphism of $O_n = C^*(S_i)$ such that $\alpha(S_i) = S_{\sigma(i)}$ for all $i$, then $\alpha$ is outer.
The proof follows from the fact that \( e_k e_k = e_{nk+i-1} \). The adjoint of \( T_i \) is defined by

\[
T_i^* e_l = \begin{cases} 
  e_{n^{-1}(l-i+1)} & \text{if } l = i - 1 \pmod{n} \\
  0 & \text{else}.
\end{cases}
\]

We have \( \sum_{i=1}^{n} T^*_i T_i e_l = e_l \) since

\[
T_i T_i^* e_l = \begin{cases} 
  e_l & \text{if } l = i - 1 \pmod{n} \\
  0 & \text{else},
\end{cases}
\]

and so \( \sum_{i=1}^{n} T^*_i T_i = 1 \). Each \( T^*_i \) is an isometry, thus \( \{T_i\}_{i=1}^{n} \) is a Cuntz family.

By Cuntz’s uniqueness theorem [7, Theorem 1.12] there is an isomorphism \( \pi \) of \( C^*(S_i) \) onto \( C^*(T_i) \) such that \( \pi(S_i) = T_i \) for all \( i \). Then \( U^* S_i U = S_{\sigma(i)} \) implies that \( \pi(U)^* T_i \pi(U) = T_{\sigma(i)} \). Since \( T_1 e_0 = e_0 \) and \( \pi(U) \in B(l^2) \) is unitary, \( T_{\sigma(i)} \) has 1 as an eigenvalue, and there is a nonzero element \( \{a_k\} \in l^2 \) such that \( T_{\sigma(i)}(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{\infty} a_k e_k \).

Then since \( T_{\sigma(i)} e_k = e_{nk+\sigma(i)-1} \) we have

\[
\sum_{k=1}^{\infty} a_k e_k = T_{\sigma(1)}(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{\infty} a_k T_{\sigma(1)} e_k = \sum_{k=1}^{\infty} a_k e_{nk+\sigma(1)-1}
\]

which implies that \( a_{nk+\sigma(1)-1} = a_k \) for \( k \geq 1 \) and \( a_m = 0 \) whenever \( m \neq nk + \sigma(1) - 1 \) for all \( k \). There are infinitely many constants \( a_m \) equal to \( a_k \) for each \( k \), so \( \sum_{m=1}^{\infty} |a_m|^2 < \infty \) forces \( a_k = 0 \) for all \( k \geq 1 \), which is a contradiction. Hence \( \alpha \) is outer.

**Proof of Theorem 6.3.3.** The first two cases are easy: the domino graph is a product graph by Lemma 6.2.4, and \( C^*(E \times F) \cong C^*(E) \otimes C^*(F) \) by [36, Corollary 3.5(iv)].

For \( n = 1 \) we get

\[
C^*(\Lambda) = C^*(K_1 \times K_1) \cong C^*(K_1) \otimes C^*(K_1) \cong C(T) \otimes C(T),
\]

since \( C^*(K_1) \cong C(T) \) [54, Example 1.23]. For \( (n,q,t) = (2,2,0) \), we get

\[
C^*(\Lambda) = C^*(K_1 \times K_2) \cong C^*(K_1) \otimes C^*(K_2) \cong C(T) \otimes \mathcal{O}_2,
\]

since \( C^*(K_2) \cong \mathcal{O}_2 \).

Otherwise, \( \Lambda \cong RA \times_\alpha \mathbb{Z} \) by Proposition 6.2.5 and \( C^*(\Lambda) \cong C^*(RA \times_\alpha \mathbb{Z}) \). Theorem 3.5 of [19] implies that \( C^*(RA \times_\alpha \mathbb{Z}) \) is isomorphic to \( C^*(RA) \times_\alpha \mathbb{Z} \). Lemma 6.3.4 implies that \( C^*(RA) \times_\alpha \mathbb{Z} \) is isomorphic to \( C^*(T_v) \times_{\gamma \alpha} \mathbb{Z} \). Applying Corollary 5.3.3 with \( N = n \mathbb{Z} \) shows that \( C^*(T_v) \times_{\gamma \alpha} \mathbb{Z} \) is isomorphic to the induced algebra \( \text{Ind}_{(n\mathbb{Z})^+}^{\mathbb{Z}}(C^*(T_v) \times_{\gamma} (\mathbb{Z}/n\mathbb{Z}), \gamma) \), which gives (6.3.1) since \( \mathbb{Z} = \mathbb{T} \) and \( (n\mathbb{Z})^+ = C_n \).

Since \( \mathbb{Z}/n\mathbb{Z} \) is amenable, the crossed product \( C^*(T_v) \times_{\gamma} (\mathbb{Z}/n\mathbb{Z}) \) is isomorphic to the reduced crossed product \( C^*(T_v) \times_{\gamma, r} (\mathbb{Z}/n\mathbb{Z}) \) (by, for example, [49, Theorem 7.7.7]). We know from Lemma 6.3.4 that \( C^*(T_v) \) is a Cuntz algebra and \( \gamma \) is not the identity (Lemma 6.2.3), so Lemma 6.3.5 implies that \( \gamma \) is outer for every \( m \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{0\} \). Since the Cuntz algebra
is simple, the reduced crossed product $C^*(T_v) \times_{\gamma,r} (\mathbb{Z}/n\mathbb{Z})$ is simple by [30, Theorem 3.1]. Hence $C^*(T_v) \times_{\gamma} (\mathbb{Z}/n\mathbb{Z})$ is simple.

Remark 6.3.6. Since $C^*(T_v) \times_{\gamma} (\mathbb{Z}/n\mathbb{Z})$ is simple, it follows from [59, Proposition 6.16] that the primitive ideal space of the induced algebra in (6.3.1) is homeomorphic to $\mathbb{T}/C_n \cong \mathbb{T}$. It is intriguing that the other family of periodic 2-graphs whose algebras have been analysed also have $C^*$-algebras with primitive ideal space $\mathbb{T}$ (see [11, §5]). The graphs in [11] are quite different: the bicoloured graphs have just one vertex but multiple edges, so there are many possible families $C$ of commuting squares. In domino graphs there is at most one edge of each colour from one vertex to another, and the bicoloured graph admits a unique family $C$ of commuting squares.

6.4. $K$-theory of domino graphs

Based on the numerical evidence in Appendix D, we made conjectures in Section 4.6 about the $K$-theory of the $C^*$-algebras of our 2-graphs, in particular, that $K_0(C^*(\Lambda))$ and $K_1(C^*(\Lambda))$ are cyclic groups of the same order. In this section we verify this for domino graphs using the identification of $C^*(\Lambda)$ as either a tensor product or a crossed product.

Proposition 6.4.1. Suppose that $\Lambda = \Lambda(n, q, t)$ is a domino graph. Then $C^*(\Lambda)$ has $K$-theory

$$K_i(C^*(\Lambda)) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 1 \\ 0 & \text{if } (n, q, t) = (2, 2, 0) \\ \mathbb{Z}/(q^{n-1} - 1)\mathbb{Z} & \text{otherwise}. \end{cases}$$

Proof. In the first case $C^*(\Lambda) = C(\mathbb{T}) \otimes C(\mathbb{T})$ by Theorem 6.3.3. Since $K_i(C(\mathbb{T})) = \mathbb{Z}$ [64, page 234], the $K$-groups of $C(\mathbb{T})$ are torsion-free and the Künneth formula [65, Theorem 1.18] gives

$$K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) = K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T})) \oplus K_1(C(\mathbb{T})) \otimes K_1(C(\mathbb{T}))$$

$$= \mathbb{Z} \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}^2$$

$$K_1(C(\mathbb{T}) \otimes C(\mathbb{T})) = K_0(C(\mathbb{T})) \otimes K_1(C(\mathbb{T})) \oplus K_1(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))$$

$$= \mathbb{Z} \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}^2.$$

In the second case $C^*(\Lambda) = C(\mathbb{T}) \otimes O_2$ by Theorem 6.3.3, and since $K_i(O_2) = 0$ [64, page 234] the Künneth formula gives

$$K_0(C(\mathbb{T}) \otimes O_2) = K_0(C(\mathbb{T})) \otimes K_0(O_2) \oplus K_1(C(\mathbb{T})) \otimes K_1(O_2)$$

$$= \mathbb{Z} \otimes 0 \oplus \mathbb{Z} \otimes 0 = 0$$

$$K_1(C(\mathbb{T}) \otimes O_2) = K_0(C(\mathbb{T})) \otimes K_1(O_2) \oplus K_1(C(\mathbb{T})) \otimes K_0(O_2)$$

$$= \mathbb{Z} \otimes 0 \oplus \mathbb{Z} \otimes 0 = 0.$$

Otherwise, we have $C^*(\Lambda) \cong C^*(T_v) \times_\alpha \mathbb{Z}$. Since $C^*(T_v)$ is a Cuntz algebra with $q^{n-1}$ generators, we deduce from [64, page 234], for example, that

$$K_1(C^*(T_v)) = 0 \text{ and } K_0(C^*(T_v)) = \mathbb{Z}/(q^{n-1} - 1)\mathbb{Z},$$
where $K_0(C^*(T_v))$ is generated by the class $[1]$ of the identity. Thus the Pimsner-Voiculescu sequence [51] for this crossed product reduces to

$$0 \rightarrow K_1(C^*(T_v) \times \tilde{\alpha} \mathbb{Z}) \rightarrow K_0(C^*(T_v)) \rightarrow K_0(C^*(T_v) \times \tilde{\alpha} \mathbb{Z}) \rightarrow 0$$

and we have $\text{id} - \tilde{\alpha}_* = 0$ because $\tilde{\alpha}_*(1) = 1$. So both $K_0(C^*(\Lambda)) = K_0(C^*(T_v) \times \tilde{\alpha} \mathbb{Z})$ and $K_1(C^*(\Lambda)) = K_1(C^*(T_v) \times \tilde{\alpha} \mathbb{Z})$ are isomorphic to $\mathbb{Z}/(q^n - 1)\mathbb{Z}$. □
APPENDIX A

Integer Partitions

A partition of an integer \( n \) is a list of positive integers with sum equal to \( n \) whose order is not significant (conventionally written from largest to smallest). The number of partitions of an integer is given by the partition function \( p \) (Sloane’s A000041 in [70]). The row lengths (or column heights) of a tile \( T \) with \( |T| = n \) correspond naturally to a partition of \( n \). Thus the number of different tiles of size \( |T| = n \) is given by \( p(n) \). For example, \( p(3) = 3 \) and the partitions of 3 are 3, 2 + 1, and 1 + 1 + 1, corresponding to the tiles \( 3 \), \( 2 + 1 \), and \( 1 + 1 + 1 \). The following table lists \( p(n) \) for the first ten values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>22</td>
<td>30</td>
<td>42</td>
</tr>
</tbody>
</table>

A partition is represented graphically using a Ferrers diagram or Young diagram. A Young diagram is the upside-down version of the drawing of a tile. For example, the partition \( 8 = 4 + 2 + 1 + 1 \) usually written \([4, 2, 1, 1]\) has Ferrers diagram (dots) and Young diagram:

Conjugate partitions are pairs of partitions of the same integer whose Young/Ferrers diagrams transform into each other when reflected about the line \( y = -x \) (taking the upper left dot as the origin). For example, \([6, 3, 3, 2, 1]\) and \([5, 4, 3, 1, 1]\) are conjugate.

A self-conjugate partition has conjugate equivalent to itself. For example, the partition \([4, 2, 1, 1]\) is self-conjugate. We call a tile self-conjugate if it is symmetric when reflected about the line \( y = x \) (taking \( 0 \in \mathbb{N}^2 \) as the origin). The sock tile \( \square \) is the smallest non-trivial self-conjugate tile. The number of self-conjugate tiles is given by Sloane’s A000700, the number of self-conjugate partitions of \( n \); the first ten values are given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{sc}(n) )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
on Young diagrams is illustrated by a Hasse diagram where the edges join tiles less than it by only one square. The Hasse diagram for tiles up to four levels is drawn below.

The lexicographic order is a total order on the set of all partitions.

Every partition of \( n \) can be written in the form \([n_1^{i_1}, n_2^{i_2}, \ldots, n_k^{i_k}]\) where \( i_1, i_2, \ldots, i_k \geq 1 \) and \( n_j > n_{j+1} \) for \( 1 \leq j < k \) satisfy \( n = \sum_{j=1}^{k} i_j n_j \). (We omit the superscript when it is a 1.) For example, in this notation the partition \([4, 2, 2, 2, 1, 1]\) is written \([4, 2^3, 1^2]\).
APPENDIX B

Necklaces and Lyndon words

A word is a finite or infinite sequence of symbols from a finite set \( A \) called the alphabet. Subsets of the set \( A^* \) of all finite words are called languages and we write \( A^n \) for the words of length \( n \). The product of two words \( u \) and \( v \) is their concatenation \( uv \), and for \( k \in \mathbb{N} \) the \( k \)th power of \( u \) is \( u^k \).

Consider the alphabet \( A = \{0, 1, \ldots, q-1\} \). Let \( \rho \) be “anticlockwise” rotation, that is, for \( a_1 \cdots a_n \in A^n \),

\[
\rho(a_1 \cdots a_n) = a_2 \cdots a_na_1.
\]

Then the cyclic subgroup \( \langle \rho \rangle \) acts on \( A^n \), and the equivalence classes under rotation are known as necklaces of length \( n \), so called because a necklace of length \( n \) can be visualised as a regular \( n \)-gon where the corners represent the “beads”, each designated one of \( q \) colours. For example, if \( q = 2 \) and \( n = 6 \), the class of 011011 is the necklace \([011011] = \{011011, 101101, 110110\} \) and is drawn as

\[
\begin{array}{c}
\begin{array}{c}
1 & 1 \\
0 & 0 \\
1 & 1
\end{array}
\end{array}
\]

The period of a necklace \([a]\) is the smallest \( d \in \mathbb{N} \) such that \( \rho^d(a) = a \), and it must divide the length of the necklace. A necklace with period equal to its length is called aperiodic and its lexicographic least representative is known as a Lyndon word.\(^1\) For example, 000011 and 000001 are Lyndon words of length 6. For every necklace \([a]\) of length \( n \) and period \( d \) there is a unique subword \( b \) of length \( d \) such that \( [b^{n/d}] = [a] \); we call \( b \) the Lyndon subword of \([a]\). For example, the necklace \([011011]\) in (B.0.2) has period 3 and Lyndon subword 011.

We identify the alphabet \( A = \{0, 1, \ldots, q-1\} \) with the commutative ring \( \mathbb{Z}/q\mathbb{Z} \), and let the trace of a word be the sum of its symbols \( \pmod{q} \). We will often consider the collection \( A^n_t \) of words with length \( n \) and trace \( t \pmod{q} \). See Table B.1 for the period, trace and Lyndon subword of each binary necklace of length 6.

By [66, Theorem 1.2] the number of Lyndon words of length \( n \) with trace \( t \pmod{q} \) over the alphabet \( \{0, 1, \ldots, q-1\} \) is given by

\[
L_q(n, t) = \frac{1}{qn} \sum_{\substack{d|n \\gcd(d,q)|t}} \gcd(d,q) \mu(d) q^{n/d},
\]

\(^1\) Lyndon words were introduced in [42] as standard lexicographic sequences.
Table B.1. The binary necklaces of length 6

<table>
<thead>
<tr>
<th>Necklaces of $A_6^0$</th>
<th>Period</th>
<th>Lyndon subword</th>
<th>Necklaces of $A_6^1$</th>
<th>Period</th>
<th>Lyndon subword</th>
</tr>
</thead>
<tbody>
<tr>
<td>[000000]</td>
<td>1</td>
<td>0</td>
<td>[000001]</td>
<td>6</td>
<td>000001</td>
</tr>
<tr>
<td>[000011]</td>
<td>6</td>
<td>000011</td>
<td>[000111]</td>
<td>6</td>
<td>000111</td>
</tr>
<tr>
<td>[000101]</td>
<td>6</td>
<td>000101</td>
<td>[001011]</td>
<td>6</td>
<td>001011</td>
</tr>
<tr>
<td>[001001]</td>
<td>3</td>
<td>001</td>
<td>[001101]</td>
<td>6</td>
<td>001101</td>
</tr>
<tr>
<td>[001111]</td>
<td>6</td>
<td>001111</td>
<td>[010101]</td>
<td>2</td>
<td>01</td>
</tr>
<tr>
<td>[010111]</td>
<td>6</td>
<td>010111</td>
<td>[010111]</td>
<td>6</td>
<td>011111</td>
</tr>
<tr>
<td>[011011]</td>
<td>3</td>
<td>011</td>
<td>[011111]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[111111]</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table B.2. Values of $L_q(k, t)$

<table>
<thead>
<tr>
<th>q</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>0</td>
<td>1</td>
<td>0,1,2,3,4</td>
<td>0</td>
<td>1,5,2,4,3</td>
</tr>
<tr>
<td>k</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>38</td>
<td>39</td>
</tr>
</tbody>
</table>

where $\mu$ is the Möbius function. A selection of values for $L_q(k, t)$ taken from [67] is given in Table B.2. Note that if $t_1, t_2 \in \mathbb{Z}/q\mathbb{Z}$ with $\gcd(q, t_1) = \gcd(q, t_2)$ then $L_q(k, t_1) = L_q(k, t_2)$. 
APPENDIX C

Magma code

Included here is Magma code which we used to compute the values $|K_i|$ in the tables in Appendix D. We have used a functional style of programming and in this appendix we explain how the methods work and how they interact. We illustrate with examples of code and output throughout.

In Section C.1, we show how to represent the basic data $(T, q, t, w)$ and calculate the vertex matrices $B$ and $R$ of the bicoloured directed graph associated to the 2-graph $\Lambda = \Lambda(T, q, t, w)$ (see Section 4.1). In Section C.2, we give code for calculating the order of the $K$-groups of $C^*(\Lambda)$ (which implements the procedures detailed in Section 4.6), and we also show by example how to verify that the $K$-groups are cyclic groups. Finally, in Section C.3, we give code for generating lists of necklaces and Lyndon words (see Appendix B) which were used to describe the structure of domino graphs in Section 6.2.

In the code, we use the following naming convention: variables are in all-lowercase and functions are in mixed case with the first letter of each internal word capitalised. Bold text is used for reserved words and text in italics indicates a function built-in to the Magma language.

C.1. Tiles

C.1.1. Representing the basic data. Elements of $\mathbb{N}^2$ are represented as a list of integers of the form $[a, b]$. A tile $T$ is then represented as a list of elements $[a, b]$ ordered lexicographically and containing at least the origin $[1, 1]$. An integer $q$ greater than or equal to 2 represents the alphabet and the trace $t$ is an integer taking a value between 0 and $q - 1$. The rule $w$ is represented as a list of $|T|$ integers respecting the lexicographic order of elements of $T$. For example, the basic data in the Ledrappier example of Example 4.1.2 would be entered as the following code.

$$T := [[1, 1], [2, 1], [1, 2]];$$
$$q := 2;$$
$$t := 0;$$
$$w := [1, 1, 1];$$

C.1.2. Generating the vertex set. Vertices of $\Lambda(T, q, t, w)$ may be identified with elements of the set $A_q^n$ of words of length $n$ over $\{0, 1, \ldots, q-1\}$. The vertex set $\Lambda^0$ consists of those words of $A_q^n$ which satisfy a condition determined by $t$ and $w$. Vertices are thus represented as lists of integers (in the same way as the rule is). There are two ways to generate the vertex set. The first method is to list all the words of $A_q^n$ and then check the condition for each word. The second method is to list the words of $A_q^{n-1}$ and to each
word append the last entry which is the solution to a modular equation determined by the condition. (Note that this method relies on at least one value of \( w \) being invertible — so that each modular equation has exactly one solution — whereas the first method does not.) Code for the first method is below.

\[
\text{Strings} := \text{function}(n, q) \\
\quad \text{strings} := []; \\
\quad \text{for } i := 0 \text{ to } q^{n-1} \text{ do} \\
\quad \quad x := \text{IntegerToSequence}(i, q); \\
\quad \quad \text{while } \#x \lt n \text{ do} \\
\quad \quad \quad \text{Append}(\sim x, 0); \\
\quad \quad \end{\text{while}}; \\
\quad \quad \text{Reverse}(\sim x); \\
\quad \quad \text{Append}(\sim \text{strings}, x); \\
\quad \end{\text{for}}; \\
\quad \text{return } \text{strings}; \\
\end{\text{function}};
\]

\[
\text{VertexSet} := \text{function}(q, t, w) \\
\quad n := \#w; \\
\quad X := \text{Strings}(n, q); \\
\quad \text{for } x \text{ in } X \text{ do} \\
\quad \quad \text{pairwiseprod} := 0; \\
\quad \quad \text{for } i := 1 \text{ to } n \text{ do} \\
\quad \quad \quad \text{pairwiseprod} + := w[i] \times x[i]; \\
\quad \quad \end{\text{for}}; \\
\quad \quad \text{if } (\text{pairwiseprod} \mod q) \neq t \text{ then} \\
\quad \quad \quad \text{Exclude}(\sim X, x); \\
\quad \quad \end{\text{if}}; \\
\quad \end{\text{for}}; \\
\quad \text{return } X; \\
\end{\text{function}};
\]

Code for the second method:

\[
\text{RingStrings} := \text{function}(n, q) \\
\quad A := \text{Integers}(q); \\
\quad V := []; \\
\quad \text{if } n \text{ eq } 1 \text{ then} \\
\quad \quad \text{for } a \text{ in } A \text{ do} \\
\quad \quad \quad \text{Append}(\sim V, [a]); \\
\quad \end{\text{for}}; \\
\end{\text{function}};
\]
else
    \( W := (n-1,q) \);
    for \( v \) in \( W \) do
        for \( a \) in \( A \) do
            \( vnew := \text{Append}(v, a) \);
            \( \text{Append}(\tilde{V}, vnew) \);
        end for;
    end for;
end if;
return \( V \);
end function;

VS:=function(q,t,w)
    \( A := \text{Integers}(q) \);
    index := 0;
    for \( i := 1 \) to \#w do
        if \( \text{IsUnit}(A!w[i]) \) then
            index := \( i \);
            break;
        end if;
    end for;
    \( U := \text{RingStrings}(\#w-1,q) \);
    \( V := [] \);
    \( wnew := \text{Remove}(w, index) \);
    for \( u \) in \( U \) do
        sum := 0;
        for \( j := 1 \) to \#u do
            sum += \( wnew[j] \ast u[j] \);
        end for;
        star := \( \text{Solution}(A!w[index], t-\text{sum}) \);
        \( v := \text{Insert}(u, index, star) \);
        \( \text{Append}(\tilde{V}, v) \);
    end for;
    return \( \text{Sort}(V) \);
end function;

Example C.1.1. For the Ledrappier data, the first method VertexSet \((2,0,[1,1,1])\); or the second method VS \((2,0,[1,1,1])\); produces the list below
\[ [0,0,0], [0,1,1], [1,0,1], [1,1,0] \]
corresponding to the vertices in (4.1.1).
C.1.3. Calculating the vertex matrices. Suppose that we have generated the vertex set using either method above and stored the result as a variable X. We want to calculate the square matrix indexed by X which is the vertex matrix of the graph with given basic data (in either the blue or red direction). First we initialise a square matrix \( M \) with zero entries of size equal to the number of elements of X. For each pair of words \( a \) and \( b \) in X we check an overlap condition to determine if they correspond to adjacent vertices and if so, set \( M(a, b) = 1 \). This procedure is implemented as the method \( \text{VertexMatrix}(T, i, X) \) which takes as parameters a tile, a variable determining the direction \( i = 1 \) for blue or \( i = 2 \) for red) and a vertex set, and returns the vertex matrix \( M \). (The vertex set must be in the list format which is returned by \( \text{VertexSet} \) or \( \text{VS} \).) The code for the method \( \text{VertexMatrix}(T, i, X) \) is given below. (Note that \( \text{VertexMatrix}(T, i, X) \) calls the method \( \text{Minusei}(1, i, k) \) which simply takes as parameters an element \( m \) of \( \mathbb{N}^2 \), direction \( i = 1 \) or 2, and an integer \( k \), and then returns \( m - ke_i \).)

Minusei:=function(I,i,k)
Inew:=[ ];
for j:=1 to #I do
  if j eq i then
    Append("Inew",I[j]−k);
  else
    Append("Inew",I[j]);
  end if;
end for;
return Inew;
end function;

VertexMatrix:=function(T,i,X)
n:=#X;
M:=ZeroMatrix(IntegerRing(),n,n);
for a in X do
  for b in X do
    matchonoverlap:=true;
    for I in T do
      if I[i] gt 1 then
        if a[Index(T,I)] ne b[Index(T,Minusei(I,i,1))] then
          matchonoverlap:=false;
          break;
        end if;
      end if;
    end for;
    if matchonoverlap then

 Example C.1.2. The blue and red matrices (stored as variables B and R) for the Ledrappier graph in (4.6.1) are produced by the following code.

\[
T := \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}; \\
q := 2; \\
t := 0; \\
w := [1, 1, 1]; \\
X := VS(q, t, w); \\
B := VertexMatrix(T, 1, X); \\
R := VertexMatrix(T, 2, X);
\]

C.1.4. Tiles and partition notation. For large tiles it is tedious to type in every element of the tile; we prefer to use partition notation (see Appendix A). In this section we give algorithms for converting between the two notations and also for generating lists of all the tiles of a given cardinality.

First we need two small methods which are called by various other methods: one to find the conjugate of a tile

\[
\textbf{Conjugate} := \textbf{function} (T) \\
\text{conjT} := []; \\
\text{for } i := 1 \text{ to } T[1] \text{ do} \\
\text{conjT}[i] := 0; \\
\text{for } j := 1 \text{ to } \#T \text{ do} \\
\text{if } T[j] \geq i \text{ then} \\
\text{conjT}[i]+ := 1; \\
\text{end if}; \\
\text{end for}; \\
\text{end for}; \\
\text{return conjT;}
\]

and one to find \(c_1 + 1\) and \(c_2 + 1\):

\[
\textbf{imax} := \textbf{function} (\text{tile}, i) \\
\text{foundmax} := \text{tile}[1][i]; \\
\text{for } j := 2 \text{ to } \#\text{tile} \text{ do}
\]
if tile[j][i] \gt foundmax then
    foundmax:= tile[j][i];
end if;
end for;
return foundmax;
end function;

Example C.1.3. If $T$ is the tile

$$
\begin{bmatrix}
1 & 1 \\
2 & 1 \\
3 & 1 \\
1 & 2
\end{bmatrix}
$$

then

$\text{imax}(T, 1)$;

$\text{imax}(T, 2)$;

$\text{Conjugate}(T)$;

displays the output

$$
\begin{bmatrix}
3 \\
2 \\
[[1,1],[2,1],[1,2],[1,3]]
\end{bmatrix}
$$

which indicates that $c_1 + 1 = 3$, $c_2 + 1 = 2$, and the conjugate of $T$ is the tile

$$
\begin{bmatrix}
1 & 1 \\
2 & 1 \\
3 & 1 \\
1 & 2
\end{bmatrix}
$$

We can easily convert a tile into its corresponding partition, that is, list the row lengths
of the tile using the method below.

\begin{verbatim}
Rows:=function (tile)
    n1:=imax(tile,1);
    h:=imax(tile,2);
    rows:=[n1];
    for j:=2 to h do
        for i:=1 to rows[j-1] do
            if [rows[j-1]-i+1,j] in tile then
                Append(rows,j-1-i+1);
                break;
            end if;
        end for;
    end for;
    return rows;
end function;
\end{verbatim}

Similarly, the list of column heights of the tile is the (row) partition corresponding to the
conjugate tile and is given by

\begin{verbatim}
Cols:=function (tile)
    return Conjugate(Rows(tile));
end function;
\end{verbatim}
Example C.1.4. For the tile \[
\begin{array}{|c|c|}
\hline
1 & 1 \\
\hline
2 & 1 \\
\hline
3 & 1 \\
\hline
1 & 2 \\
\hline
\end{array}
\]
the code
\[
\begin{align*}
T & := \begin{bmatrix} 1,1 \end{bmatrix}, \begin{bmatrix} 2,1 \end{bmatrix}, \begin{bmatrix} 3,1 \end{bmatrix}, \begin{bmatrix} 1,2 \end{bmatrix} \\
\text{Rows}(T) & ; \\
\text{Cols}(T) & ; \\
\end{align*}
\]
has output
\[
\begin{bmatrix} 3,1 \end{bmatrix}, \begin{bmatrix} 2,1,1 \end{bmatrix}
\]
Now we need to be able to convert a tile in partition notation back to the format accepted by methods such as VertexMatrix. The process is dependent upon two methods.

\text{MakeHereditary} := \text{function}(l) \\
\text{Hset} := [] \\
\text{for } I \text{ in } l \text{ do} \\
\quad \text{for } i:=1 \text{ to } I[1] \text{ do} \\
\quad\quad \text{for } j:=1 \text{ to } I[2] \text{ do} \\
\quad\quad\quad \text{Append}(\text{Hset},[i,j]) \\
\quad\end{for} \\
\end{for} \\
\text{return IndexedSetToSequence(SetToIndexedSet(SequenceToSet(Hset)))); end function;}

\text{ReverseLexOrder} := \text{function}(T) \\
\text{Tnew} := [] \\
\text{for } I \text{ in } T \text{ do} \\
\quad \text{Append}(\text{Tnew},\text{Reverse}(I)) \\
\end{for} \\
\text{Sort}(\text{Tnew}); \\
\text{for } j:=1 \text{ to } \#Tnew \text{ do} \\
\quad \text{Reverse}(\text{Tnew}[j]) \\
\end{for}; \\
\text{return Tnew; end function;}

To convert a partition into a list of points in \(\mathbb{N}^2\) we “make it hereditary” and then order the list lexicographically using the method below.

\text{RowsToTile} := \text{function}(\text{list}) \\
\text{tile} := [] \\
\text{for } i:=1 \text{ to } \#\text{list} \text{ do} \\
\quad \text{Append}(\text{tile},[\text{list}[i],i]) \\
\end{for}
Similarly, if the tile is expressed as a partition of columns (rather than rows) we use the following method instead.

\text{ColsToTile} := \text{function}\ (\text{list} )
\begin{align*}
\text{return } & \text{RowsToTile(Conjugate(list))} ;
\end{align*}
\text{end function} ;

Example C.1.5. The code
\begin{align*}
T := & [3,1] ;\\
\text{RowsToTile}(T) ;
\text{ColsToTile}(T) ;
\end{align*}
displays the output
\begin{align*}
[ [1,1], [2,1], [3,1], [1,2] ] \\
[ [1,1], [2,1], [1,2], [1,3] ]
\end{align*}
which corresponds to the tiles \rectangle{1}{1} \rectangle{2}{1} \rectangle{3}{1} and \rectangle{1}{2} \rectangle{1}{1} \rectangle{1}{2} \rectangle{1}{3}.

To list all the tiles of a given cardinality \( n \) we use \text{Partitions}(n), which is one of the functions built-in to the \text{Magma} language, to get a list of the integer partitions of \( n \) and then convert them to tiles.

\text{Tiles} := \text{function}\ (n)
\begin{align*}
P := & \text{Partitions(n) ;}\\
tiles := & [];\\
\text{for } & p \text{ in P do}\n\begin{align*}
T := & [];\\
\text{for } & j := 1 \text{ to } \#p \ do\\
\text{for } & i := 1 \text{ to } p[j] \ do\\
& \text{Append}(\tilde{T},[i,j]);\\
\text{end for} ;
\end{align*}
\text{end for} ;
\text{Append}(\tilde{\text{tiles}},T) ;
\text{end for} ;
\text{return } \text{tiles} ;
\end{align*}
\text{end function} ;

Or if we want a list which only includes one from each pair of conjugate tiles (and the self-conjugate tiles) we use the method below.
NonConjTiles := function(n)
  P := Partitions(n);
  done := [];
  tiles := [];
  for partn in P do
    if Conjugate(partn) notin done then
      k := Index(P, partn);
      newp := [];
      for j := 1 to #partn do
        Append(~newp, [partn[j], j]);
      end for;
      newp := ReverseLexOrder(MakeHereditary(newp));
      Append(~tiles, newp);
      Append(~done, partn);
    end if;
  end for;
  return tiles;
end function;

Example C.1.6. For example, Tiles(3); returns
[
  [[1,1],[2,1],[3,1]],
  [[1,1],[2,1],[1,2]],
  [[1,1],[1,2],[1,3]]
] corresponding to the tiles □, □, □, and NonConjTiles(3); returns
[
  [[1,1],[2,1],[3,1]],
  [[1,1],[2,1],[1,2]]
]

We often perform calculations for basic data in which \( w \) is the rule \( w \equiv 1 \). The method OneSequence(n) below returns the list \([1,...,1]\) of length \( n \). If \( T \) is a tile, then we can use OneSequence(#T); to represent the rule \( w \equiv 1 \). We also use this method to represent the unit vector \( e = (1,\ldots,1) \).

OneSequence := function(n)
  w := [];
  for i := 1 to n do
    Append(~w, 1);
  end for;
return w;
end function;

C.2. K-theory

C.2.1. Calculating $|K_0|$ and $|K_1|$. Once we have calculated the vertex matrices stored as variables B and R we are able to do K-theory calculations. The procedures outlined in Section 4.6 are implemented as the methods KZero(B,R) and KOne(B,R) which return the values of $|K_0|$ and $|K_1|$. The code for these methods is below.

KZero:=function (B,R)
Bt:=Transpose(B);
Rt:=Transpose(R);
Y:=HorizontalJoin(1−Bt,1−Rt);
Z:=VerticalJoin(Rt−1,1−Bt);
ColspY:=RowSpace(Transpose(Y));
Mt:=BasisMatrix(ColspY);
return Abs(Determinant(Mt));
end function;

KOne:=function (B,R)
Bt:=Transpose(B);
Rt:=Transpose(R);
Y:=HorizontalJoin(1−Bt,1−Rt);
Z:=VerticalJoin(Rt−1,1−Bt);
kerY:=NullspaceOfTranspose(Y);
Ht:=BasisMatrix(kerY);
Wt:=Solution(Ht,Transpose(Z));
return Abs(Determinant(Wt));
end function;

(Note that the matrices Y and Z are the matrices of the maps $\delta_1$ and $\delta_2$ of Section 4.6.)

Examples C.2.1. For the Ledrappier graph the following code returns the values $|K_0|=1$ and $|K_1|=1$ (more details of these calculations were given in Section 4.6).

T:=RowsToTile([2,1]);
q:=2;
t:=0;
w:=[1,1,1];
X:=VS(q,t,w);
B:=VertexMatrix(T,1,X);
R:=VertexMatrix(T,2,X);
For a more interesting example, suppose that we have basic data consisting of the tile $\begin{array}{c}
\end{array}$, $q = 3$, $t = 0$, and $w \equiv 1$. The following code returns the values $|K_0| = 8$ and $|K_1| = 8$.

```magma
T:=RowsToTile([3,2,1]);
q:=3;
t:=0;
w:=[1,1,1,1,1,1];
X:=VS(q,t,w);
B:=VertexMatrix(T,1,X);
R:=VertexMatrix(T,2,X);
KZero(B,R);
KOne(B,R);
```

C.2.2. Procedure to verify that $K_0$ and $K_1$ are cyclic groups. From previous calculations we know $|K_0|$ and now we want to identify the group $K_0$. If there is an element $v$ of $K_0$ which has order $|K_0|$ then $v$ is a generator and $K_0$ is the cyclic group. There are two situations:

1. If the unit vector $e = (1, \ldots, 1)$ is in the image of $\delta_1$ then $\delta_1$ is onto and $|K_0| = |\text{coker} \delta_1| = 1$.

2. If the unit vector $e$ is not in the image of $\delta_1$ then $e$ is a nonzero element of $\text{coker} \delta_1$.

Then the order of $e$ is the smallest $i$ between 2 and $|K_0|$ for which the vector $ie$ is in the image of $\delta_1$. (Note that we really only have to check the divisors of $|K_0|$.)

In the first situation $K_0$ is the trivial group. In the second situation we always found that $i = |K_0|$, hence $K_0$ is the cyclic group of order $|K_0|$.

Example C.2.2. For the tile $[3,1,1]$ with $q = 3$, $t = 0$, and $w \equiv 1$, the code below displays 8 as the order of the unit vector, which confirms that $K_0 = \mathbb{Z}/8\mathbb{Z}$. (In the code, the symbols “//” indicate text which forms comments and is ignored by the Magma compiler.)

```magma
// enter the basic data
tile:=RowsToTile([3,1,1]);
q:=3;
t:=0;
w:=OneSequence(#tile);

// hard code in the order of K0
order:=8;

// calculate the matrix Y (the matrix for delta 1)
X:=VS(q,t,w);
```
B := AdjacencyMatrix (tile, 1, X);  
R := AdjacencyMatrix (tile, 2, X);  
Bt := Transpose (B);  
Rt := Transpose (R);  
Y := HorizontalJoin (1 - Bt, 1 - Rt);  
e := Vector (OneSequence (#X)); // the unit vector  
// check if e is in the image of delta 1  
if IsConsistent (Transpose (Y), e) then  
    print "delta 1 is onto";  
else  // find the first i such that i * e is in the image of delta 1  
    for i := 2 to order do  
        if IsConsistent (Transpose (Y), i * e) then  
            print "the order of e is ", i;  
            break;  
        end if;  
    end for;  
end if;  

We have also calculated |K1| and now we want to identify the group K1. The procedure is as follows.

(1) Make a list L of vectors v_i which are basis vectors for ker δ_1 (the columns of the matrix H) but which are not in the image of δ_2.

(2) Let 1 ≤ k ≤ |K1|. For all pairs of vectors v_i, v_j ∈ L, determine if kv_i - v_j ∈ img δ_2.

(3) The order of v_i in K_1 is the smallest l such that lv_i ∈ img δ_2.

Usually none of the basis vectors for ker δ_1 are in img δ_2. When k = 1, we found that v_i - v_j ∈ img δ_2 for all pairs v_i, v_j ∈ L, and when k = 2, ..., |K1|, we found that kv_i - v_j ∉ img δ_2 for all pairs v_i, v_j ∈ L. So for each i, we have v_i + img δ_2 = n v_1 + img δ_2 for some n ∈ N. In every example we found that the order of each v_i is |K_1|.

**Example C.2.3.** For the tile [2, 1] with q = 5, t = 0, and w ≡ 1, the code below returns 4 as the order of each vector v_i, which confirms that K_1 = Z/4Z.

// enter the basic data  
tile := RowsToTile ([2, 1]);  
q := 5;  
t := 0;  
w := OneSequence (#tile);  

// hardcoded in the order of K1  
order := 4;
C.2. K-Theory

// calculate the matrices Y, Z, and H
// (recall that Z is the matrix for delta2)
X:=VS(q,t,w);
B:=AdjacencyMatrix(tile,1,X);
R:=AdjacencyMatrix(tile,2,X);
Bt:=Transpose(B);
Rt:=Transpose(R);
Y:=HorizontalJoin(1−Bt,1−Rt);
Z:=VerticalJoin(Rt−1,1−Bt);
Zt:=Transpose(Z);
kerY:=NullspaceOfTranspose(Y);
Ht:=BasisMatrix(kerY);

// list basis vectors of ker delta1 which are not in img delta2
// (the columns of H are a basis for ker delta1)
list := [];
for m:=1 to NumberOfRows(Ht) do
    h:=RowSubmatrixRange(Ht,m,m); // h is the mth column of H
    if not IsConsistent(Zt,h) then // h is not in img delta2
        Append(˜list,h);
    end if;
end for;
diffs := [];
others := [];
for i:=1 to #list do
    for j:=1 to #list do
        for k:=1 to order do
            x:=k*list [i]−list [j];
            if not IsConsistent(Zt,x) then // x is not in img delta2
                Append(˜diffs,k);
            else // x is in img delta2
                Append(˜others,k);
            end if;
        end for;
    end for;
end for;
print diffs; // check: this lists only numbers of [2,...,order]
print others; // check: this only lists 1s
for \( x \) in list do // find the first \( i \) with \( i \times x \) in img delta2
    for \( i := 2 \) to order do
        if IsConsistent(\( Zt, i \times x \)) then
            print "the order of \( x \) is ", i;
            break;
        end if;
    end for;
end for;

When the order of \( K_i \) is such that there is only one group of that order (for example, when \( K_i \) is a prime or equal to 1, 15, or 26), we automatically know that \( K_i \) is the cyclic group and the above procedures are unnecessary. Examples of where there is actually something to check are:

1. When \( q = 3, t = 0, w \equiv 1, \) and \( T \) is one of the tiles \([3, 1, 1], [3, 2, 1], [5, 1, 1], \) or \([3, 3, 1] \). In each case \( K_0 = K_1 = \mathbb{Z}/8\mathbb{Z} \).
2. When \( q = 5, t = 0, w \equiv 1, \) and \( T \) is one of the tiles \([2, 1], [3, 1], [2, 2], [4, 1] \) or \([3, 2] \). In each case \( K_0 = K_1 = \mathbb{Z}/4\mathbb{Z} \).
3. When \( q = 5, t = 0, w \equiv 1, \) and \( T \) is the tile \([3, 1, 1] \). Then \( K_0 = K_1 = \mathbb{Z}/24\mathbb{Z} \).

C.3. Dominos

In Section 6.2, we used necklaces and Lyndon words (see Appendix B) to describe the structure of domino graphs. Before we give code for generating necklaces and Lyndon words, we need to introduce some small useful methods. The method \( L(q, n, t) \) below is the function in (B.0.3) which returns the number of Lyndon words of length \( n \) and trace \( t \) over \( \{0, 1, \ldots, q - 1\} \).

\[
L := \text{function} \ (q, n, t) \\
\text{sum} := 0; \\
\text{for } d \text{ in } \text{Divisors}(n) \text{ do}; \\
\text{if } t \mod \text{Gcd}(d, q) \equiv 0 \text{ then} \\
\text{sum} += \text{Gcd}(d, q) \ast \text{MoebiusMu}(d) \ast q^\frac{n}{d}; \\
\text{end if}; \\
\text{end for}; \\
\text{return } \text{IntegerRing}()!\text{sum}/(q \ast n); \\
\text{end function};
\]

Thus the number of Lyndon words of length \( n \) (of any trace) over \( \{0, 1, \ldots, q - 1\} \) is given by

\[
\text{NumLyndons} := \text{function} \ (q, n) \\
\text{lyndons} := 0; \\
\text{for } t := 0 \text{ to } q - 1 \text{ do }
\]
C.3. DOMINOS

and the number of necklaces of length \( n \) over \( \{0, 1, \ldots, q - 1\} \) is given below.

\[
\text{NumNecklaces} := \text{function} (q, n) \\
\quad \text{sum} := 0; \\
\quad \text{for } d \text{ in } \text{Divisors} (n) \text{ do} \\
\quad \quad \text{sum} := \text{EulerPhi} (d) \cdot q^{(n/d)/n}; \\
\quad \text{end for}; \\
\quad \text{return } \text{IntegerRing} () ! \text{sum}; \\
\end{function}
\]

Fredericksen, Kessler, and Maiorana gave an algorithm for generating representatives of the necklaces of length \( n \) over an alphabet \( \{0, 1, \ldots, q - 1\} \) in [24, 23]. The idea is to create the list of all words from \( 0^n \) to \( (q - 1)^n \), ordered lexicographically, and at each stage check whether the word is a necklace representative. The algorithm generates words which are the lexicographic least representative of a necklace. We outline the algorithm below (note that in [24] a decreasing list is produced whereas ours is increasing).

If \( a = a_1 \cdots a_n \) is a word not equal to \( (q - 1)^n \), let \( i \) be the largest index such that \( a_i < q - 1 \). Let \( b \) be the word \( b = a_1 \cdots a_{i-1}(a_i + 1) \). Then the successor of \( a \) is the first \( n \) characters of the word \( bbb \cdots \), and it is a necklace if \( i | n \).

The FKM algorithm is implemented as the method NecklaceReps(q,n) and was used to generate the necklaces for \( q = 2 \) and \( n = 6 \) in Table B.1 in Appendix B.
found := false;
for j := 1 to n do
    if alpha[n-j+1] < q-1 then
        i := n-j+1;
        break;
    end if;
end for;
end if;
beta := [];
for r := 1 to i-1 do
    Append(~beta, alpha[r]);
end for;
Append(~beta, alpha[i]+1 mod q);
alpha := beta;
l := #beta;
for m := (l+1) to n do
    x := m mod l;
    if m mod l eq 0 then
        x := l;
    end if;
    alpha[m] := alpha[x];
end for;
if n mod i eq 0 then
    Append(~necklaceReps, alpha);
    numFound+:=1;
end if;
end while;
end for;
return necklaceReps;
end function;

From the list produced by the FKM algorithm it is easy to obtain the necklaces of a certain trace. The method Trace(q, str)

Trace := function(q, str)
    trace := 0;
    for i := 1 to #str do
        trace+:= str[i];
    end for;
    return trace mod q;
end function;
returns the sum of the symbols in a word \( \text{mod } q \), and is called by the method below to filter out the necklaces of trace \( t \).

\textbf{NecklaceRepsTrace:=} \textbf{function}(q,n,t)

\begin{verbatim}
specials := []; 
N := NecklaceReps(q,n); 
for i := 1 to \#N do
    if Trace(q,N[i]) eq t then
        Append(~specials,N[i]);
    end if;
end for;
return specials;
end function;
\end{verbatim}

From a list of necklaces we can also produce a list of their Lyndon subwords.

\textbf{LyndonSubwords:=} \textbf{function}(q,n,t);

\begin{verbatim}
N := NecklaceRepsTrace(q,n,t); 
W := N; 
for i := 1 to \#N do
    W[i] := []; 
    for j := 1 to Period(N[i]) do
        Append(~W[i],N[i][j]);
    end for;
end for;
return W;
end function;
\end{verbatim}

To generate the Lyndon words of length \( n \) over \( \{0,1,\ldots,q-1\} \) we begin with the necklaces generated by the FKM algorithm and then filter out the ones which are aperiodic using the method \textbf{Lyndons}(q,n) below. To determine whether a word is aperiodic we need the method \textbf{Period}(str) which returns the least period of a word.

\textbf{Period:=} \textbf{function}(str)

\begin{verbatim}
period := 1; 
str2 := Rotate(str,-1); 
for i := 2 to \#str do
    if str ne str2 then
        str2 := Rotate(str2,-1); 
    else
        break;
    end if;
period += 1;
\end{verbatim}
end for;
return period;
end function;

Lyndons:=function(q,n)
N:=NecklaceReps(q,n);
lyndons:=[[];
for i:=1 to #N do
    if Period(N[i]) eq n then
        Append(lyndons,N[i]);
    end if;
end for;
return lyndons;
end function;

From the list of Lyndon words it is then easy to filter out the Lyndon words of given trace.

LyndonsTrace:=function(q,n,t)
specials:=[[];
N:=Lyndons(q,n);
for i:=1 to #N do
    if Trace(q,N[i]) eq t then
        Append(specials,N[i]);
    end if;
end for;
return specials;
end function;

Example C.3.1. To illustrate the most useful methods of this section, we give the output when $q=2$, $n=3$, and $t=0$. The code NecklaceReps(2,3); returns the binary necklaces of length 3
[[0,0,0], [0,0,1], [0,1,1], [1,1,1]]
Then NecklaceRepsTrace(2,3,0); returns the ones with sum 0 (mod 2)
[[0,0,0], [0,1,1]]
and the output of LyndonSubwords(2,3,0); is
[[0], [0,1,1]]
The binary Lyndon words of length 3 are given by the code Lyndons(2,3);
[[0,0,1], [0,1,1]]
and those with sum 0 (mod 2) are given by Lyndons(2,3,0);
[[0,1,1]]
APPENDIX D

K-theory tables

The following tables contain the results of our $K$-theory calculations which we performed using the Magma code in Appendix C. We first discuss the notation used in the tables and summarise the observations which led us to certain results of the thesis.

Recall from Appendix A that a tile $T$ corresponds to an integer partition of $|T|$ and we adopt partition notation to describe tiles more easily. Tiles whose corresponding partitions are conjugate are listed together because they have the same $K$-theory. For example, the tile $\begin{array}{c} \vdots \\ \vdots \end{array}$ and its conjugate $\begin{array}{c} \vdots \\ \vdots \end{array}$ are listed together as $[3, 1]$ and $[2, 1^2]$.

Each table contains the values of $|K_0|$ and $|K_1|$ for the 2-graph $\Lambda(T, q, 0, w)$ with $w \equiv 1$. (We also performed calculations for other rules and traces but we obtained the same values of $|K_0|$ and $|K_1|$.) Observe that $|K_0| = |K_1|$ in all the examples we considered; see Section 4.6, and in particular, Theorem 4.7.10, which is a proof of this fact under certain circumstances. Note that for domino tiles (tiles of the form $[n]$) the values in the tables support the $K$-theory calculations in Proposition 6.4.1.

Blank spaces in the tables would require calculations beyond sensible computation time. The computation time increases dramatically with the number of vertices in the graph, $q^{|T|-1}$, and so only the first table, which has tiles with relatively small $|T|$, contains calculations for alphabets with $q > 2$. The results for alphabets $q > 2$ do not reveal any interesting new phenomena however. For $q = 2$, we were able to perform calculations for tiles of cardinality up to $|T| = 11$. 

99
| $|T|$ | Tile | Conjugate | $q = 2$ | $q = 3$ | $q = 4$ | $q = 5$ |
|-----|------|----------|--------|--------|--------|--------|
|     |      |          | $|K_0|$ | $|K_1|$ | $|K_0|$ | $|K_1|$ | $|K_0|$ | $|K_1|$ |
| 2   | [2]  | $[1^2]$  | 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
| 3   | [3]  | $[1^3]$  | 3      | 3      | 8      | 8      | 15     | 15     | 24     | 24     |
|     | [2,1]| self     | 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [4]  |          | 7      | 7      | 26     | 26     | 63     | 63     | 124    | 124    |
|     | [3,1]| $[2,1^2]$| 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [2,2]| self     | 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
| 5   | [5]  | $[1^5]$  | 15     | 15     | 80     | 80     | 255    | 255    | 624    | 624    |
|     | [4,1]| $[2,1^3]$| 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [3,2]| $[2^2,1]$| 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [3,1,1]| self  | 3      | 3      | 8      | 8      | 15     | 15     | 24     | 24     |
|     | [5,1]| $[2,1^4]$| 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [4,2]| $[2^2,1^2]$| 1     | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [4,1,1]| $[3,1^3]$| 1     | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [3,3]| $[2^3]$  | 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [3,2,1]| self  | 3      | 3      | 8      | 8      | 15     | 15     | 24     | 24     |
| 7   | [7]  | $[1^7]$  | 63     | 63     | 728    | 728    |        |        |        |        |
|     | [6,1]| $[2,1^5]$| 1      | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [5,2]| $[2^2,1^3]$| 1    | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [5,1,1]| $[3,1^4]$| 3     | 3      | 8      | 8      | 4      | 4      | 5      | 5      |
|     | [4,3]| $[2^3,1]$ | 1    | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [4,2,1]| $[3,2,1^2]$| 1  | 1      | 2      | 2      | 3      | 3      | 4      | 4      |
|     | [4,1,1,1]| self  | 7     | 7      | 26     | 26     | 5      | 5      | 6      | 6      |
|     | [3,3,1]| $[3,2^2]$| 3     | 3      | 8      | 8      | 15     | 15     | 24     | 24     |
| 8   | [8]  | $[1^8]$  | 127    | 127    |        |        |        |        |        |        |
|     | [7,1]| $[2,1^6]$| 1     | 1      |        |        |        |        |        |        |
|     | [6,2]| $[2^2,1^4]$| 1   | 1      |        |        |        |        |        |        |
|     | [6,1,1]| $[3,1^5]$| 1     | 1      |        |        |        |        |        |        |
|     | [5,3]| $[2^3,1^2]$| 1   | 1      |        |        |        |        |        |        |
|     | [5,2,1]| $[3,2,1^3]$| 3   | 3      |        |        |        |        |        |        |
|     | [5,1,1,1]| $[4,1^4]$| 1    | 1      |        |        |        |        |        |        |
|     | [4,4]| $[2^4]$  | 1     | 1      |        |        |        |        |        |        |
|     | [4,3,1]| $[3,2^2,1]$| 1  | 1      |        |        |        |        |        |        |
|     | [4,2,2]| $[3^2,1^2]$| 1  | 1      |        |        |        |        |        |        |
|     | [4,2,1,1]| self  | 7     | 7      |        |        |        |        |        |        |
|     | [3,3,2]| self   | 3     | 3      |        |        |        |        |        |        |

Table D.1. Table of $K$-theory calculations for $2 \leq |T| \leq 8.$
| $|T|$ | Tile | Conjugate | $q = 2$ | $|K_0|$ | $|K_1|$ |
|---|---|---|---|---|---|
| 9 | $[9]$ | $[1^9]$ | 255 | 255 |
|  | $[8,1]$ | $[2, 1^7]$ | 1 | 1 |
|  | $[7,2]$ | $[2^2, 1^5]$ | 1 | 1 |
|  | $[7,1,1]$ | $[3, 1^6]$ | 3 | 3 |
|  | $[6,3]$ | $[2^3, 1^3]$ | 1 | 1 |
|  | $[6,2,1]$ | $[3, 2, 1^4]$ | 1 | 1 |
|  | $[6,1,1,1]$ | $[4, 1^5]$ | 1 | 1 |
|  | $[5,4]$ | $[2^4, 1]$ | 1 | 1 |
|  | $[5,3,1]$ | $[3, 2^2, 1^2]$ | 3 | 3 |
|  | $[5,2,2]$ | $[3^2, 1^3]$ | 3 | 3 |
|  | $[5,2,1,1]$ | $[4, 2, 1^3]$ | 1 | 1 |
|  | $[5,1,1,1,1]$ | self | 15 | 15 |
|  | $[4,4,1]$ | $[3, 2^3]$ | 1 | 1 |
|  | $[4,3,2]$ | $[3^2, 2, 1]$ | 1 | 1 |
|  | $[4,3,1,1]$ | $[4, 2^2, 1]$ | 7 | 7 |
|  | $[3,3,3]$ | self | 3 | 3 |
| 10 | $[10]$ | $[1^{10}]$ | 511 | 511 |
|  | $[9,1]$ | $[2, 1^8]$ | 1 | 1 |
|  | $[8,2]$ | $[2^2, 1^6]$ | 1 | 1 |
|  | $[8,1,1]$ | $[3, 1^7]$ | 1 | 1 |
|  | $[7,3]$ | $[2^3, 1^4]$ | 1 | 1 |
|  | $[7,2,1]$ | $[3, 2, 1^5]$ | 3 | 3 |
|  | $[7,1,1,1]$ | $[4, 1^6]$ | 7 | 7 |
|  | $[6,4]$ | $[2^4, 1^2]$ | 1 | 1 |
|  | $[6,3,1]$ | $[3, 2^2, 1^3]$ | 1 | 1 |
|  | $[6,2,2]$ | $[3^2, 1^4]$ | 1 | 1 |
|  | $[6,2,1,1]$ | $[4, 2, 1^4]$ | 1 | 1 |
|  | $[6,1,1,1,1]$ | $[5, 1^5]$ | 1 | 1 |
|  | $[5,5]$ | $[2^5]$ | 1 | 1 |
|  | $[5,4,1]$ | $[3, 2^3, 1]$ | 3 | 3 |
|  | $[5,3,2]$ | $[3^2, 2, 1^2]$ | 3 | 3 |
|  | $[5,3,1,1]$ | $[4, 2^2, 1^2]$ | 1 | 1 |
|  | $[5,2,2,1]$ | $[4, 3, 1^3]$ | 1 | 1 |
|  | $[5,2,1,1,1]$ | self | 15 | 15 |
|  | $[4,4,2]$ | $[3^2, 2^2]$ | 1 | 1 |
|  | $[4,4,1,1]$ | $[4, 2^3]$ | 7 | 7 |
|  | $[4,3,3]$ | $[3^3, 1]$ | 1 | 1 |
|  | $[4,3,2,1]$ | self | 7 | 7 |

Table D.2. $K$-theory table for $9 \leq |T| \leq 10$. 

D. $K$-THEORY TABLES 101
| $|T|$ | Tile | Conjugate | $q = 2$ |
|-----|------|-----------|--------|
|     |      |           | $|K_0|$ | $|K_1|$ |
| 11  | [11] | $[1^{11}]$ | 1      | 1      |
|     | [10,1] | $[2,1^{10}]$ | 1      | 1      |
|     | [9,2] | $[2^2,1^9]$ | 1      | 1      |
|     | [9,1,1] | $[3,1^8]$ | 3      | 3      |
|     | [8,3] | $[2^3,1^5]$ | 1      | 1      |
|     | [8,2,1] | $[3,2,1^6]$ | 1      | 1      |
|     | [8,1,1,1] | $[4,1^7]$ | 1      | 1      |
|     | [7,4] | $[2^4,1^3]$ | 1      | 1      |
|     | [7,3,1] | $[3,2^2,1^4]$ | 3      | 3      |
|     | [7,2,2] | $[3^2,1^5]$ | 3      | 3      |
|     | [7,2,1,1] | $[4,2,1^5]$ | 7      | 7      |
|     | [7,1,1,1,1] | $[5,1^6]$ | 3      | 3      |
|     | [6,5] | $[2^5,1]$ | 1      | 1      |
|     | [6,4,1] | $[3,2^3,1^2]$ | 1      | 1      |
|     | [6,3,2] | $[3^2,2,1^3]$ | 1      | 1      |
|     | [6,3,1,1] | $[4,2^2,1^3]$ | 1      | 1      |
|     | [6,2,2,1] | $[4,3,1^4]$ | 1      | 1      |
|     | [6,2,1,1,1] | $[5,2,1^4]$ | 1      | 1      |
|     | [6,1,1,1,1,1] | self | 31     | 31     |
|     | [5,5,1] | $[3,2^4]$ | 3      | 3      |
|     | [5,4,2] | $[3^2,2^2,1]$ | 3      | 3      |
|     | [5,4,1,1] | $[4,2^3,1]$ | 1      | 1      |
|     | [5,3,3] | $[3^3,1^2]$ | 3      | 3      |
|     | [5,3,2,1] | $[4,3,2,1^1]$ | 1      | 1      |
|     | [5,3,1,1,1] | $[5,2^2,1^2]$ | 15     | 15     |
|     | [5,2,2,2] | $[4^2,1^3]$ | 1      | 1      |
|     | [4,4,3] | $[3^3,2]$ | 1      | 1      |
|     | [4,4,2,1] | $[4,3,2^2]$ | 7      | 7      |
|     | [4,3,3,1] | self | 7      | 7      |

Bibliography


