RANDOM WALKS ON GROUPS

John J. L. W. Harrison
BSc(Hons)(Tasmania)

January 2018

Supervisors: Prof. George A. Willis and Dr. Jeffrey A. Hogan

Submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy
(Mathematics)

This research was supported by an Australian Government Research Training Program (RTP) Scholarship
Statement of Originality

I hereby certify that the work embodied in the thesis is my own work, conducted under normal supervision.

The thesis contains no material which has been accepted, or is being examined, for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to the final version of my thesis being made available worldwide when deposited in the University's Digital Repository, subject to the provisions of the Copyright Act 1968 and any approved embargo.

______________________________
John J. Harrison
This thesis is concerned with random walks on solvable matrix groups, direct products of automorphism groups of trees, semi-direct products arising from totally disconnected locally compact groups and unrestricted lamplighter groups.

Brofferio and Schapira [14], described the Poisson boundary of $GL_n(\mathbb{Q})$ for measures of finite first moment with respect to adelic length. We define matrix groups $FG_n(P)$ for each natural number $n$ and finite set of primes $P$, such that every rational-valued upper triangular matrix group is a (possibly distorted) subgroup. We show that adelic length is a word metric estimate on $FG_n(P)$ by constructing another, intermediate, word metric estimate which can be easily computed from the entries of any matrix in the group. Finite first moment of a probability measure with respect to adelic length is equivalent to finite first moment with respect to word length in $FG_n(P)$.

The Poisson boundaries of finite direct products of affine automorphism groups of homogeneous trees are also considered. The Poisson boundary is a product of ends of trees with a hitting measure for spread-out, aperiodic measures of finite first moment, whose closed support generates subgroups which are not fully exceptional. The Poisson boundary of a semi-direct product, $V \rtimes \langle \alpha \rangle$, for any automorphism $\alpha$ and tidy compact open subgroup $V$ in a locally compact, totally disconnected group $G$ is also shown to be the space of ends of the tree with the hitting measure under similar assumptions. Boundary triviality is discussed in both cases. This extends work of Cartwright, Kaimanovich and Woess [16].

In the final chapter, we discuss pointwise convergence and non-trivial boundaries for unrestricted lamplighter groups. We define a rate of eschewal on the rough Cayley graph of a compactly generated, totally disconnected, locally compact group $G$. For appropriate choices of compact open subgroups, the rate of eschewal is finite and equal to the rate of escape for measures supported within the restricted lamplighter subgroup.
Acknowledgements

To my supervisor, George Willis, thank you. I have always appreciated your advice, suggestions and patience. Your insight, knowledge and thoughtful guidance has been key to my success.

I am grateful, in alphabetical order, to Ben Brawn, Tim Bywaters, Murray Elder, Nicholas Ham, Colin Reid, Matthew Tam and Stephan Tornier for friendship, helpful discussions and comments on my work. I would also like to thank my co-supervisor Dr Jeffrey Hogan for his support, particularly in the early stages of my studies. I am also indebted to the staff and students at the University of Newcastle. Completion of this work would have been more difficult without the support and friendship they have provided.

I wish to express my deepest gratitude to my family. Thank you to my father and late mother and especially my partner Xin Yi Koh for their dedication and love.

I would also like to thank Teresa Bates for her thoughtful and professional proofreading services, which were performed according to the guidelines laid out in the university-endorsed national ‘Guidelines for editing research theses’.

Finally, I would like to thank the Australian Federal Government for their funding under the Australian Postgraduate Award and the Research Training Program.
# Contents

Statement of Originality .................................................. iii

Abstract .................................................................................. v

Acknowledgements ...................................................................... vii

1 Introduction .............................................................................. 1

1.1 Basic definitions, notation and terminology .............................. 4

1.2 Locally compact groups ......................................................... 5

1.2.1 Measures and integration .................................................... 7

1.2.2 Semi-direct products .......................................................... 8

1.2.3 Gauges, gauge functions and word length ......................... 9

1.2.8 Nilpotent and solvable groups ............................................. 12

1.2.12 Totally disconnected groups ............................................ 13

1.3 Probability .............................................................................. 15

1.3.10 Independence ................................................................. 19

1.3.14 Conditional expectation ................................................... 20

1.3.20 Sequences of random variables ....................................... 22

1.4 Random walks ....................................................................... 26

1.4.1 $\mu$-harmonic functions ....................................................... 29

1.4.10 Recurrence, transience and product measures .................... 37

1.4.18 Topological boundaries .................................................... 41

1.4.24 The Poisson boundary ....................................................... 43
## CONTENTS

1.4.29 Measurable boundaries .............................................. 45  
1.4.31 The measurable Poisson boundary ................................. 46  
1.4.33 The stationary boundary ............................................ 47  
1.4.37 Group structure, geometry and the Poisson boundary .......... 51  
1.5 Summary of results ....................................................... 57  

2 Solvable matrix groups 59  
2.1 The groups, $FG_n(P)$ .................................................. 59  
2.2 A word metric estimate on $FG_n(P)$ ................................. 61  
2.3 Adelic length is a word metric estimate on $FG_n(P)$ ............ 67  
2.3.2 Upper bounds .......................................................... 71  
2.3.8 Lower bounds .......................................................... 75  
2.3.14 Conclusions ............................................................ 79  

3 Trees and products of trees 83  
3.1 Preliminaries ............................................................. 84  
3.2 Partially exceptional and fully exceptional subgroups .......... 92  
3.3 Random walks and gauge functions .................................. 95  
3.4 Actions of totally disconnected groups on trees .................. 101  

4 Unrestricted wreath products over $\mathbb{Z}^k$ 109  
4.1 Random walks ........................................................... 110  
4.2 Rate of eschewal .......................................................... 113  
4.2.2 Rate of eschewal on lamplighter groups ......................... 115  

5 Summary of results .......................................................... 121  

References 122
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>xi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>130</td>
</tr>
</tbody>
</table>