RANDOM WALKS ON GROUPS

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Statement of Originality

I hereby certify that the work embodied in the thesis is my own work, conducted under normal supervision.

The thesis contains no material which has been accepted, or is being examined, for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to the final version of my thesis being made available worldwide when deposited in the University’s Digital Repository, subject to the provisions of the Copyright Act 1968 and any approved embargo.

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John J. Harrison
Abstract

This thesis is concerned with random walks on solvable matrix groups, direct products of automorphism groups of trees, semi-direct products arising from totally disconnected locally compact groups and unrestricted lamplighter groups.

Brofferio and Schapira [14], described the Poisson boundary of $GL_n(\mathbb{Q})$ for measures of finite first moment with respect to adelic length. We define matrix groups $FG_n(P)$ for each natural number $n$ and finite set of primes $P$, such that every rational-valued upper triangular matrix group is a (possibly distorted) subgroup. We show that adelic length is a word metric estimate on $FG_n(P)$ by constructing another, intermediate, word metric estimate which can be easily computed from the entries of any matrix in the group. Finite first moment of a probability measure with respect to adelic length is equivalent to finite first moment with respect to word length in $FG_n(P)$.

The Poisson boundaries of finite direct products of affine automorphism groups of homogeneous trees are also considered. The Poisson boundary is a product of ends of trees with a hitting measure for spread-out, aperiodic measures of finite first moment, whose closed support generates subgroups which are not fully exceptional. The Poisson boundary of a semi-direct product, $V_\sim \rtimes \langle \alpha \rangle$, for any automorphism $\alpha$ and tidy compact open subgroup $V$ in a locally compact, totally disconnected group $G$ is also shown to be the space of ends of the tree with the hitting measure under similar assumptions. Boundary triviality is discussed in both cases. This extends work of Cartwright, Kaimanovich and Woess [16].

In the final chapter, we discuss pointwise convergence and non-trivial boundaries for unrestricted lamplighter groups. We define a rate of eschewal on the rough Cayley graph of a compactly generated, totally disconnected, locally compact group $G$. For appropriate choices of compact open subgroups, the rate of eschewal is finite and equal to the rate of escape for measures supported within the restricted lamplighter subgroup.
ABSTRACT
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A random walk is a time-homogeneous Markov chain for which the transition probabilities are invariant with respect to the structure of the underlying space in some natural way. Random walks are used to model stochastic processes in many scientific fields, including developing models of physical Brownian motion, heat flow, chemical diffusion, polymer formation, human reaction time, disease transmission, population genetics, options pricing and climate [2, 8, 19, 48, 54, 67, 68, 85, 75]. Computational mathematics techniques including genetic algorithms and Monte Carlo methods also employ random walks [47, 73].

It is interesting to describe the asymptotic behaviour of random walks. There are various ways to define the term ‘asymptotic’. One possible definition amounts to finding the $\sigma$-algebra of events whose probability is independent of any information known up to any given finite time. Recurrence of a random walk is a familiar example of an event with this information independence property. Describing this $\sigma$-algebra amounts to finding the tail boundary associated with each random walk. The exit boundary is the $\sigma$-algebra of events which are invariant under a certain shift map, and it gives another possible meaning of ‘asymptotic’.

This thesis is concerned with random walks which are left invariant with respect to the action of a locally compact group $G$. That is, the transition probability from an element $g \in G$ to every measurable set $E \in G$ is the same as the transition probability from $hg$ to $hE$ for any group element $h \in G$. In this setting, the tail boundary can be identified, up to sets of measure zero, with the exit boundary, and, modulo constant factors, a space of bounded $\mu$-harmonic functions called the Poisson boundary. The Poisson boundary is a maximal $\mu$-boundary – every $\mu$-boundary is an equivariant image of the Poisson boundary.

This chapter outlines the theoretical background and preliminary results required for an un-
CHAPTER 1. INTRODUCTION

derstanding of the main results presented in the later chapters of this thesis. In order, we discuss locally compact groups, probability theory, random walks in groups and real asymptotics. The section on random walks pays particular attention to harmonic functions and the Poisson boundary associated with a random walk.

Brofferio and Schapira [14] described the Poisson boundary of $GL_n(\mathbb{Q})$ for measures of finite first moment with respect to adelic length. Chapter 2 aims to relate the adelic length to the word length in finitely generated rational matrix groups. To do this, we define matrix groups $FG_n(P)$ for each $n \in \mathbb{N}$ and finite set of primes $P$. These groups contain every rational valued upper triangular matrix group as a subgroup. We show that adelic length is a word metric estimate on $FG_n(P)$ by constructing another, intermediate, word metric estimate which can be easily computed from the entries of any matrix in the group. In particular, requiring a probability measure on $FG_n(P)$ to have finite first moment with respect to adelic length is equivalent to requiring it to have finite first moment with respect to word length.

Cartwright, Kaimanovich and Woess investigated random walks on closed subgroups of the affine group of a homogeneous tree in [16]. Chapter 3 presents two extensions of their work. We first consider random walks in a finite direct product $P = \prod_{i=1}^{k} \text{Aff } T_i$ of affine automorphism groups of homogeneous trees. We define partially exceptional and fully exceptional closed subgroups of $P$, and explore the relationships between each type, the scale function, the modular function, and transience of the random walk. When the probability measure is spread-out, aperiodic, has finite first moment and its support generates a closed subgroup which is not fully exceptional, we show that the Poisson boundary is the direct product $\prod_{i=1}^{k} \partial T_i$ of the space of ends of each tree with a probability measure. We give necessary and sufficient conditions on $\mu$ for boundary triviality.

Let $\alpha$ be an automorphism of a totally disconnected locally compact group $G$. The scale of $\alpha$ is the positive integer

$$s(\alpha) := \min\{[\alpha(U) : U \cap \alpha(U)] \mid U \text{ is a compact open subgroup of } G\}.$$ 

If $s(\alpha) \neq 1$, then $G$ has a sub-quotient isomorphic to a closed subgroup of an affine group of a homogeneous tree with non-trivial degree. We investigate random walks on these groups and describe the Poisson boundary for aperiodic, spread-out probability measures of finite first moment.

There has been considerable research interest in random walks on lamplighter groups, which are the restricted wreath products of $\mathbb{Z}^k$ over finite groups. These restricted wreath products are discrete subgroups of unrestricted wreath products over the same spaces, which we call unre-
stricted lamplighter groups. It is more difficult to investigate the properties of random walks in the unrestricted case because many of the geometric tools used in the restricted case rely on the group being finitely generated and discrete.

Chapter 4 discusses random walks on unrestricted lamplighter groups. We provide a sufficient condition on probability measures for almost sure convergence of paths to limit configurations, and prove that the Poisson boundary can be non-trivial.

If \((G, \mu)\) is a random walk on a discrete group \(G\), which is finitely generated by a set \(K\), then the \(K\)-rate of escape is the limit

\[
R_K = \lim_{n \to \infty} \frac{|R_n|_K}{n} \in [0, +\infty],
\]

where \(R_n\) is the corresponding right random walk and \(| \cdot |_K\) is the word length function with respect to the word length metric \(d\). We provide a possible generalization of the \(K\)-rate of escape to compactly generated totally disconnected groups, which we call the rate of eschewal. Much as the rate of escape depends on a choice of generating set, the rate of eschewal depends on a chosen sequence of strictly decreasing compact open subgroups with trivial intersection. We show that, for appropriate sequences, the rate of eschewal is finite and equal to the rate of escape for measures supported within the restricted lamplighter subgroup.
1.1 Basic definitions, notation and terminology

We shall employ the following notation throughout this thesis:

(i) \( \mathbb{N} \) : the set of natural numbers, \( \mathbb{N} = \{1, 2, \ldots\} \),

(ii) \( \mathbb{N}_0 \) : the set of non-negative integers, \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \),

(iii) \( \mathbb{Z} \) : the ring of integers,

(iv) \( \mathbb{Q} \) : the field of rational numbers,

(v) \( \mathbb{R} \) : the field of real numbers,

(vi) \( \mathbb{Z}[1/p] \) : the set \( \mathbb{Z}[1/p] = \left\{ \frac{m}{p^n} \in \mathbb{Q} : m, n \in \mathbb{Z} \right\} \), and

(vii) \( \mathbb{Q}_p \) : the set of \( p \)-adic numbers.

Suppose that \( X \) is a totally ordered set. If \( x, y, z \in X \), then we say that \( y \) is between \( x \) and \( z \) if either \( x \leq y \leq z \) or \( z \leq y \leq x \). If \( y \) is between \( x \) and \( z \), but equal to neither \( x \) nor \( z \), then \( y \) is strictly between \( x \) and \( z \).

A sequence \( \{x_i\}_{i=1}^{\infty} \in X \) is monotone increasing if \( x_n \leq x_{n+1} \), and monotone decreasing if \( x_n \geq x_{n+1} \) for all \( n \in \mathbb{N} \). A sequence which is either monotone increasing or monotone decreasing is monotone. A monotone sequence, \( \{x_i\}_{i=1}^{\infty} \in X \), is strictly monotone if \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N} \).

Given sets \( A \) and \( B \), denote by:

(i) \( A \setminus B \) : the set containing elements of \( A \) which are not elements of \( B \),

(ii) \( A \times B \) : the Cartesian product of \( A \) and \( B \), and

(iii) \( \mathcal{P}(A) \) : the power set of \( A \).

Given an indexed family of sets \( \mathcal{A} = \{A_i\}_{i \in I} \), over a non-empty set \( I \), we denote by:

(i) \( \bigcup_{i \in I} A_i \) : the union of the sets contained in \( \mathcal{A} \),

(ii) \( \bigcap_{i \in I} A_i \) : the intersection of the sets contained in \( \mathcal{A} \), and

(iii) \( \prod_{i \in I} A_i \) : the Cartesian product of the sets contained in \( \mathcal{A} \).
If $X$ is a topological space, the Borel $\sigma$-algebra $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the set of all open subsets in $X$. Let $(X, \mathcal{B}(X), \mu)$ be a measure space. If $f : X \to \mathbb{C}$ is continuous, the support of $f$ is the set

$$\text{supp}(f) = \{x \in \Omega : f(x) \neq 0\},$$

where $\overline{S}$ denotes the closure of the set $S$.

The following function spaces are important in this thesis. All can be viewed as topological vector spaces when endowed with pointwise addition and scalar multiplication:

(i) $C(X)$ : the normed linear space of all bounded complex valued continuous functions on $X$.

(ii) $C_0(X)$ : the subspace of $C(X)$ consisting of those functions which vanish at infinity. That is, those continuous functions satisfying the condition that for every $\epsilon > 0$ there exists a compact subset $F \subset X$ such that $F \subset X$ with $|f(g)| < \epsilon$ for all $g \in X \setminus F$.

(iii) $C_00(X)$ : the space of continuous functions with compact support on $X$; the linear subspace of $C_0(X)$ consisting of functions $f$ such that the support of $f$ is contained in some compact subset $F$ of $X$.

(iv) $L^p(X, \mu)$ : the vector space of complex valued functions bounded under the semi-norm $\|\cdot\|_p$ given by

$$\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p}$$

if $p \in [1, +\infty)$ or

$$\|f\|_\infty = \text{ess sup}\{|f|\}$$

if $p$ is infinite.

(v) $L^p(X, \mu)$ : the Lebesgue space on $(X, \mu)$: the quotient normed space of $L^p(X, \mu)$ with respect to the kernel of the semi-norm $\|\cdot\|_p$ for $p \in [1, +\infty)$. Functions in this space are identified if they agree $\mu$-almost everywhere. If the measure $\mu$ is clear from the context, we will simply refer to $L^p(X, \mu)$ as $L^p(X)$.

### 1.2 Locally compact groups

In this section, we introduce those aspects of theory of locally compact groups which we require to fix our notation and terminology. Our treatment follows that of Cohn [23], Hewitt and Ross [49] and Stroppel [93] unless otherwise noted. Dummit and Foote [29] has been a source of inspiration for some of the examples throughout.
A topological group is a group endowed with a Hausdorff topology under which the inverse map is continuous and the binary operation is continuous with respect to the induced product topology on $G \times G$. Every abstract group is a topological group with respect to the discrete topology. Standard examples of topological groups include the additive group $\mathbb{R}^n$ with the Euclidean topology, the multiplicative group of points on the unit circle, $\mathbb{T}$, in the complex plane with the topology induced by the usual metric, and the $n \times n$ real matrix groups, with the topology induced by the Euclidean metric on $\mathbb{R}^{n^2}$.

Every subgroup of a topological group is a topological group when endowed with the subspace topology. Every open subgroup $G$ of a topological group is closed because the continuity of the inverse and/or product map implies that the complement of $G$ is a union of open sets (see e.g. Woess [105]). If $H$ is a closed normal subgroup of a topological group $G$, then the quotient group $G/H$ is a topological group with the quotient topology.

A locally compact group is a topological group for which the underlying topology is locally compact, that is, every point has a neighbourhood with compact closure. Closed subgroups of locally compact groups are locally compact with the subspace topology.

We adopt multiplicative notation for non-abelian groups and additive notation for abelian groups. We denote the identity of the group $G$ by $e_G$, or just $e$ if the group is clear from the context. If $G$ and $H$ are groups, and $\varphi : G \to H$ is a homomorphism, then we use $\ker \varphi$ to denote the kernel of $\varphi$ and $\im \varphi$ to denote the image of $\varphi$. We indicate that a group $H$ is a subgroup of a group $G$ by writing $H \leq G$, or, if $H$ is normal in $G$, by $H \unlhd G$. If $\{x_i\}_{i=1}^{n}$ is a finite sequence of group elements, we use $\prod_{i=1}^{n} x_i$ to denote the left-to-right product $x_1 \ldots x_n$.

If $G$ is a locally compact group, then we say that a subset $K$ of $G$ generates $G$ if every element of $G$ can be written as a finite product of elements of $K$ and their inverses. If $J \subseteq G$, then the subgroup generated by $J$ is the smallest subgroup of $G$ containing every element of $J$. If there exists a finite (compact) set $K$, such that $G$ is generated by $K$, then we say that $G$ is finitely (respectively compactly) generated, and that $K$ is a finite (respectively compact) generating set.

A locally compact group $G$ is amenable if there exists a non-negative linear functional $\lambda : L^\infty(G) \to \mathbb{R}$, satisfying $\lambda(e) = 1$ and $\lambda(g \cdot f) = \lambda(f)$ for all $f \in L^\infty(G)$ and $g \in G$, where $G$ acts by left translation. There are many equivalent definitions of amenability. The reader may wish to consult Runde [86], Paterson [77] or Pier [78] for detailed discussions of these definitions.

All closed subgroups of amenable locally compact groups are amenable. All compact topological groups, including finite groups, are amenable. If $G$ is a locally compact group, and $N$ is a
closed normal subgroup such that both $N$ and $G/N$ are amenable, then $G$ is amenable. Proofs of these useful facts may be found in Pier [78].

§ 1.2.1 Measures and integration

Suppose that $G$ is a locally compact group. Then $G$ admits a non-trivial Haar measure $\lambda_G$ on the Borel $\sigma$-algebra $\mathcal{B}(G)$, which satisfies:

(i) $\lambda_G(E) = \lambda_G(Ex)$ for all $E \in \mathcal{B}(G)$,

(ii) $\lambda_G(K) < +\infty$ for all compact $K \in \mathcal{B}(G)$,

(iii) $\lambda_G(E) = \inf\{\lambda_G(U) : U \subseteq X \text{ open}, E \subseteq U\}$ for all $E \in \mathcal{B}(G)$, and

(iv) $\lambda_G(U) = \sup\{\lambda_G(K) : K \subseteq X \text{ compact}, K \subseteq U\}$ for all $U \in \mathcal{B}(G)$.

The first property is right translation invariance. The third and fourth properties are inner regularity and outer regularity respectively. If there is no ambiguity, we write $\lambda$ instead of $\lambda_G$. The measure $\lambda$ always exists, and is unique up to scalar multiples (see Section 9.2 of Cohn [24] for proof). A particular choice of $\lambda$ is referred to as the right Haar measure. The right Haar measure allows the integration of Borel measurable functions over the group.

If $G$ is a compact group, then any measure which satisfies property (ii) is finite, and the right Haar measure is chosen so that it is a probability measure. The topology of a finite topological group is always compact. If $G$ is an infinite discrete group, then $\lambda$ is chosen to be the counting measure on $G$, so that each singleton has unit measure. The right Haar measure on the additive group $\mathbb{R}^n$ is the Lebesgue measure.

If $\lambda$ is a right Haar measure on a locally compact group $G$, then it follows from the continuity of multiplication that the left translate $g\lambda$ defined by

$$g\lambda(E) := \lambda(g^{-1}E)$$

for each Borel set $E$ and each group element $g$ is a right Haar measure on $G$. If $H$ is an open, closed, $\sigma$-compact subgroup of $G$, then the restriction $\lambda_H$ of $\lambda_G$ given by

$$\lambda_H(E) = \lambda_G(E \cap H),$$

for all Borel sets $E \subseteq H$, is a Haar measure on $H$ (see Section 2.3 of Folland [39] for a proof).
The modular function is the map $\Delta_G : G \to \mathbb{R}^+$ given by

$$\Delta_G(g) = \frac{\lambda(g^{-1}E)}{\lambda(E)}$$

for every Borel set $E$. The map is well defined and independent of the choice of $\lambda$ because the right Haar measure is unique up to scalar multiples. Since the map $E \mapsto g^{-1}E$ between Borel subsets of $G$ is a bijection, it follows that

$$\Delta_G(g) \Delta_G(h) = \frac{\lambda(g^{-1}E)}{\lambda(E)} \frac{\lambda(h^{-1}E)}{\lambda(E)} = \frac{\lambda(g^{-1}E)}{\lambda(E)} \frac{\lambda(h^{-1}g^{-1}E)}{\lambda(E)} = \frac{\lambda((gh)^{-1}E)}{\lambda(E)}.$$ 

That is, the modular function is a group homomorphism. A unimodular group is a group for which the modular function is the constant function $1$. Compact groups and abelian groups are both unimodular.

§ 1.2.2 Semi-direct products

We follow Bourbaki [12] in this section. Let $(H,e_H)$ and $(N,e_N)$ be abstract groups with identity. Denote by $\text{Aut} G$ the group of automorphisms of a group $G$. Let $\varphi$ be a homomorphism from $H$ into $\text{Aut} N$. We use $\varphi_h$ to denote the automorphism $\varphi(h)$. Let $H \ltimes N$ be the set of ordered pairs $(h,n)$ with $h \in H$ and $n \in N$, endowed with the product

$$(h_1,n_1) \cdot (h_2,n_2) = (h_1h_2,n_1\varphi_{h_1}(n_2)),$$

for each pair of elements $(h_1,n_1)$ and $(h_2,n_2) \in H \ltimes N$. Then $H \ltimes N$ is a group called the semi-direct product of $H$ over $N$ with respect to $\varphi$ (section III.2.10 in Bourbaki [12] may be consulted for a proof). The sets $\{(h,e_N) : h \in H\}$ and $\{(e_H,n) : n \in N\}$ are both subgroups of $H \ltimes N$ that can be identified with $H$ and $N$, respectively.

If $H$ and $N$ are topological groups, we place the induced topology from the product topology on $H \ltimes N$, and require that the mapping $(h,n) \mapsto \varphi_h(n)$ is continuous with respect to the product topology on $H \times N$. This topology is compatible with the group structure, and the projections onto both $H$ and $N$ are continuous.

The subgroup $N$ is always normal in $H \ltimes N$ and the intersection $H \cap N$ contains only the identity and $hnh^{-1} = \varphi_h(n)$ for all $h \in H$ and $n \in N$. Conversely, if $G$ is a group with a subgroup $H$ and a normal subgroup $N$ for which $H \cap N$ contains only the identity, then $G$ is isomorphic to a semi-direct product, where the homomorphism into $\text{Aut} N$ is given by the action of conjugation. We call the group $G$ an internal semi-direct product in this case. A group $G$ is an internal semi-direct product if and only if there is a split short exact sequence

$$1 \to N \xrightarrow{f} G \xrightarrow{g} H \to 1.$$
such that there are group homomorphisms $f : N \to H$ and $g : G \to H$ such that $\text{im}(f) = \ker(g)$, and a group homomorphism $\gamma : C \to B$ such that $g \circ \gamma$ is the identity map on $H$. In the case that $G$ is a topological group, we require in addition that $f$, $g$ and $\gamma$ are continuous and open.

Further suppose that $H$ and $N$ are locally compact topological groups which form a semi-direct product of $H$ over $N$. Let $\lambda_H$ and $\lambda_N$ be the right Haar measures on $H$ and $N$, respectively. The expansion factor is the function $E : H \to \mathbb{R}$ satisfying

$$\lambda_N(\varphi_h(E)) = E(\varphi_h)\lambda_N(E)$$

for each measurable set $E \subseteq N$ and element $h \in H$. By Lema 1.4. ma 8.3 of Fell and Doran [37], the expansion factor is a continuous homomorphism. Moreover, by Proposition 9.5 of the same text, the measure $\lambda_{H \ltimes N}$, satisfying

$$\int_{H \ltimes N} f(h,n) \, d\lambda_{H \ltimes N}(h,n) = \int_{H \ltimes N} f(h,n)E(\varphi_h)^{-1} \, d\lambda_H(h) \times \lambda_N(n)$$

for all functions $f \in C_{00}(G)$ is a right Haar measure on $H \ltimes N$. The measure $\lambda_{H \ltimes N}$ is unique, up to the choice of $\lambda_H$ and $\lambda_N$, by the Riesz Representation Theorem (see Section 7.2 of Cohn [23], for example). The modular functions of $H$, $N$ and $H \ltimes N$ satisfy

$$\Delta_{H \ltimes N}(h,n) = \Delta_H(h)\Delta_N(n)(E(\varphi_h))^{-1}$$

for all $h \in H$ and $n \in N$. In the case of a direct product, where $(\varphi_h$ is always the identity mapping, $E$ is always equal to 1. Hence

$$\Delta_{H \times N}(h,n) = \Delta_H(h)\Delta_N(n)$$

for all $h \in H$ and $n \in N$.

§ 1.2.3 Gauges, gauge functions and word length

We follow the treatment of Kaimanovich [56, 58] in this section. Suppose that $G$ is a second countable locally compact group. A gauge is an increasing sequence $\mathcal{A}$ of measurable sets $A_j$, which exhausts $G$, i.e. $G = \bigcup_{j=1}^{\infty} A_j$. A gauge function is a non-negative integer-valued function $\delta$ for which there exists a non-negative constant $K$ such that

$$\delta(gh) \leq \delta(g) + \delta(h) + K$$

for all $g$ and $h \in G$. A gauge function is subadditive if

$$\delta(gh) \leq \delta(g) + \delta(h)$$
for all \( g \) and \( h \in G \).

Let \( A = \{ A_i \}_{i=1}^\infty \) be a gauge. Then \( A \) is subadditive if the gauge map

\[
|\gamma|_A = \min \{ k \in \mathbb{N} : \gamma \in A_k \}
\]

is a subadditive gauge function. If \( \delta \) is a gauge function, then the sequence \( A^\delta = \{ A^\delta_i \}_{i=1}^\infty \) given by

\[
A_j^\delta = \{ g \in G : \delta(g) \leq j \}
\]

is a gauge. If \( \delta \) is a subadditive gauge function, then \( A_j^\delta \) is a subadditive gauge. We say that the gauge \( A \) is \( C \)-temperate, or just temperate, if \( \lambda_G(A_j) \leq e^{Cj} \) for all \( j \in \mathbb{N} \) and \( C \in \mathbb{R} \). A sequence of gauges \( A^{(j)} \) is uniformly temperate if there is a positive real number \( C \) such that \( A^{(j)} \) is \( C \)-temperate for each \( j \in \mathbb{N} \).

If \( \delta_1 \) and \( \delta_2 \) are non-negative integer-valued functions on \( G \) such that

\[
A_1 \delta_1(g) - B_1 \leq \delta_2(g) \leq A_2 \delta_1(g) + B_2 \tag{1.1}
\]

for some real constants \( A_1, A_2 > 0 \) and \( B_1, B_2 \geq 0 \), then \( \delta_1 \) is a gauge function if and only if \( \delta_2 \) is a gauge function. If \( \delta_1 \) and \( \delta_2 \) are gauge functions which satisfy Equation (1.1), then they are equivalent.

Suppose that \( G \) is a group with a compact generating set \( K \). Let \( K = \{ K_i \}_{i=1}^k \). Then \( K \) is a subadditive temperate gauge, and the word length function

\[
|g|_K := |g|_K = \min \{ n \in \mathbb{N} : g \in K^n \}
\]

is a gauge function on \( G \). If \( K' \) is some other compact symmetric generating set, then \( | \cdot |_K \) and \( | \cdot |_{K'} \) are equivalent gauge functions. A gauge function is a metric estimate if is equivalent to a word length gauge function on \( G \).

**Example 1.2.4.** Let \( G = \mathbb{Z}^k \) for \( k \geq 2 \). The word length in \( \mathbb{Z}^k \) is the norm of \( \mathbb{Z}^k \). The \( L_\infty \) norm is a metric estimate.

**Example 1.2.5.** Let \( C_2 \) be the cyclic group of order two, with identity 0 and remaining element 1. Let \( G \) be the lamplighter group \( C_2 \wr \mathbb{Z} \). This is the semi-direct product \( \mathbb{Z} \ltimes \bigoplus \mathbb{Z} C_2 \) with the multiplication

\[
(n, f)(m, g) = (n + m, f(n \cdot g)),
\]

where \( (n \cdot g)(x) := g(x - n) \).
The group is generated by $a = (0, \chi_0)$, $t = (1, 0)$ and their inverses, where $\chi_x$ is the characteristic function of $x$. Let $\delta : G \to \mathbb{Z}^+$ satisfy

$$\delta(n, f) = \max \{|n|, |d_-(f)|, |d_+(f)|\}$$

for each $(n, f) \in G$. Here $d_-(f)$ is the smallest integer $n$ satisfying $f(n) \neq 0$ whenever $f$ is not the identity, and $0$ otherwise, and $d_+(f)$ is the largest integer $n$ satisfying $f(n) \neq 0$ whenever $f$ is not the identity, and $0$ otherwise. Then $\delta$ is a metric estimate.

**Proof.** Let $(n, f) \in G$ and let $|\cdot| : C_2 \wr \mathbb{Z} \to \mathbb{N}_0$ be the word length. If $(n, f)$ is the identity, then $\delta(e) = |e| = 0$.

Suppose that $(n, f)$ is not the identity, so that $\delta(n, f) \geq 1$. Then we certainly have that $\delta(n, f) \leq |(n, f)|$. Notice that

$$\delta(n, f) = (0, f)(n, 0),$$

and that $|(n, 0)| = |n|$. Hence, by subadditivity of the word length

$$|(n, f)| \leq |(0, f)| + |n| \leq |(0, f)| + \delta(n, f). \quad (1.2)$$

We will now estimate $|(0, f)|$ in terms of $\delta(0, f)$ by writing $(0, f)$ as a product of generators.

Notice that

$$(d_-(f), 0) = t^{d_-(f)}.$$ Traversing across the support of $f$, we have

$$ (1 + d_+(f), f) = t^{d_-(f)} \prod_{r=d_-(f)} a^{f(r)} t.$$ Hence, we can write $(0, f)$ as the product

$$ (0, f) = t^{-d_-(f)} \left( \prod_{r=d_-(f)} a^{f(r)} t \right) t^{-1 - d_+(f)}.$$ Counting the number of generators in this product yields the bound

$$|(0, f)| \leq 3 + 3|d_-(f)| + 3|d_+(f)|$$

$$\leq 3 + 6 \max |d_-(f)|, |d_+(f)|. \quad (1.3)$$

Since $(n, f)$ is not the identity, $1 \leq \delta(n, f)$. Hence, combining equations (1.3) and (1.2) yields

$$|(n, f)| \leq 8 \delta(n, f),$$

as required. □
Lemma 1.2.6. Suppose that $G$ and $H$ are locally compact groups, with compact symmetric generating sets. Let $\delta_G$ be a gauge function on $G$ and $\delta_H$ be a gauge function on $H$. Let $p \geq 1$ be an extended real number, and set

$$\delta_G \oplus_p \delta_H(g, h) = \| (\delta_G(g), \delta_H(h)) \|_p$$

for each $g \in G$ and $h \in H$, where $\| \cdot \|_p$ is the ordinary $p$-norm on $\mathbb{R}^2$. Then $\delta_G \oplus_p \delta_H$ is a gauge function on the direct product $G \times H$. If $\delta_G$ and $\delta_H$ are both subadditive, then $\delta_G \oplus_p \delta_H$ is also subadditive.

Proof. Let $(g_1, h_1)$ and $(g_2, h_2)$ be elements of $G \times H$. Then, by Minkowski’s inequality,

$$\delta_G \oplus_p \delta_H ((g_1, h_2)(g_1, h_2)) = \delta_G \oplus_p \delta_H (g_1g_2, h_1g_2)$$

$$= \| (\delta_G(g_1g_2), \delta_H(h_1h_2)) \|_p$$

$$\leq \| (\delta_G(g_1) + C_1, \delta_H(h_1) + C_2) \|_p$$

$$\leq \| (\delta_G(g_1), \delta_H(h_1)) + (\delta_G(g_2), \delta_H(h_2)) + (C_1, C_2) \|_p$$

$$\leq \| (\delta_G(g_1), \delta_H(h_1)) \|_p + \| (\delta_G(g_2), \delta_H(h_2)) \|_p + \| (C_1, C_2) \|_p$$

$$\leq \delta_G \oplus_p \delta_H (g_1, h_1) + \delta_G \oplus_p \delta_H (g_2, h_2) + K,$$

where $C_1, C_2$ and $K$ are positive real constants. Suppose that $\delta_G$ and $\delta_H$ are both subadditive. Then the above inequalities are valid with $C_1, C_2$ and $K$ all equal to zero. Hence, $\delta_G \oplus_p \delta_H$ is subadditive.

Remark 1.2.7. This lemma may be extended to direct products of any finite length.

§ 1.2.8 Nilpotent and solvable groups

We make use of Palmer [76] in this section. Let $G$ be a topological group. The lower central series of $G$ is the chain of closed normal subgroups

$$G_0 = G \supseteq G_1 = [G, G] \supseteq \ldots \supseteq G_{i+1} = [G, G_i] \supseteq \ldots,$$

where $[G, G_i]$ is the closure of the subgroup generated by the commutators

$$[g, h] = g^{-1}h^{-1}gh$$

for $g \in G$ and $h \in G_i$. The topological group $G$ is nilpotent if there is a $c \in \mathbb{N}$ satisfying $G_c = \{e\}$. The smallest such $c$ is the nilpotence class of $G$. 
The derived or commutator series of $G$ is the chain of normal subgroups
\[ G^{(0)} = G \supseteq G^{(1)} = [G, G] \supseteq \ldots \supseteq G^{(i+1)} = [G^{(i)}, G^{(i)}] \supseteq \ldots \]

A topological group $G$ is solvable if there is a $c \in \mathbb{N}$ satisfying $G^{(c)} = \{e\}$. The smallest such $c$ is the solvable length of $G$.

A topological group is nilpotent if and only if it is nilpotent with respect to the discrete topology, and solvable if and only if it is solvable with respect to the discrete topology (see e.g. Stroppel [93]). Every abelian group $G$ is nilpotent because $G^{(1)} = [G, G] = \{e\}$. Every nilpotent group $G$ is solvable because $G^{(n)}$ is a subgroup of $G_n$ for each $n \in \mathbb{N}$.

**Example 1.2.9.** The permutation group $G = S_3$, endowed with the discrete topology is not nilpotent because $G_i = [G, G] = A_3$ for every $n \in \mathbb{N}$. It is solvable because $G^{(2)} = [G^{(1)}, G^{(1)}] = [A_3, A_3] = \{e\}$.

**Example 1.2.10.** Let $\mathbb{R}$ be the real additive group, and $\mathbb{R}^\times$ the multiplicative group of positive real numbers. Let $G$ be the semi-direct product $\mathbb{R} \rtimes \mathbb{R}^\times$, with the product
\[ (x_1, y_1)(x_2, y_2) = (x_1 + y_1 x_2, y_1 y_2). \]
Note that $(x_1, y_1)^{-1} = \left(-\frac{x_1}{y_1}, \frac{1}{y_1}\right)$. The group $G$ is non-abelian and solvable, but it is not nilpotent.

**Lemma 1.2.11.** If $H$ is abelian and $N$ is nilpotent, then the semi-direct product $G = H \rtimes N$ is solvable.

**Proof.** Let $(h_1, n_1), (h_2, n_2) \in G$. Then, since $H$ is abelian,
\[ [(h_1, n_1), (h_2, n_2)] = (e, n) \]
for some element $n \in N$. Consequently, $[G, G] = \{e\} \times K \cong K$ for some subgroup $K$ of $N$. But $K$ is nilpotent because it is a subgroup of $N$. It follows that the derived series of $G$ terminates at the identity. \qed

**§ 1.2.12 Totally disconnected groups**

We follow Willis [100, 101], Horodam [52], and Hewitt and Ross [49] in this section. A locally compact group $G$ is totally disconnected if its topology is totally disconnected, that is, if all connected components of $G$ are singletons. It is common for authors to adopt the abbreviation t.d.l.c for ‘totally disconnected locally compact’. We shall hereafter follow this convention. Every group
is a t.d.l.c. group with respect to the discrete topology. Such groups are called \textit{discrete groups}. In this sense, the theory of totally disconnected groups incorporates abstract group theory.

Every locally compact group $G$ can be decomposed into a connected group and a totally disconnected group: the connected component $G_0$ of the identity is a closed normal subgroup, and $G/G_0$ is totally disconnected.

Van Dantzig \cite{95} showed that every totally disconnected group has a base of neighbourhoods of the identity consisting of compact open subgroups (see Theorem 7.7 in Hewitt and Ross \cite{49}).

The behaviour of continuous automorphisms was examined by Willis \cite{100, 101}, and we mention some of the results here. Let $G$ be a t.d.l.c. group and let $\alpha$ be a continuous automorphism of $G$. For each compact open subgroup $V$ of $G$, let

$$V_+ = \bigcap_{k \geq 0} \alpha^k(V) \quad \text{and} \quad V_- = \bigcap_{k \geq 0} \alpha^{-k}(V).$$

The subgroup $V_0 = V_+ \cap V_-$ is always closed. If $V = V_+V_-$, then we say that $V$ is \textit{tidy above} with respect to $\alpha$. If the subgroup

$$V_{++} = \bigcup_{k \geq 0} \alpha^k(V_+)$$

is closed, then we say that $V$ is \textit{tidy below} with respect to $\alpha$. The subgroup

$$V_{--} = \bigcup_{k \geq 0} \alpha^{-k}(V_-)$$

is closed if and only if $V_{++}$ is closed. We say that $V$ is \textit{tidy for $\alpha$} if it is both tidy above and tidy below with respect to $\alpha$. Tidy compact open subgroups for $\alpha$ can be constructed from arbitrary compact open subgroups by employing a tidying procedure described in Willis \cite{102}.

The scale of the automorphism $\alpha$ is the positive integer

$$s(\alpha) := \min\{[\alpha(U) : U \cap \alpha(U)] \mid U \text{ is a compact open subgroup of } G\}.$$

Van Dantzig’s Theorem ensures the existence of at least one compact open subgroup $U$ of $G$. The scale of $U$ is finite because $[\alpha(U) : U \cap \alpha(U)]$ is finite by the compactness of $U$ and the continuity of $\alpha$. Any compact open subgroup $V$ satisfying $s(\alpha) = [\alpha(V) : V \cap \alpha(V)]$ is \textit{minimizing for $\alpha$}. Willis \cite{103} showed that every compact open subgroup which is tidy for $\alpha$ is also minimizing for $\alpha$. Hence, the scale of $\alpha$ may be calculated using

$$s(\alpha) = [\alpha(V) : V \cap \alpha(V)] = [\alpha(V_+) : V_+],$$

where $V$ is any compact open subgroup that is tidy for $\alpha$. Every automorphism $\alpha$ on a discrete group has scale 1 because the singleton containing the identity is compact, open and minimizing for $\alpha$. 
The scale function $s : \text{Aut}(G) \to \mathbb{Z}^+$ has the following properties:

1. $s(\alpha) = 1 = s(\alpha^{-1})$ if and only if there is a compact open subgroup $V$ satisfying $\alpha(V) = V$.

2. $s(\alpha^n) = s(\alpha)^n$ for every $n \in \mathbb{N}$, and

3. $\Delta(\alpha) = \frac{s(\alpha)}{s(\alpha^{-1})}$, where $\Delta : \text{Aut}(G) \to (\mathbb{R}^+, \times)$ is the modular function on $\text{Aut}(G)$.

Suppose that $g \in G$. The scale of $g$ is given by

$$s(g) := s(\alpha_g),$$

where $\alpha_g$ is conjugation by $g$. The scale function $s : G \to \mathbb{N}$, satisfies

$$\Delta(g) = \frac{s(g)}{s(g^{-1})},$$

where $\Delta$ is the modular function on $G$. A subset of group elements $E$ is called uniscalar if the scale function is identically 1 on $E$. All uniscalar groups are unimodular.

**Example 1.2.13.** Let $p$ be prime, let $\mathbb{Q}_p$ be the additive group of $p$-adic numbers and let $\alpha$ be the continuous automorphism on $\mathbb{Q}_p$ which satisfies

$$\alpha \left( \sum_{i=k}^{\infty} f_i p^i \right) = \left( \sum_{i=k}^{\infty} f_i p^{i+1} \right) = \left( \sum_{i=k+1}^{\infty} f_i p^i \right)$$

for each element $f \in \mathbb{Q}_p$ with base-$p$ representation $\sum_{i=k}^{\infty} f_i p^i$. Since $\alpha$ has infinite order, $\langle \alpha \rangle$ is infinite. Note that $\alpha^n(\mathbb{Z}_p) = p^n\mathbb{Z}_p$, and that $V = \mathbb{Z}_p$ is compact and open. In fact, $V$ is tidy for $\alpha$ because

$$V_+ = \bigcap_{n \geq 0} \alpha^n(V) = \{ e \},$$

$$V_- = \bigcap_{n \geq 0} \alpha^{-n}(V) = V,$$

and

$$V_{--} = \bigcup_{n \geq 0} \alpha^{-n}(V_-) = \mathbb{Q}_p.$$

Thus, $V = V_+ V_- \quad \text{and} \quad V_{--}$ is closed. The scale of $\alpha$ is 1, and the scale of $\alpha^{-1}$ is $p$.

### 1.3 Probability

In this section, we review some well-known definitions and theorems from the theory of probability and martingales. Our treatment follows Cohn [24], Durrett [30], Klenke [65], Revuz [82] and
Williams [99] unless otherwise noted. We refer the interested reader to these texts for additional detail and any omitted proofs.

Throughout, let \((\Omega, \mathcal{F}, \mu)\) be a probability space, consisting of a set \(\Omega\), a \(\sigma\)-algebra \(\mathcal{F}\) and a probability measure \(\mu\). The set \(\Omega\) is called the sample space. The measurable sets in \(\mathcal{F}\) are interpreted as events or outcomes. The probability \(\Pr(A)\) of an event \(A\) occurring is given by \(\mu(A)\). We usually employ the discrete \(\sigma\)-algebra on finite sets, and the Borel \(\sigma\)-algebra when \(\Omega\) is a topological space.

**Example 1.3.1.** A pair of fair coin flips may be modelled by

\[
\Omega = \{HH, HT, TH, TT\},
\]

\[
\mathcal{F} = \mathcal{P}(\Omega),
\]

with \(\mu\) as the normalised counting measure. The probability of two heads is

\[
\mu(\{HH\}) = \frac{1}{4},
\]

while the probability of at least one head is

\[
\mu(\{HH, HT, TH\}) = \frac{3}{4}.
\]

**Example 1.3.2.** A suitable choice of \((\Omega, \mathcal{F}, \mu)\) to model countably infinitely many fair coin flips is \(\Omega = \{H, T\}^\mathbb{N}\). We take \(\mathcal{F}\) to be the product \(\sigma\)-algebra, which is the smallest \(\sigma\)-algebra containing all subsets of \(\Omega\) of the form \(\prod_{i \in \mathbb{N}} B_i\), where \(B_k \subseteq \{H, T\}\) is not equal to \(\{H, T\}\) for at most finitely many \(k \in \mathbb{N}\), and set \(\mu\) to be the product measure

\[
\mu = \prod_{\mathbb{N}} \frac{1}{2} (\delta_H + \delta_T)
\]

over \(\mathcal{F}\).

**Example 1.3.3.** A suitable choice of \((\Omega, \mathcal{F}, \mu)\) for a process which randomly selects a real number between 0 and 1 uniformly is

\[
\Omega = [0, 1],
\]

\[
\mathcal{F} = \mathcal{B}([0, 1])
\]

and \(\mu\) the normalised Lebesgue measure over the unit interval.

**Lemma 1.3.4** (Borel–Cantelli [65]). Let \((\Omega, \mathcal{F}, \mu)\) be a probability space and let \(\{A_i\}_{i \in \mathbb{N}}\) be a sequence of events in \(\Omega\).
1.3. PROBABILITY

(i) If the sum of the probabilities of the events \( \{A_i\}_{i \in \mathbb{N}} \) is finite then the probability

\[
\mu \left( \limsup_{i \to \infty} A_i \right)
\]

that infinitely many of them occur is zero.

(ii) If \( \{A_i\}_{i \in \mathbb{N}} \) is independent (see Section 1.3.10) and

\[
\sum_{i=1}^{\infty} \mu(A_i) = +\infty
\]

then

\[
\mu \left( \limsup_{i \to \infty} A_i \right) = 1.
\]

Let \((\Psi, \mathcal{H}, \nu)\) be a probability space and let \((\Omega, \mathcal{F}, \mu)\) be the product probability space

\[
(\Omega, \mathcal{F}, \mu) = \prod_{i \in I} (\Psi, \mathcal{H}, \nu).
\]

An event \(E \in \Omega\) is said to be exchangeable if \(p(E) = E\) for every finite permutation \(p\) of the index set \(I\). The set of all exchangeable events is a \(\sigma\)-algebra called the exchange \(\sigma\)-algebra.

**Theorem 1.3.5** (Hewitt–Savage 0–1 Law [40]). Let \(I\) be a countable index set. Let \((\Psi, \mathcal{H}, \nu)\) be a probability space and let \((\Omega, \mathcal{F}, \mu)\) be the product probability space

\[
(\Omega, \mathcal{F}, \mu) = \prod_{i \in I} (\Psi, \mathcal{H}, \nu).
\]

If \(E\) is an exchangeable event, then \(\Pr(E)\) is either 0 or 1.

Given events \(A\) and \(B\), for which \(\Pr(B) > 0\), the conditional probability of \(A\) given \(B\) is defined by

\[
\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\mu(A \cap B)}{\mu(B)}.
\]

If \(S\) is a set of events, then the \(\sigma\)-algebra \(\sigma(S)\) generated by \(S\) is the intersection of all \(\sigma\)-algebras containing \(S\). The sub-\(\sigma\)-algebra generated by an event \(A\) in \(S\) is

\[
\sigma(A) := \sigma(\{A\}) = \{\emptyset, A, \Omega \setminus A, \Omega\}.
\]

A random variable is a measurable function. If \(X_1, \ldots, X_n\) are random variables from a probability space \((\Omega, \mathcal{F}, \mu)\) to a measurable space \((\Psi, \mathcal{H})\), then the \(\sigma\)-algebra generated by \(X_1, \ldots, X_n\) is the smallest \(\sigma\)-algebra which makes each \(X_i\) measurable. We denote this \(\sigma\)-algebra by \(\sigma(X_1, \ldots, X_n)\).

The \(\sigma\)-algebra \(\sigma(X_1, X_2, \ldots)\) generated by a sequence of random variables \(\{X_i\}_{i \in \mathbb{N}}\) is defined similarly.
Example 1.3.6. Let $A$ be an event in $\Omega$. Then the sub-$\sigma$-algebra generated by the indicator function $\chi_A : \Omega \to \mathbb{R}$ is equal to the sub-$\sigma$-algebra generated by the event $A$. This is because

$$\sigma(\chi_A) = \sigma\left\{\chi_A^{-1}(0), \chi_A^{-1}(1)\right\} = \sigma\{A, \Omega \setminus A\} = \sigma\{\emptyset, A, \Omega \setminus A, \Omega\},$$

which is just the $\sigma$-algebra generated by $A$.

The distribution of a random variable $X$ is the push-forward probability measure $X_*\mu = \mu \circ X^{-1}$. If $A$ is a measurable set, then the probability that $X \in A$ is given by

$$\Pr(X \in A) = X_*\mu(A).$$

If a singleton $\{x\}$ is measurable, then the probability that $X = x$ is equal to

$$\Pr(X = x) = X_*\mu(\{x\}).$$

If $X$ is a real valued random variable and $a$ is a real constant, then we write $\Pr(X > a)$ to mean $\Pr(\{x \in \mathbb{R} : x > a\})$. In this definition ‘$>$’ may be replaced with ‘$\neq$’, ‘$<$’, ‘$\geq$’ or ‘$\leq$’, with the corresponding meanings.

Example 1.3.7. Let $(\Omega, \mathcal{F}, \mu)$ be the model for tossing two coins as in Example 1.3.1, and let $X$ count the number of heads. That is,

$$X(\omega) = \begin{cases} 
2 & \text{if } \omega = HH, \\
1 & \text{if } \omega \in \{HT, TH\}, \\
0 & \text{if } \omega = TT.
\end{cases}$$

The distribution of $X$ is the measure $X_*\mu = \frac{1}{4}\delta_2 + \frac{1}{2}\delta_1 + \frac{1}{4}\delta_0$. The probability of tossing one head and one tail is $\Pr(X = 1) = \mu(\{HT, TH\}) = 0.5$. The probability of at least one head is $\Pr(X \in \{1, 2\}) = \mu(\{HH, HT, TH\}) = 0.75$.

Suppose $X$ is a real or complex-valued random variable on $\Omega$. The $m$th moment of $X$ is defined to be

$$m_r(X) = \int_{\Omega} X^m \, d\mu.$$ 

The expected value $\mathbb{E}(X)$ of $X$ is the first moment of $X$. The standard properties of integrals for measurable functions apply. It is common to abuse the notation and write $\mathbb{E}(X)$ to mean the random variable on $\Omega$ with constant value $\mathbb{E}(X)$. We do this in some instances.

The expected value of $X^2$ is the second moment of $X$. If $X$ has finite second moment, then it has finite first moment, since $|x| \leq x^2 + 1$ for all real numbers $x$. The expected value of
\( (X - \mathbb{E}(X))^2 \) is the variance of \( X \) and is denoted by \( \text{Var}(X) \). Linearity of integration implies that the variance of \( X \) is finite if and only if the second moment of \( X \) is finite. The square root of \( \text{Var}(X) \) is the standard deviation of \( X \), and is denoted by \( \text{Dev}(X) \).

**Example 1.3.8.** If \( A \) is an event, then the expected value of the indicator random variable \( \chi_A \) is given by
\[
\mathbb{E}(\chi_A) = \int_{\Omega} \chi_A \ d\mu = \int_{A} 1 \ d\mu = \mu(A).
\]

**Example 1.3.9.** The expected value of the indicator random variable \( \chi_{[0,0.5]} \) on the unit interval, endowed with the Lebesgue measure is
\[
\mathbb{E}(\chi_{[0,0.5]}) = \int_{[0,1]} \chi_{[0,0.5]} \ d\lambda = \int_{[0,0.5]} 1 \ d\lambda = 0.5.
\]
The second moment has the same value, as \( \chi_{[0,0.5]}^2(x) = \chi_{[0,0.5]}(x) \) for all \( x \in \mathbb{R} \). The variance is
\[
\text{Var}(\chi_{[0,0.5]}) = \mathbb{E}(\left(\chi_{[0,0.5]} - 0.5\right)^2) = \mathbb{E}(\chi_{[0,0.5]}^2 - \chi_{[0,0.5]} + 0.25) = 0.25.
\]
The standard deviation is 0.5.

**§ 1.3.10 Independence**

A finite collection of events \( \{A_i\}_{i \in I} \) is independent if
\[
\Pr\left(\bigcap_{i \in \mathbb{N}} A_i\right) = \prod_{i \in \mathbb{N}} \Pr(A_i).
\]

An infinite sequence of events is said to be pairwise independent if any two distinct terms from the sequence are independent. If, instead, every finite subsequence of events is independent, then the sequence is said to be mutually independent or just independent. When \( \Pr(B) \neq 0 \), it follows from the definition of conditional probability, that events \( A \) and \( B \) are independent if and only if
\[
\Pr(A \mid B) = \Pr(A).
\]

Similarly, if \( \Pr(A) \neq 0 \), then events \( A \) and \( B \) are independent if and only if
\[
\Pr(B \mid A) = \Pr(B).
\]

A finite sequence of sub-\( \sigma \)-algebras \( \{A_i\}_{i \in I} \) is independent if every finite sequence of events \( \{E_i\}_{i \in I} \), where \( E_i \in A_i \), is independent. An infinite sequence of sub-\( \sigma \)-algebras is independent (pairwise independent) if every finite subsequence of events is mutually independent (pairwise independent).

A (possibly finite) sequence of random variables is independent if the sequence of \( \sigma \)-algebras generated by them is independent. An infinite sequence of random variables is pairwise independent if the sequence of \( \sigma \)-algebras generated by those random variables is pairwise independent.
Example 1.3.11. Let $\chi_{[0,0.5]}$, $\chi_{[0,0.25]}$ and $\chi_{[0.25,0.75]}$ be indicator random variables on the unit interval. Then $\chi_{[0,0.5]}$ and $\chi_{[0.25,0.75]}$ are independent, but $\chi_{[0,0.25]}$ and $\chi_{[0.5,0.75]}$ are not because

$$0.25 = \lambda([0,0.5] \cap [0,0.25]) \neq \lambda([0,0.5]) \lambda([0,0.25]) = 0.125$$

and $\lambda(A \cap B) = \lambda(A) \lambda(B)$ for all $A \in \sigma(\chi_{[0,0.5]})$ and $B \in \sigma(\chi_{[0.25,0.75]})$.

Remark 1.3.12 (Bertsekas and Tsitsiklis [7]). Pairwise independence does not imply mutual independence. To illustrate this, let $(\Omega, F, \mu)$ be the model for tossing two coins as in Example 1.3.1. Then let $H_1 = \{HH, HT\}$ be the event corresponding to the first toss being heads, and $H_2 = \{TH, HH\}$ be the event in which the second toss is heads. Finally, let $D = \{TH, HT\}$ be the event corresponding to the first and second tosses having different results. Then $H_1$, $H_2$ and $D$ are pairwise independent, but not mutually independent.

Suppose that $\{H_i\}_{i \in \mathbb{N}}$ is a sequence of sub $\sigma$-algebras of a $\sigma$-algebra $F$. For each positive integer $n$, let $F_n$ be the future after time $n$:

$$F_n := \sigma\left(\bigcup_{i=n}^{\infty} H_i\right)$$

and let $F_\infty$ be the tail $\sigma$-algebra:

$$F_\infty := \bigcap_{i=1}^{\infty} F_i.$$

Elements of $F_\infty$ are called tail events.

Theorem 1.3.13 (Kolmogorov’s 0–1 law [30]). Suppose that $\{H_i\}_{i \in \mathbb{N}}$ is a sequence of sub $\sigma$-algebras of a $\sigma$-algebra $F$. If $\{H_i\}_{i \in \mathbb{N}}$ is independent, then all tail events occur with either probability zero or probability one.

§ 1.3.14 Conditional expectation

The conditional expectation of a random variable $X$, with respect to an event $A$ of positive probability, is

$$E(X \mid A) = \frac{E(\chi_A X)}{Pr(A)}.$$  

This definition can be extended to conditional expectation $E(X \mid H)$ over a sub-$\sigma$-algebra $H$.

Theorem 1.3.15. Suppose $X$ is a real-valued random variable on a probability space $(\Omega, F, \mu)$, satisfying $E(|X|) < +\infty$. Suppose that $H$ is a sub-$\sigma$-algebra of $F$. Then there exists a $H$-measurable random variable $E(X \mid H)$, satisfying

$$\int_H E(X \mid H) d\mu = \int_H X d\mu$$  \hspace{1cm} (1.4)

for every measurable set $H \in H$. 

Proof. We sketch the proof. The interested reader may refer to Section 4.1 in Durrett [30] for more details. The Radon–Nikodym Theorem ensures the existence of the conditional expectation. The conditional expectation is unique because, if $Y$ and $Y'$ are $\mathcal{H}$-measurable random variables satisfying
\[
\int_H Y \, d\mu = \int_H Y' \, d\mu
\]
for all $H \in \mathcal{H}$, then $Y = Y'$ almost everywhere.

The random variable $\mathbb{E}(X \mid \mathcal{H})$ is called the conditional expectation of $X$ with respect to $\mathcal{H}$, and it is unique up to modifications on a set of measure zero. If $\{Y_i\}$ is a possibly finite sequence of random variables on $(\Omega, \mathcal{F}, \mu)$, then the conditional expectation of $X$ with respect to $Y$ is given by
\[
\mathbb{E}(X \mid Y_1, Y_2, \ldots) = \mathbb{E}(X \mid \sigma(Y_1, Y_2, \ldots)).
\]

Properties such as linearity and positivity, inequalities, and convergence theorems from integration theory apply to conditional expectations. For a comprehensive list of properties and their proofs, see Section 5.5 in Ash and Doléans [1]. A few useful identities are provided in the following lemmas.

**Lemma 1.3.16.** Suppose that $X$ is a random variable on a probability space $(\Omega, \mathcal{F}, \mu)$ and that $\mathcal{H}$ is a sub-$\sigma$-algebra of $\mathcal{F}$. If $\sigma(X)$ and $\mathcal{H}$ are independent, then $\mathbb{E}(X \mid \mathcal{H})$ is the constant random variable with value $\mathbb{E}(X)$.

**Proof.** Suppose that $H \in \mathcal{H}$. As $X$ and $\chi_H$ are mutually independent,
\[
\mathbb{E}(X \chi_H) = \mathbb{E}(X) \mathbb{E}(\chi_H) = \mathbb{E}(X) \mu(H) = \int_H \mathbb{E}(X) \, d\mu.
\]
By the definition of the conditional expectation
\[
\int_H \mathbb{E}(X \mid \mathcal{H}) \, d\mu = \int_H X \, d\mu = \mathbb{E}(X \chi_H) = \int_H \mathbb{E}(X) \, d\mu,
\]
as required. 

**Lemma 1.3.17.** Suppose $X$ is a real-valued random variable on a probability space $(\Omega, \mathcal{H}, \mu)$, with $\mathbb{E}(|X|) < +\infty$. Also suppose that $\sigma(X)$ is a sub-$\sigma$-algebra of $\mathcal{H}$. Then $\mathbb{E}(X \mid \mathcal{H}) = X$.

**Proof.** The quantity $\mathbb{E}(|X|)$ is bounded by assumption. The random variable $X$ is $\mathcal{H}$-measurable since it is a sub-$\sigma$-algebra of $\mathcal{H}$ and that satisfies the integral condition.
Lemma 1.3.18 (Tower property for sub-σ-algebras). Suppose that $X$ is a random variable on a probability space $(\Omega, F, \mu)$, and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are sub-σ-algebras of $F$, satisfying $\mathcal{H}_1 \subseteq \mathcal{H}_2$. Then

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{H}_2) \mid \mathcal{H}_1) = \mathbb{E}(X \mid \mathcal{H}_1) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{H}_1) \mid \mathcal{H}_2)$$

almost everywhere.

Proof. By definition of conditional expectation,

$$\int_H \mathbb{E}(\mathbb{E}(X \mid \mathcal{H}_1) \mid \mathcal{H}_2) \, d\mu = \int_H \mathbb{E}(X \mid \mathcal{H}_1) \, d\mu = \int_H X \, d\mu$$

for all $H \in \mathcal{H}_1$. Similarly,

$$\int_H \mathbb{E}(E(X \mid \mathcal{H}_2) \mid \mathcal{H}_1) \, d\mu = \int_H \mathbb{E}(X \mid \mathcal{H}_2) \, d\mu = \int_H X \, d\mu$$

for all $H \in \mathcal{H}_1$, establishing our result \qed

Lemma 1.3.19 (Jensen’s inequality [30]). Suppose that $X$ is a real valued random variable on a probability space $(\Omega, F, \mu)$. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function, then

$$\varphi(\mathbb{E}(X \mid \mathcal{H})) \leq \mathbb{E}(\varphi(X) \mid \mathcal{H})$$

for all sub-σ-algebras $\mathcal{H}$ of $F$.

§ 1.3.20 Sequences of random variables

Let $(\Omega, F)$ and $(\Psi, \mathcal{H})$ be a measurable spaces. Choose a probability measure $\mu$ on $\Omega$, a random variable $X : \Omega \to \Psi$, and a sequence of random variables $\{X_i\}_{i \in \mathbb{N}}$ from $\Omega$ to $\Psi$. We say that $\{X_i\}_{i \in \mathbb{N}}$ converges $\mu$-almost surely ($\mu$-$a.s.$), $\mu$-almost everywhere ($\mu$-$a.e.$), or with probability 1 to $X$ if it converges pointwise everywhere except on a set of measure zero, i.e.

$$\Pr \left( \left\{ \omega \in \Omega : \lim_{i \to \infty} X_i(\omega) = X(\omega) \right\} \right) = 1.$$ 

If the measure is clear from the context, we just say that $\{X_i\}_{i \in \mathbb{N}}$ converges almost everywhere. If there exists a random variable $X$ satisfying

$$\lim_{i \to \infty} \Pr (\{\omega \in \Omega : |X_i(\omega) - X(\omega)| \geq \varepsilon\}) = 0$$

for some $\varepsilon > 0$. If there exists a random variable $X$ satisfying

$$\lim_{i \to \infty} \Pr (\{\omega \in \Omega : |X_i(\omega) - X(\omega)| \geq \varepsilon\}) = 0$$

for some $\varepsilon > 0$.
for any $\varepsilon > 0$, then $\{X_i\}_{i\in\mathbb{N}}$ is said to *converge in probability* to $X$. Finally, if $\{X_i\}_{i\in\mathbb{N}}$ satisfies

$$\lim_{i\to\infty} \|X_i - X\|_p = 0,$$

for some $p \in [1, \infty]$, then we say that *converges in $L^p$* to $X$. Since $\|X_n - X\|_1 = \mathbb{E}(|X_n - X|)$ for all $n \in \mathbb{N}$, we say that $\{X_i\}_{i\in\mathbb{N}}$ *converges in mean* to $X$ if it converges in $L^1$.

A sequence of random variables which have the same range is said to be *identically distributed* if any two distinct terms in the sequence have the same distribution. A sequence of random variables which is both independent and identically distributed is often called a sequence of *i.i.d.* random variables.

**Theorem 1.3.21** (Strong law of large numbers [89]). Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed real valued random variables with distribution $\mu$. Then the sequence of sample averages $\{\bar{X}_i\}_{i\in\mathbb{N}}$, where

$$\bar{X}_n := \frac{1}{n}(X_1 + \cdots + X_n)$$

for each natural number $n$, converges almost surely to the expected value $\mathbb{E}(X_1) = \mathbb{E}(X_n)$.

**Theorem 1.3.22** (Central Limit Theorem [30]). Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed real valued random variables that have finite first moment $\mathbb{E}(X_i)$ and finite variance. Suppose that each $X_i$ has distribution $\mu$. Let $\{\tilde{X}_i\}_{i\in\mathbb{N}}$ be the sequence of sample averages

$$\tilde{X}_n := \frac{1}{n}(X_1 + \cdots + X_n).$$

Then the sequence of random variables $\{S_i\}_{i\in\mathbb{N}}$, where

$$S_n = \sqrt{n}(\tilde{X}_n - \mathbb{E}(X_n))$$

converges in distribution to a normal distribution with mean zero and the same variance as each $X_n$.

**Theorem 1.3.23** (Multi-variable Central Limit Theorem [30]). Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{R}^n$ that each have finite first moment and finite variance. Suppose that each $X_i$ has distribution $\mu$. Let $\{\tilde{X}_i\}_{i\in\mathbb{N}}$ be the sequence of sample averages,

$$\tilde{X}_i := \frac{1}{i}(X_1 + \cdots + X_i).$$

Then the sequence of random variables $\{S_i\}_{i\in\mathbb{N}}$ satisfying

$$S_i = \sqrt{i}\left(\tilde{X}_i - \mathbb{E}(X_i)\right)$$
converges in distribution to a multivariate normal distribution $\mathcal{N}_k(0, \Sigma)$, where $\Sigma$ is a covariance matrix.

A Markov chain on $(\Omega, \mathcal{F}, \mu)$ is a sequence of identically distributed random variables $\{X_i\}_{i \in \mathbb{N}}$ satisfying the Markov property

$$\Pr(X_{n+1} | X_1, \ldots, X_n) = \Pr(X_{n+1} | X_n)$$

for all $n \in \mathbb{N}$, such that the next state of the system $X_{n+1}$ depends only on the current state. The Markov chain is time-homogeneous if

$$\Pr(X_{n+1} | X_n) = \Pr(X_n | X_{n-1})$$

for all $n \in \mathbb{N}$.

A sequence of real-valued random variables $\{X_i\}_{i \in \mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \mu)$ is a martingale with respect to a filtration $\mathcal{F}_n$ of sub-$\sigma$-algebras of $\mathcal{F}$ if, for all $n \in \mathbb{N}$,

$$E(|X_n|) < +\infty, \text{ and}$$

$$E(X_{n+1} | \mathcal{F}_n) = X_n.$$

If the filtration is not specified, we choose $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. If, in this definition ‘=’ is replaced with ‘$\leq$’ or ‘$\geq$’, respectively, then $\{X_i\}_{i \in \mathbb{N}}$ is instead called a super-martingale or sub-martingale, respectively. It is only necessary to consider sub-martingales from a theoretical point of view because if $\{X_i\}_{i \in \mathbb{N}}$ is a super-martingale, then $\{-X_i\}_{i \in \mathbb{N}}$ is a sub-martingale. A sequence of complex-valued random variables $\{X_i\}_{i \in \mathbb{N}}$ is said to be a martingale if the sequences $\{\text{Re}(X_i)\}_{i \in \mathbb{N}}$ and $\{\text{Im}(X_i)\}_{i \in \mathbb{N}}$ are both martingales under the previous definition.

**Example 1.3.24.** (Fair coin tossing game) Model an infinite sequence of fair coin flips by taking $X = \{H, T\}$, $\mu = \frac{1}{2}(\delta_H + \delta_T)$ and setting $(\Omega, \mathcal{F}, \mathbb{P}) = (X, \mathcal{P}(X), \mu)^\infty$. For each $\omega = \{\omega_i\}_{i \in \mathbb{N}} \in \Omega$, let $X_0(\omega) = 0$ and let

$$X_n(\omega) = \begin{cases} 
1 & \text{if } \omega_n = H \\
-1 & \text{if } \omega_n = T
\end{cases}$$

for each natural number $n$. Let $R_n(\omega)$ be the cumulative number of heads minus the cumulative number of tails at time $n$, that is,

$$R_n(\omega) = \sum_{j=1}^{n} X_j(\omega)$$
for each natural number \( n \). The sequence \( \{R_i\}_{i \in \mathbb{N}} \) is a martingale because

\[
\mathbb{E}(R_n \mid R_1, \ldots, R_{n-1}) = \mathbb{E}(R_{n-1} \mid R_1, \ldots, R_{n-1}) + \mathbb{E}(X_n \mid R_1, \ldots, R_{n-1}) \\
= R_{n-1} + \mathbb{E}(X_n) \\
= R_{n-1}.
\]

We have used the facts that \( \mathbb{E}(X_n) = 0 \) for all \( n \in \mathbb{N} \), and \( X_n \) is independent of both \( \sigma(X_1, \ldots, X_{n-1}) \) and the expected value of \( X_n \).

**Remark 1.3.25.** If, in the previous example, we instead took \( \mu = p\delta_H + (1 - p)\delta_T \), then

\[
\mathbb{E}(R_n \mid R_1, \ldots, R_{n-1}) = R_{n-1} + 2p - 1.
\]

Hence, \( \{R_i\}_{i \in \mathbb{N}} \) is a sub-martingale if \( p > \frac{1}{2} \) and a super-martingale if \( p < \frac{1}{2} \).

**Theorem 1.3.26** (Doob’s Martingale Convergence Theorem [30]). If \( \{X_i\}_{i \in \mathbb{N}} \) is a submartingale on \((\Omega, \mathcal{F}, \mu)\) satisfying

\[
\sup_{i \in \mathbb{N}} \mathbb{E}(X_i^+) < +\infty,
\]

where \( X_i^+ \) is the positive part of \( X_i \) for each \( i \in \mathbb{N} \), then \( \{X_i\}_{i \in \mathbb{N}} \) converges almost surely to a random variable \( X_\infty \) in \( L^1(\Omega, \mathcal{F}, \mu) \).

Given a martingale \( \{X_i\}_{i \in \mathbb{N}} \), a random variable \( T : \Omega \to \{0, 1, 2, \ldots\} \cup \{+\infty\} \) is a stopping time or hitting time if

\[
\{\omega \in \Omega : T(\omega) \leq n\} \in \sigma(X_1, \ldots, X_n)
\]

for each natural number \( n \). Given a stopping time \( T \) on \( \{X_i\}_{i \in \mathbb{N}} \), let

\[
X_n^T(\omega) := X_{\min(T(\omega), n)}(\omega)
\]

for each natural number \( n \). The sequence \( \{X_i^T\}_{i \in \mathbb{N}} \) is called the stopped process corresponding to \( \{X_i\}_{i \in \mathbb{N}} \).

**Theorem 1.3.27** (Doob’s Optional-Stopping Theorem [99]). Suppose \( \{X_i\}_{i \in \mathbb{N}} \) is a discrete-time martingale on \((\Omega, \mathcal{F}, \mu)\), with stopping time \( T \) which is bounded almost surely. That is, suppose there exists a positive real constant \( c \) such that \( T(\omega) \leq c \) for \( \mu \)-almost every \( \omega \in \Omega \). Then \( \{X_i^T\}_{i \in \mathbb{N}} \) is a martingale, and

\[
\mathbb{E}(X_n^T) = \mathbb{E}(X_1)
\]

almost surely for all \( n \in \mathbb{N} \).
Example 1.3.28. Let \( \{R_i\}_{i \in \mathbb{N}} \) be the martingale from the fair coin tossing game of Example 1.3.24. Let \( T \) be time of the first heads:

\[
T(\omega) = \min_{n \in \mathbb{N}} \{R_n(\omega) \geq R_{n-1}(\omega)\}
\]

for each \( \omega \in \Omega \). The stopping time is almost surely bounded because the probability of not stopping in \( n \) steps decays exponentially. The set

\[
\{ \omega \in \Omega : T(\omega) \leq n \} = \{ \omega \in \Omega : X_n(\omega) = H, X_i(\omega) = T \text{ for all } i < n \}
\]

is in \( \sigma(X_1, \ldots, X_n) \) for all \( n \in \mathbb{N} \). So the process \( \{R_T^i\}_{i \in \mathbb{N}} \) is a martingale, and

\[
\mathbb{E}(R_T^n) = \mathbb{E}(R_1),
\]

for all \( n \in \mathbb{N} \).

\section*{1.4 Random walks}

In this section, we introduce random walks on groups. We begin with a discussion of the spaces of \( \mu \)-harmonic functions associated with each random walk and some related spaces. Next, we discuss topological boundaries of random walks, and a special boundary, called the Poisson boundary, which captures asymptotic information about the random walk. We give an integral representation of bounded left uniformly continuous \( \mu \)-harmonic functions. We next discuss measurable boundaries, and, in particular, discuss equivalent definitions of the measurable Poisson boundary, which represents the essentially bounded \( \mu \)-harmonic functions. Finally, we discuss boundary behaviour under recurrence and transience, in addition to mentioning geometric criteria for the boundary maximality of discrete groups.

Most of our discussion relating to harmonic functions and the Poisson boundary come from Babillot [4], Furman [41] and Midjord [74]. We make use of the survey articles written by Erschler [35], Furstenberg [42], Kaimanovich and Vershik [60, 56]. Some of the discussion centred around recurrent sets is sourced from Högnös and Mukherjea [51], Revuz [82] and Woess [106].

Let \( \mu \) be a Borel probability measure on a locally compact group \( G \) with identity \( e \) and right Haar measure \( \lambda \). We call such a pair \((G, \mu)\) a random walk. We let \( \rho \) be a probability measure on \( G \), called the initial distribution of the random walk. Often, \( \rho \) is taken to be the point mass at the identity.

The space of trajectories is the set \( G^\mathbb{N} \), endowed with the product \( \sigma \)-algebra. Here \( G^\mathbb{N} \) is an infinite Cartesian product of countably many copies of \( G \). An element \( \omega \in G^\mathbb{N} \) is called a
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trajectory or path. We denote by $P^\mu$ the product measure $\mu^N$, and $P^\mu$ the pushforward of $P^\mu$ with respect to the map $S : G^N \to G^N$ satisfying

$$S(\omega_1, \omega_2, \omega_3, \ldots, \omega_k, \ldots) = (\omega_1, \omega_1 \omega_2, \omega_1 \omega_2 \omega_3, \ldots, \omega_1 \ldots \omega_k, \ldots).$$

We call the pair $(G^N, P^\mu)$ the space of increments, the measure $P^\mu$ the path measure and the pair $(G^N, P^\mu)$ the path space.

Let $\{X_i\}_{i \in \mathbb{N}_0}$ be the sequence of random variables from $G^N$ to $G$ formed by the projections

$$X_i(\omega) = \omega_i.$$

These random variables are independent, and are called the increments of the random walk. The random variable $X_i$ has distribution $\mu$ for each natural number $i$. We identify the random walk $(G, \mu)$ with a discrete time-homogeneous Markov chain $\{R_i\}_{i \in \mathbb{N}_0}$, called the right random walk, where

$$R_0 = X_0, \quad R_i = R_{i-1}X_i,$$

for each $i \in \mathbb{N}_0$. Each random variable $R_i$ may be equivalently defined to be the projections from $(G^N, P^\mu)$ to each factor given by

$$R_i(\omega) = \omega_i$$

for each non-negative $i \in \mathbb{Z}$.

The Markov chain $\{R_i\}_{i \in \mathbb{N}_0}$ has state space $G$ and transition probabilities

$$p(g, E) = \mu(g^{-1}E)$$

between $g \in G$ and $E \in B(G)$. The transition probabilities are group-invariant in the sense that for all group elements $g$ and $h \in G$, and every Borel set $E$ of $G$

$$p(g, E) = p(hg, hE).$$

The left random walk is defined similarly, but with group multiplication on the left:

$$L_0 = e, \quad L_n = X_nL_{n-1}.$$

We usually do not consider these walks individually, because the left random walk associated with $(G, \mu)$ corresponds to the right random walk associated with $(G, \check{\mu})$, where $\check{\mu}$ is the reflected measure:

$$\check{\mu}(E) := \mu(E^{-1})$$

for all measurable subsets $E$ of $G$. 
The group $G$ acts on elements of the path space, $(\omega_1, \omega_2, \ldots) \in G^N$ via

$$h \cdot (\omega_1, \omega_2, \ldots) = (h\omega_1, h\omega_2, \ldots)$$

for each $h \in G$. This action extends to an action on the probability measures on $G^N$: if $m$ is a probability measure on $G^N$, then

$$g \cdot m(E) = m(g^{-1}E)$$

for each measurable set $E$ and $g \in G$.

The support $\text{supp} \mu$ of the measure $\mu$ is the complement of the union of all open sets $E$ which satisfy $\mu(E) = 0$. The closed group $\text{gr} \mu$ generated by the support of $\mu$ is the smallest closed group containing $\text{supp} \mu$. Similarly, the closed semi-group generated by the support of $\mu$, given by

$$\text{sgr} \mu = \bigcup_{n \geq 0} \text{supp} \mu^n$$

where $\mu^n$ is the $n$th convolution power of $n$, is the smallest closed semi-group containing $\text{supp} \mu$.

The measure $\mu$ and the random walk $(G, \mu)$ are both said to be adapted, irreducible, non-degenerate or aperiodic if $\text{sgr} \mu$ is $G$, finitary if the support of $\mu$ is finite and symmetric if $\mu = \tilde{\mu}$. If $G$ is finitely generated with generators $a_1, \ldots, a_n$ and

$$\mu = \frac{1}{2n} \sum_{i=1}^{n} \left( \delta_{a_i} + \delta_{a_i^{-1}} \right),$$

where $\delta_x$ is the point mass on an element $x \in G$, then we say that $(G, \mu)$ is simple. The measure $\mu$ and the random walk $(G, \mu)$ are said to be discrete if the support of $\mu$ is a countable set, i.e. there exists a sequence of group elements $\{x_i\}_{i \in \mathbb{N}}$ and a sequence of non-negative real numbers $\{\alpha_i\}_{i \in \mathbb{N}}$, satisfying

$$\mu = \sum_{i \in \mathbb{N}} \alpha_i \delta_{x_i}$$

for all measurable sets $E$.

Let $r \geq 1$ be a real number. Suppose that $\delta$ is a gauge function on $G$. The $r$th-moment of $\mu$ with respect to $\delta$ is

$$m_r^\delta(\mu) = \int_G \delta(x)^r \ d\mu(x).$$

If $m_r^\delta(\mu)$ is finite, then $\mu$ is said to have finite $r$th moment with respect to $\delta$. The quantity $m_1^\delta(\mu)$ is also called the mean of $\mu$ with respect to $\delta$. If $\delta$ is clear from the context, we just write e.g. $m_r(\mu)$ or ‘finite $r$th moment’ without mentioning $\delta$. Often, $\delta$ is taken to be the word length

$$|g|_K = \min\{n \in \mathbb{N} : g \in K^n\},$$
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where \( K \) is a compact generating set for \( G \). If \( \delta_1 \) and \( \delta_2 \) are equivalent gauge functions on \( G \), then \( \mu \) has finite \( r \)th moment with respect to \( \delta_1 \) if and only \( \mu \) has finite \( r \)th moment with respect to \( \delta_2 \).

Let \( \eta \) and \( \nu \) be Borel measures on \( G \). Then \( \eta \) is absolutely continuous with respect to \( \nu \) if \( \eta(E) = 0 \) whenever \( \nu(E) = 0 \) for all Borel sets \( E \). The measures are mutually singular if there is a measurable set \( E \) satisfying \( \eta(E) = 0 \) and \( \nu(E^c) = 0 \). If \( \mu \) is absolutely continuous with respect to the Haar measure \( \lambda \), then it is absolutely continuous. If \( \mu \) and \( \lambda \) are mutually singular, then \( \mu \) is singular. If there is a positive integer convolution power of \( \mu \) such that \( \mu \) is not singular, then we say that \( \mu \) is spread-out.

\[ \S 1.4.1 \] \( \mu \)-harmonic functions

From now on, suppose that the locally compact group \( G \) is second countable. An essentially bounded function \( f \in L^\infty(G,\lambda) \) is called \( \mu \)-harmonic if it satisfies the convolution equation

\[ f(x) = f * \mu(x) = \int_G f(xy) \, d\mu(y) \]

for almost every \( x \in G \). We denote the space of all essentially bounded \( \mu \)-harmonic functions by \( H^\infty(G,\mu) \).

**Example 1.4.2.** Each constant function on \( G \) is \( \mu \)-harmonic for every probability measure \( \mu \) because

\[ f * \mu(x) = \int_G f(xy) \, d\mu(y) = f(x) \int_G d\mu(y) = f(x) \]

whenever \( f \) is a constant function.

**Example 1.4.3.** If \( H^\infty(G,\mu) \) contains only constant functions, then it is said to be trivial. The simple random walk over the integers, with the standard generating set and the corresponding measure \( \mu \), is an example of a random walk for which the corresponding space of harmonic functions is trivial. To see this, suppose that \( f \) is a \( \mu \)-harmonic function on \( \mathbb{Z} \) which is not necessarily bounded. Then

\[ f(x) = f * \mu(x) = \int_{\mathbb{Z}} f(xy) \, d\mu(y) = \frac{1}{2} f(x - 1) + \frac{1}{2} f(x + 1). \]

Rearranging this recurrence relation yields

\[ f(x + 1) = 2f(x) - f(x - 1). \]

If \( f(0) = a \) and \( f(1) = a + d \), then \( f(n) = a + nd \) for every integer \( n \). Therefore, the \( \mu \)-harmonic functions on the additive group of integers consist of the arithmetic sequences. Moreover, all bounded \( \mu \)-harmonic functions are constant.
The conclusion of Example 1.4.3 is true more generally. A group $G$ is Liouville if for all Borel probability measures $\mu$, satisfying $\text{sgr} \mu = G$, the essentially bounded harmonic functions are constant almost everywhere. Many groups have the Liouville property. Blackwell [10] proved a result of this type for countable abelian groups. Choquet and Deny [18] showed that all abelian groups are Liouville, because, if $G$ is abelian, the essentially bounded continuous $\mu$-harmonic functions are always constant on the cosets of $\text{gr} \mu$. Results of this type are often called Choquet and Deny theorems.

Margulis [71] showed that positive harmonic functions on finitely generated nilpotent groups are constant on the cosets of their commutator subgroups. Dynkin and Malyutov [31] obtained an earlier result for the case of a finitely generated group $G$ acting transitively on a set. Later, Chu and Hilberdink [20] extended Choquet and Deny’s argument to prove that if $f$ is a real, bounded and left uniformly continuous function on a nilpotent group $G$ which is $\mu$-harmonic so that $\mu$ is non-degenerate, then it is constant. Other necessary conditions for constant harmonic functions were given by Chu in [21] and [22]. Raugi [81] showed that all $\mu$-harmonic functions on nilpotent groups $G$ are constant whenever $\mu$ is non-degenerate.

The following example, due to Dynkin [31], shows that if $G$ is not nilpotent, then the essentially bounded $\mu$-harmonic functions need not be constant.

**Example 1.4.4.** Let $(\mathbb{F}_m, \mu)$ be the simple random walk on the free group $\mathbb{F}_m$. Here $m$ is an integer greater than 1, so that $\mathbb{F}_m$ is non-abelian. We define a function $f : \mathbb{F}_m \to \mathbb{R}$ as follows: for each element $x \in \mathbb{F}_m$ with reduced word length $|x|$, let

$$f(x) = \begin{cases} 0 & \text{if } x = e, \\ \sum_{k=0}^{|x|-1} \frac{1}{(2m-1)^k} & \text{if } a_1 \text{ is a generator, and} \\ -\sum_{k=0}^{|x|-1} \frac{1}{(2m-1)^k} & \text{if } a_1 \text{ is the inverse of a generator,} \end{cases}$$

where $a_1, \ldots, a_{|x|}$ is the reduced word corresponding to $x$. Then $f$ is a non-constant bounded $\mu$-harmonic function on $\mathbb{F}_m$.

**Proof.** Because every element of the free group has a unique reduced word, this function is well defined. It is bounded because each value of the function is a partial sum of a convergent geometric
series. Using the definition of \( f \), we may calculate

\[
0 = f(e) = (f \ast \mu)(e) = \frac{1}{2m} \sum_{i=1}^{m} (f(g_i) + f(g_i^{-1})) = \frac{1}{2m} \sum_{i=1}^{m} (1 - 1) = 0.
\]

Next, observe that, for all positive integers \( k \),

\[
\frac{2m}{(2m - 1)^k} = \frac{1}{(2m - 1)^k} + \frac{1}{(2m - 1)^{k+1}}.
\]

If \( x \in \mathbb{F}_m \), then

\[
f(x) = \pm \sum_{k=0}^{|x|-1} \frac{1}{(2m - 1)^k}
\]

\[
f(x) = \pm \frac{1}{2m} \sum_{k=0}^{|x|-1} \frac{2m}{(2m - 1)^k}
\]

\[
f(x) = \pm \frac{1}{2m} \left( \sum_{k=0}^{|x|-1} \frac{1}{(2m - 1)^k} + \frac{1}{(2m - 1)^{k+1}} \right)
\]

\[
f(x) = \pm \frac{1}{2m} \left( \sum_{k=0}^{|x|-2} \frac{1}{(2m - 1)^k} + \frac{1}{(2m - 1)^{|x|-1}} + \sum_{k=0}^{|x|-1} \frac{1}{(2m - 1)^{k+1}} \right)
\]

\[
= \pm \frac{1}{2m} \left( \sum_{k=0}^{|x|-2} \frac{1}{(2m - 1)^k} + (2m - 1) \sum_{k=0}^{|x|} \frac{1}{(2m - 1)^k} \right)
\]

\[
= (f \ast \mu)(x).
\]

It follows by induction on the word length that \( f \) is \( \mu \)-harmonic for every reduced word.

Remark 1.4.5. If \( f \) is the bounded \( \mu \)-harmonic function on \( \mathbb{F}_m \) given in Example 1.4.4, then the pointwise product \( f^2 = f \cdot f \) is not \( \mu \)-harmonic because e.g.

\[
0 = f(e) = f^2(e) \neq (f^2 \ast \mu)(e) = \frac{1}{2m} \sum_{i=1}^{m} (f^2(g_i) + f^2(g_i^{-1})) = 1.
\]

A complex-valued function \( f \) on \( G \) is called left uniformly continuous if, for every \( \varepsilon > 0 \), there exists an open neighbourhood \( U \) of the identity in \( G \) such that

\[
\sup_{g \in G} |f(ug) - f(g)| < \varepsilon
\]

for all \( u \in U \). Right uniformly continuous functions are defined similarly by substituting ‘\( ug \)’ for ‘\( gu \)’ in the above definition. If a function is both left and right uniformly continuous, then it is
uniformly continuous. We denote by $H^\infty_\rho(G, \mu)$ the space of left uniformly continuous $\mu$-harmonic functions on $(G, \mu)$. Both left and right uniformly continuous functions are continuous.

**Lemma 1.4.6.** Suppose that $\nu$ and $\eta$ are finite Borel measures on $G$ so that $\nu$ is absolutely continuous. Then the convolution $\nu * \eta$ is absolutely continuous.

**Proof.** Let $E$ be a Borel set in $G$. We compute

$$\nu * \eta(E) = \nu * \eta(E) = \int_G \nu(Ey^{-1}) d\eta(E)$$

The absolute continuity of $\nu * \eta$ follows by the right translation invariance of the right Haar measure because if $\lambda(E) = 0$, then $\lambda(Ey) = 0$ for every $y \in G$.

**Proposition 1.4.7.** Let $\mu$ be a spread-out Borel probability measure on $G$, and let $f$ be an essentially bounded $\mu$-harmonic function. Then $f$ is (uniformly) continuous.

**Proof.** We follow the proof given in Azencott [3]. Suppose that $p \in \mathbb{N}$ is such that $\mu^{*p}$ and $\lambda$ are not mutually singular. For each $n \in \mathbb{N}$, let

$$\mu^{*np} = \alpha_n + \beta_n$$

be the Lebesgue decomposition of $\mu^{*np}$ as in Theorem C in Section 32 of Halmos [46], so that $\alpha_n$ is absolutely continuous and $\beta_n$ is singular. Then

$$\alpha_n + \beta_n = (\alpha_1 + \beta_1)^n = \beta_1^n + \sigma_n,$$
where, by Lemma 1.4.6,
\[ \sigma_n = \sum_{k=0}^{n-1} \binom{n}{k} \alpha_1^{n-k} \beta_1^k \]
is absolutely continuous.

Since \( \| \alpha_n \| \geq \| \sigma_n \| \) and \( \| \beta_n \| \leq \| \beta_1 \|^n \), we have
\[
\| f - f * \alpha_n \|_\infty = \| f * \mu^{*n} - f * \alpha_n \|_\infty \\
= \| f * \beta_n \|_\infty \\
\leq \| f \|_\infty \| \beta_n \| \\
\leq \| f \|_\infty \| \beta_1 \|^n,
\]
which tends to zero as \( n \) goes to \( \infty \). Each \( f * \alpha_n \) is uniformly continuous by Theorems 19.18 and 20.16 in Hewitt and Ross [49]. It follows that \( f \) is (uniformly) continuous, being a uniform limit of (uniformly) continuous functions.

**Proposition 1.4.8.** Let \( f \) be an essentially bounded \( \mu \)-harmonic function on a group \( G \), and let \( \{R_n\}_{n \in \mathbb{N}} \) be the associated right random walk. Then the sequence \( \{f(gR_n)\}_{n \in \mathbb{N}} \) is a martingale with respect to the filtration \( \mathcal{F}_n = \sigma(R_1, \ldots, R_n) \) for every \( g \in G \).

**Proof.** Let \( n \in \mathbb{N} \). Since \( f \) is \( \mu \)-harmonic, and \( X_{n+1} \) is independent of \( R_1, \ldots, R_n \), we may calculate
\[
\int_H \mathbb{E}(f(gR_{n+1} \mid R_1, \ldots, R_n)) \, d\mu = \int_H f(gR_{n+1}) \, d\mu \\
= \int_H f(gR_nX_{n+1}) \, d\mu \\
= \int_H \int_G f(gR_n(\omega)x) \, d\mu(x) \, d\mu^\mu(\omega) \\
= \int_H f(gR_n) \, d\mu^\mu(\omega)
\]
for all events \( H \in \sigma(X_1, \ldots, X_n) \). The boundedness of \( \mathbb{E}(|f(gR_n)|) \) for each \( n \in \mathbb{N} \) follows from the fact that \( f \) is essentially bounded. We conclude that \( f(gR_n) \) is a martingale.

The Martingale Convergence Theorem, Theorem 1.3.26, implies that \( f(gR_n) \) converges almost surely to a random variable on \((G^\mathbb{N}, \mu^\mu)\) for each \( g \in G \). This function lies in \( L^\infty(G^\mathbb{N}, \mu^\mu) \) because the convergence is pointwise except on a set of measure zero, and \( f \) is essentially bounded.
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For each function $f \in H^\infty_0(G, \mu)$, we define $Z_f : G \rightarrow L^\infty(G^N, \mathbf{P}^\mu)$ to be the map satisfying

$$Z_f(g)(\omega) = \lim_{n \rightarrow \infty} f(g R_n(\omega))$$

(1.5)

for all $g \in G$. Since $f$ is $\mu$-harmonic,

$$\mathbb{E}(Z_f(g)) = \int_{G^N} Z_f(g)(\omega) \, d\mathbf{P}^\mu(\omega)$$

$$= \int_{G^N} \lim_{n \rightarrow \infty} f(g R_n(\omega)) \, d\mathbf{P}^\mu(\omega)$$

$$= \int_G \lim_{n \rightarrow \infty} f(g \tau^n(x)) \, d\mu^\ast n(x)$$

$$= f(g).$$

(1.6)

A Borel measurable function $Z : G \rightarrow L^\infty(G^N, \mathbf{P}^\mu)$ is called Borel invariant if

$$Z(g) = Z(g X_1 \circ T)$$

almost surely, where $T$ is the left shift map on $G^N$:

$$T(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots).$$

The map $Z_f$ is Borel invariant, because

$$Z_f(g)(\omega) = \left( \lim_{n \rightarrow \infty} f(g R_n) \right)(\omega) = \lim_{n \rightarrow \infty} f(g X_1(\omega) R_n(T(\omega))).$$

Let $I^\infty(G, \mu)$ be the set of all bounded Borel invariant maps from $G$ to $L^\infty(G^N, \mathbf{P}^\mu)$. The set $I^\infty(G, \mu)$ is closed under pointwise multiplication and addition. Since $\mu$ is a probability measure, it follows from Hölder’s inequality and Equation (1.6) that $f \mapsto Z_f$ is injective. It is an isometry because

$$\|f\|_\infty = \sup_{g \in G} \{|f(g)|\}$$

$$= \sup_{g \in G} \{\mathbb{E}(Z_f(g))\}$$

$$= \sup_{g \in G} \left\{ \int Z_f(g)(\omega) \, d\mathbf{P}^\mu(\omega) \right\}$$

$$\leq \sup_{g \in G} \left\{ \int |Z_f(g)(\omega)| \, d\mathbf{P}^\mu(\omega) \right\}$$

$$= \sup_{g \in G} \{\|Z_f(g)\|_1\}$$

$$\leq \sup_{g \in G} \{\|Z_f(g)\|_\infty\} = \|Z_f(g)\|_\infty$$
and, using the definition of $Z_f$ in Equation (1.5),

$$
\|Z_f(g)\|_{\infty} = \sup_{\omega \in G^N} |Z_f(g)(\omega)|
= \sup_{\omega \in G} \lim_{n \to \infty} f(g R_n(\omega))
\leq \lim_{n \to \infty} \sup_{\omega \in G} \|f(g R_n(\omega))\| = \|f\|_{\infty}.
$$

Given a bounded Borel invariant map $Z$, the function $f$ satisfying

$$f(g) = \mathbb{E}(Z(g))$$

$$= \int_{G^N} Z(g)(\omega) \, d\mathbb{P}^\mu(\omega)$$

$$= \int_{G^N} Z(g x)(\omega) \, d\mathbb{P}^\mu(\omega)$$

$$= \int_G \int_{G^N} Z(g x)(\omega) \, d\mathbb{P}^\mu \, d\mu(x)$$

$$= \int_G \mathbb{E}(Z(g x)) \, d\mu(x)$$

$$= \int_G f(g x) \, d\mu(x)$$

for all $g \in G$ is essentially bounded and $\mu$-harmonic. This means that $f \mapsto Z_f$ is a bijective isometry.

A map $Z : G \to L^\infty(G^N, \mathbb{P}^\mu)$ is left uniformly continuous if, given any $\varepsilon > 0$, there is an open neighbourhood $U$ of the identity in $G$ such that

$$\sup_{g \in G} \|Z(ug) - Z(g)\| < \varepsilon$$

for every $u \in U$. Let $I^\infty_{luc}(G, \mu)$ be the set of all bounded Borel invariant maps from $G$ to $L^\infty(G^N, \mathbb{P}^\mu)$. If $Z$ is a Borel invariant left uniformly continuous function and $f(g) = \mathbb{E}(Z(g))$, then

$$\sup_{g \in G} |f(ug) - f(g)| = \sup_{g \in G} |\mathbb{E}(Z(ug)) - \mathbb{E}(Z(g))|$$

$$\leq \sup_{g \in G} \|Z(ug) - Z(g)\|_1$$

$$\leq \sup_{g \in G} \|Z(ug) - Z(g)\|_{\infty},$$

and so $f$ is left uniformly continuous. If $f$ is left uniformly continuous, then

$$\sup_{g \in G} \|Z_f(ug) - Z_f(g)\|_{\infty} = \sup_{g \in G, \omega \in G} \left| \lim_{n \to \infty} f(ug R_n(\omega)) - f(g R_n(\omega)) \right|$$

$$\leq \sup_{g \in G} \lim_{n \to \infty} \sup_{\omega \in G} |f(ug R_n(\omega)) - f(g R_n(\omega))|$$

$$= \sup_{g \in G} |f(ug) - f(g)|,$$
and so $Z_f$ is left uniformly continuous. Thus, $f \mapsto Z_f$ is an isometric isomorphism between $H_{luc}^\infty(G, \mu)$ and $I_{luc}^\infty(G, \mu)$.

**Lemma 1.4.9.** The set $I_{luc}^\infty(G, \mu)$ is closed under pointwise multiplication of elements.

**Proof.** Let $Z_1$ and $Z_2$ be elements of $I_{luc}^\infty(G, \mu)$. Then, for all $u$ and $g \in G$,

$$
\|Z_1 \cdot Z_2(ug) - Z_1 \cdot Z_2(g)\|_\infty = \|Z_1(ug)Z_2(ug) - Z_1(ug)Z_2(ug)\|_\infty \\
\leq \|Z_1(ug)Z_2(ug) - Z_1(ug)Z_2(g)\|_\infty + \|Z_1(ug)Z_2(g) - Z_1(g)Z_2(g)\|_\infty \\
\leq \|Z_1(ug)\|_\infty \|Z_2(ug) - Z_2(g)\|_\infty + \|Z_2(g)\|_\infty \|Z_1(ug)(g) - Z_1(g)\|_\infty,
$$

which implies that $Z_1 \cdot Z_2$ is left uniformly continuous. □

We now equip $H_{luc}^\infty(G, \mu)$ with addition, multiplication and involution so that it becomes a commutative unital $C^*$-algebra with respect to the supremum norm. We take addition to be pointwise, and the involution to be pointwise complex conjugation:

$$\bar{f}(g) = E(Z_f(g)).$$

By Remark 1.4.5, $H_{luc}^\infty(G, \mu)$ is not closed under pointwise multiplication. We choose the product

$$(f \times g)(x) = E(Z_f(x)Z_g(x))$$

for all $f, g \in H_{luc}^\infty(G, \mu)$ and $x \in G$. The product is well-defined by Lemma 1.4.9, and is the pointwise limit of $(f \cdot g) * \mu^n$ as $n$ tends to infinity because

$$E(Z_f(x)Z_g(x)) = \int_{G^n} \lim_{n \to \infty} (f \cdot g)(xR_n(\omega)) dP^\mu(\omega)$$

$$= \lim_{n \to \infty} \int_G (f \cdot g)(xy) d\mu^n(y)$$

$$= \lim_{n \to \infty} ((f \cdot g) * \mu^n)(x).$$

With this multiplication, our $*$-algebra is certainly unital and commutative. The norm is sub-multiplicative because

$$\|f \times g\|_\infty = \sup_{x \in G} |(f \times g)(x)|$$

$$\leq \sup_{x \in G} |E(Z_f(x)Z_g(x))|$$

$$\leq \sup_{x \in G} \{\|Z_f(x)\|_\infty \cdot \|Z_g(x)\|_\infty\}$$

$$\leq \sup_{x \in G} \left\{\|Z_f(x)\|_\infty \cdot \sup_{x \in G} \|Z_g(x)\|_\infty\right\}$$

$$= \|f\|_\infty \cdot \|g\|_\infty.$$
We use the Cauchy-Schwartz inequality to see that, for every \( x \in G \),
\[
|f \ast \mu(x)|^2 = \left| \int_G f(xy) \, d\mu(x) \right|^2 \\
\leq \left( \int_G |f(xy)| \, d\mu(x) \right)^2 \\
\leq \int_G |f(xy)|^2 \, d\mu(x).
\]
Using this inequality, and taking the supremum yields
\[
\|f\|_\infty^2 \leq \|f \ast \mu\| \\
= \sup_{x \in G} \int_G |f(xy)|^2 \, d\mu(y) \\
\leq \|f\|_\infty^2.
\]
In particular, we can show by induction on \( n \) that
\[
\|f\|_\infty^2 = \|f \ast \mu^n\|
\]
for all \( n \in \mathbb{N} \). This gives the \( C^* \) identity, since
\[
\|f \times \tilde{f}\|_\infty = \left\| \lim_{n \to \infty} (f \cdot \tilde{f}) \ast \mu^n \right\| \\
= \lim_{n \to \infty} \|f \ast \mu^n\| \\
= \|f\|_\infty^2.
\]
We have shown that \( H_{\text{loc}}^\infty(G, \mu) \) is a commutative unital \( C^* \)-algebra with respect to the supremum norm.

§ 1.4.10 Recurrence, transience and product measures

**Lemma 1.4.11.** Let \((G, \mu)\) be a random walk, and let \( R \) be a measurable subset of \( G \). Then the following are equivalent:

(i) The walk eventually returns to \( R \) with probability 1, i.e.
\[
\Pr(\{\omega \in G^\infty : \forall N \in \mathbb{N} \exists n > N \text{ s.t. } R_n(\omega) \in R\}) = 1.
\]

(ii) The expected number of visits to \( R \) is infinite, i.e.
\[
\sum_{n=0}^{\infty} \Pr(\{\omega \in G^\infty : R_n(\omega) \in R\}) = \sum_{n=0}^{\infty} \mu^\ast_n(R) = +\infty.
\]
Proof. As the walk is time homogeneous and has the Markov property, the probability of returning \( k + 1 \) times is 1 if the probability of returning \( k \) times is 1. It follows by induction that, if the probability of a walk eventually returning to \( R \) is 1, then the expected number of visits to \( R \) is infinite.

Conversely, suppose that the expected number of visits to \( R \) is infinite, and that the probability \( p \) of returning to \( R \) at least once is less than 1. Then the probability of returning at least \( n \) times is \( p^n \). So the expected number of returns to the \( R \) is

\[
\sum_{i=1}^{\infty} i(p^i - p^{i+1}) = \sum_{i=1}^{\infty} p^i = \frac{p}{1 - p},
\]

which contradicts our assumption. So \( p \) must be 1.

We say that a measurable subset \( R \in G \) is a **recurrent set** if it satisfies either of the conditions from the previous lemma, otherwise we call \( R \) a **transient set**. If \( G \) is discrete, we say that the random walk \( (G, \mu) \) is **recurrent** if the singleton set containing the identity is a recurrent set, otherwise we call \( (G, \mu) \) **transient**. If \( G \) is any locally compact group, then we say that \( (G, \mu) \) is **recurrent** if every compact neighbourhood \( U \) of the identity \( e \) is recurrent, otherwise we say it is transient.

**Proposition 1.4.12.** Let \( G \) be a finite group, and let \( \mu \) be a probability measure satisfying \( \text{sgr} \mu = G \). Let \( S \) be a non-empty subset of \( G \). Then \( S \) is recurrent.

**Proof.** Since \( G \) is finite, it must have at least one recurrent element \( x \), which we write as a finite product of (not necessarily distinct) elements \( a_1, \ldots, a_n \in \text{supp} \mu \):

\[
x = a_1 \ldots a_n.
\]

Since \( G \) is finite, \( \mu \) is discrete, and each \( a_i \) must have some finite order \( o_i \). Moreover, there is a \( K > 0 \) satisfying

\[
\mu(a_1)^{o_1} \ldots \mu(a_n)^{o_n} = K.
\]

The probability of transitioning from \( x \) to the identity in \( o_1 + \cdots + o_n \) steps is then greater than or equal to \( K \). Since \( X_n = x \) for infinitely many \( n \), we must have \( X_n = e \) for infinitely many \( n \).

If \( S \) is a non-empty subset of \( G \), and \( y \in S \), then we can choose, not necessarily distinct, elements \( b_1, \ldots, b_n \in \text{supp} \mu \) satisfying

\[
x = b_1 \ldots b_n.
\]

The probability of transitioning from \( e \) to \( y \) is then finite, and so \( \{y\} \), and hence \( S \), is recurrent. \( \square \)
Corollary 1.4.13. Let $G$ be a compact group, and let $\mu$ be a spread-out probability measure satisfying $\text{sgv} \mu = G$. Let $S$ be any non-empty subset of $G$ with positive measure. Then $S$ is recurrent.

Proof. This result is a corollary of the Kawada-Ito Theorem [50, 63], which states that the sequence of measures $\mu^n$ is weak*-convergent to the Haar measure $\lambda$ on $G$. In particular, if $G$ is totally disconnected and if $V$ is a neighbourhood of the identity, then there is a compact open subgroup $U$ contained in $V$ with finite index in $G$. The expected number of visits to $U$ is infinite because $\lim \inf_{n \to \infty} \mu^n(U) \geq \lambda(U) \geq K$ for some positive constant $K$.

Example 1.4.14. Consider the simple random walk over the integers with the standard symmetric generating set $\{1, -1\}$. Here, the group $G$ is the additive group of integers with the discrete topology, and

$$
\mu = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1}
$$

is the probability measure corresponding to the simple random walk. A walk on this group can only return to the origin in an even number of steps. For a walk of length $2n$, there are $2^{2n}$ possible paths, and the probability of choosing 1 and $-1$ a total of $n$ times each is $\binom{2n}{n}$. The probability of being at the origin after $2n$ steps is, therefore, $\frac{1}{4^n} \binom{2n}{n}$. Stirling’s formula [83] states that $n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n$, and so the probability of being at zero after $2n$ steps is asymptotically equal to $C_1 n^{-\frac{1}{2}}$ for some positive real constant $C_1$. The sum $C_1 \sum_{i=1}^{\infty} i^{-\frac{1}{2}}$ gives the expected number of visits to the origin, which is infinite, so the walk is recurrent.

The simple random walk on $\mathbb{Z}^d$ has a probability of return to the identity which is asymptotically equal to $C_d n^{-d/2}$ for some positive real constant $C_d$ (see Woess [106] or Pólya’s [80] original paper). Therefore, this random walk is recurrent if $d$ is 1 or 2, and transient otherwise.

Kesten’s conjecture states that the only finitely generated groups which carry a recurrent random walk are those which have a finite index subgroup isomorphic to $\{e\}, \mathbb{Z}$ or $\mathbb{Z}^2$. The conjecture emerged from questions in Kesten’s Ph.D. thesis [88]. The result was first proved by Varopoulos [97].

The following proposition is true in the case where $G$ is only required to be locally compact instead of discrete. We give a proof for the more general case in Lemma 1.4.34

Proposition 1.4.15. Let $(G, \mu)$ be a recurrent, non-degenerate random walk on a discrete group $G$. Then the corresponding space of essentially bounded $\mu$-harmonic functions is trivial.
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**Proof.** Let $R_n$ be the right random walk corresponding to $(G, \mu)$. Let $g, h \in G$. Let $f$ be any essentially bounded $\mu$-harmonic function. Recall that $f(R_n)$ is a bounded martingale, so it is convergent $\mathbb{P}^{\mu}$-almost everywhere to a bounded measurable function. Since the walk is recurrent, $R_n = g$ and $R_m = h$ for infinitely many $n, m \in \mathbb{N}$. Without loss of generality, we may suppose that $f(g) \leq f(h)$. Then

$$\lim \inf_{n \to \infty} f(R_n) \leq f(g) \leq f(h) \leq \lim \sup_{n \to \infty} f(R_n)$$

$\mu$-almost surely. But $f(R_n)$ is convergent almost everywhere, so

$$\lim \inf_{n \to \infty} f(R_n) = \lim \sup_{n \to \infty} f(R_n).$$

It follows that $f$ is constant $\mu$-almost everywhere. \qed

Given a measure $\nu$ on $G$ and a measurable subset $E$ of $G$, let $\nu|_E(B) = \nu(B \cap E) = \int_B \chi_E d\mu$ be the restriction of $\nu$ to $E$. It is easily verified that the restriction of $\nu$ to $E$ is a measure.

**Lemma 1.4.16.** Let $(G, \mu)$ be a random walk and let $G_0$ be a measurable subgroup. Define the sequence of measures

$$\mu^{(0)} = \mu$$

$$\mu^{(k+1)} = \mu|_{G_0} + (\mu|_{G \setminus G_0}) * \mu.$$

Then each $\mu^{(k)}$ is a probability measure.

**Proof.** We proceed by induction. Suppose that $\mu^{(i)}$ is a probability measure. Then $\mu^{(i+1)}$ is a probability measure because

$$\mu^{(i+1)}(G) = \mu|_{G_0}^{(i)}(G) + \left(\mu^{(i)} - \mu|_{G_0}^{(i)}\right) * \mu(G)$$

$$= \mu|_{G_0}^{(i)}(G) + \int_G \mu^{(i)}(Gy^{-1}) d\mu(y) - \int_G \mu|_{G_0}^{(i)}(Gy^{-1}) d\mu(y)$$

$$= \mu|_{G_0}^{(i)}(G) + \mu^{(i)}(G) \int_G d\mu(y) - \mu|_{G_0}^{(i)}(G) \int_G d\mu(y)$$

$$= 1.$$

Since $\mu^{(0)}$ is a probability measure, $\mu^{(k)}$ is a probability measure for every $k \in \mathbb{N}$. \qed

As $G_0$ is a recurrent set, $\lim_{k \to \infty} \mu^{(k)}(G_0) = 1$ and $\lim_{k \to \infty} \mu^{(k)}(G \setminus G_0) = 0$. It follows that the sequence $\mu^{(k)}$ is Cauchy in total variation. Since the space of all probability measures on $G$ is a closed subspace of the Banach space $M(G)$, it is complete and so $\mu^{(k)}$ has a limit. This limit
is a probability measure \( \mu_0 \), called the hitting measure on \( G_0 \). Since \( \mu_0 \) is a probability measure, \( (G_0, \mu_0) \) is a random walk.

Suppose that \( (X, \mu_X) \) and \( (F, \mu_F) \) are random walks, and that \( X \) has a recurrent subgroup \( G_0 \). Let \( (\mu_X)_0 \) be the hitting measure on \( G_0 \). If \( G = X \ltimes F \) and \( \mu = \mu_X \times \mu_Y \), then \( G_0 \ltimes F \) is recurrent in \( (\mu, \mu) \), and the hitting measure \( \mu_0 \) is the product measure \( \mu_{G_0} \times \mu_F \).

**Lemma 1.4.17** (Furstenberg [42]). Let \( G_0 \) be a measurable subgroup which is a recurrent set. Let \( f \) be a \( \mu \)-harmonic function, and let \( f_0 \) be the restriction of \( f \) to \( G_0 \). Let \( \phi : H^\infty(G) \to H^\infty(G_0) \) be the function satisfying \( \phi(f) = f_0 \). Then \( f_0 \) is a \( \mu_0 \)-harmonic function, and \( \phi \) is bijective.

### § 1.4.18 Topological boundaries

If \( X \) is a topological space, then the pair \( (X, \cdot) \) is a \( G \)-space if \( G \) acts on \( X \) and the map \( (g, f) \mapsto g \cdot f \) is continuous with respect to the product topology on \( G \times X \). Given a compact \( G \)-space \( B \), the action of \( G \) on \( B \) may be extended to \( C(B) \) via the action

\[
g \cdot f(x) = f(g^{-1}x)
\]

for all elements \( g \in G \) and \( x \in B \). We view \( C(B) \) as a \( C^* \)-algebra under pointwise multiplication and addition, with complex conjugation as the involution.

Every probability measure \( \nu \) on \( B \) may be acted upon by a group element as follows

\[
g \cdot \nu(E) = \nu(g^{-1}E).
\]

Given Borel probability measures \( \mu \) on a group \( G \) and \( \nu \) on a \( G \)-space \( B \), the convolution of \( \mu \) and \( \nu \) is defined to be the probability measure \( \mu * \nu \) on \( B \) satisfying

\[
\int_B \varphi(b) \, d\mu * \nu(b) = \int_G \int_B \varphi(gb) \, d\nu(b) \, d\mu(g)
\]

for all continuous functions \( \varphi \) with compact support on \( B \). The measure \( \nu \) is called \( \mu \)-stationary if \( \mu * \nu = \nu \).

A \( G \)-space \( B \) equipped with a \( \mu \)-stationary measure \( \nu \) is called a \( (G, \mu) \)-space. Every \( G \)-space has a probability measure \( \nu \) that is \( \mu \)-stationary. This is because the space of probability measures on \( B \) forms a real Banach space that is isomorphic to a compact, convex subset of \( C(B)^* \), so that the linear map \( \nu \mapsto \nu * \mu \) has a fixed point by the Markov–Kakutani Theorem [53].

Suppose that \( (B, \nu) \) is a \( (G, \mu) \)-space and that \( B \) is compact. Let \( P_\nu \) be the Poisson transform from \( C(B) \) to \( H^\infty_{loc}(G, \mu) \) satisfying

\[
P_\nu(\varphi)(g) = \int_B \varphi(gb) \, d\nu(b)
\]
for each \( g \in G \). The function \( P_\nu(\varphi) \) is \( \mu \)-harmonic on \( G \) for every \( \varphi \in L^\infty(B, \nu) \) as, for each \( g \in G \),

\[
P_\nu(\varphi)(g) = \int_B \varphi(gb) \, d\nu(b) \\
= \int_B \varphi(gb) \, d\mu \ast \nu(b) \\
= \int_B \int_G \varphi(ghb) \, d\mu(h) \, d\nu(b) \\
= \int_G \int_B \varphi(ghb) \, d\nu(b) \, d\mu(h) \\
= \int_G P_\nu(\varphi)(gh) \, d\mu(h).
\]

Moreover, \( P_\nu(\varphi) \) is essentially bounded for each \( \varphi \in C(B) \) because the Poisson transform is norm-decreasing. In particular,

\[
\|P_\nu(\varphi)\|_\infty = \sup_{g \in G} \left\{ \left| \int_B \varphi(gb) \, d\nu(b) \right| \right\} \\
\leq \sup_{g \in G} \left\{ \int_B |\varphi(gb)| \, d\nu(b) \right\} \\
= \|\varphi\|_\infty
\]

for all \( \varphi \in C(B) \). Let \( x \in G \), and let \( \varphi \in C(B) \). Then

\[
\sup_{g \in G} \left| P_\nu(\varphi)(xg) - P_\nu(\varphi)(g) \right| = \sup_{g \in G} \left| \int_B \varphi(xgb) - \varphi(gb) \, d\nu \right| \\
= \sup_{g \in G} \left| \varphi(xg) - \varphi(g) \right|.
\]

Therefore, by Proposition 3.6 in van Dijk [96], \( P_\nu(\varphi) \) is left uniformly continuous because \( \varphi \) is continuous and compactly supported.

If the Poisson transform \( P_\nu \) on a \((G, \mu)\)-space \((B, \nu)\) is an isometric embedding, then we say that \((B, \nu)\) is a contractible \((G, \mu)\)-space.

**Proposition 1.4.19** (Babillot [4]). Let \((B, \nu)\) be a compact \((G, \mu)\)-space. Then \((B, \nu)\) is contractible if and only if \( \delta_B \) is contained in the weak* closure of the set \( \{ \delta_g \ast \nu : g \in G \} \). In particular, if the support of \( \nu \) is \( B \), then \((B, \nu)\) is contractible.

**Lemma 1.4.20** (Babillot [4]). Suppose that \((B, \nu)\) is a second countable compact \((G, \mu)\)-space. Then the sequence of probability measures \( R_n(\omega) \cdot \nu \) is weak*-convergent to a measure \( \mathcal{V}_\omega \) for \( \mathbb{P}_\mu \)-almost every \( \omega \in G^\mathbb{N} \). Furthermore,

\[
\int_{G^\mathbb{N}} \int_B \varphi(gb) \, d\mathcal{V}_\omega(b) \, d\mathbb{P}_\mu(\omega) = \int_B \varphi(gb) \, d\nu(b)
\]

for all \( \varphi \in C(B) \).
Let $B$ be a second countable $G$-space and $(B, \nu)$ be a $(G, \mu)$-space. Then $(B, \nu)$ is said to be a \textit{\mu-boundary} if there exists a random variable $\text{bnd} : G^\mathbb{N} \rightarrow B$, called the \textit{boundary map}, such that $R_n(\omega) \cdot \nu$ converges in the weak* topology to a point measure $\delta_{\text{bnd}(\omega)}$ for almost every $\omega \in G^\mathbb{N}$.

**Proposition 1.4.21.** Suppose that $(B, \nu)$ is a $\mu$-boundary, and that $\text{bnd}$ is the boundary map. Then $\nu$ is the distribution of the random variable $\text{bnd}$.

Recall that, whenever $(B, \nu)$ is a second countable $(G, \mu)$-space, we may view $C(B)$ as a commutative unital $C^*$-algebra with respect to pointwise addition and multiplication, when endowed with the complex conjugation norm. Moreover, $H^\infty_{\text{loc}}(G, \mu)$ is also a commutative unital $C^*$-algebra with respect to the multiplication defined on page 36.

**Proposition 1.4.22** (Babillot [4]). A second countable $(G, \mu)$-space $(B, \nu)$ is a $\mu$-boundary if and only if Poisson transformation $P_\nu$ is a $*$-homomorphism.

A proof of Proposition 1.4.23 (below) may be found in Kaimanovich [58] or Proposition 2.1 in Brofferio and Schapira [14].

**Proposition 1.4.23.** Let $B$ be a compact, separable $G$-space and $\nu$ be a $\mu$-stationary probability measure on $B$. Suppose that $(B, \nu)$ is a $(G, \mu)$-space. Let $U$ be the map satisfying

$$U(\omega)_i = \omega_1^{-1}\omega_i$$

for all $\omega \in G^\mathbb{N}$. Suppose that there exists a random variable $z : G^\mathbb{N} \rightarrow B$, such that

$$X_1(\omega) \cdot z(U\omega) = z(\omega)$$

for almost all $\omega \in G^\mathbb{N}$. Then $(B, \nu)$ is a $\mu$-boundary and $z$ is the associated boundary map.

§ 1.4.24 The Poisson boundary

A continuous map $\gamma$ from a $G$-space $B$ to a $G$-space $B'$ is \textit{equivariant} if $\gamma(g \cdot b) = g \cdot \gamma(b)$ for all $b \in B$ and $g \in G$. If $(B, \nu)$ and $(\bar{B}, \bar{\nu})$ are $\mu$-boundaries of the random walk $(G, \nu)$, then $(\bar{B}, \bar{\nu})$ is an \textit{equivariant image} of $(B, \nu)$ if there exists an equivariant map $\gamma : B \rightarrow B'$ for which the pushforward measure $\gamma_*\nu$ is equal to $\bar{\nu}$.

Given a random walk $(G, \mu)$, there is a $\mu$-boundary $(\Pi_\mu, \nu)$, defined in terms of the $C^*$-algebra of $\mu$-harmonic functions, called the \textit{topological Poisson boundary}, for which the Poisson map $P_\nu$ is an isometric $*$-isomorphism, and every other $\mu$-boundary $(B, \eta)$ is an equivariant image of $(\Pi_\mu, \nu)$. 
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The spectrum $\sigma(A)$ of a commutative unital Banach algebra $A$ is the set of all multiplicative complex-valued functionals on $A$. It is a closed subset of the closed unit ball $B \in A^*$ in the weak* topology. So, by Alaoglu’s Theorem, $\sigma(A)$ is a compact Hausdorff space. If $x \in A$, define $\hat{x}$ to be the function on $\sigma(A)$ by

$$\hat{x}(h) = h(x).$$

for each $h \in \sigma(A)$. The Gelfand transform $\Gamma$ is the map $x \mapsto \hat{x}$ on $A$. The Gelfand-Naimark Theorem states that if $A$ is a commutative unital Banach algebra, then the Gelfand transform is an isometric *-isomorphism from $A$ onto $C(\sigma(A))$. The Poisson space $\Pi_\mu$ corresponding to $\mu$ is the spectrum of the commutative $C^*$-algebra $H^\infty_{luc}(G, \mu)$. See Chapter 1 in Folland [39] for more information about the Gelfand transform and the Gelfand–Naimark Theorem.

**Theorem 1.4.25** (Furstenberg [42]). Let $\mu$ be a Borel probability measure on $G$. Let $\Pi_\mu$ be the Poisson space. Then there exists a $\mu$-stationary measure $\nu$ on $\Pi_\mu$, called the Poisson kernel such that $(\Pi_\mu, \nu)$ is a $\mu$-boundary, and the Poisson transform, $P_\nu : C(\Pi_\mu) \to H^\infty_{luc}(G, \mu)$ is an isometric *-isomorphism. In particular, every left uniformly continuous bounded $\mu$-harmonic function $f$ admits a Poisson representation

$$f(g) = \int_{\Pi_\mu} \hat{f}(gb) \ d\nu(b),$$

where $\hat{f}$ is an essentially bounded left uniformly continuous $\nu$-measurable function on $\Pi_\mu$. The topological Poisson boundary corresponding to the random walk $(G, \mu)$ is the pair $(\Pi_\mu, \nu)$.

The $C^*$-algebras $C(X)$ and $C(Y)$ are *-isomorphic if and only if $X$ and $Y$ are homeomorphic (see II.2.6 in Blackadar [9]). Consequently, any compact $(G, \mu)$-space $(B, \eta)$ for which the Poisson map is an isometry from $C(B)$ onto $H^\infty_{luc}(G, \mu)$ is homeomorphic to $\Pi_\mu$.

**Lemma 1.4.26** (Babillot [4]). Suppose that $\text{sgr} \mu = G$, and let $(\Pi_\mu, \nu)$ be the Poisson boundary of $(G, \mu)$. Then $\text{supp} \nu = \Pi_\mu$.

**Proposition 1.4.27.** Suppose that $\text{sgr} \mu = G$. Let $(B, \nu)$ be a compact $(G, \mu)$-space. Then $\text{supp} \nu$ is an invariant set under the action of $G$, i.e. $g \text{supp} \nu = \text{supp} \nu$ for all $g \in G$.

**Theorem 1.4.28** (Furstenberg [42]). Assume that the closed semi-group generated by the support of $\mu$ is all of $G$, and let $(B, \eta)$ be a $\mu$-boundary such that $\text{supp} \eta = B$. Suppose that $(\Pi_\mu, \nu)$ is the Poisson boundary of $(G, \mu)$. Then there exists a continuous, surjective, $G$-equivariant map $\gamma : \Pi_\mu \to B$ such that $\gamma_* \nu = \eta$. The map $\gamma$ is uniquely defined, up to sets of measure zero.
§ 1.4.29 Measurable boundaries

If $G$ is a topological group with an action $\cdot$ on a measure space $X$, then the pair $(X, \cdot)$ is a measurable $G$-space if $G$ acts on $X$ and $g \mapsto g \cdot f$ is measurable with respect to the product $\sigma$-algebra on $G \times X$. Here we endow $G$ with the Borel $\sigma$-algebra.

A pair $(B, \nu)$, where $B$ is a measurable $G$-space and $\nu$ is a $\mu$-stationary measure, forms a weak-measurable $(G, \mu)$-space. If $(B, \nu)$ is a weak-measurable $(G, \mu)$-space, then the Poisson transform $P_{\nu} : L^\infty(B, \nu) \to H^\infty(G, \mu)$ is given by

$$P_{\nu}(\varphi)(g) = \int_B \varphi(g \cdot b) \, d\nu(b)$$

for each $g \in G$ and each $\varphi \in L^\infty(B, \nu)$. The argument from Subsection 1.4.18 shows that this Poisson transform is norm decreasing, and maps essentially bounded functions in $L^\infty(B, \nu)$ to harmonic functions in $H^\infty(G, \mu)$.

If $(B, \nu)$ is a weak-measurable $(G, \mu)$-space and there exists a random variable \(\text{bnd} : G^\infty \to (B, \nu)\) called the boundary map, satisfying

$$\lim_{n \to \infty} P_{\nu}(\varphi)(gR_n(\omega)) = \lim_{n \to \infty} \int_B \varphi(gR_n(\omega) \cdot b) \, d\nu(b) = \varphi(g \cdot \text{bnd}(\omega))$$

for $P^\mu_n$-almost every $(g, \omega) \in G \times G^\infty$ and all $\varphi \in L^\infty(B, \nu)$, then $(B, \nu)$ is a measurable $\mu$-boundary.

**Proposition 1.4.30.** Suppose that $(B, \nu)$ is a measurable $\mu$-boundary, and that $z : G^\infty \to B$ satisfies

$$\lim_{n \to \infty} \int_B \varphi(gR_n(\omega)b) \, d\nu(b) = \varphi(g \cdot z(\omega))$$

for $P^\mu_n$-almost every $(g, \omega) \in G \times G^\infty$ and all $\varphi \in L^\infty(B, \nu)$. Then $\nu$ is the distribution of the random variable $z$.

**Proof.** Let $\varphi \in L^\infty(B, \nu)$. Then, since $\nu$ is $\mu$-stationary,

$$\int_{G^\infty} \varphi(b) \, d\nu^\mu(b) = \int_{G^\infty} \varphi(gz(\omega)) \, d\nu^\mu(\omega) = \int_{G^\infty} \lim_{n \to \infty} \int_B \varphi(R_n(\omega)b) \, d\nu(b) \, d\nu^\mu(\omega) = \int_{G^\infty} \lim_{n \to \infty} \int_B \varphi(xb) \, d\nu(b) \, d\mu^\ast n(x) = \lim_{n \to \infty} \int_B \varphi(x) \, d\nu \ast \mu^\ast n(x) = \int_B \varphi(x) \, d\nu(x).$$

Since $\varphi$ was arbitrary, $z_* \nu^\mu = \nu$, establishing our result.
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§ 1.4.31 The measurable Poisson boundary

Let $\rho$ be a probability measure on $G$ which is equivalent to the Haar measure $\lambda$. Denote by $H^\infty_{\rho}(G, \mu)$ be the space of equivalence classes mod $\rho$ of $\mu$-harmonic functions. Let $P^\mu_{\rho}$ be the measure $\rho \times P^\mu$ on $G \times G^N$, and let $P^\mu_{\rho}$ be the pushforward of $P^\mu_{\rho}$ with respect to the map $S_{\rho} : G \times G^N \to G \times G^N$ given by

$$S_{\rho}(g, \omega_1, \omega_2, \omega_3, \ldots, \omega_k, \ldots) = (g, g\omega_1, g\omega_1\omega_2, g\omega_1\omega_2\omega_3, \ldots, g\omega_1 \ldots \omega_k, \ldots).$$

Let $\{X_i\}_{i \in \mathbb{N}_0}$ be the sequence of random variables from $G \times G^N$ to $G$ given by the projections

$$X_i(g, \omega) = \begin{cases} 
\omega_i & \text{if } i > 0 \\
g & \text{if } i = 0
\end{cases}.$$

Also, let

$$R_0 = X_0, \quad R_n = R_{n-1}X_n$$

for each non-negative integer $n$.

As in Proposition 1.4.8, every function $f \in H^\infty_{\rho}(G, \mu)$ and element $g \in G$ gives rise to a bounded martingale $f(gR_n)$. As before, the Martingale Convergence Theorem implies an almost sure limit of the process $f(gR_n) \in L^\infty(G^N, P^\mu_{\rho})$ for each $g \in G$. Equivalently, for each function $f \in H^\infty_{\rho}(G, \mu)$, there is a function $Z_f \in L^\infty(G \times G^N, P^\mu_{\rho})$ satisfying

$$Z_f(g, \omega) = \lim_{n \to \infty} f(gR_n(\omega)) \quad (1.7)$$

We now discuss a related space of essentially bounded invariant maps $I^\infty_{\rho}(G, \mu)$. The definitions and properties are similar to those given for $I^\infty(G, \mu)$ in Subsection 1.4.1.

A Borel function $Z \in L^\infty(G \times G^N, P^\mu)$ is invariant if it satisfies

$$Z(g, \omega) = Z(gX_1(\omega), T(\omega))$$

$P^\mu_{\rho}$-almost surely, where $T$ is the shift map $T(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$. The function $Z_f$ is invariant because

$$Z_f(g, \omega) = \lim_{n \to \infty} f(gR_n(\omega)) = \lim_{n \to \infty} f(gX_1(\omega)R_n(T(\omega)))$$

$P^\mu_{\rho}$-almost surely. A modification of the proof given in Subsection 1.4.1 for $I^\infty(G, \mu)$ shows that $f \mapsto Z_f$ is an isometric bijection from $H^\infty_{\rho}(G, \mu)$ to the space of essentially bounded invariant maps $I^\infty_{\rho}(G, \mu)$. 
Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the space of bounded linear operators on $\mathcal{H}$. The weak operator topology is the weakest topology on $\mathcal{B}(\mathcal{H})$ for which the maps $\langle Tx, y \rangle$ are continuous for all vectors $x, y \in \mathcal{H}$. A net $\{T_\alpha\}_{\alpha \in A}$ converges in the weak operator topology to a point $T$ if and only if, for all $x \in \mathcal{H}$ and $y \in \mathcal{H}$, the net $\{\langle T_\alpha x, y \rangle\}$ converges to $\langle Tx, y \rangle$.

A von Neumann algebra is a sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$ that is closed with respect to the weak operator topology and contains an identity. If $(X, \eta)$ is a measure space such that $X$ is $\sigma$-finite, then $L^\infty(X, \eta)$, considered as operators acting by pointwise multiplication on the Hilbert space $L^2(X, \eta)$, is a commutative von Neumann algebra. Every commutative von Neumann algebra is isometrically $*$-isomorphic to the $*$-algebra $L^\infty(X, \eta)$ for some measure space $(X, \eta)$. The reader may wish to consult an introductory text such as Blackadar [9] or Dixmier [28] for more information about the theory of von Neumann algebras.

**Proposition 1.4.32** (Babillot [4]). The set $I^\infty_\rho(G, \mu)$ is a closed sub-$*$-algebra of $L^\infty(G \times G^\mathbb{N}, \mathbf{P}_\rho^\mu)$ in the weak operator topology, and hence it is a commutative von Neumann algebra, when regarded as a space of operators acting by pointwise multiplication on $L^2(\Pi^\rho_\mu, \nu^\rho)$. In particular, $I^\infty_\rho(G, \mu)$ is isometrically $*$-isomorphic to $L^\infty(\Pi^\rho_\mu, \nu^\rho)$, where $(\Pi^\rho_\mu, \nu^\rho)$ is a probability space called the measurable Poisson boundary. In this context, $\Pi^\rho_\mu$ is called the Poisson space, and the measure $\nu^\rho$ is the Poisson kernel.

The measurable Poisson boundary is a measurable $\mu$-boundary and, as in the topological case, has a universal property. Namely, for every measurable $\mu$-boundary $(B, \eta)$, there is an equivariant map $\gamma : \Pi_\mu \to B$ satisfying $\gamma \ast \nu = \eta$. In order to prove these facts, we now discuss another equivalent construction of the boundary, due to Kaimanovich and Vershik [60], called the stationary boundary.

**§ 1.4.33 The stationary boundary**

Probability spaces $(\Omega, \mathcal{F}, \nu)$ and $(\Theta, \mathcal{H}, \eta)$ are said to be isomorphic mod 0 if there are measurable sets $F \in \mathcal{F}$ and $H \in \mathcal{H}$ and a bijective measurable and measure-preserving map between the probability spaces $\Omega \setminus F$ and $\Theta \setminus H$, endowed with the restrictions of the measures $\mu$ and $\nu$ and the natural $\sigma$-algebras (see Section 9.2 in Bogachev [11]).

Let $\bar{T} : G \times G^\mathbb{N} \to G \times G^\mathbb{N}$ and $\bar{\theta} : G \times G^\mathbb{N} \to G \times G^\mathbb{N}$ be the maps satisfying

$$ \bar{T}(g, \omega_0, \omega_1, \ldots) = (\omega_0, \omega_1, \ldots) $$

and

$$ \bar{\theta}(g, \omega_0, \omega_1) = (g\omega_0, \omega_1, \omega_2, \ldots), $$
respectively. A function $Z$ is in $I^\infty_\rho(G,\mu)$ if it belongs to $L^\infty(G \times G^\mathbb{N}, \mathbb{P}_\rho^\mu)$ and satisfies $Z = Z \circ \overline{\theta}$.

Thus, $I^\infty_\rho(G,\mu)$ is isomorphic mod 0 to the function space $L^\infty(G \times G^\mathbb{N}, \mathcal{I}_T, \mathbb{P}_\rho^\mu)$, where $\mathcal{I}_T$ is the invariant $\sigma$-algebra of $T$-invariant Borel sets:

$$\mathcal{I}_T = \{A \in \bar{A} : A = T^{-1}(A)\},$$

and $\bar{A}$ is the completion of $A$ with respect to $\mathbb{P}_\rho^\mu$.

**Lemma 1.4.34.** Let $G$ be a locally compact group. Let $\mu$ be a non-degenerate probability measure on $G$, with respect to which the random walk $(G,\mu)$ is recurrent. Then the invariant $\sigma$-algebra of $T$-invariant Borel sets is trivial.

**Proof.** In Proposition 1.4.15, we showed that the essentially bounded $\mu$-harmonic functions associated with a recurrent random walk are trivial. We have already seen that this is equivalent to triviality of the stationary boundary.

An alternate, and more direct proof is possible using the classical Kolmogorov’s 0–1 law, quoted in Theorem 1.3.13. Let $\{R_i\}_{i \in \mathbb{N}}$ be the right random walk with increments $\{X_i\}_{i \in \mathbb{N}}$. Then for $\mathbb{P}_\rho^\mu$-almost every path $\omega$, the set of integers

$$I(\omega) = \{i \in \mathbb{Z} : R_i(\omega) = e\}$$

is countably infinite. Each excursion, $E_{i,j} = R_i(\omega), R_{i+1}(\omega), \ldots, R_j(\omega)$, where $i, j \in I(\omega)$ are consecutive elements, is independent and identically distributed. It then follows from Kolmogorov’s 0–1 law that the tail $\sigma$-algebra is trivial.

The measure space $(G \times G^\mathbb{N}, \mathcal{I}, \mathbb{P}_\rho^\mu)$ does not separate points in general, so it is not a Borel measurable $G$-space. We now construct a more useful quotient space with the same essentially bounded measurable functions.

To do this, define an equivalence relation $\sim$ on $G \times G^\mathbb{N}$ by $x \sim y$ if and only if there are $m, n \in \mathbb{N}$ satisfying $T^m x = T^n y$. Let $\mathcal{S}$ be the quotient $\mathcal{S} = G \times G^\mathbb{N} / \sim$. Note that the action of $G$ on $G \times G^\mathbb{N}$ preserves the equivalence classes of $\mathcal{S}$. Let $\pi : G \times G^\mathbb{N} \to \mathcal{S}$ be the projection which takes each element to its equivalence class. This map is $G$-equivariant with respect to the action, i.e.

$$g \cdot \pi(x) = \pi(g \cdot x)$$

for $\rho$-almost every $g \in G$ and $\rho$-almost every $x \in G \times G^\mathbb{N}$.

Let $\mathcal{F}$ be the $\sigma$-algebra on $\mathcal{S}$ given by

$$\mathcal{F} = \{A \subseteq \mathcal{S} : \pi^{-1}(A) \in \mathcal{I}_T\}.$$
If \( a \in G \times G^N \) and \( x \in \pi(a) \), then \( x \sim a \), which means that \( Tx \sim a \), and that \( Tx \in \pi(a) \).

This implies that there is an \( x \in T^{-1}(\pi(a)) \) such that \( \pi(x) \in \mathcal{I}_T \). Now, \( \mathcal{F} \) separates points in \( S \) because if \( \pi(x) \) and \( \pi(y) \) are distinct elements of \( S \), then \( \{\pi(x)\} \in \mathcal{F} \) contains \( \pi(x) \), but not \( \pi(y) \).

Suppose that \( x \) and \( y \) are distinct points of \( G \times G^N \) satisfying \( x \sim y \). Without loss of generality, we may write \( T^m x = T^k \circ T^n(x) \) for \( m, n \in \mathbb{N} \). Thus, if \( x \) is in some \( \mathcal{I}_T \)-measurable set \( A \), then \( y \in A \). Therefore, every measurable set in \( \mathcal{I}_T \) can be written as the union of disjoint \( \sim \)-equivalence classes. In particular, the sets in \( \mathcal{I}_T \) do not separate \( \sim \)-equivalence classes, so every \( \mathcal{I}_T \)-measurable function is constant on each equivalence class \( \mathbb{P}^\mu \)-almost surely. Hence, given an element of \( L^\infty(G \times G^N, \mathcal{I}_T, \mathbb{P}^\mu) \), there is a unique element in \( L^\infty(S, \mathcal{F}, \alpha_\rho) \), where \( \alpha_\rho \) lies in the image \( \pi_* \mathbb{P}^\mu \).

The space \( (S, \alpha_\rho) \) is the stationary boundary of \( (G, \mu) \). We can write

\[
\alpha_\rho(E) = \int_G \pi_\rho E \mu \mathbb{P}^\mu d\rho(g) = \int_G \pi_\rho \mu(E) \mathbb{P}^\mu d\rho(g),
\]

and since \( \mu \ast \mathbb{P}^\mu = \mathbb{P}^\mu = T_* \pi_* \mathbb{P}^\mu \), which coincides with \( \mathbb{P}^\mu \) on \( \mathcal{F} \),

\[
\alpha_\rho(A) = \int_G g \cdot \pi_\rho \mathbb{P}^\mu(A) d\rho(g)
= \int_G g \cdot \pi_\rho \mu \ast \mathbb{P}^\mu(A) d\rho(g)
= \int_G \int_G \mathbb{P}^\mu \circ \pi^{-1}(g x^{-1} A) \mu(x) d\rho(g)
= \int_G \alpha_\rho(x^{-1} A) d\mu(x)
= \mu \ast \alpha_\rho.
\]

That is, \( \alpha_\rho \) is \( \mu \)-stationary and so \( (S, \alpha_\rho) \) is a weak measurable \( (G, \mu) \)-space. The Poisson transform is an isometry, since

\[
P_{\alpha_\rho}(\varphi)(g) = \int_S \varphi(g \cdot b) d\alpha_\rho(b)
= \int_G \varphi \circ \pi(g \cdot b) d\mathbb{P}^\mu(b)
\]

for every \( \varphi \in L^\infty(S, \alpha_\rho) \).

**Theorem 1.4.35.** The stationary boundary \( (S, \alpha_\rho) \) of a random walk \( (G, \mu) \) is a measurable \( \mu \)-boundary.

**Proof.** Let \( \varphi \in L^\infty(S, \alpha_\rho) \). We will show that the choice \( z(\omega) = \pi \circ S(e, \omega) \), where \( S(g, \omega) = (g, g \omega_1, g \omega_1 \omega_2, \ldots) \), satisfies the definition of a measurable \( \mu \)-boundary. Let \( f \) be the element of
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\[ H^\infty(G, \mu) \text{ which satisfies} \]
\[ f(g) = P_{\alpha_\rho}(\varphi)(g) = \int_B \varphi(g \cdot b) \, d\alpha_\rho(b), \]

and, as in the discussion in Section 1.4.31, let \( Z_f \) be the function in \( L^\infty(G \times G^\mathbb{N}, \mathbb{P}^\mu) \) satisfying

\[ \lim_{n \to \infty} \int_B \varphi(g R_n(\omega) \cdot b) \, d\alpha_\rho(b) = \lim_{n \to \infty} f(g R_n(\omega)) = Z_f(g, \omega) \]

for \( \mathbb{P}^\mu \)-almost every \( (g, \omega) \in G \times G^\mathbb{N} \). It remains to show that \( Z_f(g, \omega) \) is equal \( \mathbb{P}^\mu \)-almost everywhere to \( \varphi(g \cdot z(\omega)) \). But this is because

\[ \int_{G^\mathbb{N}} Z_f(g, \omega) \, d\mathbb{P}^\mu(\omega) = P_{\alpha_\rho}(\varphi)(g) = \int_B \varphi(g \cdot b) \, d\alpha_\rho(b) \]
\[ = \int_{G^\mathbb{N}} \varphi \circ \pi(g \cdot b) \, d\mathbb{P}^\mu(b) \]
\[ = \int_{G^\mathbb{N}} \varphi(g \cdot \pi(b)) \, d\mathbb{P}^\mu(b) \]
\[ = \int_{G^\mathbb{N}} \varphi(g \cdot \pi \circ S(\omega)) \, d\mathbb{P}^\mu(\omega) \]
\[ = \int_{G^\mathbb{N}} \varphi(g \cdot z(\omega)) \, d\mathbb{P}^\mu(\omega) \]

and the map \( f \mapsto Z_f \) is a bijection. \( \square \)

We are now ready to state the universal property.

**Theorem 1.4.36.** Assume that the closed semi-group generated by the support of \( \mu \) is all of \( G \), and let \( (B, \nu) \) be a measurable \( \mu \)-boundary such that \( \text{supp} \nu = B \). Suppose that \( (S, \alpha_\rho) \) is the stationary boundary of \( (G, \mu) \). Then there exists a surjective, \( G \)-equivariant map which maps \( \alpha_\rho \) to \( \nu \).

**Proof.** Let \( z : G \times G^\mathbb{N} \to B \) correspond to \( (B, \nu) \) as per the definition of a measurable \( \mu \)-boundary. Let \( Z : G \times G^\mathbb{N} \to B \) be given by \( Z(g, \omega) = g \cdot z(\omega) \). As before, let \( \theta(g, \omega) = (g X_1(\omega), T(\omega)) \). By definition,

\[ \varphi \circ Z \circ \theta(g, \omega) = \varphi \circ Z(g R_1(\omega), T(\omega)) \]
\[ = \lim_{n \to \infty} \int_B \varphi(g R_1(\omega) R_n \circ T(\omega) \cdot b) \, d\nu(b) \]
\[ = \lim_{n \to \infty} \int_B \varphi(g R_n(\omega) \cdot b) \, d\nu(b) \]
\[ = \varphi \circ Z(g, \omega) \]

for all \( \varphi \in L^\infty(B, \nu) \). Hence, \( Z = Z \circ \theta \) almost everywhere. As in the discussion of the stationary boundary, let \( \pi : G \times G^\mathbb{N} \to S \) be the projection which maps each element to its equivalence class.
Let $\tilde{Z} : (S, \alpha_p) \to (B, \nu)$ satisfy

$$\tilde{Z} \circ \pi(h, y) = Z \circ S^{-1}(h, y)$$

for each $(h, y) \in G \times G^N$. Suppose that $(h, y) \in G \times G^N$. Then $(h, y)$ is $\mathbb{P}^\mu_p$ almost surely equal to $(h, h\omega_0, h\omega_0\omega_1, \ldots)$ for some $\omega \in G^N$. Furthermore, $\tilde{Z}$ is $G$-equivariant, as by definition of $\tilde{Z}$ and $Z$,

$$g \cdot (\tilde{Z} \circ \pi)(h, y) = g \cdot (Z \circ S^{-1})(h, y)$$

$$= g \cdot Z(h, \omega)$$

$$= gh \cdot Z(e, \omega)$$

$$= Z(gh, \omega)$$

$$= Z \circ S^{-1}(gh, gy)$$

$$= \tilde{Z}(gh, gy)$$

for every $g \in g$, where

$$S(g, \omega) = (g, g\omega_1, g\omega_1\omega_2, \ldots)$$

and

$$S^{-1}(g, \omega) = (g, g^{-1}\omega_1, g^{-1}\omega_1\omega_2, g^{-1}\omega_2\omega_3, \ldots)$$

correspond to $(S, \alpha_p)$ as in the definition of a measurable $\mu$-boundary. By Proposition 1.4.30, $\nu$ is the distribution of $z$. Since $\alpha_p$ is the distribution of $z'$, it is sufficient to show that $z = \tilde{Z}(z')$. This is the case because

$$\tilde{Z}(z'(\omega)) = \tilde{Z} \circ \pi(e, \{ R_i(\omega) \}_{i \in \mathbb{N}}) = Z \circ S^{-1}(e, \{ R_i(\omega) \}_{i \in \mathbb{N}}) = Z(e, \omega) = z(\omega)$$

for every path $\omega$. \qed

§ 1.4.37 Group structure, geometry and the Poisson boundary

This section surveys some known results about Poisson boundaries of random walks and some techniques used to prove them.

For some groups, the measurable Poisson boundary is trivial, consisting of only a single point for every non-degenerate choice of probability measure. This is the case for nilpotent groups because the only essentially bounded harmonic functions are constant.

A group $G$ is polycyclic if it admits a series of subgroups $G = G_n \triangleright \cdots \triangleright G_2 \triangleright G_1 \triangleright G_0 = \{0\}$ for which all of the factor groups $G_{i+1}/G_i$ are cyclic. If $G$ is polycyclic and $\mu$ is a symmetric
probability measure on $G$ with finite first moment, then the measurable Poisson boundary is trivial [56]. Rosenblatt [84], and later Kaimanovich and Vershik [60] showed that if $\supp \mu$ generates a non-amenable group, then the measurable Poisson boundary is non-trivial.

Kaimanovich and Vershik [60] gave examples of solvable, hence amenable, groups with non-degenerate probability measures for which the boundary is non-trivial. Rosenblatt [84], and later Kaimanovich and Vershik [60], showed that every countable amenable group has a non-degenerate symmetric probability measure with respect to which the boundary of the corresponding random walk is trivial.

Erschler [34] showed that if $G$ is a finitely generated solvable group, then it admits a symmetric measure with non-trivial measurable Poisson boundary if and only if the group is not virtually nilpotent. Gromov’s Theorem [44] then gives that if $G$ is a finitely generated solvable group, it admits a symmetric measure with non-trivial measurable Poisson boundary if and only if the group does not have polynomial growth.

The work of Dynkin and Malyutov [31] allowed the computation of the boundary for the simple random walk on the free group with two (or more) generators. Furstenberg showed that there is a measure $\mu$ on the free group $F_\infty$, with a countable set of generators, such that $(F_\infty, \mu)$ has a measurable Poisson boundary which may be identified with the measurable Poisson boundary of the simple random walk on the free group with two generators. The argument is based on the fact that $F_\infty$ may be identified with the commutator subgroup of $F_2$ and that random walks on $\mathbb{Z}^2$ are recurrent.

Furstenberg [42] investigated random walks on the non-amenable special linear groups $\text{SL}_2(\mathbb{R})$ and $\text{SL}_3(\mathbb{R})$ as well as discrete subgroups of $\text{SL}_r(\mathbb{R})$. He was able to describe the boundary for discrete subgroups of semi-simple lie groups for a particular class of measures. Ledrappier [69] was able to describe the boundary on discrete subgroups of $\text{SL}_d(\mathbb{R})$ for a wider class of measures.

Élie [33] computed the measurable Poisson boundary of the real affine group. The real affine group is amenable.

Kaimanovich [60] considered the symmetric group $S_\infty$ of finite permutations of a countable set. The support of any finitary measure on $S_\infty$ generates a finite group, so the boundary is trivial for those measures. He showed there exists a probability measure of finite entropy on $S_\infty$, for which the boundary is non-trivial.

Let $\mathbb{Z}[1/2]$ denote the dyadic rationals. The affine group $\text{Aff}(\mathbb{Z}[1/2])$ over the dyadic rationals
is the matrix group

$$\text{Aff}(\mathbb{Z}[1/2]) = \left\{ \begin{pmatrix} 2^k & m \\frac{m}{2^n} \\ 0 & 1 \end{pmatrix} : k, m, n \in \mathbb{Z} \right\}. $$

This group is amenable, solvable, not nilpotent and isomorphic to the Baumslag–Solitar group BS(1, 2). Let \( \mu \) be a probability measure with finite first moment such that the group generated by \( \text{supp} \mu \) is non-abelian. Denote by \( \mu_\mathbb{Z} \) the image of the measure \( \mu \) on \( \mathbb{Z} \) under the homomorphism

\[
\begin{pmatrix} 2^k & m \\ 0 & 1 \end{pmatrix} \mapsto k
\]

from \( \text{Aff}(\mathbb{Z}[1/2]) \) to \( \mathbb{Z} \). Let the drift of \( \mu \) be the mean of the measure \( \mu_\mathbb{Z} \). In [56] Kaimanovich showed that the drift of \( \mu \) is finite because \( \mu \) has finite first moment, and that the measurable Poisson boundary is trivial if and only if the drift of \( \mu \) is 0. If \( \mu \) has finite first moment and \( \mu \) has negative drift, then the boundary may be identified with \( (\mathbb{R}, \gamma) \) for a measure \( \gamma \). Similarly, if \( \mu \) has finite first moment and positive drift, then the boundary may be identified with \( (\mathbb{Q}_2, \gamma) \) for a measure \( \gamma \). Their argument is easily adapted to \( \text{Aff}(\mathbb{Z}[1/p]) \) for any prime \( p \).

The paper also included the following useful criterion for maximality of the Poisson boundary:

**Theorem 1.4.38 (Kaimanovich’s ray criterion [55]).** Let \( \mu \) be a probability measure of finite first moment on a finitely generated discrete group \( G \), and let \( (B, \nu) \) be a \( \mu \)-boundary of \( (G, \mu) \). Let \( d \) be the word length metric corresponding to some finite generating set for \( G \). Let \( \text{bnd} \) be the boundary map associated with \( B \). If there exists a sequence of measurable approximation maps \( \Pi_m : B \to G \) satisfying

\[
\frac{1}{m} d(\Pi_m(\text{bnd}(\omega)), \omega_1 \omega_2 \ldots \omega_m) \to 0
\]

for almost every path \( \omega = (\omega_1, \omega_2, \ldots) \). Then \( (B, \nu) \) is the Poisson boundary of the pair \( (G, \mu) \).

Brofferio later extended the description of \( \text{Aff}(\mathbb{Z}[1/p]) \) to finitely generated subgroups of \( \text{Aff}(\mathbb{Q}) \) in [13] for measures with finite first moment with respect to an adelic length on the group. In this case, the boundary is a product of \( p \)-adic fields with the hitting measure. He was later able to give more information in the finitely generated case in [13], proving maximality with Kaimanovich’s strip criterion.

Cartwright, Kaimanovich and Woess [16] obtained a description of the measurable Poisson boundary of the affine group of closed subgroups of the homogeneous tree \( T_d \) for certain non-exceptional measures and all \( d \in \mathbb{N} \) satisfying \( d \geq 2 \). As a direct application of this result, they described the boundary of group

\[
\text{Aff}(\mathscr{F}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathscr{F}, a \neq 0 \right\}
\]
over non-archimedian local fields $\mathcal{F}$ of characteristic zero (e.g. the $p$-adic numbers $\mathbb{Q}_p$). The group is locally compact, totally disconnected, amenable, but not unimodular. A formulation of the ray criterion for topological groups was given in this paper without proof. A preprint containing the proof has not yet been published (personal communication with Kaimanovich).

In [14], Brofferio and Schapira described the Poisson boundary of $GL_n(\mathbb{Q})$ for measures of finite first moment with respect to a left-invariant pseudometric on the group. In this case, the boundary is a product of flag manifolds over $p$-adic fields. Their proof required the following generalization of Kaimanovich’s ray criterion.

**Theorem 1.4.39** (Brofferio and Schapira’s generalisation of the ray criterion [14]). Let $\mu$ be a probability measure on a discrete group $G$ with finite entropy, and let $(B, \nu)$ be a $\mu$-boundary. Let $\text{bnd} : G^\mathbb{N} \to G$ be the boundary map associated with $B$, as defined in Proposition 1.4.22. Suppose that, for every positive real $\delta$ and each $n \in \mathbb{N}$, there exists a measurable map $C^{(m)} : B \to \mathcal{P}$, satisfying

$$\lim_{m \to \infty} \mathbb{P}^\mu(\omega_1 \omega_2 \ldots \omega_n \in C^{(m)}(\text{bnd}(\omega))) \text{ and } \lim_{m \to \infty} \frac{1}{m} \ln |C^{(m)}(z)| \leq \delta$$

for every positive real $\delta$ and almost every path $\omega = (\omega_1, \omega_2, \ldots)$. Then $(B, \nu)$ is the Poisson boundary of the pair $(G, \mu)$.

**Theorem 1.4.40** (Kaimanovich’s ray criterion for topological groups [16, 57]). Let $G$ be a second countable Hausdorff topological group. Let $\mu$ be an aperiodic, spread-out probability measure on $G$ with finite first moment with respect to a subadditive gauge $A$ on $G$. Suppose that $(B, \nu)$ is a $\mu$-boundary with boundary map $\text{bnd}$. If, for $\nu$ almost every point $b \in B$, there is a uniformly temperate sequence of gauges $G^n = G^n(b)$ satisfying

$$\frac{1}{n} |\omega_n|_{G^n(\text{bnd}(\omega))} \to 0$$

for almost every path $\omega = (\omega_1, \omega_2, \ldots)$, then $(B, \nu)$ is the Poisson boundary of the pair $(G, \mu)$.

**Corollary 1.4.41.** Let $G$, $\mu$, $B$, $\nu$ and $\gamma$ be as in Theorem 1.4.40. Suppose that $G$ is generated by a compact set $K$. Let $d$ be the word length metric. If there exists a sequence of measurable approximation maps $\Pi_m : B \to G$ satisfying

$$\frac{1}{n} d(\Pi_m(\text{bnd}(\omega)), \omega_1 \omega_2 \ldots \omega_n) \to 0$$

for almost every path $\omega = (\omega_1, \omega_2, \ldots)$, then $(B, \nu)$ is the Poisson boundary of the pair $(G, \mu)$.

Kaimanovich [58] showed that the measurable Poisson boundary of word hyperbolic groups can be naturally identified with the hyperbolic boundary whenever $\mu$ has finite entropy and finite
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first logarithmic moment. In the same paper, he introduced an improved version of the ray criterion (below) and used it to give identifications of the measurable Poisson boundary for discontinuous groups of isometries of Gromov hyperbolic spaces, groups with infinitely many ends, cocompact lattices in Cartan-Hadamard manifolds and discrete subgroups of semi-simple Lie groups.

Theorem 1.4.42 (Improved ray criterion for countable groups [58]). Let $\mu$ be a probability measure on a countable group $G$, and $(B, \nu)$ be a $\mu$-boundary. Suppose that the entropy

$$H(\mu) = -\sum_{g \in G} \mu(g) \ln \mu(g)$$

of $\mu$ is finite. Let $\text{bnd}$ be the boundary map associated with $B : G^\mathbb{N} \to G$. If, for $\nu$-a.e. point $b \in B$, there is a uniformly temperate sequence of gauges $G^n = G^n(b)$, satisfying

$$\frac{1}{n} |\omega_n|_{G^n(\text{bnd}(\omega))} \to 0$$

for almost every path $\omega = (\omega_1, \omega_2, \ldots)$, then $(B, \nu)$ is the Poisson boundary of the pair $(G, \mu)$.

Corollary 1.4.43. Let $\mu$ be a probability measure on a countable group $G$, and $(B, \nu)$ be a $\mu$-boundary. Suppose that the entropy

$$H(\mu) = -\sum_{g \in G} \mu(g) \ln \mu(g)$$

of $\mu$ is finite. Let $\text{bnd}$ be the boundary map associated with $B : G^\mathbb{N} \to G$. If there exists a temperate gauge $B$ and a sequence of measurable maps $\Pi_m : B \to G$, satisfying

$$\frac{1}{n} |\Pi_m(\text{bnd}(\omega))^{-1} \omega_n|_B \to 0$$

for almost every path $\omega = (\omega_1, \omega_2, \ldots)$, then $(B, \nu)$ is the Poisson boundary of the pair $(G, \mu)$.

Kaimanovich and Woess [61] also studied the Poisson boundaries of permutation groups $G$ of a countable set $X$, where $G$ acts transitively on $X$, $G$ is closed in the topology of pointwise convergence in the semigroup of all self-maps of $X$ and the $G$-stabilizer of every point in $X$ has all orbits finite. The group $G$ is always locally compact and totally disconnected under these conditions.

They established a one-to-one correspondence between $G$-invariant Markov operators on $X$ and bi-$K$-invariant probability measures on $G$ satisfying

$$\mu(gK) = \mu(kgK)$$

for all $g \in G$ and $k \in K$, where $K$ is the stabilizer of a distinguished vertex in $X$. They studied the Poisson boundary of these operators—defined as a representation of bounded harmonic
functions—and established many analogues to the results for the discrete group case, including versions of the ray and strip criteria.

Vershik and Malyutin [98] described the measurable Poisson boundary of the braid group in terms of the hyperbolic boundary of a special free subgroup. Braid groups are non-amenable. Applications included results on groups of automorphisms acting on Diestel-Leader graphs.

Under certain conditions on the measure, Karlsson and Woess [62] were able to give an explicit description of the boundary for the wreath product $C_r \wr G_d$, where $G_d$ is a finitely generated group, with Cayley graph corresponding to the homogeneous tree $T_d$ of degree $d + 1 \geq 3$, and $C_r$ is a finite cyclic group. In general, $G_q$ is a free product of copies of $C_2$ and the infinite cyclic group.

Let $G_k$ be the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}_k$, and let $\mu$ be a probability measure on $G_k$. Each $G_k$ is solvable. Kaimanovich [56] showed that the measurable Poisson boundary of $(G_k, \mu)$ is non-trivial if and only if the random walk on $(\mathbb{Z}^k, \mu_{\mathbb{Z}^k})$ is transient and the subgroup generated by $\text{supp} \mu$ is non-abelian. In particular, if $\mu$ is non-degenerate and symmetric, with finite first moment, then the measurable Poisson boundary of $(G_k, \mu)$ is trivial for $k=1,2$, and non-trivial for $k \geq 3$. He conjectured that the measurable Poisson boundary could be identified with a certain space of limit configurations whenever $\mu$ had a finite first moment, but could only prove this in the case of finitely supported measures and some special measures on $G_1$. Erschler [36] provided a partial answer to this question, showing, if $\mu$ has a finite third moment and if the support of $\mu$ generates $G_k$ then the result holds for $k \geq 5$. Lyons and Peres [70] further extended this result, showing that the result is true for all $k \geq 3$ and measures with finite second moment.

Deroin [27] computed the measurable Poisson boundary of a locally discrete subgroup of diffeomorphisms of the circle. Fernós [38] showed that if $X$ is a finite dimensional CAT(0) cube complex, $G$ is a group admitting a non-elementary proper action on $X$ and $\mu$ is a probability measure of finite entropy and finite first logarithmic moment which generates $G$, then there is a $\mu$-stationary measure on the Roller boundary of $X$ making it the measurable Poisson boundary.

Cuno [25] considered random walks on non-amenable Baumslag–Solitar groups $BS(p, q)$ for integers $p$ and $q$ satisfying $1 \leq p < |q|$. He showed that the measurable Poisson boundary is either trivial, or isomorphic to $(\partial T, \nu_{\partial T})$ or $(\partial T \times \partial \mathbb{H}, \nu_{\partial T \times \partial \mathbb{H}})$, where $\partial T$ is the space of ends of the corresponding Bass-Serre tree and $\partial \mathbb{H}$ is the hyperbolic boundary of the Poincaré disk.
1.5. **Summary of results**

Brofferio and Schapira’s [14] description of the Poisson boundary of $GL_n(\mathbb{Q})$ is for measures of finite first moment with respect to adelic length. Chapter 2 relates adelic length to the word length in finitely generated rational matrix groups. To do this, we define matrix groups $FG_n(P)$ for each $n \in \mathbb{N}$ and finite set of primes $P$. These groups contain every rational valued upper triangular matrix group as a (possibly distorted) subgroup. We show that adelic length is a word metric estimate on $FG_n(P)$ by constructing another, intermediate, word metric estimate which can be easily computed from the entries of any matrix in the group. In particular, requiring a probability measure on $FG_n(P)$ to have finite first moment with respect to adelic length is equivalent to requiring it to have finite first moment with respect to word length.

Chapter 3 presents two possible extensions of the work of Cartwright, Kaimanovich and Woess in [16]. Firstly, we consider random walks finite direct products of affine automorphism groups of homogeneous trees. When the probability measure is spread-out, aperiodic, has finite first moment and its support generates a closed subgroup which is not fully exceptional, we show that the Poisson boundary is the direct product of the space of ends of each tree with a probability measure. We give necessary and sufficient conditions for boundary triviality. Secondly, we investigate random walks on closed sub-quotients in totally disconnected locally compact groups which are isomorphic closed subgroup of an affine group of a homogeneous tree with non-trivial degree. A description of the Poisson boundary for aperiodic, spread-out probability measures of finite first moment is given in that case.

In Chapter 4 we consider random walks on unrestricted wreath products, provide sufficient conditions for almost sure convergence of paths to limit configurations, and prove that the Poisson boundary can be non-trivial. We also examine a possible generalization of the $K$-rate of escape to compactly generated totally disconnected groups, which we call the *rate of eschewal*. Much as the rate of escape depends on a choice of generating set, the rate of eschewal depends on a chosen sequence of strictly decreasing compact open subgroups with trivial intersection. For appropriate sequences, the rate of eschewal is finite and equal to the rate of escape for measures supported within the restricted lamplighter subgroup.
Chapter 2

Solvable matrix groups

This chapter discusses the solvable matrix groups $\text{FG}_n(P)$, where $n \in \mathbb{N}$ and $P$ is a finite set of primes. In Section 2.1, we define these groups and prove some basic properties. In particular, we show that every finitely generated group of upper triangular matrices with rational entries is a subgroup of $\text{FG}_n(P)$ for suitable choices of $n$ and $P$. In Section 2.2, we give a metric estimate of word length on $\text{FG}_n(P)$. Unlike the word length, our estimate can be efficiently computed from the entries of a given group element.

Brofferio and Schapira [14] described the Poisson boundary of $\text{GL}_n(\mathbb{Q})$ for measures of finite first moment with respect to adelic length. The adelic length is defined in terms of matrix norms. In Section 2.3, we show that the adelic length is a word metric estimate on $\text{FG}_n(P)$. We also discuss finite moment conditions on probability measures with respect to adelic length and word length.

2.1 The groups, $\text{FG}_n(P)$

For each rational number $q$, let $\theta_{rs}(q)$ be the matrix satisfying

$$(\theta_{rs}(q))_{ij} = \begin{cases} q & \text{if } (r, s) = (i, j) \\ 1 & \text{if } i = j \text{ and } (r, s) \neq (i, j) \\ 0 & \text{if } i \neq j \text{ and } (r, s) \neq (i, j) \end{cases}$$

for all $r, s, i, j \in \{1, \ldots, n\}$.

Throughout, let $P$ be a finite, non-empty set of primes and let $k$ be the cardinality of $P$. Set

$$U := \{\theta_{rs}(1), \theta_{rs}(1)^{-1} : r, s \in \mathbb{N}, 1 \leq r < s \leq n\}$$
and
\[ J_P := \left\{ \theta_r(p), \theta_r(p)^{-1} : p \in P, r \in \mathbb{N}, 1 \leq r \leq n \right\}. \]

Let \( FG_n(P) \) be the subgroup of \( GL_n(\mathbb{Q}) \) generated by \( K_P := U \cup J_P \) and let \( |\cdot|_{n,P} \) the word length function on \( FG_n(P) \) with respect to \( K_P \). Identify each prime \( p \) with the set \( \{p\} \), so that, for example, if \( p \) is prime, then
\[ J_p = \left\{ \theta_r(p), \theta_r(p)^{-1} : r \in \mathbb{N}, 1 \leq r \leq n \right\}, \]
and \( FG_n(p) \) is the subgroup of \( GL_n(\mathbb{Q}) \) generated by \( K_p := U \cup J_p \).

**Lemma 2.1.1.** Each element \( f \in FG_n(P) \) is upper triangular. Moreover, for \( r \in \{1, \ldots, k\} \) and \( 1 \leq i \leq j \leq n \), there are integers \( r_{ij} \) and \( c^{(r)}_{ij} \) satisfying
\[ f_{ij} = \begin{cases} \prod_{r=1}^{k} p_r^{c^{(r)}_{ij}} & \text{if } i = j \text{ and } \prod_{r=1}^{k} p_r^{c^{(r)}_{ij}} \text{ if } i < j. \end{cases} \]
Conversely, every matrix of this form is an element of \( FG_n(P) \).

**Proof.** The forward direction follows from the well known fact that \( U \) generates \( UT_n(\mathbb{Z}) \) (see Elder, Elston and Ostheimer [32], for example). To prove the converse, suppose that \( f \) is an upper triangular matrix with entries
\[ f_{ij} = \begin{cases} \prod_{r=1}^{k} p_r^{c^{(r)}_{ij}} & \text{if } i = j, \text{ and } \prod_{r=1}^{k} p_r^{c^{(r)}_{ij}} & \text{if } i < j. \end{cases} \]
Let \( g \) be the diagonal matrix satisfying \( g_{ii} = f_{ii} \) for all \( i \in \{1, \ldots, n\} \). Then the matrix \( g^{-1}f \) is strictly upper unitriangular and
\[ (g^{-1}f)_{ij} = r_{ij} \prod_{r=1}^{k} p_r^{c^{(r)}_{ij} - c_i^{(r)}} \]
whenever \( i < j \). With this in mind, let \( M = \min_{i \leq j} \left\{ c^{(r)}_{ij} - c_i^{(r)} \right\} \), and let \( A \) be the diagonal matrix with entries
\[ A_{jj} = \begin{cases} b_{p,j}^M & \text{if } j < n, \text{ and } \end{cases} \]
\[ 1 & \text{if } j = n. \]
for all \( j \in \{1, \ldots, n\} \). Then \( z = A g^{-1} f A^{-1} \in UT_n(\mathbb{Z}) \) and \( f = g A^{-1} z A \). Since \( g A^{-1} \) and \( A \) are diagonal matrices, which may be written as a finite product of the diagonal generators of \( J_P \), and we know that \( U \) generates \( UT_n(\mathbb{Z}) \), we have shown that \( f \) is a finite product of elements of \( FG_n(P) \). Moreover, \( f \) is upper triangular because every generator of \( FG_n(P) \) is upper triangular. \( \square \)
Corollary 2.1.2. Suppose that $G$ is a finitely generated group of upper triangular matrices with rational entries. Then $G$ is a subgroup of $\text{FG}_n(P)$ for some set of primes $P$.

Proof. Let $P = \{p_r\}_{r=1}^k$ be the minimal set of primes such that each generator $f \in G$ can be expressed in the form

$$f_{ij} = \begin{cases} 
\prod_{r=1}^k p_r^{c_{ij}^{(r)}} & \text{if } i = j, \\
\prod_{r=1}^k p_r^{c_{ij}^{(r)}} & \text{if } i < j
\end{cases}$$

as in Lemma 2.1.1. Then $P$ is finite because $G$ is finitely generated and every rational number has a unique prime factorisation.

2.2 A word metric estimate on $\text{FG}_n(P)$

Let $b > 1$ be a natural number. For any non-zero rational number $x$ with finite base $b$ representation, $\sum_{i=k}^l a_i b^i$, let

$$d_b^+(x) = k,$$

$$d_b^-(x) = l$$

and

$$[x]_b = 1 + \max\{|d_b^-(x)|, |d_b^+(x)|\}.$$  

If $x$ is zero, let $[x]_b = 0$. Kaimanovich [56] defined $[\cdot]_2$ to construct a word metric estimate on $\text{Aff}(\mathbb{Z}[\frac{1}{2}])$; moreover, base 2 versions of some of the identities below were derived.

Let $x, y \in \mathbb{Q}$ have finite base $b$ representations. It is easy to see, by considering $b$ representations that

$$d_b^-(x) = -\log_b |x|_b,$$

and that

$$d_b^+(x) \leq \log_b |x| \leq 1 + d_b^+(x), \quad (2.1)$$

where $|\cdot|_p$ is the $p$-adic absolute value. Here, $b$ is not necessarily a prime. Such considerations also allow us to easily see that

$$d_b^+(xy) \leq \log_b |xy|$$

$$= \log_b |x| + \log_b |y|$$

$$\leq 2 + d_b^+(x) + d_b^+(y),$$

$$d_b^+(x + y) \leq 1 + \max\{d_b^+(x), d_b^+(y)\}$$
and
\[ d_+^b (x + y) \geq \min\{d_+^b (x), d_+^b (y)\}. \]

**Lemma 2.2.1.** Suppose that \( b > 1 \) is a natural number. Then
\[ [x + y]_b \leq [x]_b + [y]_b, \]
whenever \( x \) and \( y \) are rational numbers with finite base \( b \) representations. The equality holds if \( x \) or \( y \) is zero.

**Proof.** If \( x \) or \( y \) is zero, then the result is certainly true. Suppose that \( x \) and \( y \) are both non-zero. In this case, the definition of \([\cdot]_b\) gives
\[ [x + y]_b = 1 + \max\{|d_+^b (x + y)|, |d_+^b (x + y)|\}. \]

Suppose that
\[ |d_+^b (x + y)| \geq |d_+^b (x + y)|. \]
Then, by considering addition in terms of the base \( b \) expansions of \( x \) and \( y \),
\[ [x + y]_b \leq 1 + |d_+^b (x + y)| \]
\[ \leq 2 + \max(|d_+^b (x)|, |d_+^b (y)|) \]
\[ < 1 + |d_+^b (x)| + 1 + |d_+^b (y)| \]
\[ \leq [x]_b + [y]_b. \]

Similarly, if
\[ |d_+^b (x + y)| < |d_+^b (x + y)|, \]
then
\[ [x + y]_b \leq 1 + |d_+^b (x + y)| \]
\[ \leq 1 + \max(|d_+^b (x)|, |d_+^b (y)|) \]
\[ < 1 + |d_+^b (x)| + 1 + |d_+^b (y)| \]
\[ \leq [x]_b + [y]_b. \]

\[ \Box \]

**Lemma 2.2.2.** Suppose that \( b > 1 \) is a natural number. Then
\[ [xy]_b \leq 3([x]_b + [y]_b), \]
whenever \( x \) and \( y \) are rational numbers with finite base \( b \) representations.
2.2. A WORD METRIC ESTIMATE ON $\text{FG}_N(P)$

**Proof.** The statement is true if $x = 0$ or $y = 0$. Suppose that $x \neq 0$ and $y \neq 0$. Then

$$[xy]_b = 1 + \max\{|d_b^-(xy)|, |d_b^+(xy)|\}$$

$$\leq 1 + \max\{2 + |d_b^-(x)| + |d_b^-(y)|, 2 + |d_b^+(x)| + |d_b^+(y)|\}$$

$$\leq 3 + |d_b^-(x)| + |d_b^-(y)| + |d_b^+(x)| + |d_b^+(y)|$$

$$\leq 3 + 2 \max\{|d_b^-(x)|, |d_b^+(x)|\} + 2 \max\{|d_b^-(y)|, |d_b^+(y)|\}$$

$$\leq 3([x]_b + [y]_b),$$

establishing our result. \qed

**Lemma 2.2.3.** Let $b_1, b_2 > 1$ be natural numbers. Suppose that $x$ is a rational number with finite base $b_1$ representation such that $d_{b_1}^+(x)$ is positive. Let $b = b_1b_2$. Then $x$ has a finite base $b$ representation, and

$$|d_b^+(x)| \leq 2 + \left(\frac{\ln(b)}{\ln(b_1)}\right)\left(1 + |d_b^+(x)|\right).$$

**Proof.** Changing base in Equation (2.1) yields

$$|d_b^+(x)| \leq 1 + \log_b |x|$$

$$= 1 + \left(\frac{\ln(b)}{\ln(b_1)}\right)\log_b |x|$$

$$\leq 2 + \left(\frac{\ln(b)}{\ln(b_1)}\right)\left(1 + |d_b^+(x)|\right),$$

which is the desired inequality. \qed

The proof of the following Lemma is straightforward.

**Lemma 2.2.4.** Suppose that $x$ is a non-zero rational number with finite base $b$ representation and that $k$ is a non-zero integer. Let $y = kx$. Then $d_b^+(y) \geq d_b^+(x)$.

**Corollary 2.2.5.** Let $P = \{p_r\}_{r=1}^k$ be a finite, non-empty set of primes. Suppose that $x$ is a non-zero rational number of the form

$$x = y \prod_{r=1}^k p_r^{c_r}$$

for some integer $y$ which does not divide any prime in $P$. Here, for each natural number $r \leq k$, $c_r$ is an integer. Then

$$d_r^p(x) \geq -\max_{1 \leq r \leq k}\{|c_r|\}.$$
Proof. For brevity, write \( M = -\max_{r=1}^{k} \{ |c_r| \} \). Notice that \( y \left( \prod_{r=1}^{k} p_r^{c_r-M} \right) \) is an integer, and

\[
x = y \left( \prod_{r=1}^{k} p_r^{c_r-M} \right) (b_P)^M,
\]

where \( b_P \) is the product of all primes in \( P \). Applying Lemma 2.2.4 yields

\[
d_{\mathbb{Z}}(x) \geq d_{\mathbb{Z}}((b_P)^M) = M,
\]

which completes our proof. \( \square \)

Proposition 2.2.6 (Metric estimate on \( \mathrm{FG}_n(P) \)). Let \( n \in \mathbb{N} \), \( P = \{ p_r \}_{r=1}^{k} \) be a non-empty set of primes, and let \( b_P \) be the product of all primes in \( P \). Suppose that \( f \in \mathrm{FG}_n(P) \) has diagonal entries

\[
f_{ii} = \prod_{r=1}^{k} p_r^{c_{ii}^{(r)}},
\]

where \( i \in \{1, \ldots, n\} \) and \( c_{ii}^{(r)} \in \mathbb{Z} \). Set

\[
\|f\|_{n,P} = \sum_{i=1}^{n} \sum_{r=1}^{k} |c_{ii}^{(r)}| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |f_{ij}|_P,
\]

where \(| \cdot |\) is the absolute value. Then \( \| \cdot \|_{n,P} \) is a word metric estimate on \( \mathrm{FG}_n(P) \). In particular, there is a positive real constant \( J \) such that

\[
\frac{1}{J} \|f\|_{n,P} \leq |f|_{n,P} \leq J \|f\|_{n,P},
\]

where \(|f|_{n,P}\) is the word length of \( f \) with respect to the generating set \( K_P \).

Proof. If \( f \) is the identity, then

\[
|f|_{n,P} = 0 = \|f\|_{n,P}.
\]

Otherwise, if \( n = 1 \), then

\[
|f|_{1,P} = \sum_{r=1}^{k} |c_{11}^{(r)}| = \|f\|_{n,P}
\]

because \( \mathrm{FG}_1(P) \) is isomorphic to \( \mathbb{Z}^k \).

Suppose \( n > 1 \). We use induction to establish an upper bound. For each \( q \in \mathbb{N} \), identify \( \mathrm{FG}_q(P) \) with its isomorphic subgroup in \( \mathrm{FG}_{q+1}(P) \), consisting of all elements of \( \mathrm{FG}_{q+1}(P) \) for which the first row and column are the same as those of the identity matrix \( I_{q+1} \). Suppose, for induction, that there is a positive constant \( A_q \) such that

\[
|g|_{q,P} \leq A_q \|g\|_{q,P}
\]
for every \( g \in \text{FG}_q(P) \). Let \( f \in \text{FG}_{q+1}(P) \), and write \( f = gh \) for \((q + 1) \times (q + 1)\) matrices \( f \) and \( g \) which are the same as the identity, except that \( f \) and \( h \) agree on the first row and \( f \) and \( g \) agree on all remaining rows. Subadditivity of the word length function implies that

\[
|f|_{q+1,P} = |gh|_{q+1,P} \\
\leq |g|_{q+1,P} + |h|_{q+1,P} \\
= |g|_{q,P} + |h|_{q+1,P}.
\]

Hence, by the inductive hypothesis,

\[
|f|_{q+1,P} \leq J_q \|g\|_{q,P} + |h|_{q+1,P} \\
= J_q \|g\|_{q+1,P} + |h|_{q+1,P}. \tag{2.2}
\]

The word length of \( h \) is bounded by \( \sum_{j=0}^{q} |\theta_{1,q+1-j} (h_{1,q+1-j})|_{q+1,P} \) because

\[
h = \prod_{j=0}^{q} \theta_{1,q+1-j} (h_{1,q+1-j}),
\]

where the product is taken from left to right. We will now bound the word length of each term in this product. We can write \( \theta_{11} (h_{11}) \) in terms of the generators as

\[
\theta_{11} (h_{11}) = \prod_{\epsilon \in \{0, \ldots, b_P - 1\}} \theta_{11} (p_{\epsilon})^{-c_{11}^{(\epsilon)}}. \tag{2.3}
\]

Hence, the word length of \( \theta_{11} (h_{11}) \) is bounded above by \( \sum_{r=1}^{k} |c_{11}^{(r)}| \).

We now bound the word length of \( \theta_{1j} (h_{1j}) \) in the case where \( j \geq 2 \). If \( h_{1j} = 0 \), then \( \theta_{1j} (h_{1j}) \) is the identity, so \( |\theta_{1j} (h_{1j})| = 0 \). Let \( b_P^{t_j} \prod_{\epsilon \in \{0, \ldots, b_P - 1\}} \epsilon_j (i) b_P^i \) be the unique base \( b_P \) expansion of \( |h_{1j}| \), where \( \epsilon_j (i) \in \{0, \ldots, b_P - 1\} \) for each \( i \). Suppose \( h_{1j} \) is positive. Then, traversing across the base \( b_P \) representation of the integer \( \sum_{i=0}^{t_j} \epsilon_j (i) b_P^i \), as in Example 1.2.5, or the proof of Lemma 5.1 in Kaimanovich [56], we have

\[
\theta_{1j} \left( \sum_{i=0}^{t_j} \epsilon_j (i) b_P^i \right) = \prod_{i=0}^{t_j} \theta_{1j} (1)^{\epsilon_j (i)} \theta_{11} (b_P) \theta_{11} (b_P)^{-t_j - 1}.
\]

Hence, conjugating by \( \theta_{11} (b_P)^{n_j} \),

\[
\theta_{1j} (h_{1j}) = \theta_{11} (b_P)^{n_j} \left( \prod_{i=0}^{t_j} \theta_{1j} (1)^{\epsilon_j (i)} \theta_{11} (b_P) \right) \theta_{11} (b_P)^{-n_j - t_j - 1}. \tag{2.4}
\]

Each term \( \theta_{11} (b_P) \) is equal to \( \prod_{p \in \mathbb{P}} \theta_{11} (p) \), which is a product of \( k \) generators. Each term \( \theta_{1j} (1)^{\epsilon_j (i)} \) is a product of at most \( b_P - 1 \) generators. Hence,

\[
|\theta_{1j} (h_{1j})|_{q+1,P} \leq (2n_j + t_j + 1) k + (b_P - 1 + k)(t_j + 1) \\
= 2kn_j + (b_P - 1 + 2k)(t_j + 1).
\]
By uniqueness of the base $b_P$ expansion of $h_{1j}$, we have $n_j = d^{b_P}_-(h_{1j})$ and
\[ t_j = d^{b_P}_+(h_{1j}) - d^{b_P}_-(h_{1j}). \]

Hence,
\[
|\theta_{1j}(h_{1j})|_{q+1,P} \leq 2kd^{b_P}_-(h_{1j}) + (b_P - 1 + 2k)(d^{b_P}_+(h_{1j}) - d^{b_P}_-(h_{1j}) + 1) \\
\leq (b_P + 4k) \left( 1 + |d^{b_P}_-(h_{1j})| + |d^{b_P}_+(h_{1j})| \right) \\
\leq (2b_P + 8k) |h_{1j}|_{b_P}.
\]

The case where $h_{1j}$ is negative is similar, except that we replace $\theta_{1j}(1)$ everywhere in Equation (2.4) with $\theta_{1j}(-1)$. Therefore, we have the upper bound,
\[
|h|_{q+1,P} \leq q(2b_P + 8k) \left( \sum_{j=2}^{q+1} |h_{1j}|_{b_P} + \sum_{r=1}^{k} |c_{11}^{(r)}| \right) = q(2b_P + 8k) \|h\|_{q+1,P}.
\]

In conjunction with Equation (2.2), we have
\[
|f|_{q+1,P} \leq J_q \|g\|_{q+1,P} + q(2b_P + 8k) \|h\|_{q+1,P} \\
\leq J_{q+1} \left( \|g\|_{q+1,P} + \|h\|_{q+1,P} \right) \\
= J_{q+1} \|f\|_{q+1,P},
\]
where $J_{q+1} = \max(q(2b_P + 8k), J_q)$. Hence, by induction
\[
|f|_{n,P} \leq (n - 1)(2b_P + 8k) \|f\|_{n,P},
\]
which is an upper bound for $|f|_{n,P}$ in terms of $\|f\|_{n,P}$.

We will now find a lower bound for $|f|_{n,P}$. Without loss of generality, we may assume that $f$ is not the identity, so that $1 \leq |f|_{n,P}$. Notice that any product of generators $g$ with
\[ g_{ii} = f_{ii} = \prod_{r=1}^{k} p_{ii}^{c_{ii}^{(r)}} \]
must contain at least $|c_{ii}^{(r)}|$ copies of $\theta_{ii}(p_r)$ for each $r \in \{1, \ldots, k\}$. Hence,
\[
\sum_{r=1}^{k} |c_{ii}^{(r)}| \leq |f|_{n,P}
\]
for each $i \in \{1, \ldots, n\}$.

If $g$ is any element of $\text{FG}_n(P)$, then
\[
\frac{1}{b_P}|(gx)_{ij}| \leq |g_{ij}| \leq b_P|(gx)_{ij}|
\]  
(2.5)
for every generator $x \in \text{FG}_n(P)$ and $1 \leq i < j \leq n$. It follows that

$$|d^{bp}_+(f_{ij})| \leq |f|_{n,P}, \quad (2.6)$$

where $f_{ij}$ is a strictly upper diagonal entry of the matrix $f$. Hence, if $d^{bp}_+(f_{ij})$ is not positive, then

$$|d^{bp}_+(f_{ij})| \leq |d^{bp}_-(f_{ij})| \leq |f|_{n,P} \cdot (2.7)$$

It also follows from Equation (2.5) that

$$|f_{ij}| \leq b_P |f|_{n,P}.$$  

Hence, if $d^{bp}_+(f_{ij})$ is positive,

$$|d^{bp}_+(f_{ij})| \leq \log_{b_P} |f_{ij}| \leq |f|_{n,P} \cdot (2.8)$$

for all strictly upper off-diagonal entries $f_{ij}$ of the matrix $f$. Therefore,

$$|f_{ij}|_{b_P} \leq 1 + 2 |f|_{n,P} \leq 3 |f|_{n,P}.$$

As there are $\frac{n(n+1)}{2}$ entries in the upper triangular part of $f$,

$$\|f\|_{n,P} \leq \frac{3}{2} |f|_{n,P} \cdot$$

which means that

$$\frac{1}{J} \|f\|_{n,P} \leq |f|_{n,P} \leq J \|f\|_{n,P},$$

where $J = \max \{ \frac{3}{2}n(n+1), n(2b_P + 8k) \}$. This completes the proof.  

\[\square\]

---

### 2.3 Adelic length is a word metric estimate on $\text{FG}_n(P)$

Let $\mathcal{P}^*$ be the set of all primes. Let $\mathcal{P} = \mathcal{P}^* \cup \{\infty\}$. For each vector $v = (v_1, \ldots, v_n) \in \mathbb{Q}^n_p$, and each prime $p$, let

$$|v|_p = \max_{1 \leq i \leq n} |v_i|_p.$$  

For each vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, let

$$|v|_\infty = \sqrt[n]{\sum_{i=1}^n |v_i|^2}.$$  

If $f \in \text{GL}_n(\mathbb{Q})$, define the operator norm

$$\|f\|_p = \sup_{|v|_p = 1} |fv|_p.$$
for each $p \in \mathcal{P}$. Note that $\|\cdot\|_\infty$ is just the $l_2$ operator norm. For $p \in \mathcal{P}$ and $f, g \in GL_n(\mathbb{Q})$, set
\[ d_p^a(f, g) = \ln^+ \| f^{-1} g \|_p + \ln^+ \| g^{-1} f \|_p, \]
where $\ln^+$ is the positive part of the natural logarithm function. The map $d_p$ is symmetric, satisfies the triangle inequality and is left-invariant.

For $f, g \in GL_n(\mathbb{Q})$, let
\[ d^a(f, g) = \sum_{p \in \mathcal{P}} d_p^a(f, g). \]
Notice that $d_p^a(f, g) = 0$ for all but finitely many elements $p \in \mathcal{P}$, and hence this sum converges. The map $d^a$ is a left invariant pseudometric on $GL_n(\mathbb{Q})$, called the adelic pseudometric. To see that $d^a$ does not separate points because $d^a(f, g) = 0$ whenever $f$ and $g$ are permutation matrices.

We can use the adelic pseudometric to define the adelic length of $f \in GL_n(\mathbb{Q})$ as follows
\[ [f]_{n,a} = d^a(I_n, f) = \sum_{p \in \mathcal{P}} \left( \ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p \right), \]
where $I_n$ is the $n \times n$ identity matrix.

Let $V$ be a vector space. Take the tensor product $V \otimes V$ of two copies of $V$. The exterior product of $V$ with itself, denoted by $V \wedge V$, is the subspace of $V \otimes V$ consisting of linear combinations of antisymmetric vectors of the form
\[ v_1 \wedge v_2 := v_1 \otimes v_2 - v_2 \otimes v_1, \]
for $v_1, v_2 \in V$. The map $\wedge$ is called the wedge product. The wedge product satisfies
\[ v_1 \wedge v_2 = -v_1 \wedge v_2 \quad (2.9) \]
\[ (\lambda v_1) \wedge v_2 = \lambda (v_1 \wedge v_2) \quad (2.10) \]
\[ (v_1 + v_2) \wedge v_3 = v_1 \wedge v_3 + v_1 \wedge v_2 \quad (2.11) \]
for $v_1, v_2, v_3 \in V$. Equation (2.9) implies that
\[ v \wedge v = 0 \quad (2.12) \]
for all $v \in V$.

The exterior product of $k$ copies of $V$, denoted by $\bigwedge_k V$ is the vector space spanned by expressions of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_k$, where $v_1, \ldots, v_k \in V$. If $V$ has dimension $n$ and basis \{ $e_1, \ldots, e_n$ \}, then $\bigwedge_k V$ has dimension $\binom{n}{k}$ and basis \[ \{ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : i_1 < i_2 < \cdots < i_k \leq n \}, \]
2.3. **ADELIC LENGTH IS A WORD METRIC ESTIMATE ON** \( \text{FG}_N(P) \)  

where \( i_1, i_2, \ldots, i_k \in \mathbb{N} \). For each linear operator \( g \) on \( V \), denote by \( \wedge^k g \) the linear operator on \( \wedge_k V \) satisfying 

\[
(\wedge^k g)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = (gv_1 \wedge gv_2 \wedge \cdots \wedge gv_k).
\]

See Winitzki [104] for more information about exterior products and wedge products.

Let \( \mu \) be a probability measure on \( \text{GL}_n(\mathbb{Q}) \). For each \( p \in \mathcal{P} \) and each \( k \in \{1, \ldots, n\} \), let \( \lambda_k(p) \) be the Lyapunov coefficient of \( \mu \). This is the real number defined recursively by

\[
\sum_{i=1}^{k} \lambda_i(p) = \lim_{r \to \infty} \frac{1}{r} \int_{\text{GL}_n(\mathbb{Q})} \ln \|\wedge^k g\|_p d\mu^{*r}(g),
\]

where \( \wedge^k g \) is the operator on \( \wedge^k \mathbb{R}^n \) associated with \( g \). Let

\[
B_p = \text{GL}_n(\mathbb{K}_p/P_p),
\]

be the associated **flag manifolds**. Here \( \mathbb{K}_p = \mathbb{Q}_p \) if \( p \) is prime or \( \mathbb{R} \) if \( p = \infty \), and \( P_p \) is the normal subgroup of \( \text{GL}_n(\mathbb{K}_p) \) consisting of all matrices whose entries \( f_{ij} \) are zero whenever \( \lambda_i(p) < \lambda_j(p) \).

Brofferio and Schapira described the adelic pseudometric in [14]. They showed that if a measure \( \mu \) on \( \text{GL}_n(\mathbb{Q}) \) has finite first moment with respect to the adelic length, i.e.

\[
\int_{\text{GL}_n(\mathbb{Q})} \|f\|_{n,a} d\mu(f) < +\infty,
\]

then there is a probability measure \( \nu \) supported on the product

\[
\mathcal{B} = \prod_{p \in \mathcal{P}} B_p,
\]

such that \( (\mathcal{B}, \nu) \) is the Poisson boundary of \( (\text{GL}_n(\mathbb{Q}), \mu) \).

**Example 2.3.1.** Let \( G = \text{GL}_2(\mathbb{Q}) \). Let \( \mu = \delta_A \), where \( A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \). Then, using Equation (2.13) with \( k = 1 \),

\[
\lambda_1(2) = \lim_{r \to \infty} \frac{1}{r} \int_{\text{GL}_2(\mathbb{Q})} \ln \|\wedge^1 g\|_2 d\mu^{*r}(g)
\]

\[
= \lim_{r \to \infty} \frac{1}{r} \int_{\text{GL}_2(\mathbb{Q})} \ln \|g\|_2 d\mu^{*r}(g)
\]

\[
= \lim_{r \to \infty} \frac{1}{r} \ln \|A^r\|_2(g).
\]

A computation shows that \( \|A^r\|_2 = |1|_2 \) for all \( r \in \mathbb{N} \). Hence,

\[
\lambda_1(2) = \lim_{r \to \infty} \frac{1}{r} \ln \|A^r\|_2 = \lim_{r \to \infty} \frac{1}{r} \ln(1) = 0.
\]
Using Equation (2.13) again with \( k = 2 \),
\[
\lambda_2(2) = \lim_{r \to \infty} \frac{1}{r} \int_{\text{GL}_2(\mathbb{Q})} \ln \| \wedge^2 g \|_2 \, d\mu^G(g).
\]
If \( \{ e_1, e_2 \} \) is the standard basis for \( \mathbb{Q}^2 \), then \( \{ e_1 \wedge e_2 \} \) is a basis for \( \wedge_k \mathbb{Q}^2 \). A calculation from the definition shows that \( (\wedge^2 g)(v_1 \wedge v_2) = \det(g)v_1 \wedge v_2 \). Hence,
\[
\lambda_2(2) = \lim_{r \to \infty} \frac{1}{r} \ln \| \det(A^r) \|_2,
\]
where \( \det(A^r) \) is identified with the linear operator which maps \( e_1 \wedge e_2 \) to \( e_1 \wedge e_2 \det(A^r) \in \wedge_k \mathbb{Q}^2 \). Computing the limit, we obtain
\[
\lambda_2(2) = \ln(2) + \lim_{r \to \infty} \ln \frac{1}{r} = \ln(2) + \lim_{r \to \infty} \ln \frac{1}{r} = 0.
\]
Similar computations show that \( \lambda_1(\infty) = \ln 2 \), that \( \lambda_2(\infty) = 0 \) and that
\[
\lambda_1(p) = 0 = \lambda_2(p)
\]
for every prime \( p \). It follows that \( B \) consists of a single point, and, hence, that \( (\text{GL}_2(\mathbb{Q}), \mu) \) has trivial Poisson boundary.

In the remainder of this section, we show that the adelic length is a word metric estimate on \( \text{FG}_n(P) \). Many of our arguments require the estimation of the adelic length using the word metric estimate \( \| \cdot \|_{n,P} \), discussed in previous section. The adelic length is defined in terms of matrix norms, whereas our metric estimate from the last section was defined in terms of matrix entries. We present our argument in the next three subsections of this chapter. The first subsection contains the technical lemmas and propositions needed to construct an upper bound of the form
\[
|f|_{n,P} \leq S + T \| f \|_{n,a},
\]
for positive real constants \( S \) and \( T \). In the second subsection, we seek to construct a lower bound
\[
Q + R \| f \|_{n,a} \leq |f|_{n,P},
\]
where \( Q \) and \( R \) are positive real constants. The final section combines the results from the two earlier subsections to complete the proof that adelic length is a word metric estimate on \( \text{FG}_n(P) \).

We will often make use of the max norm inequality (see, for example, Equation (2.3.8) of Golub [43]):
\[
\max_{i,j} |f_{ij}| \leq \| f \|_\infty \leq \sqrt{mn} \max_{i,j} |f_{ij}|,
\]
where \( f \) is any real valued \( m \times n \) matrix and each maximum is taken across all \( i, j \in \{1, \ldots, n\} \). This inequality relates the \( l_2 \) operator norm of matrix \( f \) to the absolute value of its entries.
2.3. ADELIC LENGTH IS A WORD METRIC ESTIMATE ON $\text{FG}_N(P)$

§ 2.3.2 Upper bounds

**Lemma 2.3.3.** Let $P = \{p_r\}_{r=1}^k$ be a finite, non-empty set of primes. Let $b_P$ be the product of all primes in $P$. Suppose that $x$ is a non-zero rational number

$$x = y \prod_{r=1}^{k} p_r^{c_r}$$

for some integer $y$, such that no prime in $P$ divides $y$. Then

$$d_{b_P}^F(x) = \min_{1 \leq r \leq k} \{c_r\} = \min_{1 \leq r \leq k} \{-\log_{p_r} |x|_{p_r}\} = \min_{1 \leq r \leq k} \{d_{p_r}^F(p_r^{c_r})\},$$

and $x$ has a finite base $b_P$ representation.

**Proof.** We consider the case where $k = 2$, so that

$$x = yp_1^{c_1}p_2^{c_2}.$$  

Suppose that $c_1 \leq c_2$. Then

$$x = yp_2^{c_2-c_1}b_P^{c_1}$$

and $yp_2^{c_2-c_1}$ is an integer which is not divisible by $p_1$, so

$$d_{b_P}^F(x) = c_1.$$  

A similar argument shows that

$$d_{b_P}^F(x) = c_2$$

if $c_2 \leq c_1$. Hence,

$$d_{b_P}^F(x) = \min \{c_1, c_2\} = \min \{-\log_{p_1} |x|_{p_1}, -\log_{p_2} |x|_{p_2}\}.$$  

It is easy to see that $x$ has finite base $b_P$ representation. The remaining cases, where $k \neq 2$, follow by a similar argument.

**Corollary 2.3.4.** Let $P = \{p_r\}_{r=1}^k$ be a finite, non-empty set of primes. Let $b_P$ be the product of all primes in $P$. Suppose that $x$ is a non-zero rational number

$$x = y \prod_{r=1}^{k} p_r^{c_r}$$

for some integer $y$, such that no prime in $P$ divides $y$. If $d_{b_P}^F(x) \leq 0$, then

$$|d_{b_P}^F(x)| \leq \frac{1}{\ln 2} \sum_{p \in \mathbb{P}^*} \ln^+ |x|_p.$$  

(2.15)
Furthermore, if \( d_+^{b_P} (x) \leq 0 \), then \( d_+^{b_P} (x) \leq 0 \) and
\[
|d_+^{b_P} (x)| \leq |d_+^{b_P} (x)|,
\]
so Equation (2.15) holds with \( d_+^{b_P} (x) \) in the place of \( d_+^{b_P} (x) \).

**Proof.** By the previous corollary, there is an \( r \in \{1, \ldots, k\} \) such that
\[
|d_+^{b_P} (x)| = -d_+^{b_P} (x) = \log_{b_P} |x|_{p_r}.
\]
Since \( \ln 2 < \ln p \) for all primes \( p \),
\[
\log_{b_P} |x|_{p_r} \leq \frac{\ln^+ |x|_{p_r}}{\ln p_r} \leq \frac{\ln^+ |x|_{p_r}}{\ln 2} \leq \frac{1}{\ln 2} \sum_{p \in \mathcal{P}^*} \ln^+ |x|_p,
\]
which gives the desired inequality.

**Lemma 2.3.5.** Let \( P = \{p_r\}_{r=1}^k \) be a finite, non-empty set of primes. Let \( b_P \) be the product of all the primes in \( P \). Suppose that \( x \) is a non-zero rational number and that \( d_+^{b_P} (x) \geq 0 \). Then
\[
|d_+^{b_P} (x)| \leq \log_{b_P} (e) \ln^+ |x| \quad (2.16)
\]
for every \( p \in P \). Furthermore, if \( d_-^{b_P} (x) \geq 0 \), then \( d_+^{b_P} (x) \geq 0 \) and
\[
|d_-^{b_P} (x)| \leq |d_+^{b_P} (x)|,
\]
so Equation (2.16) holds with \( d_-^{b_P} (x) \) in the place of \( d_+^{b_P} (x) \).

**Proof.** By consideration of base \( b_P \) representations, we have
\[
|d_+^{b_P} (x)| = d_+^{b_P} (x) \leq \log_{b_P} |x| \leq \log_{b_P} |x| = \log_{b_P} (e) \ln^+ |x|.
\]
The right hand equality is just the change of base formula.

**Proposition 2.3.6.** Let \( P = \{p_r\}_{r=1}^k \) be a finite, non-empty set of primes. Let \( b_P \) be the product of all primes in \( P \). Suppose that \( f \in \mathrm{FG}_n(P) \). Then there are positive real constants \( K \) and \( M \) such that
\[
[f_{ij}]_{b_P} \leq K + M \sum_{p \in \mathcal{P}} \ln^+ \|f\|_p
\]
for every pair of \( i, j \in \mathbb{N} \) with \( i < j \leq n \).

**Proof.** Let \( i \) and \( j \) satisfy \( i < j \leq n \). First, we relate \([f_{ij}]_{b_P}\) to \( \ln^+ |f_{ij}|_p \) for each \( p \in \mathcal{P} \). From the definition of \([\cdot]_{b_P}\),
\[
[f_{ij}]_{b_P} = 1 + \max \{ |d_+^{b_P} (f_{ij})|, |d_-^{b_P} (f_{ij})| \}.
\]
2.3. ADELIC LENGTH IS A WORD METRIC ESTIMATE ON $FG_N(P)$

We now seek an upper bound on both $|d_P^{-}(f_{ij})|$ and $|d_P^{+}(f_{ij})|$. There are four cases to consider, depending on the signs of $d_P^{-}(f_{ij})$ and $d_P^{+}(f_{ij})$. If they are both non-positive, then we can use Corollary 2.3.4. Similarly, if they are both non-negative, we can use Lemma 2.3.5. If $d_P^{+}(f_{ij})$ is non-negative and $d_P^{-}(f_{ij})$ is non-positive, then a combination of those lemmas will provide our bound. The final case, where $d_P^{+}(f_{ij}) < d_P^{-}(f_{ij})$, is impossible by the definitions of $d_P^{+}$ and $d_P^{-}$.

In any case,
\[
[f_{ij}]_{b_p} \leq 1 + \frac{\log b_p}{\ln 2} (e) \sum_{p \in P} \ln^+ |f_{ij}|_p. \tag{2.17}
\]

Now we will relate $\ln^+ |f_{ij}|_p$ to $\|f\|_p$ for each $p \in P$. The max norm inequality (2.14) gives
\[
\ln^+ |f_{ij}| \leq \ln \sqrt{n} + \ln^+ \|f\|_\infty. \tag{2.18}
\]

Let $p$ be a prime, and let $w = (w_1, \ldots, w_n) \in \mathbb{Q}_p^n$ be the vector satisfying
\[
w_t = \begin{cases} 
1 & \text{if } t = j, \\
0 & \text{otherwise},
\end{cases}
\]
for every $t \in \{1, \ldots, n\}$. Then $|w|_p = 1$, so
\[
\|f\|_p = \sup_{|v|_p = 1} |fv|_p \\
\geq \sup_{|v|_p = 1} \max_{s} \left| \sum_{t=1}^n f_{st} v_t \right|_p \\
\geq \sup_{|v|_p = 1} \left| \sum_{t=1}^n f_{it} v_t \right|_p
\]
for each $i \in \{1, \ldots, n\}$. Hence,
\[
\|f\|_p \geq \left| \sum_{t=1}^n f_{it} w_t \right|_p = |f_{ij}|_p.
\]

It follows that
\[
\ln^+ \|f\|_p \geq \ln^+ |f_{ij}|_p. \tag{2.19}
\]

Substituting equations (2.18) and (2.19) into Equation (2.17) gives
\[
[f_{ij}]_{b_p} \leq K + M \sum_{p \in P} \ln^+ \|f\|_p,
\]
where
\[
K = 1 + \ln \sqrt{n}
\]
and $M = \frac{\log b_p (e)}{\ln 2}$, establishing our result. \qed
Lemma 2.3.7. Let $P = \{p_r\}_{r=1}^k$ be a finite set of primes, let $f \in \text{FG}_n(P)$, and let $c_{ii}^{(r)}$ be integers, as in Lemma 2.1.1, such that

$$f_{ii} = \prod_{r=1}^k p_r^{c_{ii}^{(r)}}.$$ 

Then

$$\sum_{r=1}^k |c_{ii}^{(r)}| \leq \frac{3 \ln n}{2 \ln 2} + \frac{3}{\ln 2} \sum_{p \in P} \ln^+ \|f\|_p.$$ 

Proof. We begin by observing that, for each $i \in \{1, \ldots, n\}$,

$$\ln^+ |f_{ii}| \geq \ln |f_{ii}| = \ln \left( \prod_{r=1}^k p_r^{c_{ii}^{(r)}} \right) = -\sum_{r=1}^k \ln \left( p_r^{c_{ii}^{(r)}} \right) = -\sum_{r=1}^k \ln |f_{ii}|_{p_r},$$

Since $\ln^+ |f_{ii}|_p$ is zero for all primes not in $P$,

$$\sum_{p \in P} \ln^+ |f_{ii}|_p = \ln^+ |f_{ii}| + \sum_{r=1}^k \ln^+ |f_{ii}|_{p_r} \geq -\sum_{r=1}^k \ln |f_{ii}|_{p_r} + \sum_{r=1}^k \ln^+ |f_{ii}|_{p_r} \geq -\sum_{r=1}^k \ln |f_{ii}|_{p_r} \geq -\sum_{r=1}^k \ln |f_{ii}|_{p_r} \quad \text{(2.20)}$$

where $\ln^-$ is the negative part of the natural logarithm function.

Now,

$$-\sum_{r=1}^k \ln |f_{ii}|_{p_r} = \sum_{r=1}^k c_{ii}^{(r)} = \sum_{r=1}^k |c_{ii}^{(r)}| - 2 \sum_{1 \leq r \leq k \text{ s.t. } c_{ii}^{(r)} < 0} |c_{ii}^{(r)}|.$$ 

Rearranging the last expression and applying Equation (2.20) yields

$$\sum_{r=1}^k |c_{ii}^{(r)}| \leq \sum_{p \in P} \ln^+ |f_{ii}|_p + 2 \sum_{1 \leq r \leq k \text{ s.t. } c_{ii}^{(r)} < 0} |c_{ii}^{(r)}|. \quad \text{(2.21)}$$
2.3. ADELIC LENGTH IS A WORD METRIC ESTIMATE ON \( \text{FG}_N(P) \)

Notice that \(-c_{ij}^{(r)} \ln(p_r) = \ln |f_{ii}|_{p_r}\). Hence, if \(c_{ij}^{(r)}\) is negative,

\[
\frac{\ln |f_{ii}|_{p_r}}{\ln 2} \geq |c_{ij}^{(r)}| = -c_{ij}^{(r)},
\]
as 2 is the smallest prime. Returning to Equation (2.21), we have

\[
\sum_{r=1}^{k} |c_{ii}^{(r)}| \leq \frac{3}{\ln 2} \sum_{p \in p} \ln^+ |f_{ii}|_p.
\]
The argument used in the proof of Proposition 2.3.6 to derive equation (2.18) and (2.19) shows that

\[
\ln^+ |f_{ii}|_p \leq \ln^+ \|f\|_p,
\]

for every prime \(p\) and

\[
\ln^+ |f_{ii}| \leq \ln \sqrt{n} + \ln^+ \|f\|_{\infty}.
\]

Hence,

\[
\sum_{r=1}^{k} |c_{ii}^{(r)}| \leq \frac{3}{\ln 2} \sum_{p \in p} \ln^+ |f_{ii}|_p \leq \frac{3 \ln n}{2 \ln 2} + \frac{3}{\ln 2} \sum_{p \in p} \ln^+ \|f\|_p,
\]
as required.

\[\Box\]

§ 2.3.8 Lower bounds

**Lemma 2.3.9.** Let \(P = \{p_r\}_{r=1}^{k}\) be a finite set of primes, let \(f \in \text{FG}_n(P)\) and let \(r_{ij}\) and \(c_{ij}^{(r)}\) be integers, as in Lemma 2.1.1, such that

\[
f_{ij} = \begin{cases} 
\prod_{r=1}^{k} p_r^{c_{ij}^{(r)}} & \text{if } i = j, \text{ and} \\
 r_{ij} \prod_{r=1}^{k} p_r^{c_{ij}^{(r)}} & \text{if } i < j.
\end{cases}
\]

Then

\[
\ln^+ \|f\|_{p_r} \leq \max_{1 \leq i \leq j \leq n} \left\{|c_{ij}^{(r)}| \ln (p_r)\right\}
\]

for all \(r \in \{1, \ldots, k\}\).

**Proof.** Suppose that \(v = (v_1, \ldots, v_n)\) is a vector in \(\mathbb{Q}_{p_r}\). Suppose that \(|v|_{p_r} = 1\). Then the ultrametric inequality and the definition of the \(p_r\)-adic norm imply that

\[
|fv|_{p_r} = \max_{1 \leq i \leq j \leq n} \left| p_r^{c_{ij}^{(r)}} v_j \right|_{p_r} \leq \max_{1 \leq i \leq j \leq n} p_r^{-c_{ij}^{(r)}} \leq \max_{1 \leq i \leq j \leq n} p_r^{c_{ij}^{(r)}}.
\]

Since \(p_r^{c_{ij}^{(r)}}\) is always greater than or equal to 1,

\[
\ln^+ \|f\|_{p_r} = \ln^+ \left( \sup_{|v|_{p_r} = 1} |fv|_{p_r} \right) \leq \ln^+ \left( \max_{1 \leq i \leq j \leq n} p_r^{c_{ij}^{(r)}} \right) = \max_{1 \leq i \leq j \leq n} \left\{|c_{ij}^{(r)}| \ln (p_r)\right\},
\]
establishing our result. \[\Box\]
Proposition 2.3.10. Let \( f \in \text{FG}_n(P) \) where \( P = \{p_r\}_{r=1}^k \) is a non-empty, finite set of primes. Then
\[
\sum_{p \in P} \left( \ln^+ \|f\|_p + \ln^+ \|f^{-1}\|_p \right) \leq K \|[f]_{n,P}\n
\]
for some positive real constant \( K \).

Proof. Let \( r_{ij} \) and \( c_{ij}^{(r)} \) be integers, as in Lemma 2.1.1, such that
\[
f_{ij} = \begin{cases} 
\prod_{r=1}^k p_r^{c_{ij}^{(r)}} & \text{if } i = j, \text{ and} \\
r_{ij} \prod_{r=1}^k p_r^{c_{ij}^{(r)}} & \text{if } i < j.
\end{cases}
\]

The result is true if \( f \) is the identity \( I_n \). Suppose that \( f \) is not the identity. We may apply Lemma 2.3.9 to show
\[
k \sum_{r=1}^k \ln(p_r) \sum_{i=1}^n |c_{ii}^{(r)}| + \sum_{1 \leq i < j \leq n} \sum_{r=1}^k |c_{ij}^{(r)}| \ln(p_r).
\]

Let \( M_{ij} = -\max_{r=1}^k \{|c_{ij}^{(r)}|\} \). Then, by Corollary 2.2.5,
\[
\sum_{r=1}^k |c_{ij}^{(r)}| \ln(p_r) \leq \sum_{r=1}^k |M_{ij}| \ln(p_r)
\]
\[
= |M_{ij}| \sum_{r=1}^k \ln(p_r)
\]
\[
\leq |d_{bp}(f_{ij})| \sum_{r=1}^k \ln(p_r).
\]

Hence,
\[
\sum_{r=1}^k \ln^+ \|f\|_p \leq \left( \sum_{r=1}^k \ln(p_r) \right) \left( \sum_{i=1}^n \sum_{r=1}^k |c_{ii}^{(r)}| + \sum_{1 \leq i < j \leq n} [f_{ij}]_{bp} \right)
\]
\[
= \left( \sum_{r=1}^k \ln(p_r) \right) \|[f]_{n,P}\n
\]
where \( b_P \) is the product of all the primes in \( P \). Since \( \|[\cdot]_{n,P}\n \) is a word metric estimate,
\[
\frac{1}{J} \|[f^{-1}]_{n,P}\n \leq |f^{-1}|_{n,P} = |f|_{n,P} \leq J \|[f]_{n,P}\n
\]
where \( J \) is the positive real constant from Proposition 2.2.6. This yields the desired inequality, namely,

\[
\sum_{p \in P} \left( \ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p \right) \leq K \| f \|_{n,P},
\]

where \( K = \left( \sum_{r=1}^{k} \ln(p_r) \right) \left( 1 + J^2 \right) \).

**Lemma 2.3.11.** Let \( P = \{ p_r \}_{r=1}^{k} \) be a finite set of primes. If the rational number \( f \) is a product of powers of primes from \( P \), that is,

\[
f = \prod_{r=1}^{k} p_r^{c(r)},
\]

then

\[
\hat{d}_P(f) \leq 1 + \sum_{r=1}^{k} |c(r)|,
\]

where \( b_P \) is the product of all the primes in \( P \).

**Proof.** We begin by noting that

\[
\hat{d}_P(f) \leq \hat{d}_P \left( \prod_{r=1}^{k} p_r^{c(r)} \right) \leq 1 + \ln b_P \left( \prod_{r=1}^{k} \prod_{1 \leq i \leq j \leq n} |f_{ij}| \right).
\]

Applying the logarithm laws,

\[
\hat{d}_P(f) = 1 + \sum_{r=1}^{k} |c(r)| \ln p_r.
\]

Since \( b_P \) is the product of all the primes in \( P \), \( \ln b_P (p_r) \leq 1 \) for each \( r \) in the above summation. So,

\[
\hat{d}_P(f) \leq 1 + \sum_{r=1}^{k} |c(r)|,
\]

and our result is established.

**Proposition 2.3.12.** Let \( f \in \text{FG}_n(P) \) for a finite, non-empty set of primes \( P = \{ p_r \}_{r=1}^{k} \). Then

\[
\ln^+ \| f \|_\infty + \ln^+ \| f^{-1} \|_\infty \leq L + M \| f \|_{n,P},
\]

where \( L \) and \( M \) are positive real constants.

**Proof.** The max norm inequality (2.14) yields

\[
\ln^+ \| f \|_\infty \leq \ln \sqrt{n} + \ln \left( \max_{1 \leq i \leq j \leq n} |f_{ij}| \right),
\]
where $| \cdot |$ is the ordinary absolute value. Hence,

\[
\ln^+ \| f \|_\infty \leq \ln \sqrt{n} + \ln(b_P) \left( 1 + d_{p+}^b \left( \max_{1 \leq i \leq j \leq n} |f_{ij}| \right) \right)
\]

\[
\leq \ln \sqrt{n} + \ln(b_P) \left( \sum_{i=1}^n \left( d_{p+}^b(|f_{ii}|) \right) + \sum_{1 \leq i < j \leq n} \left( d_{p+}^b(|f_{ij}|) \right) \right).
\]

Applying Lemma 2.3.11,

\[
\ln^+ \| f \|_\infty \leq \ln \sqrt{n} + \ln(b_P) \left( n + \sum_{i=1}^n \sum_{r=1}^k |c_{ii}^{(r)}| + \sum_{1 \leq i < j \leq n} \left( d_{p+}^b(|f_{ij}|) \right) \right)
\]

\[
\leq \frac{L}{2} + \ln(b_P) \| f \|_{n,P},
\]

where $L = \ln n + 2 \ln(b_P)$. Let $J$ be as in Proposition 2.2.6. Then, since $\| f \|_{n,P}$ is a word metric estimate,

\[
\ln^+ \| f^{-1} \|_\infty \leq \frac{L}{2} + \| f \|_{n,P}
\]

\[
\leq \frac{L}{2} + J \| f^{-1} \|_{n,P}
\]

\[
= \frac{L}{2} + J \| f \|_{n,P}
\]

\[
\leq \frac{L}{2} + J^2 \| f^{-1} \|_{n,P}.
\]

Hence,

\[
\ln^+ \| f \|_\infty + \ln^+ \| f^{-1} \|_\infty \leq L + M \| f \|_{n,P},
\]

where $M = 1 + J^2$, concluding the proof. \(\square\)

**Lemma 2.3.13.** Let $f \in FG_n(P)$ for a finite, non-empty set of primes $P = \{p_r\}_{r=1}^k$, and let $\mu$ be a probability measure on $FG_n(P)$ that has finite first moment with respect to word length. Then

\[
\ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p = 0
\]

unless $p \in P$ or $p = \infty$.

**Proof.** Suppose $p$ is any prime not in $P$. Let $r_{ij}$ and $c_{ij}^{(r)}$ be integers, as in Lemma 2.1.1, such that

\[
\begin{align*}
\text{if } i = j, \quad & f_{ij} = \prod_{r=1}^k p_r^{c_{ij}^{(r)}} \quad \text{and} \\
\text{if } i < j, \quad & f_{ij} = r_{ij} \prod_{r=1}^k p_r^{c_{ij}^{(r)}}
\end{align*}
\]

Suppose that $v = (v_1, \ldots, v_n)$ is a vector in $\mathbb{Q}_p$ with $|v|_p = 1$. The ultrametric inequality and
the definition of the $p$-adic norm imply that
\[
|f v|_p \leq \max_{1 \leq i, j \leq n} |f_{ij} v_j|_p \\
= \max_{1 \leq i, j \leq n} |f_{ij}|_p |v_j|_p \\
\leq \max_{1 \leq i, j \leq n} |f_{ij}|_p.
\]
Since $p \notin P$, we see that each $|f_{ij}|_p$ is less than or equal to 1. Hence,
\[
\|f\|_p = \sup_{|v|_p = 1} |f v|_p \leq 1,
\]
which implies that $\ln^+ \|f\|_p$ is zero. The same argument shows that $\ln^+ \|f^{-1}\|_p = 0$ because $f^{-1}$ is also a group element.

\section*{2.3.14 Conclusions}

We will now prove that the adelic length is a word metric estimate.

\begin{proposition}
Let $P = \{p_r\}_{r=1}^k$ be a finite set of primes. Suppose that $f \in \text{FG}_n(P)$. Then
\[
Q + R \|f\|_{n,a} \leq |f|_{n,P} \leq S + T \|f\|_{n,a}
\]
for some positive real constants $Q, R, S$ and $T$.

\end{proposition}

\begin{proof}
We will first prove the upper bound using the results from Subsection 2.3.8. By Proposition 2.2.6, there is a real, positive constant $J$ such that
\[
|f|_{n,P} \leq J \|f\|_{n,P}
\]
\[
= J \sum_{i=1}^n \sum_{r=1}^k |c_i^{(r)}| + J \sum_{1 \leq i < j \leq n} [f_{ij}]_{b_P}.
\]
Lemma 2.3.7 gives that
\[
\sum_{r=1}^k |c_i^{(r)}| \leq \frac{3 \ln n}{2 \ln 2} + \frac{3}{\ln 2} \sum_{P \in P} \ln^+ \|f\|_P.
\]
Since there are $\frac{(n-1)(n-2)}{2}$ upper off-diagonal entries in $f$, Proposition 2.3.6 implies that
\[
\sum_{1 \leq i < j \leq n} [f_{ij}]_{b_P} \leq \frac{(n-1)(n-2)}{2} \left( K + M \sum_{P \in P} \ln^+ \|f\|_P \right)
\]
for positive real constants $K$ and $M$. Therefore, if we let
\[
S = JK \frac{(n-1)(n-2)}{2} + J \frac{3 \ln n}{2 \ln 2},
\]
\[
R = \frac{3 \ln n}{2 \ln 2},
\]
\[
Q = \frac{3 \ln n}{2 \ln 2},
\]
\[
T = S = JK \frac{(n-1)(n-2)}{2} + J \frac{3 \ln n}{2 \ln 2}.
\]
and
\[ T = J \frac{3}{\ln 2} + JM \frac{(n-1)(n-2)}{2}, \]
then
\[ |f|_{n,P} \leq S + T \| f \|_{n,a}, \]
which is the upper bound.

Now we will establish the lower bound, using the results from Subsection 2.3.2. By definition, the adelic length of \( f \) is
\[ \| f \|_{n,a} = \sum_{p \in P} \left( \ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p \right), \]
where \( P = \mathbb{P}^* \cup \{\infty\} \) and \( \mathbb{P}^* \) is the set of all primes. By Lemma 2.3.13,
\[ \ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p \]
is zero unless \( p \in P \) or \( p = \infty \). Hence,
\[ \| f \|_{n,a} = \sum_{p \in P \cup \{\infty\}} \left( \ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p \right). \]
By Proposition 2.3.10, there is a positive real constant \( K \) such that
\[ \sum_{p \in P} \left( \ln^+ \| f \|_p + \ln^+ \| f^{-1} \|_p \right) \leq K \| f \|_{n,P}, \]
and by Proposition 2.3.12, there are positive real constants \( L \) and \( M \) such that
\[ \ln^+ \| f \|_\infty + \ln^+ \| f^{-1} \|_\infty \leq L + M \| f \|_{n,P}. \]
As before, let \( J \) be as in Proposition 2.2.6. Then, taking \( Q = \frac{L}{2} \) and \( R = \frac{K+M}{2} \),
\[ Q + R \| f \|_{n,a} \leq |f|_{n,P}, \]
which completes the proof.

\[ \square \]

**Corollary 2.3.16.** Let \( P = \{p_r\}_{r=1}^k \) be a finite, non-empty set of primes. Let \( \mu \) be a probability measure on \( \text{FG}_n(P) \). Then \( \mu \) has finite first moment with respect to word length if and only if it has finite first moment with respect to adelic length.

**Proof.** Suppose that \( \mu \) has finite first moment with respect to adelic length. The measure \( \mu \) has finite first moment with respect to word length if
\[ \int_{\text{FG}_n(P)} |f|_{n,P} \, d\mu(f) < +\infty. \]
2.3. **ADELIC LENGTH IS A WORD METRIC ESTIMATE ON $\text{FG}_N(P)$**

By Proposition 2.3.15, this quantity is bounded above by

$$S + T \int_{\text{FG}_n(P)} \|f\|_{n,a} \, d\mu(f),$$

which is finite by assumption. The same proposition gives a similar lower bound.

The corollary above allows us to restate Theorem 1.1 in Brofferio and Schapira [14] for $\text{FG}_n(P)$ with adelic length replaced by the word metric. Recall the definition of $\mathcal{B}$ from Section 2.3.

**Theorem 2.3.17.** Let $\mu$ be a probability measure on $\text{FG}_n(P)$ of finite first moment with respect to word length. Then there exists a unique probability measure $\nu$ on $\mathcal{B}$ such that $(\mathcal{B}, \nu)$ is the Poisson boundary of $(\text{GL}_d, \mu)$.

**Example 2.3.18** (Affine group over the dyadic rationals). Let $\text{Aff}(\mathbb{Z}[1/2])$ denote the affine group over the dyadic rationals:

$$\text{Aff}(\mathbb{Z}[1/2]) = \left\{ \begin{pmatrix} 2^n f & 0 \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}, f \in \mathbb{Z}[1/2] \right\}.$$

The group $\text{Aff}(\mathbb{Z}[1/p])$ is isomorphic to the solvable Baumslag-Solitar group $BS(1, p)$ (see McLaury [72]). It is a subgroup of $\text{FG}_n(2)$ for every $n \geq 2$, and is generated by the elements $\theta_{11}(2)$ and $\theta_{12}(1)$. Hence, a probability measure $\mu$ on $\text{Aff}(\mathbb{Z}[1/2])$ has finite first moment with respect to word length if and only if it has finite first moment with respect to adelic length.

Let $H$ be a finitely generated upper triangular group of matrices with rational entries. We have seen that $H$ is a subgroup of $\text{FG}_n(P)$ for suitable $n$ and $P$. Suppose that $\mu_H$ is a Borel probability measure on $H$. Then finite first moment of $\mu_H$ with respect to the word length taken over the generating set of $H$ is, in general, not equivalent to finite first moment with respect to the word length taken over a generating set of $\text{FG}_n(P)$. Equivalent conditions can be found by computing the *distortion* of $H$ in $\text{FG}_n(P)$ (see Davis and Olshanskii [26] for details).

**Example 2.3.19.** According to Burillo and Platón [15], the map $E : \text{UT}_n(\mathbb{Z}) \to \mathbb{N}$ satisfying

$$E(f) = \sum_{1 \leq i < j \leq n} |f_{ij}|^{1/j-1}$$
is a word metric estimate on that group. If $f$ is a matrix in $\text{UT}_n(\mathbb{Z})$, then

$$\ln E(f) \leq \sum_{\substack{1 \leq i < j \leq n \mid \nexists j \neq i \mid f_{ij} \neq 0}} \frac{1}{j - i} \ln (|f_{ij}|)$$

$$\leq \sum_{\substack{1 \leq i < j \leq n \mid \nexists j \neq i \mid f_{ij} \neq 0}} (1 + d_+(f_{ij}))$$

$$= \|f\|_{n,P}.$$ 

Since both $\|\cdot\|_{n,P}$ and $\|\cdot\|_{n,a}$ are word metric estimates, a probability measure $\mu$ on $\text{UT}_n(\mathbb{Z})$ has finite first logarithmic moment with respect to word length if it has finite first moment with respect to adelic length.
Chapter 3

Trees and products of trees

An automorphism $\varphi$ of a homogeneous tree $T$ is a permutation of the vertex set of $T$ satisfying

$$d(\varphi(v), \varphi(w)) = d(v, w)$$

for all vertices $v$ and $w \in T$. The set of all automorphisms of $T$ forms a group under function composition, called the automorphism group of $T$, and denoted by Aut $T$. Following the terminology of Cartwright, Kaimanovich and Woess [16], we call the closed subgroup Aff $T$ of all automorphisms of $T$ that fix a distinguished end $\omega$ the affine group of the homogeneous tree.

Cartwright, Kaimanovich and Woess studied random walks on Aff $T$ in [16]. They established a law of large numbers and a central limit theorem. In addition, they described the Poisson boundary for measures which have finite first moment with respect to a gauge function $|\cdot|_T$ and whose support generates a closed subgroup which is non-exceptional. We recall relevant conventions, definitions and results of this paper, including the definition of $|\cdot|_T$, in the preliminary section. We extend the work from [16] to closed subgroups in a direct product of automorphism groups of trees and to closed sub-quotients of t.d.l.c. groups.

In Sections 3.2 and 3.3, we consider closed subgroups of finite direct products of affine automorphism groups of homogeneous trees $P = \prod_{i=1}^k \text{Aff } T_i$. We define partially exceptional and fully exceptional closed subgroups of $P$, and explore the relationships between these properties, the modular function, transience of the random walk and the scale function. To ensure transience, we restrict our analysis to random walks associated with probability measures on $P$ whose supports generate subgroups that are not fully exceptional.

In Section 3.4, we consider the case where a probability measure $\mu$ on $P$ is spread-out, aperiodic, has finite first moment with respect to a subadditive gauge function $|\cdot|_P$, and where $\Gamma = \text{sgn } \mu$ is a closed subgroup of $P$ which is not fully exceptional. We show that the Poisson boundary
$(B, \nu)$ of $(G, \mu)$ is the direct product of the space of ends of each tree with a hitting measure, and
give necessary and sufficient conditions for boundary triviality. We conclude by showing that a
transitive action of $\Gamma$ on the ends is equivalent to $\Gamma$ being compactly generated. This simplifies the
proof that $(B, \nu)$ is the Poisson boundary in the transitive action case.

In the final section, we consider groups of the form $V \rtimes \langle \alpha \rangle$, where $V$ is a subgroup which
is tidy for an automorphism $\alpha$ of a t.d.l.c. group. According to Baumgartner and Willis [6], these
groups act naturally on an infinite homogeneous tree whose degree is dependent on the scale of $\alpha$. We describe the Poisson boundary of random walks on these groups in the case where the
probability measure generates a subgroup which is not fully exceptional, spread-out and has finite
first moment with respect to a gauge function closely related to $| \cdot |_T$.

\section*{3.1 Preliminaries}

Let $T$ be the homogeneous tree of valency $d$, that is, the unique connected graph with no cycles for
which every vertex has $d$ neighbours. For brevity, we set $q = d - 1$. We suppose that $d$ is at least
3, and use the term tree to mean a homogeneous tree of some valency. Let $o$ be a distinguished
vertex of $T$.

A path in a tree is a sequence of successive neighbouring vertices without backtracking. Paths
may be finite, singly infinite or doubly infinite. We adopt the convention that the length of the finite
path $v = \{v_i\}_{i=j}^k$ is denoted by $|v|$ and equals $k - j$. The length of an infinite path is $+\infty$. Let $v$
and $w$ be vertices of the tree $T$. The geodesic segment between $v$ and $w$ is the unique finite path $vw$
from $v$ to $w$. The distance $d(v, w)$ between $v$ and $w$ is the number of edges, $|vw| - 1$, between
the two vertices. The distance from the distinguished vertex $o$ to a vertex $v$ in $T$ is $|v| = d(o, v)$.

A geodesic ray in a tree is a singly infinite sequence of successive neighbours without back-
tracking. Two geodesic rays are equivalent if their intersection is infinite. This forms an equiva-
ence relation. Each equivalence class of rays is an end. The set of ends of a tree $T$ is denoted by
$\partial T$ and the disjoint union of the tree with its ends by $T \cup \partial T$. If $\xi \in \partial T$ and a vertex $v \in T$, then
there is a unique geodesic ray $v\xi$ starting at $v$ which represents $\xi$. Let $\omega$ be a distinguished end of
the tree $T$. The set of all ends except $\omega$ is denoted by $\partial^* T$.

Let $u, v \in T \cup \partial T$. If $u = v \in \partial T$, set $u \wedge v = u$. Let $u \wedge v$ denote the final vertex on the
path common to the geodesics \( \overline{uv} \) and \( \overline{ov} \), unless \( u = v \in \partial T \). Let

\[
\theta(u, v) = \begin{cases} 
q^{-|u \wedge v|} & \text{if } u \neq v, \\
0 & \text{if } u = v.
\end{cases}
\]

where \( |u \wedge v| \) is the length of the common path.

**Lemma 3.1.1.** The map \( \theta \) is an ultrametric on \( T \cup \partial T \). With the topology induced from \( \theta \), \( T \cup \partial T \) is a totally disconnected, compact space. The set \( T \) is an open, dense and discrete subspace of \( T \cup \partial T \), and \( \partial T \) is a closed compact subset of \( T \cup \partial T \).

**Proof.** The ultrametric inequality follows from the identity

\[
|u \wedge v| \geq \min\{|u \wedge w|, |v \wedge w|\}
\]

for all \( u, v \) and \( w \in T \cup \partial T \). Every ultrametric space is totally disconnected.

If \( u \in \partial T \), then, for every \( k \in \mathbb{N} \),

\[
B_{q^{-k}} \cap T = \{v \in T, \theta(u, v) \leq q^{-k}\}
= \{v \in T, |u \wedge v| \geq k\}
\]

is always non-empty. Hence, \( T \) is dense in \( T \cup \partial T \).

Suppose that \( v = \{v_i\}_{i=1}^{\infty} \) is a sequence in \( T \cup \partial T \). If the sequence \( \{|v_i|\}_{i=1}^{\infty} \) is bounded, then the sequence visits only finitely many points, so it has a convergent subsequence. If \( \{|v_i|\}_{i=1}^{\infty} \) is unbounded, then there is an end \( w \) such that for every \( k \in \mathbb{N} \), the open ball \( U_k = B_{q^{-k}}(w) \) of radius \( q^{-k} \) around \( w \) contains infinitely many points from \( v \). Choose a sequence of natural numbers \( \{n_i\}_{i=1}^{\infty} \) such that \( v_{n_k} \in U_k \). Then \( \{v_{n_i}\}_{i=1}^{\infty} \) is a subsequence of \( v \) that converges to \( w \).

So \( T \cup \partial T \) is sequentially compact and, hence, compact.

Suppose that \( u \in T \). Let \( k \in \mathbb{N} \) be greater than \( |u| \). Then

\[
B_{q^{-k}}(u) = \{v \in T \cup \partial T : |u \wedge v| \geq k\} \cup \{u\}.
\]

Suppose that \( v \in T \cup \partial T \) is distinct from \( u \). If \( |u \wedge v| = 0 \), then \( v \notin B_{q^{-k}}(u) \).

Suppose that \( |u \wedge v| \neq 0 \). Let \( w \) be any end \( \partial T \) such that \( u \in \overline{ow} \). Then, either \( u \wedge v = u \wedge w = u \) or \( u \wedge v = v \wedge w \). In the first case,

\[
k > |u \wedge v| = |u|
\]

and in the second case,

\[
k > |u \wedge w| = |u \wedge v|,
\]
So \( \{ v \in T \cup \partial T : |u \wedge v| \geq k \} \) is empty and \( B_{q^k}(u) \) is just the singleton containing \( u \). Since \( u \) was an arbitrary vertex, \( T \) is discrete and open. Since \( \partial T \) is the complement of an open set, it is a closed, compact subset of \( T \cup \partial T \).

Tits [94] showed that every automorphism of a tree is either elliptic, fixing a vertex or inverting an edge, or hyperbolic, acting as a translation along a doubly infinite path. Every automorphism of \( T \) extends naturally to a homeomorphism of \( T \cup \partial T \).

The automorphism group of \( T \) is a t.d.l.c. group in the compact-open topology. The open neighbourhoods of this topology are unions of sets of the form

\[
U(\alpha, F) = \{ \beta \in \text{Aut} T : \alpha v = \beta v \text{ for all } v \text{ in } F \},
\]

where \( F \) is any finite set of vertices and \( \alpha \) is any automorphism of \( T \). The compact open topology is also known as the topology of pointwise convergence, because a sequence of automorphisms \( \alpha_n \) converges to \( \alpha \) if and only if, for every finite set of vertices \( F \), \( \alpha_n(v) \) and \( \alpha(v) \) eventually agree for all \( v \in F \). With the compact-open topology, \( \text{Aut} \ T \) is Hausdorff, locally compact, \( \sigma \)-compact and second countable. Since \( \text{Aut} \ T \) is not compact, any right Haar measure on \( \text{Aut} \ T \) is an infinite measure.

We will later make use of the following technical lemma.

**Lemma 3.1.2** (Cartwright and Soardi [17], Lemma 2.2). Let \( T \) be an infinite homogeneous tree with finite vertex degree. Let \( o \) be a distinguished vertex of \( T \). Let \( \{ \varphi_i \}_{i=1}^{\infty} \) be a sequence in \( \text{Aut} T \) such that \( \{ \varphi_i o \}_{i=1}^{\infty} \) converges to a point \( \xi \in \partial T \) and \( \{ \varphi_i^{-1} o \}_{i=1}^{\infty} \) converges to a point \( \eta \in \partial T \) in the topology induced by the ultrametric \( \theta \). Then \( \{ \varphi_i v \}_{i=1}^{\infty} \) converges to \( \xi \) for every element \( v \in T \cup \partial T \setminus \{ \eta \} \).

Suppose that \( \omega \) is a distinguished end of \( T \). Let the Busemann function \( h \) be the map from the vertices of \( T \) to \( \mathbb{Z} \) given by

\[
h(v) = d(v, c) - d(o, c),
\]

for each vertex \( v \), where \( c \) is the first common vertex on the geodesic rays \( \overline{xo} \) and \( \overline{oc} \). The horocyclic map \( \phi : \text{Aff} T \to \mathbb{Z} \) given by

\[
\phi(\alpha) = h(\alpha o) = h(\alpha v) - h(v)
\]

for any vertex \( v \in T \) is a well defined homomorphism because automorphisms of \( T \) are distance preserving under \( d \); hence,

\[
h(u) - h(v) = h(\alpha u) - h(\alpha v)
\]
for all automorphisms \( \alpha \in \text{Aut } \mathcal{T} \) and vertices \( u \) and \( v \in \mathcal{T} \). A horocycle is a level set of \( h \),

\[
H_m = \{ v \in \mathcal{T} : h(v) = m \}.
\]

The horocyclic group, denoted by \( \text{Hor } \mathcal{T} \), is the kernel of \( \phi \). That is, the closed subgroup of \( \text{Aut } \mathcal{T} \) which preserves each horocycle \( H_m \). An element of \( \text{Hor } \mathcal{T} \) is a horocyclic automorphism. For every vertex \( v \in \mathcal{T} \), every \( \alpha \in \text{Hor } \mathcal{T} \) fixes the common ancestor of \( v \) and \( \alpha v \).

If \( u, v \) are vertices in \( \mathcal{T} \) and \( h(u) < h(v) \), then \( u \) is said to be an ancestor of \( v \) and \( v \) is said to be a descendant of \( u \). Every horocyclic automorphism \( \alpha \) fixes the common ancestor of \( v \) and \( \alpha v \) for every vertex \( v \in \mathcal{T} \).

The affine group of the homogeneous tree \( \mathcal{T} \) is the closed subgroup of all automorphisms of \( \mathcal{T} \) that fix the distinguished end \( \omega \), endowed with the subspace topology. We denote it by \( \text{Aff } \mathcal{T} \). The subspace topology on \( \text{Aff } \mathcal{T} \) is second countable, t.d.l.c. and \( \text{Aff } \mathcal{T} \) is \( \sigma \)-compact.

The affine group of \( \mathcal{T} \) is an internal semi-direct product, isomorphic to \( \mathbb{Z} \ltimes \text{Hor } \mathcal{T} \). To see this, fix \( \sigma \in \text{Aff } \mathcal{T} \) such that \( \sigma(o) \) is a descendant of \( o \) and \( \phi(\sigma) = 1 \). The intersection of \( \langle \sigma \rangle \cong \mathbb{Z} \) and \( \text{Hor } \mathcal{T} \) contains only the identity, and every element \( \gamma \in \text{Aff } \mathcal{T} \) can be written as

\[
\gamma = \left( \gamma \sigma^{-\phi(\gamma)} \right) \sigma^{\phi(\gamma)},
\]

where \( \gamma \sigma^{-\phi(\gamma)} \in \text{Hor } \mathcal{T} \) and \( \sigma^{\phi(\gamma)} \in \langle \sigma \rangle \cong \mathbb{Z} \). 

Figure 3.1: Horocycles of a homogeneous tree with vertex degree 3, with the vertex \( c \), common to \( x_\omega \) and \( o_\omega \) marked.
CHAPTER 3. TREES AND PRODUCTS OF TREES

Given $\gamma \in \text{Aff} \ T$, let $|\gamma|_T = |\gamma o|$. The map $| \cdot |_T$ is a subadditive gauge function because

$$|\alpha \beta|_T = d(o, \alpha \beta o)$$
$$\leq d(o, \alpha o) + d(\alpha o, \alpha \beta o)$$
$$= d(o, \alpha o) + d(o, \beta o)$$
$$= |\alpha|_T + |\beta|_T$$

for all automorphisms $\alpha$ and $\beta \in \text{Aff} \ T$. The relation

$$|\gamma|_T = d(o, \gamma o) = d(\gamma^{-1} o, o) = |\gamma^{-1}|_T$$

is also satisfied for all $\gamma \in \text{Aff} \ T$. Other properties of this gauge function are given in Lemma 4 of Cartwright, Kaimanovich and Woess [16].

**Lemma 3.1.3.** Let $v$ be a vertex in $T$. Then

$$|v| = d(o, v) \geq |h(v)|.$$  

**Proof.** Let $c$ be the first common vertex on the geodesic rays $v\omega$ and $o\omega$, so that

$$h(v) = d(v, c) - d(o, c).$$

If $v \in \overline{\omega}$, then $c = v$, so $h(v) = -d(o, v)$. If $v$ is a descendant of $o$, then $c = o$, so $|h(v)| = d(o, v) = |v|$. Suppose that $v \notin \overline{\omega}$ and that $v$ is not a descendant of $o$, then

$$d(o, v) = d(o, c) + d(v, c).$$

If $d(o, c) > d(v, c)$, then $d(o, v) > d(o, c) - d(v, c) = |h(v)|$. Similarly, if $d(v, c) > d(o, c)$, then $d(o, v) > d(v, c) - d(o, c) = |h(v)|$, establishing our result.  

**Corollary 3.1.4.** Suppose that $\mu$ is a probability measure on $\text{Aff} \ T$ with finite first moment with respect to $| \cdot |_T$. The pushforward measure $\phi_* \mu$ has finite first moment with respect to the ordinary absolute value on the integers.

**Proof.** Because the first moment of $\mu$ is finite, Theorem 3.6.1 in Bogachev [11] applies, and

$$m_1(\phi_* \mu) = \int_{\mathbb{Z}} |z| d\phi_* \mu(z)$$
$$= \int_{\text{Aff} \ T} |\phi(\gamma)| d\mu(\gamma).$$
Hence,

$$m_1 (\phi_* \mu) \leq \int_{\text{Aff} \mathcal{T}} |h(\gamma o)| \, d\mu(\gamma) \leq \int_{\text{Aff} \mathcal{T}} |\gamma o| \, d\mu(\gamma) = \int_{\text{Aff} \mathcal{T}} |\gamma| \, d\mu(\gamma),$$

which is finite. \qed

A sequence of vertices \( \{v_i\}_{i=1}^\infty \in \mathcal{T} \) is regular if there is an end \( \xi \in \partial \mathcal{T} \) and a non-negative real number

$$a = \lim_{n \to \infty} \frac{1}{n} d(v_n, \xi_{\lfloor an \rfloor}),$$
called the rate of escape, where \( \xi_{\lfloor an \rfloor} \) is the \( n \)th vertex on the geodesic \( o\xi \), and \( \lfloor \cdot \rfloor \) is the floor function. Notice that if \( a \) is non-zero, then \( v_n \) converges and \( \xi \) is unique.

**Lemma 3.1.5** (Cartwright, Kaimanovich and Woess [16], Lemma 3). A sequence of vertices \( \{v_i\}_{i=1}^\infty \in \mathcal{T} \) is regular with rate of escape \( a \) if and only if

(i) \( \lim_{n \to \infty} \frac{1}{n} d(v_n, v_{n+1}) = 0 \), and

(ii) \( \lim_{n \to \infty} \frac{1}{n} |v_n| = a \).

**Proposition 3.1.6** (Cartwright, Kaimanovich and Woess [16], Proposition 1). Let \( h \) be the Busemann function with respect to a distinguished end \( \omega \) of \( \mathcal{T} \). A sequence of vertices \( v = \{v_i\}_{i=1}^\infty \in \mathcal{T} \) is regular if and only if

(i) \( \lim_{n \to \infty} \frac{1}{n} d(v_n, v_{n+1}) = 0 \), and

(ii) the limit \( a_h = \lim_{n \to \infty} \frac{1}{n} h(v_n) \) exists.

In this case, the rate of escape of \( v \) is \( |a_h| \), and the following holds:

(i) If \( a_h < 0 \), then \( v \) converges to \( \omega \).

(ii) If \( a_h = 0 \) then \( |v_n| \in o(n) \).

(iii) If \( a_h > 0 \), then \( v \) converges to \( \partial^* \mathcal{T} \).

(iv) \( \lim_{n \to \infty} \frac{1}{n} |v_n| = a \).
Following Cartwright, Kaimanovich and Woess [16], we call a closed subgroup of $\text{Aff} \, T$ with distinguished end $\omega$ exceptional if it is either contained within $\text{Hor} \, T$, or there is an end distinct from $\omega$ which is fixed by every element.

**Proposition 3.1.7.** Let $\Gamma$ be an exceptional closed subgroup of $\text{Aff} \, T$, with distinguished end $\omega$. Then $\Gamma$ is uniscalar.

**Proof.** Suppose that $\Gamma$ is contained in $\text{Hor} \, T$, and let $\gamma \in \Gamma$, and $\alpha$ be conjugation by $\gamma$. Then $\alpha$ fixes the element $o \wedge \gamma o$. Let

$$U = \text{stab}_\Gamma(o \wedge \gamma o).$$

Then $U$ is compact and open and $\alpha(U) = U$, so

$$s(\gamma) = [\alpha(U) : U \cap \alpha(U)] = 1.$$

Suppose, instead, that $\Gamma$ fixes $\xi \in \partial^* T$. Then $\gamma$ acts as a translation along the doubly infinite path $\xi \omega$. Let $v \in \xi \omega$, and let

$$V = \text{stab}_\Gamma(v).$$

Then $V$ is compact and open. Notice that $V$ fixes every vertex on the path $\xi \omega$. Hence, $\xi \omega$ is fixed by $\gamma$, and

$$\alpha(V) = \text{stab}_\Gamma(\gamma \cdot v) = \text{stab}_\Gamma(v) = V.$$

It follows that

$$s(\alpha) = [\alpha(V) : V \cap \alpha(V)] = 1$$

in this case too. It follows that $\Gamma$ is uniscalar.

**Theorem 3.1.8.** Suppose that $\Gamma$ is a closed subgroup of $\text{Aff} \, T$. Then $\Gamma$ is unimodular if and only if it is exceptional.

**Proof.** By Proposition 3.1.7, if $\Gamma$ is exceptional, it is uniscalar, and, hence, unimodular. The remainder of the argument is the same as the one in Cartwright, Kaimanovich and Woess [16].

Suppose that $\Gamma$ is non-exceptional. Then $\Gamma \setminus \text{Hor} \, T$ is non-empty. It follows that the horocyclic map $\phi$ is non-trivial and that $\phi(\Gamma) = r \mathbb{Z}$, where

$$r = \min \{ \phi(\gamma) > 0 : \gamma \in \Gamma \}.$$

Let $\gamma \in \Gamma$ satisfy $\phi(\gamma) = r$. Then $\gamma$ acts by translation on a doubly infinite path $\xi \omega$, where $\xi$ is the unique fixed point of $\gamma \in \partial^* T$. Since $\Gamma$ is non-exceptional, $\Gamma$ does not fix $\xi$, hence, there is an $\alpha \in \Gamma$ for which $\alpha \xi \neq \xi$. 

Reference: Cartwright, Kaimanovich and Woess [16].
3.1. PRELIMINARIES

Now, \( \phi(\alpha) = lr \) for some integer \( l \). Let \( \beta = \gamma^{-l} \alpha \), and let \( \zeta = \beta \xi \). Then \( \phi(\beta) = 0 \), and \( \zeta \in \partial^* T \setminus \{\xi\} \). Let \( x = \xi \land \zeta \) be the least common ancestor of \( \xi \) and \( \zeta \in T_k \). Then \( \gamma \cdot \xi \) and \( (\beta \circ \gamma) \cdot x \) are distinct elements, and \( \beta \) fixes \( x \). Lemma 5 in Woess [105] gives that

\[
\Delta(x) = \frac{|\text{stab}_T(\gamma x) \cdot x|}{|\text{stab}_T(x) \cdot (\gamma x)|}.
\]

Therefore, \( \Gamma \) is not unimodular: \( \text{stab}_T(x) \cdot (\gamma x) \) contains \( \gamma x \) and \( \beta \gamma x \), and \( \text{stab}_T(\gamma x) \cdot x \) contains only \( x \), since \( \omega_k \) is fixed by \( \Gamma \).

**Corollary 3.1.9** (Cartwright, Kaimanovich and Woess [16]). Suppose that \( \mu \) is a probability measure on \( \text{Aff} T \). Suppose that \( \text{sgr} \mu \) is non-exceptional. Then \( (\text{Aff} T, \mu) \) is a transient random walk.

**Proof.** Since \( \text{sgr} \mu \) is non-exceptional, it is not unimodular. Non-unimodularity implies transience of the random walk (see Guivarc’h, Keane and Roynette [45]).

**Theorem 3.1.10** (Cartwright, Kaimanovich and Woess [16], Theorem 4). Suppose that \( \mu \) is a probability measure on \( \text{Aff} T \). Let \( R_n \) be the right random walk associated with \( (\text{Aff} T, \mu) \). Suppose \( \mu \) has finite first moment with respect to \( |\cdot|_T \). Then,

\[
\lim_{n \to \infty} d(R_n o, R_{n+1} o) = 0
\]

almost surely, and

\[
\lim_{n \to \infty} \frac{1}{n} |R_n|_T = |m_1 (\phi \mu)|
\]

almost surely and in \( L_1 \)-norm.

**Proof.** The existence of the limit can be demonstrated using the argument given in proof of Theorem 8.14 in Woess [106]. If \( n, m \in \mathbb{N} \) and \( n > m \), then

\[
|R_n(\omega)|_T = d(e, R_{n+m}(\omega)) \leq d(e, R_m(\omega)) + d(R_m(\omega), R_{n+m}(\omega))
\]

\[
\leq d(e, R_m(\omega)) + d(e, R_m(\omega) \cdot \ldots \cdot X_{n+m}(\omega))
\]

\[
= d(e, R_m(\omega)) + d(e, R_n(T^m \omega))
\]

for \( \mu^\mathbb{N} \)-almost all \( \omega \in G^\mathbb{N} \), where \( T \) is the measure preserving left shift map we defined in Section 1.4.1.

It follows by the Kingman Subadditive Ergodic Theorem in the form given by Steele in [92] that, if \( \mu \) has finite first moment with respect to \( |\cdot|_T \), then the limit \( \lim_{n \to \infty} \frac{1}{n} |R_n|_T \) exists and is well defined.
By the Monotone Convergence Theorem,
\[ 0 = \lim_{n \to \infty} \frac{1}{n} \int_{\text{Aff } T} |x| \, d\mu(x) = \lim_{n \to \infty} \frac{1}{n} \int_{\text{Aff } T} \frac{1}{n} |X_{n+1}(\gamma)| \, d\mu^\gamma(\gamma) = \lim_{n \to \infty} \frac{1}{n} \int_{\text{Aff } T} d(R_n(\gamma)o, R_{n+1}(\gamma)o) \, d\mu^\gamma(\gamma). \]
Since \( d \) is always non-negative,
\[ \lim_{n \to \infty} d(R_no, R_{n+1}o) = 0 \]
\( \mathbb{P}^{\mu} \)-almost surely. Therefore, \( R_no \) is regular by Lemma 3.1.5 and Proposition is satisfied with \( a_h = |m_1(\phi_\ast \mu)|. \)

\[ \square \]

**Theorem 3.1.11** (Cartwright, Kaimanovich and Woess [16], Theorem 2). Let \( \mu \) be a probability measure on \( \text{Aff } T \) and associated right random walk \( \{R_i\}_{i \in \mathbb{N}} \). Suppose that \( \text{sgr } \mu \) is non-exceptional. Let \( o \) be a distinguished vertex of \( T \), and let \( \omega \) be the fixed end.

(i) If \( m_1(\phi_\ast \mu) \) is finite and the mean of \( \phi_\ast \mu \) is negative, then \( R_no \) converges almost surely to \( \omega \).

(ii) If \( m_1(\mu) \), with respect to \( |\cdot|_T \), is finite and the mean of \( \phi_\ast \mu \) is positive, then \( R_no \) converges to an end in \( \partial^* T \) almost surely.

(iii) If \( m_1(\mu) \), with respect to \( |\cdot|_T \), is finite and the mean of \( \phi_\ast \mu \) is zero, and
\[ \mathbb{E} \left( |o \wedge R_{i-1}o| q^{o \wedge R_{i+1}o} \right) \]
is finite, then \( R_no \) converges to the end \( \omega \) fixed by every element of \( \text{Aff } T \) almost surely.

### 3.2 Partially exceptional and fully exceptional subgroups

Let \( P = \prod_{i=1}^k \text{Aff } T_i \) be a finite direct product of affine automorphism groups of trees with the product topology. For each affine group \( \text{Aff } T_j \), let \( \omega_j \) be the fixed end, let \( o_j \) be the distinguished vertex and let \( d_j \) be the vertex degree. Let \( \phi_j : \text{Aff } T_j \to \mathbb{Z} \) be the horocyclic map, \( \omega = \{\omega_i\}_{i=1}^k \), \( o = \{o_i\}_{i=1}^k \) and \( \phi = \prod_{i=1}^k \phi_i \). Let \( \pi_j : P \to \text{Aff } T_j \) be the projection map. Given \( \alpha \in P \), we write \( \alpha_j \) for \( \pi_j(\alpha) \). Similarly, if \( v \in \prod_{i=1}^k T_i \), we write \( v_j \) to mean the projection of \( v \) onto \( T_j \).

Notice that \( P \) is \( \sigma \)-compact because it is a product of \( \sigma \)-compact spaces, and that the topology is t.d.l.c. and second countable. The action of each factor \( \text{Aff } T_j \) on \( \partial T_j \cup T_j \) extends to an action of \( P \) on the Cartesian product \( \prod_{i=1}^k (\partial T_i \cup T_i) \).
Suppose that $\Gamma$ is a closed subgroup of $P$. If $\pi_j(\Gamma)$ is an exceptional subgroup of $\pi_j(P)$ for every $j \in \{1, \ldots, k\}$, then we say that $\Gamma$ is *fully exceptional*. If there is at least one $j \in \mathbb{N}$ for which $\pi_j(\Gamma)$ is a exceptional subgroup of $\pi_j(P)$, then we say that $\Gamma$ is *partially exceptional*. If $\Gamma$ is fully exceptional, then it is partially exceptional. A partially exceptional subgroup need not be uniscalar or unimodular.

**Proposition 3.2.1.** Let $\Gamma$ be a closed subgroup of a finite product $P = \prod_{i=1}^{k} \text{Aff} \ T_i$ of affine automorphism groups of trees. Then $\Gamma$ is fully exceptional if and only if it is uniscalar.

**Proof.** Suppose that $\Gamma$ is fully exceptional. Let 

$$E = \{ j \in \{1, \ldots, k\} : \pi_j(\Gamma) \text{ is non-exceptional} \}$$

Let 

$$E' = \{ j \in \{1, \ldots, k\} : \pi_j(\Gamma) \text{ is exceptional} \}$$

set 

$$H = \{ j \in \{1, \ldots, k\} : \pi_j(\Gamma) \subset \text{Hor} \ T_j \}$$

and set $H' = E' \setminus H$.

Suppose that $\gamma$ is an automorphism of $\Gamma$. For each $i$ in $H$ there is an element $v_i := o_i \land \gamma_i o_i \in T_i$ fixed by $\gamma_i$. For each $i \in H'$, let $v_i \in \partial^* T_i$ be the element fixed by $\pi_i(\Gamma)$, and let $v_i \in \overline{\nu_{\gamma_i}}$. Then $v_i$ is translated by the action of $\gamma_i$ for each $i \in H'$. Let 

$$V = \text{stab}_\Gamma (v),$$

where $v := \{v_i\}_{i=1}^{k} \in \prod_{i=1}^{k} T_i$.

It is easy to see that $V$ is compact and open. If $i$ is in $H$ then $\gamma_i(V_i) = V$ by choice of $v_i$. If $i \in H'$ then $V_i$ fixes $\overline{\nu_{\gamma_i}}$ and $\gamma_i(V_i) = V$ too. It follows that $\gamma(V) = V$, and that 

$$s(\gamma) = [\alpha(V) : V \cap \alpha(V)] = 1.$$ 

Since $\gamma$ was an arbitrary automorphism, $\Gamma$ is uniscalar.

Suppose, instead, that $\Gamma$ is not fully exceptional. Then $E$ is non-empty. We construct a compact open subgroup $U$ which is tidy for an automorphism $\alpha$ of $P$ of scale greater than one.

Let $j$ be a particular element of $E$. Then $\pi_j(\Gamma)$ is non-exceptional, $\phi_j(\Gamma)$ is a non-trivial homomorphism, and $\phi_j(\Gamma) = r_j \mathbb{Z}$, where 

$$r_j = \min \left\{ \phi_j(\gamma_j) > 0 : \{\gamma_i\}_{i=1}^{k} \in \Gamma \right\}.$$
Choose $\gamma \in \Gamma$ satisfying $\phi_j(\gamma) = -r_j$. Then $\gamma$ acts by translation on $T_i$. Let $v_j$ be a vertex in $T_j$ on the axis of translation of $\gamma_j$. For each $i \in \mathbb{N} \cap (E \setminus \{j\})$, choose $v_i$ to be the vertex $o_i \wedge \gamma_i o_i \in T_i$ fixed by $\gamma_i$ in $T_i$ if $\gamma_i$ is elliptic, and choose $v_i$ to be a vertex on the axis of translation if $\gamma_i$ is hyperbolic. For each $i \in H$, let $v_i := o_i \wedge \gamma_i o_i \in T_i$, which is fixed by the action of $\gamma$. For each $i \in H'$, let $v_i$ be a vertex on the axis fixed by $\gamma_i(\Gamma)$. Let $\alpha$ be conjugation by $\gamma$, and let $U$ be the compact open subgroup
\[
U = \text{stab}_\Gamma (v),
\]
where $v := \{v_i\}_{i=1}^k$. Then
\[
U \cap \alpha(U) = \text{stab}_P \{x_i\}_{i=1}^k,
\]
where
\[
x_r = \begin{cases} 
  v_r & \text{if } r \in E, \\
  v_r & \text{if } r \in E' \text{ and } \phi_r(\gamma) \geq 0, \\
  \gamma_r v_r & \text{otherwise}
\end{cases}
\]
for each $r \in \{1, \ldots, k\}$. Hence,
\[
[\alpha(U) : U \cap \alpha(U)] \geq d_j^r
\]
by choice of $\gamma$. Since $\omega_1$ is fixed by $\phi_i(\Gamma)$ for each $i \in \{1, \ldots, k\}$, $U = U_+ U_-$, and $U$ is tidy above. It is tidy below because $U_+ U_+$ is open and therefore closed. Hence, $\Gamma$ is not uniscalar.

**Proposition 3.2.2.** Suppose that $\mu$ is a probability measure on the finite product $P = \prod_{i=1}^k \text{Aff } T_i$ of affine automorphism groups of trees. Suppose that $\text{sgn } \mu$ is not fully exceptional. Then $(P, \mu)$ is transient.

**Proof.** Since $\text{sgn } \mu$ is not fully exceptional, there is a $j \in \{1, \ldots, k\}$ for which $\pi_j (gr \mu)$ is a non-exceptional subgroup of $\text{Aff } T_j$. Let $E$ be a compact subset of $\text{sgn } \mu$. Then $\pi_j (E)$ is compact because $\pi_j$ is continuous. Corollary 3.1.9 implies that the projected random walk $(\text{Aff } T_j, (\pi_j)_* \mu)$ is transient, that is the right random walk corresponding to $(\text{Aff } T_j, \pi_j_* \mu)$ leaves $\pi_j (E)$ almost surely after finitely many steps. It follows that the right random walk associated with $(P, \mu)$ eventually leaves $E$ almost surely.

**Lemma 3.2.3.** A closed subgroup $\Gamma \in P$ can be unimodular, but not uniscalar.

**Proof.** We demonstrate how to construct an example for the case where $k = 2$ and both trees have degree $d$. The same type of construction can be used if $k$ is larger and/or the trees have unequal vertex degrees.
3.3. RANDOM WALKS AND GAUGE FUNCTIONS

Let \( \nu_1 \in \partial^* \mathcal{T}_2 \) and let \( \nu_2 \in \partial^* \mathcal{T}_2 \). Let \( \sigma_1 \in \text{Aff} \mathcal{T}_1 \) by translation along \( \nu_1 \) away from \( \omega_1 \) and satisfy \( \phi_1(\sigma_1) = 1 \). Similarly, let \( \sigma_2 \in \text{Aff} \mathcal{T}_2 \) act by translation towards \( \omega_2 \) along \( \nu_2 \) and satisfy \( \phi_2(\sigma_2) = -1 \). Let

\[
\Gamma = \{(\sigma_1^{k_1} h_1, \sigma_2^{k_2} h_2) : k_1, k_2 \in \mathbb{Z}, h_1 \in \text{Hor} \mathcal{T}_1, h_2 \in \text{Hor} \mathcal{T}_2 \}
\]

Lemma 5 in Woess [105] yields

\[
\Delta_P(h_1, h_2) = \frac{|\text{stab}_G(o_1, o_2)(h_1, h_2)|}{|\text{stab}_G(h_1 o_1, h_2 o_2)(o_1, o_2)|} = 1,
\]

and

\[
\Delta_P(\sigma_1, \sigma_2) = \frac{|\text{stab}_G(o_1, o_2)(\sigma_1, \sigma_2)|}{|\text{stab}_G(\sigma_1 o_1, \sigma_2 o_2)(o_1, o_2)|} = 1.
\]

As \( \Delta_P \) is a homomorphism, it follows that \( \Gamma \) is a unimodular subgroup.

To see that \( \Gamma \) is not uniscalar, let \( \alpha = (\alpha_1, \alpha_2) \) be conjugation by \( (\sigma_1, \sigma_2) \). Let \( O \subseteq \Gamma \) be the stabilizer of a pair of vertices \( (o_1, o_2) \). Then \( \alpha(O) = \text{stab}_P(\sigma_1 o_1, \sigma_2 o_2) \). Since \( \omega_1 \) and \( \omega_2 \) are both fixed,

\[
O \cap \alpha(O) = \text{stab}_P(\sigma_1 o_1, o_2).
\]

Hence, by assumption,

\[
[\alpha(O) : O \cap \alpha(O)] = d > 1.
\]

We need to check that \( O \) is tidy above and tidy below. Since \( (\omega_1, \omega_2) \) is fixed, \( O = O_+ O_- \) and \( O \) is tidy above. It is tidy below because the subgroup

\[
O_{++} = \bigcup_{k=0}^{\infty} \text{stab}_P(\sigma_1^k o_1, \sigma_1^k o_2)
\]

is a open, and therefore closed. It follows that \( \alpha \) has scale \( d \), and \( \Gamma \) is not uniscalar. \( \square \)

### 3.3 Random walks and gauge functions

For each element \( \varphi \in P \), recall that

\[
|\varphi|_P = \sum_{i=1}^{k} d(o_i, \varphi_i o_i) = \sum_{i=1}^{k} |\varphi_i|_{\mathcal{T}_i}.
\]

By Lemma 1.2.6, \( |\cdot|_P \) is a subadditive gauge function, and the sequence

\[
\mathcal{A}^P = \left\{ \left\{ \gamma \in \Gamma : \sum_{i=1}^{k} d(o_i, \gamma_i o_i) \leq j \right\} \right\}_{j=1}^{\infty}
\]
is the corresponding subadditive gauge.

Let \( \mu \) be an aperiodic, spread-out probability measure on a finite direct product \( P = \prod_{i=1}^{k} \text{Aff} \, T_i \) of affine automorphism groups of trees. Suppose that \( \mu \) has finite first moment with respect to \( \cdot \mid_P \), and that \( \Gamma = \text{sgn} \, \mu \) is a closed subgroup of \( P \) which is not fully exceptional. For each \( j \in \{1, \ldots, k\} \), let \( \mu_j \) be the pushforward measure \( \mu_j(E) = \phi_j \ast \mu \).

**Lemma 3.3.1.** Suppose that \( \mu \) is a probability measure on \( P \). Let \( j \in \{1, \ldots, k\} \). If \( \mu \) has finite first moment with respect to \( \cdot \mid_P \), then \( \pi_j \ast \mu \) has finite first moment with respect to \( \cdot \mid_{T_j} \), and \( \mu_j \) has finite first moment with respect to the ordinary absolute value on the integers.

**Proof.** If \( m_1(\mu) \) is finite, then \( m_1(\pi_j \ast \mu) \) is finite, since

\[
m_1(\mu) = \int_{\Gamma} |x|_P \, d\mu(x) = \int_{\Gamma} \sum_{i=1}^{k} |x_i|_{T_i} \, d\mu(x) = \sum_{i=1}^{k} \int_{\Gamma} |x|_{T_i} \, d\pi_i \ast \mu(x_i) = \sum_{i=1}^{k} m_1(\pi_i \ast \mu) \geq m_1(\pi_j \ast \mu).
\]

Hence, \( \pi_j \ast \mu \) is finite whenever \( m_1(\mu) \) is finite.

**Corollary 3.3.2.** Suppose that \( \mu \) is a probability measure on \( P \). Let \( \{R_i\}_{i \in \mathbb{N}} \) be the right random walk associated with \( (P, \mu) \). Suppose that \( \mu \) has finite first moment with respect to \( \cdot \mid_P \), and that \( j \in \{1, \ldots, k\} \).

(i) If \( m_1(\mu_j) \) is negative, then \( \{\pi_j(R_i) \omega_j\}_{i=1}^{\infty} \) converges almost surely to \( \omega \).

(ii) If \( m_1(\mu_j) \) is positive, then \( \{\pi_j(R_i) \omega_j\}_{i=1}^{\infty} \) converges to a random end in \( \partial^* T_j \).

**Proof.** This is a special case of Theorem 3.1.11.

Let \( B = \prod_{i=1}^{k} \partial T_i \), endowed with the product topology, where each factor \( \partial T_j \) has the subspace topology induced from the ultrametric on \( T_j \cup \partial T_j \). Then \( B \) is compact because each factor \( \partial T_j \) is closed in the compact set \( T_j \cup \partial T_j \). Let \( P \) act on \( B \) via the continuous action satisfying

\[
\pi_j(g \cdot b) = \pi_j(g) \cdot \pi_j(b)
\]
for each \( g \in P, \ b \in B \) and \( j \in \{1, \ldots, k\} \). Here \( \pi_j(g) \in \text{Aff} T_j \) acts on \( \pi_j(b) \in \partial T_j \) by taking the restriction to \( \partial T_j \) of the extension of \( \pi_j(g) \) to a compatible homeomorphism of \( T_j \cup \partial T_j \). Let \( R_\infty : \Gamma^N \to B \) satisfy

\[
\pi_j(R_\infty(\gamma)) = \begin{cases} 
\omega_j & \text{if } m_1(\mu_j) < 0, \\
\lim_{n \to \infty} \pi_j(R_n(\gamma)) o_j & \text{if } m_1(\mu_j) \geq 0
\end{cases}
\]

for almost all paths \( \gamma \). The limit exists almost everywhere by Corollary 3.3.2. Let \( \nu = (R_\infty)_* \mu \) be the hitting measure on \( B \). Then \( \nu \) is \( \mu \)-stationary because

\[
\int_P \int_B f(xb) \, d\nu(b) \, d\mu(x) = \int_P \int_{\text{pri}} f(xR_\infty(\gamma)) \, d\mu(\gamma) \, d\mu(x)
\]

\[
= \int_P f(R_\infty(T\gamma)) \, d\mu(\gamma)
\]

\[
= \int_P f(R_\infty(\gamma)) \, d\mu(\gamma)
\]

\[
= \int_B f(b) \, d\nu(b),
\]

where \( T \) is the left shift map and \( f \in C(B) \). Notice that \( R_\infty \) is shift invariant in the sense of Proposition 1.4.23. It follows that \((B, \nu)\) is a \( \mu \)-boundary of \( (P, \mu) \).

**Proposition 3.3.3.** Let \( \mu \) be an aperiodic, spread-out probability measure on a finite direct product \( P = \prod_{i=1}^k \text{Aff} T_i \) of affine automorphism groups of trees. Suppose that \( \mu \) has finite first moment with respect to \( | \cdot |_P \), and that \( \Gamma = \text{sg}_\mu \mu \) is a closed subgroup of \( P \) which is not fully exceptional. Let \( u \in B, \ n \in \mathbb{N} \) and

\[
A^{(r)}(u) = \left\{ \{ \gamma \in \Gamma : \sum_{i=1}^k d\left( (\sigma_i u_i)_{nm_1(\mu_i)}, \gamma o_i \right) \leq j \} \right\}_{r=1}^\infty \setminus j = 1.
\]

Then \( \{A^{(r)}(u)\}_{r=1}^\infty \) is a uniformly temperate sequence of gauges on \( \Gamma \), and

\[
|\varphi|_{A^{(n)}(u)} = \sum_{i=1}^k d\left( (\sigma_i u_i)_{nm_1(\mu_i)}, \gamma o_i \right)
\]

is the corresponding gauge function on \( P \).

**Proof.** Each gauge \( A^{(n)} \) is well defined because the finite first moment condition on \( \mu \) and Lemma 3.3.1 ensure that \( m_1(\mu_i) \) is always finite. If \( \gamma \in \Gamma \), then the distance from \( \gamma o_i \) to any vertex is finite. It follows that

\[
\Gamma = \bigcup_{j=1}^\infty \left\{ \gamma \in \Gamma : \sum_{i=1}^k d\left( (\sigma_i u_i)_{nm_1(\mu_i)}, \gamma o_i \right) \leq j \right\}.
\]
Let $\gamma, \phi \in \Gamma$. The gauge map corresponding to $A^{(n)}(u)$ is

\[
|\gamma|_{A^{(n)}(u)} = \min \{ j \in \mathbb{N} : \gamma \in A_j \} = \sum_{i=1}^k d \left( (\sigma_i u_i)_{|nm_1(\mu_i)} , \gamma_i o_i \right).
\]

Set

\[
K = |e|_{A^{(n)}} = \sum_{i=1}^k d \left( (\sigma_i u_i)_{|nm_1(\mu_i)} , o_i \right),
\]

where $e \in \Gamma$ is the identity element. By repeated use of the triangle inequality,

\[
|\gamma \phi|_{A^{(n)}} = \sum_{i=1}^k d \left( (\sigma_i u_i)_{|nm_1(\mu_i)} , \gamma_i \phi_i o_i \right) \leq \sum_{i=1}^k d (\gamma_i o_i , \phi_i o_i) + K \leq \sum_{i=1}^k d (\gamma_i^{-1} o_i , o_i) + \sum_{i=1}^k d (o_i , \phi_i o_i) + K \leq \sum_{i=1}^k d (o_i , \gamma_i o_i) + \sum_{i=1}^k d (o_i , \phi_i o_i) + K = |\gamma|_{A^{(n)}} + |\phi|_{A^{(n)}} + K.
\]

So $|\cdot|_{A^{(n)}}$ is a gauge function. For each $j \in \mathbb{N}$, let

\[
M_j = \lambda_{\Gamma} \left( \bigg\{ \gamma \in \Gamma : \sum_{i=1}^k d \left( (\sigma_i u_i)_{|nm_1(\mu_i)} , \gamma_i o_i \right) \leq j \bigg\} \right),
\]

where $\lambda_{\Gamma}$ is a right Haar measure on $\Gamma$, normalised so the compact open stabilizer of $o$ has measure 1. For each $x$ in $\prod_{i=1}^k T_i$, let $\Gamma^x = \{ \gamma \in \Gamma : x = \gamma o \}$. Then $\lambda_{\Gamma}(\Gamma^o) = 1 = \lambda_{\Gamma}(\Gamma^x)$ whenever $\Gamma^x$ is non-empty. It follows that there is a real constant $C$ satisfying

\[
M_j \leq C \left( \max_{1 \leq i \leq k} \{ d_i \} \right)^j,
\]

where $d_i$ is the degree of $T_i$. Hence, the sequence $\{A^{(r)}(u)\}_{r=1}^\infty$ is uniformly temperate. \qed

**Theorem 3.3.4.** Let $\mu$ be an aperiodic, spread-out probability measure on a finite direct product $P = \prod_{i=1}^k \text{Aff } T_i$ of affine automorphism groups of trees. Suppose that $\mu$ has finite first moment with respect to $|\cdot|_P$, and that $\Gamma = \text{sgx } \mu$ is a closed subgroup of $P$ which is not fully exceptional. Then $(B, \nu)$ is the Poisson boundary of $(P, \mu)$. 


Proof. Suppose that $j \in \{1, \ldots, k\}$. Let $u \in B$ and $n \in \mathbb{N}$. Theorem 3.1.10 implies that

$$
\lim_{n \to \infty} d(\pi_j(R_n)\alpha_j, \pi_j(R_{n+1})\alpha_j) = 0
$$

almost surely, and

$$
\lim_{n \to \infty} \frac{1}{n} |\pi_j(R_n)| = |m_1(\phi_\mu)|
$$

almost surely and in $L_1$-norm. Hence, $\pi_j(R_n)\alpha_j$ is almost surely a regular sequence of vertices in $\mathcal{T}_j$, with rate of escape $|m_1(\mu_j)|$. Thus, for almost every path $\psi$ in the random walk, there is an end $u_j(\psi) \in \partial \mathcal{T}_j$ satisfying

$$
\lim_{n \to \infty} d\left(\pi_j(R_n(\psi))\alpha_j, (\alpha_j u_j(\psi))|_{m_1(\mu_j)n}\right) = |m_1(\mu_j)|,
$$

If $|m_1(\mu_j)| \neq 0$, then $\pi_j(R_n)\alpha_j \to u_j(\psi)$, and so $u_j(\psi)$ is unique. By Theorem 3.1.11, $u_j(\psi) = \omega_j$ if $m_1(\mu_j) < 0$, and $u_j(\psi) \in \partial^* \mathcal{T}_p$ if $m_1(\mu_j) > 0$. If $|m_1(\mu_j)| = 0$, then $u_j(\psi)$ is arbitrary and the statement reduces to

$$
\lim_{n \to \infty} d(\pi_j(R_n(\psi))\alpha_j, \alpha_j) = 0.
$$

Hence, for almost all paths $\gamma$,

$$
\lim_{n \to \infty} \frac{1}{n} |R_n(\gamma)|_{A^{(n)}(R_\infty(\gamma))} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^k d\left(\pi_i(R_n(\gamma))\alpha_i, \left((\alpha_i(R_\infty(\gamma)))|_{m_1(\mu_i)n}\right)\right)
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^k |m_1(\mu_i)|
$$

$$
= 0.
$$

where $A^{(n)}(u)$ is the gauge from Proposition 3.3.3:

$$
A^{(n)}(u) = \left\{\gamma \in \Gamma : \sum_{i=1}^k d\left((\gamma_i u_i)|_{m_1(\mu_i)}\right) \leq r\right\}_{r=1}^\infty.
$$

By assumption, $\mu$ has finite first moment with respect to the subadditive gauge $A^P$. Hence, by Theorem 1.4.40, $(B, \nu)$ is the Poisson boundary.

Proposition 3.3.5. Let $\mu$ be an aperiodic, spread-out probability measure on a finite direct product $P = \prod_{i=1}^k \text{Aff} \mathcal{T}_i$ of affine automorphism groups of trees. Suppose that $\mu$ has finite first moment with respect to $|\cdot|_P$, and that $\Gamma = \text{sgr} \mu$ is a closed subgroup of $P$ which is not fully exceptional. Let $(B, \nu)$ be the Poisson boundary associated with $(G, \mu)$. If there is $j \in \{1, \ldots, k\}$ such that $m_1(\mu_j) > 0$,

then $\nu$ is a continuous measure supported on an uncountable set.
Proof. We adapt the proofs of Proposition 2 and Theorem 3 in Cartwright, Kaimanovich and Woess [16] to products of trees.

Let $\partial \Gamma$ be the set of accumulation points of an orbit $\Gamma v \in B$ for some $v \in \prod_{i=1}^{k} T_i$. Since $\Gamma$ is a subgroup, this orbit is not dependent on the particular choice of $v$. Choose any $j \in \mathbb{N}$ such that $m_1(\mu_j) > 0$.

Let $\partial_j \Gamma$ be the set of accumulation points of an orbit $\pi_j(\Gamma) v_j \in \pi_j(B)$. Since $\pi_j(\Gamma)$ is non-exceptional, there exists $\alpha \in \Gamma$ such that

$$\phi_j(\alpha) = \min \{ \phi_j(\gamma) : \gamma \in \Gamma \}. $$

It follows that $\alpha_j$ is hyperbolic, and so $\alpha$ acts on $T_j$ by translation along a doubly infinite sequence $\pi_j u_j$, where $u_j \in B_j$ is distinct from $\omega_j$. Since $\pi_j(\Gamma)$ is non-exceptional, there is a $\beta \in \Gamma$ such that $\beta_j u_j$ is an end in $\partial T_j$, distinct from $u_j$.

Let $\xi = \beta \alpha \beta^{-1}$. Choose any sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$. For each $i \in \{1, \ldots, k\}$, let

$$x_i^{(k)} = \prod_{r=1}^{i} \alpha^{k_r} \xi^{k_r} \in \Gamma.$$

Choose $v$ to be the common ancestor of $u_j$ and $\beta_j u_j \in T_j$.

The sequence $x_i^{(k)} o_j$ is convergent to an end for each choice of $k$ because it is Cauchy with respect to the ultrametric $\theta$, and $(x_i^{(k)} \wedge x_{i+1}^{(k)}) v$ becomes arbitrarily large as $i$ tends to infinity. Furthermore, different choices of $k$ result in sequences that are eventually contained in disjoint open balls, and result in convergence to a distinct end in $\partial T_j$. We conclude that $\partial_j \Gamma$ is uncountable. Since $B$ is a closed subset of a product of compact second countable spaces, there is a subsequence $\{x_{n_i}^{(k)}\}_{i=1}^{\infty}$ such that $\{x_{n_i}^{(k)} o_{r} \}_{r=1}^{\infty}$ is convergent for all $r \in \{1, \ldots, k\}$. Hence, $\partial \Gamma$ is also uncountable.

Suppose that $\alpha = \{\alpha^{(i)}\}_{i=1}^{\infty}$ is a sequence in $\pi_j(\Gamma)$ such that $\alpha^{(i)} o_j$ converges to $v \in \partial T_j \setminus \{\omega_j\}$. Let

$$\beta^{(i)} = \left(\alpha^{(i)}\right)^{-1}.$$ 

Then $\beta^{(i)} o_j$ converges to $\omega_j$, and $\omega_j \in \partial_j \Gamma$. Taking $\varphi_r = \beta^{(r)}$, $\xi = \omega_j$ and $\eta = v$ in the statement of Lemma 3.1.2, we see that $\alpha^{(i)} u$ converges to $\omega_j$ for every $u \in \partial T_j \setminus \{\omega, v\}$. Hence, $\omega_j$ is an accumulation point of $\pi_j(\Gamma) u$ for every $u \in \partial T_j \setminus \{\omega_j\}$. Applying Lemma 3.1.2 again, with $\varphi_r = \alpha^{(r)}$, $\xi = v$ and $\eta = \omega_j$, we see that $\beta^{(i)} u$ converges to $v$ for every $u \in \partial T_j \setminus \{\omega_j\}$.

Given some $u \in \partial T_j \setminus \{\omega_j\}$, choose $\eta \in \Gamma$ such that $\eta_j u$ and $u$ are distinct ends. Then either $\{\alpha^{(i)} u\}_{i=1}^{\infty}$ or $\{\alpha^{(i)} \eta u\}_{i=1}^{\infty}$ has an infinite subsequence of elements, distinct from $v$. Hence, $v$ is a
3.4. ACTIONS OF TOTALLY DISCONNECTED GROUPS ON TREES

limit point of \( \Gamma u \setminus \{v\} \). Thus, for each \( u \in B \), such that \( u_j \) is not equal to \( \omega_j \), the orbit \( \pi_j(\Gamma)u_j \) is dense in \( \partial_j \Gamma \).

Theorem 3.1.11 implies that \( \pi_j(R_\infty) \in \partial^n \mathcal{T}_j \setminus \{\omega\} \) almost surely; hence, \( \nu(\pi^{-1}(\{\omega_j\})) \) is zero. Let \( M = \max \{\nu(\{u\}) : u \in B\} \). Suppose for contradiction that \( \nu \) is not a continuous measure. Then \( M \) is positive and the set

\[
S = \{u \in B : \nu(\{u\}) = M\}
\]

is non-empty and finite because \( \nu \) is a probability measure. Since \( \nu \) is \( \mu \)-stationary, \( \gamma S = S \) for all \( \gamma \in \text{supp} \mu \). But then \( \Gamma S = S \). So \( \pi_j(\Gamma)\pi_j(S) = \pi_j(S) \), which contradicts the fact that \( \pi_j(\Gamma)\pi_j(S) \) is dense in \( \partial_j \Gamma \). Hence, \( \nu \) is continuous.

\[\square\]

Corollary 3.3.6. Let \( \mu \) be an aperiodic, spread-out probability measure on a finite direct product \( P = \prod_{i=1}^{k} \text{Aff} \mathcal{T}_i \) of affine automorphism groups of trees. Suppose that \( \mu \) has finite first moment with respect to \( |\cdot|_P \), and that \( \Gamma = \text{sgr} \mu \) is a closed subgroup of \( P \) which is not fully exceptional. Then the Poisson boundary of \( (P, \mu) \) is trivial if and only if

\[m_1(\mu_j) \leq 0\]

for all \( j \in \{1, \ldots, k\} \).

**Proof.** If \( m_1(\mu_j) \leq 0 \) for all \( j \in \{1, \ldots, k\} \), then the boundary is a single point by definition. If there is a \( j \in \mathbb{N} \) such that \( m_1(\mu_j) > 0 \), then Proposition 3.3.5 implies that \( \nu \) is not a point measure. \[\square\]

3.4 Actions of totally disconnected groups on trees

We will now discuss the tree representation theory of Baumgartner and Willis [6] for totally disconnected, locally compact groups in the context of random walks. For the convenience of the reader, we sometimes refer to the results from previous sections, but because our groups act on a single tree, the arguments of Cartwright, Kaimanovich and Woess [16] are sufficiently general. We follow Horodam [52] in the opening paragraphs of this section.

Suppose that \( G \) is a totally disconnected, locally compact group. Let \( \alpha \) be an automorphism of \( G \). Let \( V \) be a compact open subgroup of \( G \) which is tidy for \( \alpha \). Suppose that the order of \( \alpha \) is infinite. Let \( V_- , V_-, V_+, V_++ , V_0 \) and the scale function \( s \) be as in Section 1.2.12.
Consider the semi-direct product $V_+ \rtimes \langle \alpha \rangle$. Identify $V_+$ with the subgroup $V_+ \times \{e\}$ and $V_-$ with the subgroup $V_- \times \{e\}$. For each $v \in V_-$ and each integer $m$, let $(v, m)$ be the left coset $v\alpha^m(V_-)$ of $V_- \in V_- \rtimes \langle \alpha \rangle$. Baumgartner and Willis described an action of $V_- \rtimes \langle \alpha \rangle$ on a homogenous tree $T$ in [6]. The tree $T$ has a vertex for each left coset of $v\alpha^m(V_-)$ and a directed edge from $(v, m)$ to $(w, n)$ if and only if $n = m + 1$ and $w \in v\alpha^m(V_-)$.

**Lemma 3.4.1.** Every vertex of $T$ has degree $s(\alpha^{-1}) + 1$. Each vertex receives one inward facing edge and emits $s(\alpha^{-1})$ outward facing edges.

**Proof.** Suppose $(v, n)$ is any vertex of the tree $T$. Let $(w_1, m_2)$ and $(w_1, m_2)$ be vertices which emit edges ending at $(w, n)$. Then $m_1 = m_2 = n - 1$. Set $m := n - 1$. Then $v$ must lie in the intersection of $w_1\alpha^m(V_-)$ and $w_2\alpha^m(V_-)$. As the left cosets of $V_- \in V_- \rtimes \langle \alpha \rangle$ must be equal or disjoint,

$$w_1\alpha^m(V_-) = w_2\alpha^m(V_-),$$

that is $(w_1, m_1)$ and $(w_2, m_2)$ are equal. Each vertex of $T$ therefore has only one inward facing edge.

For the outward facing edges, suppose that $(w_2, \alpha^{n_1}), (w_2, \alpha^{n_2}) \in V_-$ represent the same left coset of $V_-$. Set $n = n_1$. Then $n = n_1 = n_2$ and $(w_2, n) = (w_2, n)$, that is $w_2^{-1}w \in \alpha^n(V_-)$. Suppose that there is an edge from a vertex $(v, m)$ to $(w_2, n)$. Then both $w_2$ and $w_2 \in v\alpha^{n-1}(V_-)$, and hence $w_2^{-1}w_2 \in v\alpha^{n-1}(V_-)$. Therefore, there are $[\alpha^{n-1}(V_-) : \alpha^n(V_-)]$ choices of $(w, n)$ which result in distinct edges. Since

$$s(\alpha) = [\alpha(V_+) : V_+]$$

and

$$s(\alpha^{-1}) = [\alpha^{-1}(V_-) : V_-] = [\alpha^m(V_-) : \alpha^{m+1}(V_-)],$$

each vertex has $s(\alpha^{-1})$ outward facing edges, and every vertex of $T$ has degree $s(\alpha^{-1}) + 1$. 

The double path $P = \{(e, n)\}_{n \in \mathbb{Z}}$ is infinite as the order of $\alpha$ is infinite. Therefore, $T$ is infinite. Let $-\omega$ be the end of $T$ corresponding to the infinite path $\{(e, n)\}_{n \in \mathbb{N}}$, and let $\omega$ be the end corresponding to the infinite path $\{(e, -n)\}_{n \in \mathbb{N}}$. If $(v, m)$ and $(w, n)$ are vertices of $T$, then the paths $\{(v, m + k)\}_{k \in \mathbb{N}}$ and $\{(w, n + k)\}_{k \in \mathbb{N}}$ both eventually ascend to an element of the path $P$, which is connected, and so $T$ is connected. There are no cycles in $T$ because any cycle without backtracking must be a directed path, but the power of $\alpha$ increases strictly along directed edges, and there is only one edge directed into each vertex.
Because a vertex \( v \) is a parent of a vertex \( w \) if and only if \( v \) is the second element in the path from \( w \) to \( \omega \), we lose no information by regarding \( T \) as an undirected graph. We treat \( V_- \) as a distinguished vertex, and label it \( o \) as in our previous sections.

Suppose that the scale of \( \alpha \) is not one, so that the degree of each vertex in \( T \) is at least 3. It is easy to verify that the left action of \( V_- \rtimes \langle \alpha \rangle \) is a bijection on the vertex set of \( T \) that preserves adjacency of vertices. The element \((e, 1)\) of \( V_- \rtimes \langle \alpha \rangle \) acts by translation by one edge on \( P \) away from the distinguished end \( \omega \) towards \(-\omega\). Since every path descends from \( \omega \), it is fixed by the action of \( V_- \rtimes \langle \alpha \rangle \). The action is transitive on the other ends (see Baumgartner and Willis [6]):

As before, let \( \text{Aut} \, T \) denote the group of all automorphisms of \( T \) with the compact-open topology, and let \( \text{Aff} \, T \) denote the closed subgroup of automorphisms which fix \( \omega \), and let \( \text{Hor} \, T \) denote the subgroup of elliptic elements. Let \( \pi : V_- \rtimes \langle \alpha \rangle \to \text{Aut} \, T \) be the representation of the action of \( V_- \rtimes \langle \alpha \rangle \) on \( T \). Then

(i) \( \pi \) is continuous,

(ii) the image of \( \pi \) is a closed subgroup of \( \text{Aff} \, T \),

(iii) the kernel of \( \pi \) is the largest compact, normal, \( \alpha \)-stable subgroup of \( V_- \), and

(iv) the image of \( V_- \) under \( \pi \) is contained in \( \text{Hor} \, T \).

Let \( \eta = \phi \circ \pi \), so that if \((v, n) \in V_- \rtimes \langle \alpha \rangle\), then \( \eta(v, n) = n \). Let \( \rho : V_- \rtimes \langle \alpha \rangle \to V_- \rtimes \langle \alpha \rangle / \ker \pi \). The representation on \( V_- \rtimes \langle \alpha \rangle / \ker \pi \in \text{Aff} \, T \) is faithful. Every closed subgroup of the quotient group \( V_- \rtimes \langle \alpha \rangle / \ker \pi \) isomorphic to a closed subgroup of \( \text{Aff} \, T \).

The reader may wish to review the statements made about amenability in Section 1.2. The group \( \text{Aff} \, T \) is amenable. Furthermore, the quotient \( V_- \rtimes \langle \alpha \rangle / \ker \pi \) is amenable because it is isomorphic to a closed subgroup of \( \text{Aff} \, T \). The kernel of the map \( \pi \) is amenable because it is compact; hence, \( V_- \rtimes \langle \alpha \rangle \) is amenable. Amenability may also been shown directly from the structure of the group.

We will shortly describe the Poisson boundary of \((V_- \rtimes \langle \alpha \rangle, \mu)\) for the case where \( \mu \) is an aperiodic, spread-out Borel probability measure on \( V_- \rtimes \langle \alpha \rangle \), and \( \text{sgn} \, \mu \) is a closed subgroup of \( V_- \rtimes \langle \alpha \rangle \). The argument is similar to the proof of Theorem 3.3.4 where \( P \) is taken to be single affine automorphism group of a tree.

Kaimanovich [59] states that if \( G \) is any locally compact group with an amenable, closed, normal subgroup \( H \) and \( q : G \to G/H \) is the quotient homomorphism, then for any Borel
probability measure \( \mu' \) on \( G/H \) there is a Borel probability measure \( \mu \) on \( G \) such that \( q_* \mu = \mu' \) and the Poisson boundaries of \((G, \mu)\) and \((G/H, \mu')\) are isomorphic. That is if \( \mu' \) is a Borel probability measure on \( V_- \rtimes \langle \alpha \rangle \) then there is a Borel probability measure \( \mu \) on \( V_- \rtimes \langle \alpha \rangle \) such that \( \pi_* \mu = \mu' \), and so that the Poisson boundaries of \((V_- \rtimes \langle \alpha \rangle, \mu)\) and \((V_- \rtimes \langle \alpha \rangle / \ker \pi, \mu')\) are isomorphic. Our argument to describe the Poisson boundary of \((V_- \rtimes \langle \alpha \rangle, \mu)\) shows that \( \mu' = \pi_* \mu \).

If \( V_- \) is compact, then it is equal to \( \ker \pi \), hence \((V_- \rtimes \langle \alpha \rangle / \ker \pi, \pi_* \mu)\) is a random walk on the integers and \((V_- \rtimes \langle \alpha \rangle, \mu)\) has trivial Poisson boundary.

For each element \( (v, n) \in V_- \rtimes \langle \alpha \rangle \), set

\[
|(v, n)|_{V_- \rtimes \langle \alpha \rangle} = |\pi(v, n)|_\mathcal{T} = d(V_-, v\alpha^n(V_-)).
\]

**Lemma 3.4.2.** The map \( | \cdot |_{V_- \rtimes \langle \alpha \rangle} \) is a subadditive gauge function, and \( \mu \) has finite first moment with respect to \( | \cdot |_{V_- \rtimes \langle \alpha \rangle} \) if and only if the pushforward measure \( \pi_* \mu \) has finite first moment with respect to \( | \cdot |_\mathcal{T} \).

**Proof.** Let \((v_1, n_1)\) and \((v_2, n_2)\) be elements of \( V_- \rtimes \langle \alpha \rangle \). Then, according to Baumgartner and Willis [6],

\[
|(v_1, n_1)(v_2, n_2)|_{V_- \rtimes \langle \alpha \rangle} = d(V_-, (v_1, n_1)(v_2, n_2) \cdot V_-)
\]

\[
= d((v_1, n_1)^{-1} \cdot V_-, (v_2, n_2) \cdot V_-)
\]

\[
\leq d((v_1, n_1)^{-1} \cdot V_-, V_-) + d(V_-, (v_2, n_2)V_-)
\]

\[
= |(v_1, n_1)|_{V_- \rtimes \langle \alpha \rangle} + |(v_2, n_2)|_{V_- \rtimes \langle \alpha \rangle}.
\]

So \( | \cdot |_{V_- \rtimes \langle \alpha \rangle} \) is a subadditive gauge function. Since \( | \cdot |_{V_- \rtimes \langle \alpha \rangle} = |\pi(\cdot)|_\mathcal{T} \),

\[
\int_{\text{Aff } \mathcal{T}} |x| \, d\pi_* \mu(x) = \int_{\text{Aff } \mathcal{T}} |x| \, d\pi_* \mu(x)
\]

\[
= \int_{V_- \rtimes \langle \alpha \rangle} |x| \, d\mu(x),
\]

establishing the second statement. \( \square \)

Suppose that \( \Gamma \) is a closed subgroup of \( V_- \rtimes \langle \alpha \rangle \). Then \( \rho(\Gamma) \) is closed and isomorphic to \( \pi(\Gamma) \). Hence, \( \pi(\Gamma) \) is closed. We say that \( \Gamma \) is non-exceptional if it has a non-exceptional image under \( \pi \).

We now restate Theorems 3.1.10 and 3.1.11 in terms of \( V_- \rtimes \langle \alpha \rangle \) and \( \eta \).
Theorem 3.4.3. Let \( R_m = (v^{(m)}, r^{(m)}) \) be the right random walk associated with \( (V_- \rtimes \alpha, \mu) \). Suppose \( \mu \) has finite first moment with respect to \( | \cdot |_{V_- \rtimes \alpha} \). Then

\[
\lim_{m \to \infty} d(R_m V_-, R_{m+1} V_-) = 0
\]

almost surely, and

\[
\lim_{n \to \infty} \frac{1}{n} |R_m|_{V_- \rtimes \alpha} = |m_1(\eta_* \mu)|
\]

almost surely and in \( L_1 \)-norm.

Theorem 3.4.4. Let \( \mu \) be a Borel probability measure such that the closure of the semigroup generated by \( \mu \) is a closed, non-exceptional subgroup \( \Gamma \) of \( V_- \rtimes \alpha \). Let \( \{R_m\}_{m=1}^{\infty} \) be the right random walk associated with \( (V_- \rtimes \alpha, \mu) \). Let \( V_- \) be a distinguished vertex of \( T \) with fixed end \( \omega \).

(i) If \( m_1(\eta_* \mu) \) is finite and the mean of \( \eta_* \mu \) is negative, then \( R_m V_- \) converges almost surely to \( \omega \).

(ii) If \( m_1(\mu) \), with respect to \( |(v, n)|_{V_- \rtimes \alpha} \), is finite and the mean of \( \eta_* \mu \) is positive, then \( R_m V_- \) converges to a random end in \( \partial^* T \) almost surely.

(iii) If \( m_1(\mu) \), with respect to \( |(v, n)|_{V_- \rtimes \alpha} \), is finite and the mean of \( \eta_* \mu \) is zero, and

\[
\mathbb{E} \left( |V_- \wedge R_m^{-1} V_-| q_{V_- \wedge R_m V_-} \right)
\]

is finite, then \( R_m V_- \) converges to \( \omega \) almost surely.

From now on, suppose that \( \mu \) is a spread-out Borel probability measure on \( V_- \rtimes \alpha \) which is non-exceptional and has finite first moment with respect to \( | \cdot |_{V_- \rtimes \alpha} \).

Proposition 3.4.5. Let \( \mu \) be a Borel probability measure such that the closure of the semigroup generated by \( \mu \) is a closed, non-exceptional subgroup \( \Gamma \in V_- \rtimes \alpha \). Suppose that \( \mu \) has finite first moment with respect to \( | \cdot |_{V_- \rtimes \alpha} \). Let \( \Gamma \) be the image of \( V_- \rtimes \alpha \) under \( \pi \), \( R_m \) be the right random walk associated with \( (V_- \rtimes \alpha, \mu) \), \( V_- \) be a distinguished vertex of \( T \) with fixed end \( \omega \). Set

\[
B^{(n)}(u) = \left\{ g \in V_- \rtimes \alpha : d \left( (V_- u)_{[nm_1(\eta_* \mu)]}, gV_- \right) \leq j \right\}_{j=1}^{\infty},
\]

for each \( j \in \mathbb{N}, u \in B \) and \( n \in \mathbb{N} \). Then \( \{B^{(r)}(u)\}_{r=1}^{\infty} \) is a uniformly temperate sequence of gauges on \( \Gamma \), and

\[
|g|_{B^{(n)}(u)} = d \left( (V_- u)_{[nm_1(\eta_* \mu)]}, gV_- \right)
\]

is a gauge function on \( V_- \rtimes \alpha \).
**Proof.** The argument is similar to the proof of Proposition 3.3.3. If \( g \in V_- \rtimes \langle \alpha \rangle \), then the distance from \( gV_- \) to any vertex is always finite. It follows that

\[
V_- \rtimes \langle \alpha \rangle = \bigcup_{j=1}^{\infty} \left\{ g \in V_- \rtimes \langle \alpha \rangle : d\left( (V_-u)_{[nm_{1}(\eta, \mu)]}, gV_- \right) \leq j \},
\]

that is \( B^{(n)}(u) \) eventually exhausts \( V_- \rtimes \langle \alpha \rangle \). Let \( R_{\infty} : (V_- \rtimes \langle \alpha \rangle)^{N} \to \partial T \) be given by

\[
R_{\infty}(\gamma) = \begin{cases} 
\omega & \text{if } m_{1}(\eta_{*}\mu) \leq 0, \\
\lim_{n \to \infty} R_{m}(\gamma)V_- & \text{otherwise.}
\end{cases}
\]

Then \( \nu \) is \( \mu \)-stationary, because

\[
\int_{V_- \rtimes \langle \alpha \rangle} \int_{B} f(xb) d\nu(b) d\mu(x) = \int_{V_- \rtimes \langle \alpha \rangle} \int_{(V_- \rtimes \langle \alpha \rangle)^{N}} f(x R_{\infty}(\gamma)) d\mathbb{P}^{\mu}(\gamma) d\mu(x) \\
= \int_{(V_- \rtimes \langle \alpha \rangle)^{N}} f(R_{\infty}(T\gamma)) d\mathbb{P}^{\mu}(\gamma) \\
= \int_{(V_- \rtimes \langle \alpha \rangle)^{N}} f(R_{\infty}(\gamma)) d\mathbb{P}^{\mu}(\gamma) \\
= \int_{B} f(b) d\nu(b),
\]

where \( T \) is the left shift map, \( f \in C(B) \) and \( \nu = R_{\infty*}\mu \). The \( (G, \mu) \) space \( (B, \nu) \) is a \( \mu \)-boundary because \( R_{\infty} \) is shift invariant in the sense of Proposition 1.4.23. It follows that \( (\partial T, \nu) \) is a \( \mu \)-boundary of \( (V_- \rtimes \langle \alpha \rangle, \mu) \).

\[ \square \]

**Theorem 3.4.6.** Let \( \mu \) be a Borel probability measure such that the closure of the semigroup generated by \( \mu \) is a closed, non-exceptional subgroup \( \Gamma \in V_- \rtimes \langle \alpha \rangle \). Suppose that \( \mu \) is aperiodic, spread-out and has finite first moment with respect to \( | \cdot |_{V_- \rtimes \langle \alpha \rangle} \). Then \( (\partial T, \nu) \) is the Poisson boundary of \( (V_- \rtimes \langle \alpha \rangle, \mu) \).

**Proof.** The argument is similar to the proof of Theorem 3.3.4. Let \( u \in \partial T \), let \( n \in \mathbb{N} \) and let the gauge from Proposition 3.4.5, that is,

\[
B^{(n)}(u) = \left\{ (v, r) \in V_- \rtimes \langle \alpha \rangle : d\left( (V_-u)_{[nm_{1}(\eta, \mu)]}, (v, r)V_- \right) \leq j \} \right\}_{j=1}^{\infty}.
\]

Theorem 3.4.3 tells us that

\[
\lim_{m \to \infty} d\left( R_{m}V_-, R_{m+1}V_- \right) = 0
\]

almost surely, and

\[
\lim_{m \to \infty} \frac{1}{m} |R_{m}|_{V_- \rtimes \langle \alpha \rangle} = |m_{1}(\eta_{*}\mu)|
\]
almost surely and in $L_1$-norm. Hence, the conditions of Lemma 3.1.5 are satisfied, that is, $R_mV_-$ is almost surely a regular sequence of vertices in $\mathcal{T}$, with rate of escape $|m_1(\eta_*\mu)|$. Thus, for almost every path in the random walk, there is an end $u \in \partial \mathcal{T}$ satisfying

$$\lim_{n \to \infty} d\left(R_mV_-, (V_-u(\psi))[m_1(\eta_*\mu)]\right) = |m_1(\eta_*\mu)|.$$

If $|m_1(\eta_*\mu)| \neq 0$, then $R_mV_- \to u(\psi)$, and so $u$ is unique. By Theorem 3.1.11, $u = \omega$ if $m_1(\eta_*\mu) < 0$ and $u \in \partial^* \mathcal{T}_p$ if $m_1(\eta_*\mu) > 0$. If $|m_1(\eta_*\mu)| = 0$, then $u_j(\psi)$ is arbitrary and the statement reduces to

$$\lim_{n \to \infty} d\left(R_mV_-, V_-\right) = 0.$$

Hence, for almost all paths $\gamma$,

$$\lim_{n \to \infty} \frac{1}{n} |R_m(\gamma)|_{g(n)(R_\infty(\gamma))} = \lim_{n \to \infty} \frac{1}{n} d\left(R_mV_-, \left(V_-R_\infty(\gamma)\right)[nm_1(\eta_*\mu)]\right) = 0.$$

By assumption, $\mu$ has finite first moment with respect to the subadditive gauge associated with $|\cdot|_{V_- \times (\alpha)}$. Hence, by Kaimanovich’s ray criterion for topological groups, $(\partial \mathcal{T}, \nu)$ is the Poisson boundary.

**Corollary 3.4.7.** Let $\mu$ be a Borel probability measure such that the closure of the semigroup generated by $\mu$ is a closed, non-exceptional subgroup $\Gamma \in V_- \rtimes \langle \alpha \rangle$. Suppose that $\mu$ is aperiodic, spread-out and has finite first moment with respect to $|\cdot|_{V_- \times (\alpha)}$. Then

$$m_1(\eta_*\mu) \leq 0$$

if and only if the Poisson boundary of $(V_- \rtimes \langle \alpha \rangle, \mu)$ is trivial.

**Proof.** The proof is similar to that of Proposition 3.3.5. If $m_1(\eta_*\mu) \leq 0$, then it follows from the definition of $\nu$ that the boundary is a single point. If $m_1(\eta_*\mu) > 0$, then $\nu$ is not a point measure. \qed
Chapter 4

Unrestricted wreath products over $\mathbb{Z}^k$

Let $G$ and $H$ be locally compact groups. The direct product of $G$ over $H$, denoted by $\prod_H G$, is the group of functions from $G$ to $H$, endowed with pointwise multiplication. Elements of a direct product are called configurations. The subgroup of configurations with finite support is called the direct sum of $G$ over $H$ and denoted by $\bigoplus_H G$. The unrestricted wreath product of $G$ over $H$, denoted by $G \text{Wr} H$, is the semi-direct product $H \ltimes \prod_H G$ with the multiplication,

$$(x, f)(y, g) = (xy, fgx)$$

for all $(x, f), (y, g) \in G \text{Wr} H$, and the translation action $g_x(z) = g(\text{zx}^{-1})$. The inverse of $(x, f) \in G \text{Wr} H$ is $(x^{-1}, f^{-1}x)$. The restricted wreath product of $G$ over $H$, denoted by $G \wr H$, is the semi-direct product $H \ltimes \bigoplus_H G$ with the same action.

Let $C_2$ be the cyclic group of order two, with identity $0$ and distinguished element $1$. Let $k \in \mathbb{N}$. A lamplighter group is a restricted wreath product of $C_2$ over $\mathbb{Z}^k$, whereas an unrestricted lamplighter group is an unrestricted wreath product $C_2 \text{Wr} \mathbb{Z}^k$. Since $\mathbb{Z}^k$ and $C_2$ are both abelian groups, we use additive notation. Hence, if $(x, f)$ and $(y, g)$ are elements in either a restricted or an unrestricted lamplighter group, then

$$(x, f)(y, g) = (x + y, f + g_x), \text{ and } (x, f)^{-1} = (-x, (-f)_{-x}).$$

As mentioned in Example 3.2 of Shalom and Willis [90], unrestricted lamplighter groups arise as relative profinite completions of restricted lamplighter groups.

In Section 4.1 we discuss random walks on unrestricted lamplighter groups, provide a sufficient condition on probability measures for almost sure convergence of paths to configurations, and prove that the Poisson boundary can be non-trivial.
CHAPTER 4. UNRESTRICTED WREATH PRODUCTS OVER $\mathbb{Z}^K$

If $(G, \mu)$ is a random walk on a discrete group $G$, which is finitely generated by a set $K$, then the $K$-rate of escape is the limit

$$R_K = \lim_{n \to \infty} \frac{|R_n|_K}{n} \in [0, +\infty],$$

where $R_n$ is the corresponding right random walk and $|\cdot|_K$ is the word length function with respect to the word length metric $d$.

In section 4.2.2, we discuss a possible generalization of the $K$-rate of escape to compactly generated totally disconnected groups, which we call the rate of eschewal. Much as the rate of escape depends on a choice of generating set, the rate of eschewal depends on a chosen sequence of strictly decreasing compact open subgroups with trivial intersection. We show that, for appropriate sequences, the rate of eschewal is finite and equal to the rate of escape for measures supported within the restricted lamplighter subgroup.

### 4.1 Random walks

Kaimanovich [56] showed that if $\mu$ is a non-degenerate and symmetric probability measure on $C_2 \wr \mathbb{Z}^k$ with finite first moment, then the Poisson boundary of $(C_2 \wr \mathbb{Z}^k, \mu)$ is trivial for $k = 1$ or $2$, and non-trivial for $k \geq 3$. He conjectured that the Poisson boundary could be identified with the space of limit configurations whenever $\mu$ was of finite first moment, but only proved this for finitely supported measures and some special measures on $G_1$. Erschler [36] provided a partial answer to Kaimanovich conjecture, showing if $\mu$ has finite third moment and the support of $\mu$ generates $C_2 \wr \mathbb{Z}^k$, then the statement is true for $k \geq 5$. Lyons and Peres [70] further extended this result, proving it for all $k \geq 3$ and measures with finite second moment.

Under certain conditions on the measure, Karlsson and Woess [62] gave an explicit description of the boundary for wreath products of the form $C_r \wr G_q$, where $G_q$ is a free product of copies of $C_2$ and the infinite cyclic group, and $C_r$ is a finite cyclic group.

Let $\mu$ be a probability measure on $C_2 \wr \mathbb{Z}^k$, whose support generates a non-abelian group. For brevity, write $F$ for $\prod_{\mathbb{Z}^k} C_2$. Let $\mu_F$ be the pushforward of $\mu$ under the map $(x, f) \mapsto f$, and let $\mu_{\mathbb{Z}^k}$ be the pushforward of $\mu$ under the map $(x, f) \mapsto x$. Let $D(\mu)$ be the mean displacement of $\mu$, that is,

$$D(\mu) = \int_{\mathbb{Z}^k} z \, d\mu_{\mathbb{Z}^k}(z).$$

If $\mu_{\mathbb{Z}^k}$ has finite first moment, then $D(\mu)$ is finite. The random walk $(\mathbb{Z}^k, \mu_{\mathbb{Z}^k})$ is recurrent if and only if $D(\mu)$ is the zero vector and the support of $\mu_{\mathbb{Z}^k}$ has dimension less than 3 (see Section 1.4.10
for a discussion, or Theorem 8.1 in Spitzer [91] for a more detailed proof). If \( \mu_{Z^k} \) is a recurrent random walk, then the Poisson boundary is trivial: Lemma 1.4.17 allows us to identify it with the Poisson boundary of the projected random walk on the compact group \( F \).

**Lemma 4.1.1.** Let \( \mu \) be a measure on \( C_2 \text{ Wr } Z^k \) whose support generates a non-abelian group. Suppose that the mean displacement vector \( D(\mu) \) is non-zero and that the support of \( \mu_{Z^k} \) is non-empty and finite outside some open half plane

\[
H = \left\{ v - \alpha D(\mu) \in \mathbb{R}^k : v \cdot w > 0 \right\},
\]

where \( \alpha \geq 0 \) is fixed. Also suppose that \( w \in \mathbb{R}^k \) has positive dot product \( w \cdot D(\mu) > 0 \). If \( \mu_{Z^k} \) has finite second moment and the random walk \((Z^k, \mu_{Z^k})\) is transient, then \((C_2 \text{ Wr } Z^k, \mu)\) converges pointwise to a limit configuration.

**Proof.** We show that every point \( z \in Z^k \) changes configuration only finitely many times. Let \((y_n, \varphi_n) = (x_1, f_1), \ldots, (x_n, f_n)\) be a path in the random walk. Let \( R = \text{supp } \mu_{Z^k} \). Then

\[
\Pr[\varphi_n(z) \neq \varphi_{n+1}(z)] = \Pr[f_{n+1}(z - y_i) \neq 0] = \Pr[z - y_n \in R] = \Pr[z - ns_n + nD(\mu) - nD(\mu) \in R] = \Pr[z - \sqrt{n}(s_n - D(\mu)) \in R + nD(\mu)] = \Pr[\sqrt{n}(s_n - D(\mu)) \in -n^{-\frac{1}{2}}(R + nD(\mu) - z)] = \Pr \left[ \sqrt{n}(s_n - D(\mu)) \in -Rn^{\frac{1}{2}} - n^{\frac{1}{2}}D(\mu) + n^{-\frac{1}{2}}z \right],
\]

where \( s_n = \frac{1}{n}y_n \). As \( \mu_{Z^k} \) has finite second moment, the Multi-Variable Central Limit Theorem (Theorem 1.3.23) states that \( \sqrt{n}(s_n - D(\mu)) \) converges in distribution to a multivariate normal distribution \( \mathcal{N}_k(0, \Sigma) \), where \( \Sigma \) is a covariance matrix. This means that

\[
\Pr[\varphi_n(z) \neq \varphi_{n+1}(z)] = \Pr \left[ \sqrt{n}(s_n - D(\mu)) \in -Rn^{\frac{1}{2}} - n^{\frac{1}{2}}D(\mu) + n^{-\frac{1}{2}}z \right]
\]

almost surely converges to

\[
\int_{-Rn^{\frac{1}{2}} - n^{\frac{1}{2}}D(\mu) + n^{-\frac{1}{2}}z}^\infty \frac{1}{\sqrt{(2\pi)^k|\Sigma|}} \exp \left( -\frac{1}{2}x^\top \Sigma^{-1}x \right) d\lambda(x),
\]

where \( \lambda \in \mathbb{R}^k \).

The limit of this expression almost surely decays exponentially to zero as \( n \) tends to infinity because \( \mu \) is transient and so the intersection of \(-Rn^{\frac{1}{2}} - n^{\frac{1}{2}}D(\mu) + n^{-\frac{1}{2}}z\) with every closed, bounded ball centred at the identity in \( \mathbb{R}^k \) is almost surely eventually empty for sufficiently large
Lemma 1.3.4 (The Borel–Cantelli Lemma) implies that \( \varphi_n(z) \) almost surely changes only a finite number of times, and that \( \varphi_n \) converges pointwise almost surely to a limit configuration \( \varphi_\infty \) as \( n \) goes to infinity.

Figure 4.1: Two possible choices for \( H \) are shaded for a measure on \( U_2 \) which has infinite support above the line \( y = -x \), and finite support below it.

Figure 4.2: A measure on \( U_2 \) with infinite support in the horizontal strip \( |y| \leq 2 \). The sequence of configurations associated with each path converges pointwise almost surely whenever the vertical component of the drift \( D \) is non-zero. Any suitable choice of \( H \) contains this strip. A possible choice for \( H \) is shaded.

Let \( B \) be the space of limit configurations, let \( \varphi_\infty(\gamma) \) be the limit configuration associated with almost every path \( \gamma \), and let \( \nu = \varphi_\infty \ast \mathbb{P}^\mu \) be the hitting measure on \( B \). Then \( C_2 \) \( \text{Wr} \mathbb{Z}^K \) acts continuously on \( B \) by translation and pointwise addition, that is,

\[
(x, f)b = f(b)_x
\]
for \((x, f) \in C_2 \text{ Wr } Z^k\) and \(b \in B\). The measure \(\nu\) is \(\mu\)-stationary since

\[
\int_B f(u) \, d\mu * \nu(u) = \int_{C_2 \text{ Wr } Z^k} \int_B f(gu) \, d\nu(u) \, d\mu(g)
\]

\[
= \int_{C_2 \text{ Wr } Z^k} \int_{(C_2 \text{ Wr } Z^k)^N} f(g \varphi(\infty)(\omega)) \, d\mathbb{P}^\mu(\omega) \, d\mu(g)
\]

\[
= \int_{(C_2 \text{ Wr } Z^k)^N} f(\varphi(\infty)(\omega)) \, d\mathbb{P}^\mu(\omega)
\]

\[
= \int_{(C_2 \text{ Wr } Z^k)^N} f(\varphi(\infty)(T \omega)) \, d\mathbb{P}^\mu(\omega)
\]

\[
= \int_{B} f(u) \, d\nu(u),
\]

where \(T\) is the left shift map and \(f \in C(B)\). Furthermore, \((B, \nu)\) is a \(\mu\)-boundary of \((G, \mu)\), with boundary map \(\varphi(\infty)\) because \(\varphi(\infty)\) is shift invariant in the sense of Proposition 1.4.23.

If \(\text{supp} \mu\) generates a non-abelian group, minor modifications to an argument of Kaimanovich [56] show that the limit configuration is not the same for almost every pair of paths: If \(\text{supp} \mu\) generates a non-abelian group, then there are two non-commuting group elements in \(\text{supp} \mu\) with commutator \((0, f)\), for some non-zero configuration \(f\). Suppose, for contradiction, that \(\varphi_\infty\) is the same for almost every pair of paths. Then

\[(0, f) \varphi_\infty = f \varphi_\infty = \varphi_\infty,
\]

which is impossible. Therefore, \(B\) contains more than one element. Consequently, \(B\) is a non-trivial measurable \(\mu\)-boundary, and the Poisson boundary of the random walk \((C_2 \text{ Wr } Z^k, \mu)\) is non-trivial.

### 4.2 Rate of eschewal

There are many geometric tools available to study random walks on finitely generated and discrete groups. Most of these tools exploit the geometric structure of the Cayley graphs associated with the walk. Each finite generating set for a group produces a Cayley graph which is quasi-isometrically invariant.

Compactly generated groups have quasi-isometrically invariant graphs associated with them. Krön and Möller [66] defined the rough Cayley graph of a topological group \(G\) to be a connected graph \(X\), acted on transitively by \(G\), for which the stabilizers of the vertices are compact open subgroups. They showed that every compactly generated t.d.l.c. group has a connected locally
finite rough Cayley graph. We hereafter adopt the terminology of Baumgartner, Möller and Willis [5] and Salmi [87], and refer to rough Cayley graphs as relative Cayley graphs.

One construction of the relative Cayley graph, given by Krön and Möller, is as follows. Let $G$ be a compactly generated t.d.l.c. group with a compact open subgroup $O$. Then there is a finite subgroup $H$ of $G$ which acts transitively on $X$; moreover, every element of $G$ is a finite product of the form $h_1 h_2 \ldots h_i u$, where $h_1, \ldots, h_i \in H$ and $u \in O$. Any such pair $(O, H)$ satisfying these properties is called a good generating set. If $(O, H)$ is a good generating set, then the relative Cayley graph $\text{Cay}(G, O, H)$ is the graph with vertex set $G/O$ and edge set $\{(v, hv) : v \in G/O, h \in H\}$. This graph is connected and locally finite, and any two locally finite relative Cayley graphs of a group $G$ are quasi-isometric.

Given a good generating set $(O, H)$ on a compactly generated, totally disconnected, locally compact group $G$, let $\ell_{O,H} : G \to \mathbb{N}_0$ map each element to the length of the shortest word $h_1, \ldots, h_i \in H$ such that $g = h_1 \ldots h_i u$ for some $u \in O$. We call $\ell_{O,H}$ the $(O, H)$-word length of $g$. The $(O, H)$-word length is well defined because there is always at least one word of finite length for each element, and the minimum is taken over $\mathbb{N}$.

If $(G, \mu)$ is a random walk on a discrete group $G$, which is finitely generated by a set $K$, then the $K$-rate of escape is the limit

$$R_K = \lim_{n \to \infty} \frac{|R_n|_K}{n} \in [0, +\infty],$$

where $R_n$ is the corresponding right random walk and $|\cdot|_K$ is the word length function with respect to the word length metric $d$. If the generating set is clear from the context, we refer simply to the rate of escape. If $m, n \in \mathbb{N}$ and $n > m$, then

$$|R_n(\omega)|_K = d(e, R_{n+m}(\omega)) \leq d(e, R_m(\omega)) + d(R_m(\omega)e, R_{n+m}(\omega))$$

$$\leq d(e, R_m(\omega)) + d(e, X_m(\omega) \ldots X_{n+m}(\omega))$$

$$= d(e, R_m(\omega)) + d(e, R_n(T^m \omega))$$

for almost all $\omega \in G^\mathbb{N}$, where $T$ is the measure preserving shift map defined in Section 1.4.1. It follows that, if $\mu$ has finite first moment, then the rate of escape exists and is well defined. To see this, apply the Kingman Subadditive Ergodic Theorem in the form given by Steele [92] (see also Theorem 8.14 of Woess [106]).

Suppose now that $G$ is totally disconnected and locally compact. Let $H$ be a finite subset of $G$, and let $O = \{O_i\}_{i \in \mathbb{N}}$ be a strictly decreasing sequence of compact open subgroups such that $(O_i, H)$ is a good generating set and the subgroup $\bigcap_{i \in \mathbb{N}} O_i$ is trivial for every $i \in \mathbb{N}$. Let $R^*_{O,H}$
4.2. RATE OF ESCHEWAL

be the \((O, H)\)-rate of eschewal, given by

\[ R^{*}_{O,H} = \lim_{i \to \infty} \sup_{n \leq i} \frac{\ell_{O,H}(R_n)}{i}. \]

If the good generating set \((O, H)\) is clear from the context, we simply call \(R^{*}_{O,H}\) the rate of eschewal and write it as \(R^*\). Since \(R^*\) is the limit superior of a sequence of non-negative real numbers, it exists, but might be infinite.

**Lemma 4.2.1.** Suppose that \(G\) is a compactly generated t.d.l.c. group with a subgroup \(K\), generated by a finite set \(H\). Suppose that \(O = \{O_i\}_{i \in \mathbb{N}}\) is a sequence of compact open subgroups in \(G\) such that, for any \(i \in \mathbb{N}\), \((O_i, H)\) is a good generating set for \(G\). Let \(\mu\) be a probability measure on \(G\) whose support is contained in \(K\). Then the \((O, H)\)-rate of eschewal \(R^*\) is greater than or equal to the \(H\)-rate of escape \(R\).

**Proof.** Let \(R_n\) be the right random walk. For each path \(\omega\), let \(w_1(\omega), \ldots, w_q(\omega)(\omega)\) be the shortest word in \(K\) whose product is equal to \(R_n(\omega)\). Then

\[ w_1(\omega) \cdots w_q(\omega)(\omega)(0, e) = R_n(\omega) \]

and \((0, e)\) is contained in every subgroup of \(G\), so

\[ \ell_{O,H}(R_n(\omega)) \leq |R_n(\omega)|_H \]

for all \(i \in \mathbb{N}\). It follows that

\[ R = \lim_{n \to \infty} \frac{R_n|_H}{n} \geq \lim_{i \to \infty} \sup_{n \leq i} \frac{\ell_{A_k,r,i,H}(R_n)}{i} = R^* \]

almost surely. \(\Box\)

§ 4.2.2 Rate of eschewal on lamplighter groups

The remainder of this chapter investigates rates of eschewal on \(C_2 \text{ Wr } \mathbb{Z}^k\). There are many possible choices of decreasing sequences of compact open subgroups. We focus on sequences of compact open subgroups \(A_{k,r} = \{A_{k,r,i}\}_{i \in \mathbb{N}}\), where \(r\) is a positive real number and

\[ A_{k,r,i} = \{(0, f) \in C_2 \text{ Wr } \mathbb{Z}^k : f(x) = 0 \text{ whenever } \|x\|_\infty \leq i^r\}. \]

Each subgroup \(A_{k,r,i}\) is open and therefore a closed, hence compact subgroup of \(C_2 \text{ Wr } \mathbb{Z}^k\) for all \(i, k \in \mathbb{N}\) and \(r \in \mathbb{R}\).

For each \(j \in \mathbb{N}\), let \(e_j\) be the element of \(\mathbb{Z}^k\) which has 1 as its \(j\)th coordinate and 0 as every other coordinate, so that
is the usual symmetric generating set for $\mathbb{Z}^k$. Set $h_j = (e_j, id)$ for each $j \in \{1, \ldots, k\}$, where $id$ is the identity configuration. Let $H$ be the subset of $C_2 \wr \mathbb{Z}^k$ given by

$$H = \{ s, h_j : j \in \{1, \ldots, k\} \},$$

where $s = (0, \delta_0)$, and $\delta_0$ is the configuration with value 1 at the identity in $\mathbb{Z}^k$ and 0 everywhere else.

Let $i, k \in \mathbb{N}$ and let $r \in \mathbb{R}^+$. If $(y, h)$ is an element of $C_2 \wr \mathbb{Z}^k$, then

$$(y, h)\Lambda_{k,r,i} = \{ (y, f) \in C_2 \wr \mathbb{Z}^k : f(x) = h(x) \text{ whenever } \|x - y\|_\infty \leq i^r \}.$$ 

and so the group generated by $E_k$ acts transitively on $C_2 \wr \mathbb{Z}^k / \Lambda_{k,r,i}$. Every element of $C_2 \wr \mathbb{Z}^k$ is a finite product $h_1 h_2 \ldots h_k u$, where $h_1, \ldots, h_i \in H$ and $u \in \Lambda_{k,r,i}$. That is, $(\Lambda_{k,r}, H)$ is a good generating set for $C_2 \wr \mathbb{Z}^k$. Moreover, the intersection $\bigcap \Lambda_{k,r,i}$ contains only the identity element.

**Lemma 4.2.3.** Let $i, k \in \mathbb{N}$, and let $s, t \in \mathbb{R}^+$ satisfy $s \leq t$. Then

$$\Lambda_{k,t,i} \subseteq \Lambda_{k,s,i},$$

with equality if and only if $s = t$.

**Proof.** This is a simple consequence of the fact that if $i \in \mathbb{N}$, then $\|x\|_\infty \leq i^t$ whenever $\|x\|_\infty \leq i^s$ for all $x \in \mathbb{Z}^k$. \hfill $\Box$

A *path of length* $n$ is a finite sequence of group elements $\{X_i\}_{i \in \{1, \ldots, n\}}$ which contains each group element at most once. Given a path $\{X_i\}_{i \in \{1, \ldots, n\}}$ of length $n$, the finite sequence $\{X_i^{-1} X_{i+1}\}_{i=1}^{n-1}$ is the corresponding *sequence of increments*, itself a path of length $n - 1$.

We write $X_1 \star X_2$ to mean the concatenation of two finite sequences of elements $X_1$ and $X_2 \in G$, and identify each generator with the single term sequence containing it. For example, $x_1 \star x_2 \star x_3$ is identified with the finite sequence $x_1, x_2, x_3$. The concatenation of a finite sequence $X_1, \ldots, X_n$ of finite increments of group elements in $G$ is written as

$$\star_{i=1}^n X_i = X_1 \star X_2 \star \ldots \star X_n.$$

**Lemma 4.2.4.** Let $B^k_r$ be the ball of radius $r$ in the $L^\infty$ norm centred at the identity in $\mathbb{Z}^k$, and let

$$E_k = \{ e_1, \ldots, e_k, e_{-1}^1, \ldots, e_{-1}^k \}$$

is the usual symmetric generating set for $\mathbb{Z}^k$. Set $h_j = (e_j, id)$ for each $j \in \{1, \ldots, k\}$, where $id$ is the identity configuration. Let $H$ be the subset of $C_2 \wr \mathbb{Z}^k$ given by

$$H = \{ s, h_j : j \in \{1, \ldots, k\} \},$$

where $s = (0, \delta_0)$, and $\delta_0$ is the configuration with value 1 at the identity in $\mathbb{Z}^k$ and 0 everywhere else.

Let $i, k \in \mathbb{N}$ and let $r \in \mathbb{R}^+$.

If $(y, h)$ is an element of $C_2 \wr \mathbb{Z}^k$, then

$$(y, h)\Lambda_{k,r,i} = \{ (y, f) \in C_2 \wr \mathbb{Z}^k : f(x) = h(x) \text{ whenever } \|x - y\|_\infty \leq i^r \}.$$ 

and so the group generated by $E_k$ acts transitively on $C_2 \wr \mathbb{Z}^k / \Lambda_{k,r,i}$. Every element of $C_2 \wr \mathbb{Z}^k$ is a finite product $h_1 h_2 \ldots h_k u$, where $h_1, \ldots, h_i \in H$ and $u \in \Lambda_{k,r,i}$. That is, $(\Lambda_{k,r}, H)$ is a good generating set for $C_2 \wr \mathbb{Z}^k$. Moreover, the intersection $\bigcap \Lambda_{k,r,i}$ contains only the identity element.

**Lemma 4.2.3.** Let $i, k \in \mathbb{N}$, and let $s, t \in \mathbb{R}^+$ satisfy $s \leq t$. Then

$$\Lambda_{k,t,i} \subseteq \Lambda_{k,s,i},$$

with equality if and only if $s = t$.

**Proof.** This is a simple consequence of the fact that if $i \in \mathbb{N}$, then $\|x\|_\infty \leq i^t$ whenever $\|x\|_\infty \leq i^s$ for all $x \in \mathbb{Z}^k$. \hfill $\Box$

A *path of length* $n$ is a finite sequence of group elements $\{X_i\}_{i \in \{1, \ldots, n\}}$ which contains each group element at most once. Given a path $\{X_i\}_{i \in \{1, \ldots, n\}}$ of length $n$, the finite sequence $\{X_i^{-1} X_{i+1}\}_{i=1}^{n-1}$ is the corresponding *sequence of increments*, itself a path of length $n - 1$.

We write $X_1 \star X_2$ to mean the concatenation of two finite sequences of elements $X_1$ and $X_2 \in G$, and identify each generator with the single term sequence containing it. For example, $x_1 \star x_2 \star x_3$ is identified with the finite sequence $x_1, x_2, x_3$. The concatenation of a finite sequence $X_1, \ldots, X_n$ of finite increments of group elements in $G$ is written as

$$\star_{i=1}^n X_i = X_1 \star X_2 \star \ldots \star X_n.$$

**Lemma 4.2.4.** Let $B^k_r$ be the ball of radius $r$ in the $L^\infty$ norm centred at the identity in $\mathbb{Z}^k$, and let

$$E_k = \{ e_1, \ldots, e_k, e_{-1}^1, \ldots, e_{-1}^k \}$$
be the usual generating set for \( I^k \). Then, for each \( r \in \mathbb{N} \), there is a path \( a(r) = \{a(r)_i\}_{i=1}^k \) between \((-r, \ldots, -r)\) and \((r, \ldots, r)\) in \( \mathbb{Z}^k \) which visits every element of \( B^k_r \) only once, whose increments are elements of \( E_k \), and which satisfies

\[
|a(r)|_H \in O(r^k).
\]

**Proof.** For each non-negative integer \( d \) less than \( k \), identify \( \mathbb{Z}^d \) with the subgroup of \( \mathbb{Z}^k \) consisting of all elements whose last \( k - d \) coordinates are all zero.

If \( k = 1 \), then the first element of the path is \(-r\). The increments of the path are all equal to \( e_1 \), and the length of the path is \( 2r \) since it finishes at \( r \). Let \( I_1 \) be the finite sequence of increments:

\[
I_1 = \star_{i=1}^{2r} e_1,
\]

and let \( I_1^{-1} \) be the sequence of the same length with \( e_1^{-1} \) for each term instead of \( e_1 \), that is,

\[
I_1^{-1} = \star_{i=1}^{2r} e_1^{-1}.
\]

For each \( k \in \mathbb{N} \), let

\[
I_{k+1} = \left( \star_{i=1}^r \left( I_{s} \star e_{k+1} \star I_k^{-1} \star e_{k+1} \right) \right) \star I_k.
\]

Let \( J_k = (-r, \ldots, -r) I_k \). Then \( J_k \) has \((2r + 1)^k\) elements, hence it is in \( O(r^k) \).

![Figure 4.3: The path starting at \((-r, -r)\) with increments \( I_2 \), in the case where \( r = 3 \).](image)

The next three lemmas show that the rate of eschewal is strongly dependent on the choice of good generating set, and, in fact, it can be finite, infinite or converge to a constant value, depending on the choice made.
**Lemma 4.2.5.** Suppose that $\mu$ is a probability measure on $C_2 \wr \mathbb{Z}^k$. Further, suppose that $\mu_F$ has infinite support and that the random walk $(C_2 \wr \mathbb{Z}^k, \mu)$ converges to a limit configuration. Then there are strictly decreasing sequences of compact open subgroups $O_k = \{O^r_k\}_{r \in \mathbb{N}}$ for which the $(O_k, H)$-rate of eschewal is zero.

**Proof.** Choose $r \in \mathbb{R}$ such that $0 < r < \frac{1}{k}$. Let $\varepsilon > 0$ satisfy $r = \frac{1}{k} + \varepsilon$, let $\Lambda_{k,r,i}$ be defined as before, and let

$$(y_n, \varphi_n) = (x_1, f_1) \ldots (x_n, f_n)$$

be the $n$th element of a path in the right random walk.

Let $P$ be the path $P$ from $(-\lfloor i^r \rfloor, \ldots, -\lfloor i^r \rfloor)$ to $(\lfloor i^r \rfloor, \ldots, \lfloor i^r \rfloor) \in \mathbb{Z}^k$ as described in the proof of Lemma 4.2.4. Then, for each $i \in \mathbb{N}$,

$$(y_n, \varphi_n) = \left( \prod_{j=1}^k h_j^{-\lfloor i^r \rfloor} \right) \left( \prod_{z \in P} e_z \right) \left( \prod_{j=1}^k h_j^{y_n(j)-\lfloor i^r \rfloor} \right) \lambda$$

for some $\lambda \in \Lambda_{k,r,i}$, where

$$e_z = \begin{cases} 
s & \text{if } \varphi_n(z) = 1, \\ 
e & \text{if } \varphi_n(z) = 0 \end{cases}$$

and

$$H = \{ s, h_j : j \in \{ 1, \ldots, k \} \},$$

as defined at the beginning of this section. We now note that there is an upper bound on $\ell_{\Lambda_{k,r,i}}(y_n, \varphi_n)$. Theorem 1.3.21 (The Strong Law of Large Numbers) implies that the word $\left( \prod_{j=1}^k h_j^{y_n(j)} \right)$ has length in $o(n)$. By Lemma 4.2.4, $(\prod_{z \in P} e_z)$ is a word whose length is in $O(i^r) = O\left( \frac{k}{i^r} \right)$. The length of the word $\left( \prod_{j=1}^k h_j^{-\lfloor i^r \rfloor} \right)$ is not in $o(i^r)$. In other words, the $(\Lambda_{k,r}, H)$-rate of eschewal $R^*$ is zero for our chosen sequence of compact open subgroups. \qed

**Lemma 4.2.6.** Suppose that $\mu$ is a probability measure on $C_2 \wr \mathbb{Z}^k$ with finite second moment and non-zero mean displacement. In addition, suppose that $\operatorname{sgn} \mu_Z = \mathbb{Z}$, the support of $\mu_F$ is infinite and that the random walk $(C_2 \wr \mathbb{Z}^k, \mu)$ converges to a limit configuration. Then there are strictly decreasing sequences of compact open subgroups $O_k = \{O^r_k\}_{r \in \mathbb{N}}$ such that the $(O_k, H)$-rate of eschewal is infinite.

**Proof.** Choose a positive real number $r > \frac{1}{k}$, and let $\varepsilon > 0$ be such that $r = \frac{1}{k} + \varepsilon$. Let $z \in \mathbb{Z}$. Since the walk is convergent, $\varphi_n(z)$ almost surely converges to $\varphi_\infty(z)$ with value either 0 or 1.
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The Multi-Variable Central Limit Theorem implies that \( \Pr(\varphi_{\infty}(z) = \bar{1}) \) converges to zero as the dot product \( z \cdot D(\mu) \) becomes increasingly negative. For the details, see the proof of convergence in Section 4.1. In other words, for every positive \( \delta \), there exists an \( N \in \mathbb{N} \) such that

\[
\Pr(\varphi_{\infty}(z) = \bar{1}) < \delta
\]

for all integers \( z \) satisfying

\[
z \cdot D(\mathbb{Z}, \mu) < -N.
\]

Similarly, as \( z \cdot D(\mu) \) increases, the expected number of changes of a given configuration becomes arbitrarily large, so that for all sufficiently large \( z \) and every positive \( \delta \),

\[
|\mathbb{P}^{\mu}(\varphi_{\infty}(z) = \bar{1}) - 0.5| < \delta
\]

almost surely. In other words,

\[
\left| \left\{ z \in B_r^k : \varphi_{\infty}(z) = \bar{1} \right\} \right| \sim \frac{1}{4} \left| B_r^k \right| \in o(r^k)
\]

\( \mathbb{P}^{\mu} \)-almost surely, where \( B_r^k \) is the ball of radius \( r \) centred at the identity in \( \mathbb{Z}^k \) in the \( L^\infty \) norm. This allows us to construct a lower bound for \( \ell_{\Lambda_{k,r,i},H}(y_n, \varphi_n) \).

Suppose that \( x_1, \ldots, x_k \) is a finite sequence of elements in \( H \) and \( u \in \Lambda_{k,r,i} \), satisfies

\[
(y_n, \varphi_n) = x_1 \ldots x_j u.
\]

There are two cases to consider. Either \( u = e \) or \( u \neq e \). If \( u = e \), then \( x_1 \ldots x_j \) includes at least \( o(i^rk) \) copies of the generator \( s \). If \( u \neq e \) and the allowable \( \| \cdot \|_1 \) translation distance to a function in \( o(i) \) is bounded, then at least \( o(i^rk) \) copies of the generator \( s \) are included in any suitable word. In either case, \( \ell_{\Lambda_{k,r,i},H}(y_n, \varphi_n) \) grows faster than any linear function almost surely, and so the \((\Lambda_{k,r}, H)\)-rate of eschewal converges to \(+\infty\).

**Lemma 4.2.7.** Suppose that \( \mu \) is a probability measure on \( C_2 \text{ Wr } \mathbb{Z}^k \), with finite second moment and non-zero mean displacement. In addition, suppose that \( \text{sgn } \mu_{\mathbb{Z}} = \mathbb{Z} \), the support of \( \mu_F \) is infinite and that the random walk \((C_2 \text{ Wr } \mathbb{Z}^k, \mu)\) converges to a limit configuration. Then there are strictly decreasing sequences of compact open subgroups \( O_k = \{O^r_k\}_{r \in \mathbb{N}} \) such that the \((O_k, H)\)-rate of eschewal is finite.

**Proof.** Choose \( r = \frac{1}{k} \) and consider \( L = \ell_{\Lambda_{k,r,i},H}(y_n, \varphi_n) \). Minimal modifications to the argument used to prove Lemma 4.2.6 show that \( L \) is almost surely in \( o(i^rk) \). The argument of Lemma 4.2.5 gives a similar lower bound. It follows that the rate of eschewal is almost surely finite. \( \square \)
We have shown that the $(\Lambda_{k,r}, H)$-rate of eschewal is either zero in the case that $r < \frac{1}{k}$, finite if $r = \frac{1}{k}$ or infinite if $r = \frac{1}{k}$. We saw that there is a subsequence of $\Lambda_{k,r}$, denoted by $\Lambda^*_{k,r}$, such that the $(\Lambda^*_{k,r}, H)$-rate of eschewal agrees with the rate of escape for measures supported in $C_2 \mathrm{Wr} \mathbb{Z}^k$ whenever $r \geq \frac{1}{k}$. The choice $r = \frac{1}{k}$ therefore yields a more natural good generating set for $C_2 \mathrm{Wr} \mathbb{Z}^k$. A key feature of $\Lambda_{k,r}$ in this case is the linear growth, in a sense that shall be made more precise shortly, of the number of points fixed by its elements. Any sequence of good generating points with a similar property converges to a finite value, and agrees with the rate of escape.

**Proposition 4.2.8.** Suppose that $\mu$ is a probability measure on $C_2 \mathrm{Wr} \mathbb{Z}^k$. Suppose that $\mu$ has finite first moment and non-zero mean displacement, that $\mu_F$ has infinite support and that random walk $(C_2 \mathrm{Wr} \mathbb{Z}^k, \mu)$ converges to a limit configuration. Let $F^i_k = \{F^i_k\}_{i \in \mathbb{N}}$ be any sequence of finite subsets of $\mathbb{Z}^k$, for which there exist $C, D > 0$ and $N \in \mathbb{N}$ such that

$$B^k_{C_i} \subseteq F^i_k \subseteq B^k_{D_i}.$$ 

Finally, let $O^i_k = \{O^i_k\}_{i \in \mathbb{N}}$ be the sequence of strictly decreasing sequences of compact open subgroups given by

$$O^i_k = \{(0, f) \in C_2 \mathrm{Wr} \mathbb{Z}^k : f(x) = 0 \text{ whenever } x \in F^i_k\}.$$ 

Then the $(O^i_k, H)$-rate of eschewal is finite, and, if the support of $\mu_F$ is finite, agrees with the $H$-rate of escape.

**Proof.** Let $r = \frac{1}{k}$. The hypotheses of this lemma are equivalent to the existence of positive real constants $E$ and $F$ such that for all sufficiently large $i \in \mathbb{N},$

$$\Lambda_{k,r,\lfloor Ei \rfloor} \subset O^i_k \subset \Lambda_{k,r,\lfloor Fi \rfloor}$$

and such that

$$\ell_{\Lambda_{k,r,\lfloor Ei \rfloor}} \leq \ell_{O^i_k} \leq \ell_{\Lambda_{k,r,\lfloor Fi \rfloor}},$$

where $\lfloor \cdot \rfloor$ is the floor function. The argument is then similar to Lemma 4.2.7: a similar argument to that given in proof of Lemma 4.2.5 shows that $\ell_{\Lambda_{k,r,\lfloor Fi \rfloor}}$ is bounded above by a linear function, while the argument from Lemma 4.2.6 shows that $\ell_{\Lambda_{k,r,\lfloor Ei \rfloor}}$ is bounded below by a linear function.

**Remark 4.2.9.** The proof of Proposition 4.2.8 shows that if $0 < r < \frac{1}{k}$, and $B \subset F$, then the rate of eschewal is infinite. Similarly, if $r < \frac{1}{k}$ and we require $F \subset B$, the rate of eschewal is zero.
Chapter 5

Summary of results

In Chapter 2, we related adelic length to the word length in finitely generated rational matrix groups. Brofferio and Schapira’s [14] described the Poisson boundary of $GL_n(\mathbb{Q})$ for measures of finite first moment with respect to adelic length. To relate the two notions of length, we defined matrix groups $FG_n(P)$ for each $n \in \mathbb{N}$ and finite set of primes $P$. These groups contained every rational valued upper triangular matrix group as a subgroup. We show that adelic length is a word metric estimate on $FG_n(P)$ by constructing another, intermediate, word metric estimate which can be easily computed from the entries of any matrix in the group. In particular, requiring a probability measure on $FG_n(P)$ to have finite first moment with respect to adelic length is equivalent to requiring it to have finite first moment with respect to word length.

Chapter 3 presented two possible extensions of the work of Cartwright, Kaimanovich and Woess in [16]. Firstly, we considered random walks on finite direct products of affine automorphism groups of homogeneous trees. When the probability measure is spread-out, aperiodic, has finite first moment and its support generates a closed subgroup which is not fully exceptional, we showed that the Poisson boundary is the direct product of the space of ends of each tree with a probability measure. We gave necessary and sufficient conditions for boundary triviality and investigated random walks on closed sub-quotients in totally disconnected locally compact groups which were isomorphic closed subgroups of an affine group of a homogeneous tree with non-trivial degree. A description of the Poisson boundary for aperiodic, spread-out probability measures of finite first moment is given in that case.

In Chapter 4 we considered random walks on unrestricted wreath products, provide sufficient conditions for almost sure convergence of paths to limit configurations, and proved that the Poisson boundary can be non-trivial. We also examined a possible generalization of the $K$-rate of escape to compactly generated totally disconnected groups, which we called the rate of eschewal. Much as
the rate of escape depends on a choice of generating set, the rate of eschewal depends on a chosen sequence of strictly decreasing compact open subgroups with trivial intersection. For appropriate sequences, it was shown that the rate of eschewal is finite and equal to the rate of escape for measures supported within the restricted lamplighter subgroup.


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