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Accessed from: http://hdl.handle.net/1959.13/1315547
AIMD in a discrete time implementation or with a non-constant shared resource

S. Stüdi1, R. H. Middleton1, J. H. Braslavsky2, and R. Shorten3

Abstract—The additive increase multiplicative decrease (AIMD) algorithm, that is commonly used for congestion avoidance in communication networks, has recently been suggested in other fields such as load management in electric power networks. As for congestion avoidance, in such systems a large number of agents are required to share a given resource. In recent work by Shorten, Wirth and Leith on congestion control in networking a stochastic model has been developed to analyse AIMD algorithms. However, the analysis assumes a continuous implementation of the algorithm and a constant available resource. These assumptions are no longer useful if the AIMD algorithm is applied in fields such as load management in electric power networks, where a discrete implementation is often required, and the available resource shared may be inherently variable. In this paper we develop a disturbed AIMD model based on the model introduced by Shorten et al. that includes discrete time implementation and time varying resource availability. Further, we use that model to bound the influence of these disturbances, caused by either a discrete implementation or small variations in the available resource.

I. INTRODUCTION

The additive increase multiplicative decrease (AIMD) algorithm, originally suggested in [1], has been successfully applied in congestion avoidance in communication networks. The algorithm has proven itself robust and reliable. Nowadays, various extensions and generalisations exist to the basic AIMD algorithm to accommodate, for example, high-speed networks, see [3], [4], [7]. Further, extensive literature is concerned with the algorithm’s behaviour in regards to its fairness and stability [6], [8]. Often, a linear system representation is used for the analysis, for example in [10]–[12], [14], in regards to stability, fairness of the algorithm, and higher moments including second moments. Also, various extensions of the basic AIMD algorithm are studied using this system representation [2], [5], [9], [15].

Recently, the AIMD algorithm has been suggested for load management in power grids, where some problems can be treated as a congestion avoidance problem [13]. The analysis using the linear system representation in [11] however, assumes a continuous implementation of the algorithm and that the shared resource is constant. In these new areas however these assumptions are no longer valid. For example, in load management scenarios there will be non-stationary disturbances caused by uncontrollable loads. The main motivation of the present paper is to use a stochastic model to give worst case bounds for the error caused by such disturbances or by a discrete-time implementation of the AIMD algorithm.

In Section II we introduce a linear system representation of the AIMD algorithm as it is commonly investigated and state some important properties that have been shown using such a model. In Section III, we adapt this model to allow for specific disturbances that allow the modelling of a discrete implementation of the AIMD algorithm and variations in the available resource. Afterwards, in Section IV we develop an upper bound on the region around the defined equilibrium point to which the disturbed AIMD algorithm converges to using this model. These results are illustrated on a numerical simulation example in Section V. Finally, in Section VI we conclude the paper.

II. AIMD ALGORITHM - PRELIMINARIES

First, we describe the AIMD algorithm as it is commonly investigated using a linear system [11], [12], [14]. The AIMD algorithm, which generally operates in continuous time, consists of two phases: the additive increase (AI) phase and the multiplicative decrease (MD) phase. During the AI phase each agent linearly increases its share of the resource continuously until the available resource is completely used. The complete usage of the available resource is denoted a capacity event (CE). Upon such a CE a broadcast signal is sent to all agents which execute the MD phase upon the receipt of such a signal. While the AI phase is continuous, the MD phase is instantaneous and reduces the share of each agent multiplicatively. Although in the system representation in [11] the MD phase is random in the sense that not all agents react, we assume here a deterministic behaviour, where all agents react to a CE. Let the available resource be $P$ and let $p \in \mathbb{R}^N$ be the vector containing each agent’s share of the resource. Then, a CE is triggered whenever

$$I^T p(t) = P(t),$$

where $I$ is the column vector containing all 1, i.e. $I^T = [1 \cdots 1]$. Further, let $\tau_k$ be the time at which the $k$-th CE is triggered. At this time instance the MD phase is executed where the share is multiplied instantaneously by a constant multiplicative factor $\beta_i$ with $0 < \beta_i < 1$ for all $i$. This introduces a discontinuity in the share at this instance. During the AI phase the share increases continuously, such
that we find that for time \( t \), with \( \tau_k < t \leq \tau_{k+1} \) the share evolves according to

\[
p_i(t) = \beta_i p_i(\tau_k) + \alpha_i(t - \tau_k)
\]  

(2)

where \( \alpha_i \) is a positive additive factor. For the analysis in [11] it is assumed that the available resource \( \bar{P} \) remains constant. To investigate convergence and fairness of the AIMD algorithm, [11] investigates the system at CEs directly before the multiplicative decrease, i.e. the share at times \( \tau_k \). Let \( \alpha \) be the vector containing the individual additive factors \( \alpha_i \), and \( \beta \) be the vector containing the individual multiplicative factors \( \beta_i \) of the agents, i.e.

\[
\alpha = \begin{bmatrix} \alpha_1 & \cdots & \alpha_N \end{bmatrix}^T \quad \text{and} \quad \beta = \begin{bmatrix} \beta_1 & \cdots & \beta_N \end{bmatrix}^T
\]  

(3)

(4)

Then, according to [11], the basic AIMD from one CE at \( \tau_k \) to the next at \( \tau_{k+1} \) maps a vector \( p(\tau_k) \in \mathbb{P} \) into a vector \( p(\tau_{k+1}) \in \mathbb{P} \) such that

\[
p(\tau_{k+1}) = Ap(\tau_k),
\]  

(5)

where the set \( \mathbb{P} \) is defined as \( \mathbb{P} = \left\{ p \in \mathbb{R}^N | 1^T p = \bar{P} \right\} \) and the matrix \( A \) is

\[
A = \text{diag}(\beta) + (1^T \alpha)^{-1} \alpha (1 - \beta)^T.
\]  

(6)

This linear system model of the AIMD algorithm in Equations (5) and (6) is then used in [11] to show convergence to a unique equilibrium point given by

\[
p^* = C \begin{bmatrix} \frac{\alpha_1}{1 - \beta_1} & \cdots & \frac{\alpha_N}{1 - \beta_N} \end{bmatrix}^T
\]  

(7)

where \( C \in \mathbb{R} \) is a scaling factor such that \( 1^T p^* = \bar{P} \). Note that \( p^* \) is the Perron eigenvector of \( A \).

### III. Developing the Disturbance Model

In this section, the aforementioned linear system model of the AIMD algorithm is adapted to allow for a specific disturbance. In fact, we allow that a CE no longer is triggered when Equation (1) holds. Let \( \gamma_k \) be a scalar that varies with CE. Further, it is bounded by \( \underline{\gamma} \) and \( \bar{\gamma} \) such that for all \( k \)

\[
\underline{\gamma} \leq \gamma_k \leq \bar{\gamma}
\]  

(8)

For any \( k \), let the CE at \( \tau_k \) be triggered when

\[
1^T p(\tau_k) = \bar{P} + \gamma_k.
\]  

(9)

Assume then that the evolution of the state can be represented by

\[
p(\tau_{k+1}) = Ap(\tau_k) + \alpha(\gamma_{k+1} - \gamma_k).
\]  

(10)

In the following, we show how a discrete implementation of the algorithm or the variation of the available resource can be mapped to a problem with such a disturbance.

### A. Discrete AIMD

Here, we investigate a discrete implementation of the AIMD algorithm. This means that the AI phase is no longer continuous nor is the MD phase instantaneous. Rather, we assume that the agent at each defined time step is either executing a modified AI or MD phase depending on whether a CE has been triggered. Hence, if at time step \( s \) a CE has been triggered, agent \( i \) adapts its share according to

\[
p_i(s + 1) = \beta_i p_i(s),
\]  

(11)

and if no CE has been triggered, according to

\[
p_i(s + 1) = p_i(s) + \bar{\alpha}_i.
\]  

(12)

The parameters \( \beta_i \) and \( \bar{\alpha}_i \) are as before the individual multiplicative and additive factors, respectively. Note that while previously in Equation (2) the additive factor was multiplied by the duration since the last CE occurrence, in the discrete implementation in Equation (12), we define \( \bar{\alpha}_i \) as a fixed step increase. Hence, the two parameters \( \alpha_i \) and \( \bar{\alpha}_i \) are not identical. Let \( \delta t \) be the duration of a time step, then the two factors can be mapped into each other using \( \bar{\alpha}_i = \alpha_i \delta t \). The most immediate impact of such an implementation is that at a CE Equation (1) does not hold anymore, rather

\[
\bar{P}(\tau_k) \leq 1^T p(\tau_k) < \bar{P}(\tau_k) + 1^T \bar{\alpha}
\]  

(13)

where \( \bar{\alpha} \) is the vector containing the individual additive factors \( \bar{\alpha}_i \). Naturally, smaller additive factors \( \bar{\alpha}_i \) lead to a behaviour that is closer to the continuous time situation. In fact a smaller additive factor \( \bar{\alpha}_i \) corresponds to a smaller duration of the time steps \( \delta t \). We assume that the additive factor \( \bar{\alpha}_i \) is chosen sufficiently small such that after one CE the aggregated share of the agents is below the available resource \( \bar{P} \).

We investigate the share at two consecutive CE namely, \( \tau_k \) and \( \tau_{k+1} \). Let \( \zeta_k \) be a varying scalar with

\[
0 \leq \zeta_k < 1^T \bar{\alpha}
\]  

(14)

for all \( k \) such that

\[
1^T p(\tau_k) = \bar{P} + \zeta_k
\]  

(15)

holds. Note that such scalars exist due to Equation (13). We denote the number of time steps between these two consecutive CEs \( \Delta k \), which is non-negative. Then, using Equations (11) and (12) we find the evolution of the share from one CE to the next to be

\[
p_i(\tau_{k+1}) = \beta_i p_i(\tau_k) + \bar{\alpha}_i \Delta k.
\]  

(16)

Summing over all agents and applying Equation (15) we find \( \Delta k \) to be

\[
\Delta k = \frac{1}{1^T \bar{\alpha} \delta t} \sum_{i=1}^{N} \left( p_i(\tau_{k+1}) - \beta_i p_i(\tau_k) \right)
\]  

\[
= \frac{1}{1^T \bar{\alpha} \delta t} \left( \sum_{i=1}^{N} (p_i(\tau_k) - \beta_i p_i(\tau_k)) + \zeta_{k+1} - \zeta_k \right).
\]  

(17)
By inserting Equation (17) in Equation (16) we find the share of each agent to evolve according to

\[ p_i(\tau_{k+1}) = \frac{\bar{\alpha}_i}{\bar{\alpha}} \sum_{i=1}^{N} ((1 - \beta_i) p_i(\tau_k)) + \zeta_{k+1} - \zeta_k \]  \hspace{1cm}  \text{(18)}

Using the definitions of \( \bar{\alpha}, \beta \), and \( A \) in Equations (3), (4) and (6) this can be written in matrix form

\[ \mathbf{p}(\tau_{k+1}) = A \mathbf{p}(\tau_k) + \left( \mathbf{1}^T \bar{\alpha} \right)^{-1} \alpha (\zeta_{k+1} - \zeta_k). \]  \hspace{1cm}  \text{(19)}

Note that this leads to the general system definition in Equations (8) to (10) where \( \gamma_k = \zeta_k, \gamma = 0 \), and \( \bar{\gamma} = \mathbf{1}^T \bar{\alpha} \).

B. Varying resource

While the convergence analysis in [11] implicitly assumes that the available resource \( \bar{P}(t) \) is constant, there are other applications, where this is rarely the case. We assume, however, that variations in the available resource can be bounded such that the following assumption holds.

**Assumption III.1.** Let \( \bar{P}, \psi, \bar{\psi} \) be scalars. Then, \( \bar{P}(t) = \bar{P} + \psi(t) \), with \( \bar{\psi} \leq \psi(t) \leq \psi \) for all \( t \geq 0 \).

The principle of such variations is depicted in Figure 1. Note that the relations of the curves do not depict proper size relation. We also implicitly assume that the variations are small such that the aggregated share falls below \( \bar{P} + \bar{\psi} \) after at most one CE event.

\[ \begin{array}{c}
\text{Power} \\
\bar{T}^\Delta \bar{T}^\Delta \bar{P} + \bar{\psi} \bar{P} + \psi(t) \bar{P} + \psi \mathbf{p}(t) \end{array} \]

Fig. 1: Principle of the AIMD algorithm with a varying available resource.

As before, we investigate the evolution at two consecutive CEs \( \tau_k \) and \( \tau_{k+1} \). Assume at time step \( \tau_k \) the value of the variation in the available resource \( \psi(\tau_k) \) is given. As the variation is bounded at each time step we can compute the minimum and maximum time to the next CE event from \( \tau_k \). Let \( \Delta T_L \) and \( \Delta T_U \) be the minimum and maximum inter CE time, respectively. We know that the minimum time occurs if the variation in the resource is constantly at its minimum \( \psi \). Hence, the minimum time can be computed by solving

\[ \bar{P} + \bar{\psi} = \sum_{i=1}^{N} (\beta_i p_i(\tau_k) + \alpha_i \Delta T_L). \]  \hspace{1cm}  \text{(20)}

Since \( \psi(\tau_k) \) is given the following holds

\[ \mathbf{1}^T \mathbf{p}(\tau_k) = \bar{P} + \psi(\tau_k). \]  \hspace{1cm}  \text{(21)}

Inserting Equation (21) into Equation (20) yields

\[ \Delta T_L = \frac{1}{\mathbf{1}^T \alpha} \left( \sum_{i=1}^{N} (1 - \beta_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right). \]  \hspace{1cm}  \text{(22)}

Similarly, the maximum inter CE time occurs if the variation is at its maximum for all time. Similar steps lead then to

\[ \Delta T_U = \frac{1}{\mathbf{1}^T \alpha} \left( \sum_{i=1}^{N} (1 - \beta_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right). \]  \hspace{1cm}  \text{(23)}

From Equation (2) we see that the power consumption of each agent increases linearly with \( t - \tau_k \). Since \( \Delta T \) indicates the time between CEs, this means that the power consumption increases linearly with \( \Delta T \). Hence, inserting \( \Delta T_L \) and \( \Delta T_U \) into Equation (2), respectively, delivers an upper and lower bound such that

\[ p_i(\tau_{k+1}) \geq \beta_i p_i(\tau_k) + \frac{\alpha_i}{\mathbf{1}^T \alpha} \left( \sum_{i=1}^{N} (1 - \beta_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right). \] \hspace{1cm}  \text{(24)}

and

\[ p_i(\tau_{k+1}) \leq \beta_i p_i(\tau_k) + \frac{\alpha_i}{\mathbf{1}^T \alpha} \left( \sum_{i=1}^{N} (1 - \beta_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right). \] \hspace{1cm}  \text{(25)}

There exists a scalar \( \zeta \) with \( \bar{\psi} \leq \zeta \leq \bar{\psi} \) such that

\[ \mathbf{p}(\tau_{k+1}) = A \mathbf{p}(\tau_k) + \left( \mathbf{1}^T \alpha \right)^{-1} \alpha (\zeta - \psi(k)). \] \hspace{1cm}  \text{(26)}

where \( A \) is the AIMD matrix defined in Equation (6). Hence, variations in the available resource that fulfil Assumption III.1 can be represented using the general model in Equations (8) to (10) where \( \gamma_k = \psi(k), \gamma_{k+1} = \zeta, \gamma = \psi, \) and \( \bar{\gamma} = \bar{\psi} \).

IV. ANALYSIS OF THE DISTURBANCE MODEL

For the general disturbed system we find the following important property. This property is applied later for the two special cases of a discrete implementation and variations in the available resource.

**Lemma IV.1.** Consider the system defined by Equations (8) to (10). Then,

\[ ||\mathbf{p}(\tau_{k+1}) - \mathbf{p}^||_1 \leq \max_i (\beta_i) ||\mathbf{p}(\tau_k) - \mathbf{p}^*||_1 \]

\[ + \left( \max_i (\beta_i) + 1 \right) |\gamma_k| + |\gamma_{k+1} - \gamma_k|, \] \hspace{1cm}  \text{(27)}

where \( \mathbf{p}^* \) is the Perron eigenvector of \( A \) such that \( \mathbf{1}^T \mathbf{p}^* = \bar{P} \).

The proof of Lemma IV.1 is moved to Appendix A. Note that the Perron eigenvector of \( A \) is given by Equation (7).
A. Implications for the discrete time implementation

In Section III-A we showed how the discrete implementation of the AIMD algorithm can be represented by the disturbed model in Equations (8) to (10). Using Lemma IV.1 we find a region around the equilibrium point $p^*$ to which the state converges. Note that this equilibrium point is identical to the one from a continuous implementation. This means that a discrete implementation of the AIMD algorithm will lead eventually to a share close to the one of the continuous implementation.

**Theorem IV.2.** We consider the evolution in Equation (19) of the discrete implementation of the AIMD. Then,

$$
\limsup_{k \to \infty} \|p(\tau_k) - p^*\|_1 \leq \frac{(2 + \max_i (\beta_i)) T \alpha}{1 - \max_i (\beta_i)}.
$$

(28)

The proof of this follows directly from Section III-A and Lemma IV.1.

B. Implications for variations of the available resource

A similar result can be found for the case where the available resource varies.

**Theorem IV.3.** Consider the evolution in Equation (26) of the AIMD algorithm where the available resource varies such that Assumption III.1 holds. Then,

$$
\limsup_{k \to \infty} \|p(\tau_k) - p^*\|_1 \\
\leq \frac{(1 + \max_i (\beta_i)) \max(|\psi|, |\bar{\psi}|) + \bar{\psi} - \psi}{1 - \max_i (\beta_i)}.
$$

(29)

As for the discrete implementation the proof of Theorem IV.3 follows directly from Section III-B and Lemma IV.1.

V. EXAMPLE

A. Numerical Example

For the convenience of the reader, we here give a small numerical example for the discrete time implementation with a time step length of one second that illustrates the above results. Assume that two agents participate with an identical multiplicative factor of 0.8 and with identical additive factors of 0.001. The available resource is set to 1. Given the above values we find that the equilibrium point is $p^* = [0.5 \ 0.5]$. This means that the share of the two agents will converge to a ball around this equilibrium point with radius 0.028. For example, a value that could occur at a CE is $[0.514 \ 0.487]$. By reducing the additive factor such that it is equal to 0.0001 for both agents the radius of the ball is also reduced to 0.0028. Note that the equilibrium point itself is not affected by this change. Note that this is a conservative bound since it is based upon a Lyapunov argument.

B. Simulation Example

We illustrate the above findings on two examples where a total of 20 agents participate. In the first test we use a discrete implementation of the algorithm where the individual additive and multiplicative factors are drawn randomly at the beginning of the simulation from uniform distributions, such that the additive factors lie between 0.0005 and 0.001, and the multiplicative factors lie between 0.5 and 0.8. The maximum multiplicative decrease parameter is 0.798. The available resource is set to 40. In the second test the same individual parameters are used for a continuous implementation, i.e. very small time steps are chosen. However, this time the available resource is varying randomly with $\bar{\psi} = \bar{\psi} = 0.2$.

Figure 2 shows the results of these simulations. Not only is the bound met in these particular cases but the bound seems to be very conservative. This is expected since our results are based on a Lyapunov argument.

VI. CONCLUSION

We developed a model of a disturbed AIMD that is able to capture disturbances that arise either through a discrete
implementation of the algorithm or through allowing small variations in the available resource. This model is used to find a region to which the algorithm converges in either case. While the result is very conservative it allows a good worst case estimation.

APPENDIX

A. Proof of Lemma IV.1

We consider the system defined in Equations (8) to (10). Let \( p^* \) be the Perron eigenvector of \( A \) such that

\[
I^T p^* = \bar{p}.
\]  

(30)

We consider the norm \( ||A p(\tau_k) - p^*||_1 \). Then, using the definition of \( A \) and applying the triangle inequality yields

\[
||A p(\tau_k) - p^*||_1 = \sum_{i=1}^{N} |\beta_i (p_i(\tau_k) - p^*)|
\]

\[
+ \sum_{i=1}^{N} \left( \frac{\alpha_i}{\alpha_\ell} \right) \sum_{i=1}^{N} (1 - \beta_\ell)(p_i(\tau_k) - p^*_\ell)
\]

\[
\leq \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i|
\]

\[
+ \sum_{i=1}^{N} \left( \frac{\alpha_i}{\alpha_\ell} \right) \sum_{i=1}^{N} (1 - \beta_\ell)(p_i(\tau_k) - p^*_\ell)
\]

\[
= \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i|
\]

\[
+ \sum_{i=1}^{N} (1 - \beta_\ell) (p_i(\tau_k) - p^*_\ell)
\]

We define the two sets

\[
S^+ = \{ i \in \{1, \ldots, N\} | p_i(\tau_k) - p^*_i \geq 0 \} \quad \text{(31)}
\]

\[
S^- = \{1, \ldots, N\} \setminus S^+ \quad \text{(32)}
\]

Then, using these definitions and the triangle inequality, the bound on the norm can be rewritten as

\[
||A p(\tau_k) - p^*||_1 \leq \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i| + \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i|
\]

\[
+ \sum_{i=1}^{N} \left( \frac{\alpha_i}{\alpha_\ell} \right) \sum_{i=1}^{N} (1 - \beta_\ell)(p_i(\tau_k) - p^*_\ell)
\]

\[
\leq \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i| + \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i|
\]

\[
+ \sum_{i=1}^{N} \left( \frac{\alpha_i}{\alpha_\ell} \right) \sum_{i=1}^{N} (1 - \beta_\ell)(p_i(\tau_k) - p^*_\ell)
\]

\[
= \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i|
\]

\[
+ \sum_{i=1}^{N} \left( \frac{\alpha_i}{\alpha_\ell} \right) \sum_{i=1}^{N} (1 - \beta_\ell)(p_i(\tau_k) - p^*_\ell)
\]

We now distinguish two cases:

\[
\sum_{i=1}^{N} (-\beta_i) (p_i(\tau_k) - p^*_i) \geq \sum_{i=1}^{N} (-\beta_i) (p_i(\tau_k) - p^*_i)
\]

and

\[
\sum_{i=1}^{N} (-\beta_i) (p_i(\tau_k) - p^*_i) < \sum_{i=1}^{N} (-\beta_i) (p_i(\tau_k) - p^*_i)
\]

In the first case the norm can be bounded by

\[
||A p(\tau_k) - p^*||_1 \leq \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i|
\]

\[
+ \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i| + \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i)
\]

\[
\leq 2 \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i| + \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i)
\]

Hence,

\[
||A p(\tau_k) - p^*||_1 \leq \sum_{i=1}^{N} \beta_i |p_i(\tau_k) - p^*_i| + \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i)
\]

Note that due to Equations (9) and (30), the following holds

\[
\sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) = \gamma_k - \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i)
\]

(34)

Inserting Equation (34) yields

\[
||A p(\tau_k) - p^*||_1 \leq \sum_{i=1}^{N} \beta_i \left( \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) + \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) \right)
\]

\[
+ \sum_{i=1}^{N} p_i(\tau_k) - p^*_i
\]

\[
= \sum_{i=1}^{N} \beta_i \left( \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) + \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) \right)
\]

\[
+ \sum_{i=1}^{N} \beta_i \gamma_k + \sum_{i=1}^{N} p_i(\tau_k) - p^*_i
\]

\[
= \beta_i \left( \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) + \sum_{i=1}^{N} (p_i(\tau_k) - p^*_i) \right)
\]

\[
+ \sum_{i=1}^{N} \beta_i \gamma_k + \sum_{i=1}^{N} p_i(\tau_k) - p^*_i
\]

(35)
Using Equations (9) and (30) leads to the bound
\[ \| A\mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 \leq \max_i (\beta_i) \left( \sum_{i \in S^+} |\mathbf{p}_i(\tau_k) - \mathbf{p}_i^*| + \sum_{i \in S^-} |\mathbf{p}_i(\tau_k) - \mathbf{p}_i^*| \right) \]
\[ + \max_i (\beta_i) \gamma_k + |\gamma_k| \]
\[ = \max_i (\beta_i) \| \mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 + \max_i (\beta_i) \gamma_k + |\gamma_k|. \]

(36)

Considering the second case, a similar argument leads to
\[ \| A\mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 \leq \max_i (\beta_i) \| \mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 - \max_i (\beta_i) \gamma_k + |\gamma_k|. \]

(37)

Hence, we have
\[ \| A\mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 \leq \max_i (\beta_i) \| \mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 + \left( \max_i (\beta_i) + 1 \right) |\gamma_k|. \]

(38)

We now consider the norm \( \| \mathbf{p}(\gamma_k+1) - \mathbf{p}^* \|_1 \). Using the triangular inequality and Equations (9), (30) and (38), we find
\[ \| \mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 \]
\[ \leq \| A\mathbf{p}(\tau_k) - \mathbf{p}^* + \left( T^\alpha \right)^{-1} \alpha (\gamma_k+1 - \gamma_k) \|_1 \]
\[ \leq \| A\mathbf{p}(\tau_k) - \mathbf{p}^* \|_1 + \left( \max_i (\beta_i) + 1 \right) |\gamma_k| + \left( \max_i (\beta_i) + 1 \right) |\gamma_k| + |\gamma_k+1 - \gamma_k|. \]

(39)

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