Distributed load management supporting power injection and reactive power balancing

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Doctor of Philosophy (PhD)

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The University of Newcastle

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Statement of Originality

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Abstract

In recent years active control over selected electric loads has become increasingly important to reduce the peak demand and allow integration of intermittent renewable generation into the power grid. Two groups of loads are especially interesting in this regard due to their expected numbers and large freedom in scheduling: electric vehicles and so called thermostatically controlled loads, such as refrigerators, hot water heaters, or air conditioners. These electric loads allow their power consumption to be adapted depending on the needs of the distribution grid with a minimal impact on the customers. We consider two types of load power control abilities: binary and continuously controllable power.

In this project we propose a load management scheme to deal with such electric loads. The load management scheme allows the usage of two algorithms: one for binary and one for continuously controllable loads. The proposed load management scheme relies on broadcast signals that are sent by a central management unit to all the agents connected. This means that the communication load is low and that simultaneous management over different load types is possible, due to the identical set up. Further, as there is no data transmitted from the controllable loads to the central management unit, there are no data protection or privacy issues present.

For loads participating with binary controllable power consumption, we propose a binary automaton algorithm that uses stochastic decisions made by the agents to govern the power consumption. This algorithm’s behaviour is analysed and shows promising behaviour in simulations. The algorithm we propose for handling continuously controllable loads is the additive increase multiplicative decrease (AIMD) algorithm. This algorithm is commonly used for congestion control in communications networks and has shown to be very flexible and reliable. Its behaviour has been investigated in detail. While there are some adaptations needed to apply this algorithm in a load management case, we can apply many of the existing results found for the AIMD algorithm as it is applied in congestion control. We extend the analysis where necessary for our case.
## Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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<tr>
<td>AI</td>
<td>additive increase.</td>
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<tr>
<td>AIMD</td>
<td>additive increase multiplicative decrease.</td>
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<tr>
<td>AINMD</td>
<td>additive increase non-linear multiplicative decrease.</td>
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<td>BA</td>
<td>binary automaton.</td>
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<td>CCCV</td>
<td>constant current constant voltage.</td>
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<td>CE</td>
<td>capacity event.</td>
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<tr>
<td>CRF</td>
<td>Charge Rate Fairness.</td>
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<tr>
<td>CTF</td>
<td>Charge Time Fairness.</td>
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<td>DAIMD</td>
<td>dual additive increase multiplicative decrease.</td>
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<tr>
<td>EV</td>
<td>electric vehicle.</td>
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<tr>
<td>FEV</td>
<td>full electric vehicle.</td>
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<tr>
<td>G2V</td>
<td>grid to vehicle.</td>
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<tr>
<td>GOF</td>
<td>Global Optimum Fairness.</td>
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<tr>
<td>HEV</td>
<td>hybrid electric vehicle.</td>
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<tr>
<td>IFS</td>
<td>iterated function system.</td>
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<tr>
<td>KKT</td>
<td>Karush-Kuhn-Tucker.</td>
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<tr>
<td>MD</td>
<td>multiplicative decrease.</td>
</tr>
<tr>
<td>NAIMD</td>
<td>non-linear additive increase multiplicative decrease.</td>
</tr>
<tr>
<td>Acronyms</td>
<td>Definition</td>
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<td>---------</td>
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<tr>
<td>OFF</td>
<td>turn off.</td>
</tr>
<tr>
<td>ON</td>
<td>turn on.</td>
</tr>
<tr>
<td>PHEV</td>
<td>plug-in hybrid electric vehicle.</td>
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<tr>
<td>PO</td>
<td>prioritised optimisation.</td>
</tr>
<tr>
<td>REF</td>
<td>Required Energy Fairness.</td>
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<tr>
<td>RET</td>
<td>Renewable Energy Target.</td>
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<tr>
<td>SWER</td>
<td>single-wire earth return.</td>
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<tr>
<td>TCL</td>
<td>thermostatically controlled load.</td>
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<tr>
<td>TCP</td>
<td>Transmission Control Protocol.</td>
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<tr>
<td>V2G</td>
<td>vehicle to grid.</td>
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Symbols

$\beta^{(1)}$ The first multiplicative factor used in the AIMD algorithm.
$\beta^{(2)}$ The second multiplicative factor used in the AIMD algorithm.
$E$ The energy that is required by an EV to fully charge its battery.
$L$ The number of priorities in the Charge Time Fairness scenario.
$N$ The number of controllable agents.
$T$ The time step at which an agent connects to the grid.
$\Psi$ The time duration an EV is connected to the grid.
$\alpha$ The individual additive factor in the AIMD algorithm.
$\bar{P}$ The available power.
$\beta$ The actual multiplicative factor used in the AIMD algorithm.
$\gamma$ The priority that is assigned to an EV in the Charge Time Fairness scenario.
$\hat{p}$ The desired active power consumption of a controllable agent.
$\kappa$ The time an EV remains in the same priority group in the Charge Time Fairness scenario.
$\lambda$ The probability to choose the first multiplicative factor during the AIMD algorithm.
$S$ The set in which the active power consumption of the controllable load can lie.
$\mu$ The probability to turn off when using the BA algorithm.
$\nu$ The probability to turn on when using the BA algorithm.
$\overline{\pi}$ The global additive factor in the AIMD algorithm.
$\overline{\vartheta}$ The maximum inside temperature allowed of a controllable refrigerator.
$\overline{\pi}$ The global additive factor for the reactive power AIMD.
$\overline{p}$ The maximum active power consumption a controllable agent can handle.
$\overline{\sigma}$ The maximum apparent power a controllable agent can draw.
$\rho$ The average active power consumption of controllable loads, either it is a long term average since connection or a short term average over a period with predefined length.
$\tilde{E}$ The battery capacity of an EV.
$\tilde{p}$ The aggregated active power consumption of uncontrollable agents.
$\tilde{q}$ The aggregated reactive power consumption of uncontrollable agents.
$\overline{\beta}$ The minimum decrease enforced during the MD phase.
Symbols

ϑ The minimum inside temperature allowed of a controllable refrigerator.

p The minimum power consumption a controllable agent draws.

υ The indicator whether an agent is in G2V or V2G operation for the active power when applying the extended AIMD algorithm.

Θ The temperature inside a controllable refrigerator.

ξ The probability to choose the first multiplicative factor during the reactive power AIMD.

ζ The indicator whether an agent is in G2V or V2G operation for the reactive power when applying the DAIMD algorithm.

a The individual additive factor for the reactive power AIMD.

b(1) The first multiplicative factor used in the reactive power AIMD algorithm.

b(2) The second multiplicative factor used in the reactive power AIMD algorithm.

b The expected multiplicative factor in the AIMD algorithm.

g The individual cost function of an EV in the Global Optimum Fairness scenario.

m The number of steps used for the turn off probability during the BA algorithm.

n The number of steps used for the turn on probability during the BA algorithm.

p The active power consumption of controllable loads.

q The reactive power consumption of controllable loads.
Math Notation

\( A, B, \ldots \) Sets.
\( \mathbb{R} \) The set of real numbers.
\( \mathbb{R}_+ \) The set of non-negative real numbers.
\( \mathbb{N} \) The set of natural numbers.
conv \( \mathcal{X} \) The convex hull of a set \( \mathcal{X} \), it may be defined as the smallest convex set containing \( \mathcal{X} \).
\( |x| \) The absolute value of \( x \in \mathbb{R} \).
\( \lfloor x \rfloor \) The integer part of \( x \in \mathbb{R} \).
\( (x \mod y) \) The modular operation of two integers \( x \) and \( y \), i.e. \( x \) modulo \( y \).
\( x \) A column vector with elements in \( \mathbb{R} \).
\( x_i \) The \( i \)-th element of the vector \( x \).
\( e_i \) The \( i \)-th canonical basis vector.
\( \mathbf{1} \) The column vector of all ones.
\( x \neq y \) There exists an index \( i \) with \( x_i \neq y_i \).
\( A \) A matrix \( A \) with elements in \( \mathbb{R} \).
\( \mathbf{I}_n \) The identity matrix of dimension \( n \times n \).
\( ||x||_1 \) The \( l_1 \) norm of the vector \( x \), i.e. \( \sum_{i=1}^{n} |x_i| \).
\( \text{dist}_1 (x, \mathcal{X}) \) The distance of a point \( x \) to a set \( \mathcal{X} \) with respect to the \( l_1 \) norm, i.e. \( \min_{z \in \mathcal{X}} (||x - z||_1) \).
\( B_1 (x, \delta) \) The closed ball of radius \( \delta \) around \( x \) with respect to the \( l_1 \) norm.
\( ||x||_{H,r} \) The norm of a vector \( x = \begin{bmatrix} x_1^T & \cdots & x_T^T \end{bmatrix} \) defined by \( \min_{i=1, \ldots, T} ||x_i||_1 \).
\( \text{dist}_H (x, y) \) The Hilbert metric between two vectors, i.e. \( \max_{i} \log \frac{x_i}{y_i} - \min_{j} \log \frac{x_j}{y_j} \).
\( e^{\delta_H} (x, y) \) The Hilbert metric with the logarithm removed, i.e. \( \max_{i} \frac{x_i}{y_i} \cdot \min_{j} \frac{x_j}{y_j} \).
\( A \otimes B \) The Kronecker product of the matrices \( A \) and \( B \).
\( A \oplus B \) The Kronecker sum of two square matrices \( A \) and \( B \), i.e. \( A \otimes I_m + I_n \otimes B \) if the dimensions of \( A \) and \( B \) are \( n \times n \) and \( m \times m \), respectively.
\( A^T, x^T \) The transpose of the matrix \( A \) or the vector \( x \).
\( A^{-1} \) The inverse of the matrix \( A \).
\( \Pr [A] \) The probability that event \( A \) occurs.
\( E [X] \) The expected value of a random variable \( X \).
Chapter 1

Introduction

1.1 Motivation

The reduction of greenhouse gases has become more important due to its impact on climate change and increased public interest in environmental sustainability. One important step to reduce greenhouse gases is to increase the share of renewable sources in the energy sector. For example, the European Union set a community target that 20% of all consumed energy should be from renewable sources by 2020 [111]. Due to this community target, each individual member country has set an individual target. In Ireland, this is 16%, while in Sweden a target of 49% is set [46], [111]. The sectors that are most affected by adoption of these targets are transportation, heat, and electricity.

Particularly in the electricity sector, an increase in renewable power generation, often solar or wind power, is supposed to help achieve a reduction in carbon emissions. Ireland, for example, planned in 2010 to receive 40% of their energy from renewable sources by 2020, from which about 86% is expected to be produced by wind farms [39], [46]. Germany also plans to increase the share of renewable electricity up to 35%, 50%, 65%, and 80% by 2020, 2030, 2040, and 2050, respectively. Wind energy is also in Germany considered as an important part of the renewable generation that is needed to achieve the targets set [28]. Similarly, Australia’s Renewable Energy Target (RET) aims to ensure that 20% of the electricity supply is delivered from renewable sources by 2020 [12]. A recent review of the RET published in 2014, finds that the RET was successful in encouraging renewable energy generation, both for large and small scale installations. However, the high costs of the scheme led to the suggestion by the panel to make changes to the plan in its current form [34].

Renewable power generation is often intermittent, meaning that without any additional battery storage devices or control, the actual power output changes rapidly depending on weather conditions. This intermittent nature of the power delivery imposes new challenges on the distribution grid [39], [92]. In fact, it limits the deployment of renewable energy generation without any stabilising measures [92]. For example, Horizon Power limits the total amount of renewable energy that can be produced by customers and requires strict regulations regarding the rate of
change in the power output [45]. Complying with these requirements requires energy storage that can flatten the power output of renewable energy generation. This, in turn increases the cost of such generation, which could decrease the level of deployment.

For the reasons mentioned above, it is critical to find a way to lessen those effects and so allow a high penetration of renewable energy generation. “Load Management” and “Demand Side Management” strategies to support the distribution grid are regarded as a key part of the solution to this severe problem [20], [21], [59].

1.2 Load Management

The basic idea behind “Load management” and “Demand Side Management” is that some loads connected to the power grid change their power consumption depending on the needs of the distribution grid and to fulfil some ancillary services to the grid [55]. Usually only a small subset of loads support such services, we refer to these loads as controllable agents. Most loads that are connected to the power grid are solely controlled by the owner without considering the distribution grid. Hence, these loads are not controllable by the service provider and we refer to them as uncontrollable agents. In this context, “peak shaving” and “load tracking” are two primary support services that concern the demand side management community. These two services are investigated especially in regard to renewable energy generation. The basic idea here is that load management can be used to counteract the intermittent energy production of renewable sources. Also, renewable energy is not necessarily produced at times where it is required and so there are often times when there is a surplus of energy produced compared to the demand. In this situation, the scheduling of load to meet the production of renewable energy can consume such surplus energy that otherwise would not be used. In this form the loads fulfil a similar task to storage devices. However, both support services have the ability to support the distribution grid in other ways, such as delaying infrastructural investments or reducing costs [57].

Peak shaving is an instance of time-shifting energy demand. Usually, the total energy consumed by the controllable agents throughout the day remains unchanged and only the time allocation, when the power is drawn, is adapted. This adaptation is done in a way to avoid times with a high aggregate demand and shift adaptable demand to other periods of the day with a lower aggregated demand. Times where the demand is high are denoted peaks. The time when such peaks arise and their duration depends mostly on the season and the location, but they follow a daily repetitive pattern. Figure 1.1 reproduced from [10] shows the demand in New South Wales during three days for winter from 19 July to 22 July in 2011 midweek. As can be seen there are periods during the day with considerably higher demand than others.

During such peaks the stress on the distribution grid is generally higher since it works closer to its physical limitations. Also, during this period the electricity is generally more expensive compared to the rest of the day. Hence, the reduction of peak demand supports the operation of the distribution grid by keeping currents and temperatures in power lines and transformers within their limits. It also contributes to delay network infrastructure upgrades, since the capacity of the network, including that of the total available generation required, is determined by the peak
1.2. Load Management

Figure 1.1: Demand data for winter from 19 July to 22 July in 2011 in New South Wales. This figure is reproduced from [10] and can be found in that report as Figure 2-9.

Demand. While the current is proportional to the load, the losses are basically proportional to the square of the current. This means that transmission losses are increased during peak demand. Hence, an even load distribution throughout the day minimises the transmission losses, while transferring the same amount of energy. Similarly, the generation costs are higher during peak load as more expensive generators have to deliver the additional power. Hence, the reduction of the peak demand decreases those factors and thereby improves the network utilisation [2], [21], [41], [59], [87]. This means, for example, to use excess wind energy generated at night instead of increasing the power output of a conventional coal power plant during the day [79]. Closely related to peak shifting is “valley filling”, which shifts the demand of controllable loads to times with low power demands. While peak shaving emphasises reduction of the peak demand, valley filling focuses on increasing the demand during periods with low power demands. Both services drive the demand pattern towards an even pattern instead of the load pattern seen in Figure 1.1.

Load tracking is a network service where controllable loads are governed such that the aggregated demand follows a given varying power signal. At any time the total amount of power
generated and consumed in the power grid should be equal to avoid voltage and frequency deviations which disturb the operation of other appliances. Normally, in the power grid this balance between demand and generation is governed by adapting the generation to meet the demand. In this context, load tracking is particularly useful to support the achievement of this balance by inverting the task, i.e. the demand is adapted to follow the generation. This approach is also very useful to follow the fluctuating power generated from renewable sources, which cannot be controlled [20]. In this form, load tracking contributes to stabilising the capacity of renewable energy generation by lowering the demand when gaps in the generation arise, and increasing it when power is available. This stabilisation of the capacity is often referred to as capacity firming [35]. In particular, solar electricity generation is a leading target for capacity firming since the electricity is generated at times of high demand (shoulder and peak) but is characterised by fast intermittency. Fast intermittency at times of high demand increases the need for equally fast dispatchable power sources to compensate the drops in the renewable power capacity [35]. Such fast dispatchable generation may not be locally available, an issue that has driven some utilities in Australia to limit photovoltaic installations by imposing stringent ramping requirements, see for example the technical requirements in [45]. However, storage systems or load tracking can replace some of the required dispatchable generation.

The concept of load tracking is however not limited to the above example. By covering different time scales it can also be used to fulfil ancillary services like: 1. Regulation, 2. Load Following, 3. Frequency Responsive Spinning Reserve, 4. Supplemental Reserve, and 5. Backup Supply Plan. [55]. Those five services all cover the task of maintaining a balance between the power that is generated and absorbed. Each of those tasks is working on a specific time scale and in specific situations. For example, Regulation works on a minute basis, while Load Following works on an hourly scale [55].

The two services, peak shaving and load tracking, change the demand pattern of a single load. As the primary task of the loads is not the delivery of these services, it has to be guaranteed that the control has no negative impact on customer satisfaction. Hence, not all loads are suitable for load management schemes. The key requirements for their feasibility, as investigated in [20], [21], [39], are:

- **Freedom in scheduling**, i.e. during long periods they are able to reduce their power consumption or to turn off completely without a significant effect on the end usage.

- **Predictable energy requirements**, i.e. it is predictable how much energy is required in the near future.

- **Availability**, i.e. the more loads are available at any time the larger is the power that can be controlled.

Some loads are controversial in regards to their suitability for load management. For example, the work [20] suggests that the control of air-conditioners will have a negative effect on the customer, while [21] uses them as an example for load management due to their high freedom of
scheduling. [59] argues for control over televisions and lights by defining a utility function that captures the satisfaction of the owners, afterwards the schedule of the devices is optimised via a price signal. However, such a control requires a lot of cooperation amongst the customers as the schedule might influence their behaviour. Hence, it is questionable how many customers will allow such a control, which complicates the deployment of a widespread load management scheme. There is a general consensus that thermostatically controlled loads (TCLs), such as refrigerators or hot-water storage systems, and electric vehicles (EVs), that are connected to the distribution grid for recharging purposes, will play an important role [20], [21], [39]. In fact, both types are able to store energy in a form such that their power consumption is highly flexible in time. Also, those two load types provide precise quantification of the impact to end-use by employing load management. In the case of TCLs, this consists of the achieved temperature in comparison to the desired value and the allowed band. Especially, it is possible to prevent any negative impact on the customer by forcing the temperature to remain within some given bounds [70]. For EVs the state of charge at the end of the charging period is a good indication of the impact on the end-user.

In addition to the two primary support services that are described above, some loads could support the power grid by allowing reactive power exchange with the grid [22], [55]. The controllable loads in that case either consume or inject reactive power into the grid to compensate for the reactive power required locally where they are connected. In particular, areas close to a load with high reactive power needs, for example an industry site, are suited for such services. Allowing this service however requires a more advanced control and control capabilities at the loads. This will most likely increase the price of the appliances. Hence, we estimate that it will mostly be applied by loads that generally already support the ability by their design, e.g. loads that contain an active rectifier, or for loads with very large power capacities.

### 1.2.1 Electric Vehicles

Most car manufacturers have one or multiple vehicles which no longer rely solely on an internal combustion engine but also on an electric drive system. These vehicles can be divided in three groups:

- **Hybrid electric vehicles (HEVs)** consist of both a combustion engine and an electric motor. The electric engine supports the combustion engine to improve fuel economy and/or performance. There are two categories of HEVs. Vehicles from the first category use a combustion engine and an electric motor to propel the car. In some cases, a further distinction between a “full” and a “mild” HEV is made, where the full HEV can run on either the electric or the combustion engine, while the mild HEV uses the electric motor to mainly assist the petrol engine rather than being able to propel the vehicle by itself. Most HEVs fall in this category, including the Toyota Prius, and various models from Honda [81], [83]. Vehicles in the second category also consist of both an electric and a combustion engine. Unlike vehicles from the first category, the combustion engine of the vehicles is not used for driving but is used to produce electricity to either charge the batteries or directly power the electric motor, for example the Holden Volt uses this concept even though it actually is a plug-in hybrid.
electric vehicle [80]. The reduction in fuel consumption for passenger vehicles is between 25% – 35% [112].

- **Plug-in hybrid electric vehicles (PHEVs)** are similar to HEVs. Compared to a HEV the battery is larger and they are equipped such that the battery can be charged at appropriate charging systems. They still contain a combustion engine, where we can distinguish between the same two categories as for the HEVs. For example, the Honda Accord comes as a PHEV option and the Holden Volt also belongs in this category [80], [81], [69] predicts that PHEVs can reduce the total carbon emissions remarkably. However, the reduction is dependent on the electricity generation during the charging. More renewable energy deployment increases the positive effects on the environment by using PHEVs.

- **Full electric vehicles (FEVs)** or pure electric vehicles only use an electric engine and do not have a combustion engine included. The battery is usually larger than in plug-in hybrids to give a larger range. Examples are the Nissan ZE0 Leaf, the Honda Fit EV, the BMW I01 i3, the Renault Kangoo ZE, and the Tesla roadster [13], [81], [82]. The reduction of carbon emissions for those vehicles is dependent on the energy production that is utilised during charging. The more renewable energy generation is deployed during charging of the vehicles the less carbon emissions are related to the driving of these vehicles. Another useful capacity of FEV is that the location where the generated pollution is released is shifted from where the vehicle is driven to where the electricity is produced. This can especially have a positive effect on general health in very densely populated areas, since power stations are normally located in rural areas, while a lot of traffic occurs in cities.

We are only interested in the two later categories, i.e. PHEVs and FEVs, which require battery charging. To refer to vehicles in either of the two categories, we use the general term EVs. Such vehicles are not yet widely spread in public. However, the general trend to increasing fuel prices and the necessity to push the transportation towards environmental and CO₂ friendly technologies drives their deployment. Though the exact expected penetration levels vary in different countries, many authors agree that the penetration levels are most likely to increase in the near future [4], [25], [44], [109], [113]. The variations arise from different circumstances such as fuel price developments, government incentives, vehicle costs, household incomes and the area. For example, [44] predicts that by 2030 the market share of EVs in Victoria lies between 34% and 50% depending on the incentives that are in place.

A high penetration level of EVs in turn increases the demand for charging them, which means that the energy demand on the grid increases. Depending on the time of the day when the charging occurs this might increase the peak demand and in the worst case problems such as voltage drops or overload in lines or transformers can occur. Different studies investigated the effect that charging of EVs will have on the grid regarding peak demand and voltage deviations [1], [25], [66], [84], [89]. [84] argues that the distribution grid can cope even with high penetration levels of up to 50%, [66] estimates a maximum penetration of 10% without any control. Note that the allowed voltage deviations in [84] are twice as large as in [66], which explains the large difference in the maximum penetration levels. The expected problems also highly depend on the grid structure,
which means that they can occur locally with small penetration levels [1], [89]. In general, it is expected that large penetration levels of EVs cause voltage deviations, increase losses, and have negative effects on transformer life and hence cause the need for reinforcement of the infrastructure if their charging is not controlled [4], [25], [42], [66], [84], [85], [87], [88]. This makes it clear that control mechanisms have to be developed to allow wide spread deployment of EVs.

As EVs are expected to be parked most of the day [22] allowing them to charge over a long period, their charging can be scheduled relatively freely. Hence, they are able to act like a controllable load with energy storage capabilities. This is an essential property for participation in load management schemes. In that case, the charging infrastructure has to be available not only at home, but also in the workplace and parking areas. In particular, vehicles with small battery sizes are more likely to charge often and also at work, shopping places, and on the street [94]. Adding charging at workplaces and shopping centres could reduce the probability that vehicles run out of energy while driving and shift the load partly away from the evening peak [9]. While most studies concentrate on applying load management schemes in residential settings [24], [25], [29], [41], [85], [118], some have the potential to adapt to other settings [25], [41], [85].

With this capability to act as energy storage, EVs may also be utilised to inject power into the grid, often referred to as vehicle to grid (V2G) operation. Especially during day-times this can lower the peak demand and help regulate the supply frequency, see for example [2], [67]. When injecting power into the grid the needs of the EV owner have to be taken into account. This includes firstly that the owner receives the energy needed in time but also that there is enough energy left at all times for unexpected trips. Secondly, such an operation increases the stresses on the battery by forcing additional cycles on it or by depleting it to low energy levels. Such stresses can reduce the lifetime of the battery which is not in the interest of the customer [61]. If the lifetime of the battery is reduced this means that the capacity is reduced and the customer needs a replacement earlier, which increases his costs. These costs naturally have to be compensated for in some form. Also, the higher costs of the power electronics that allow for bi-directional power flows will have to be covered. One way to take the owners needs into account is to limit the energy that is allowed to be used for V2G operation [2]. Other possibilities are to limit the time when the vehicles may operate in V2G mode or the duration of V2G operations. Note that even without other regulations, the duration is limited by the energy that is stored in the vehicle’s battery. This mostly means that the power is only available for short periods of time before the vehicle requires charging again. Hence, EVs are most useful for short time control of the demand. Due to the higher costs and the possible impact on the customer, such as the reduction of the battery lifetime and capacity or missing energy in the battery if the vehicle is needed, the use of V2G has to be carefully considered. Also, when using the energy of vehicles it might influence the environmental impact in ways that are not usually considered. For example, at one point in time the vehicle has to be recharged. If this necessary recharging occurs at a period where even more energy is used produced by less environmentally friendly power plants the inverse environmental effect might be higher than if no V2G operation was used [104].

Further, EVs are ideal candidates for reactive power balancing as a third service to the grid. This means that the EVs consume or provide reactive power additionally to their consumption or
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provision, in case of V2G capabilities, of active power. The necessary charging infrastructure, e.g. an active rectifier will most likely be used anyway for the control of the power factor. In these cases, there is no additional hardware required to permit reactive power balancing. Regular power converters have a current limit, which means that when reactive power is consumed or provided the active power that can be used for charging the EV is reduced. However, as long as the active power consumption is reduced due to the participation in a load management scheme for the active power, the EV is able to provide some reactive power before the current limit would be reached and so participate in reactive power balancing. Such balancing during controlled charging can occur without an impact on the charge rate of the single EVs [22]. Naturally, as soon as the EVs are allowed to charge with their maximum capacity, they will not be able to provide this service without giving up some of their charge rate. In some cases it might be possible to encourage the owners to allow a small reduction in their charge rate to provide this service, however we assume in this thesis that the reactive power balancing is an additional service that should not affect the charge rate in any way. This restricts the times and the amount of reactive power a single vehicle can provide significant.

Naturally, by using EVs as controllable loads, they introduce some new limitations as to how far they are capable of supporting the grid. Those limitations are:

- **Battery size**: Each EV itself can only consume or provide a limited amount of energy before the battery is full and empty, respectively. This problem is reduced when the penetration level of EVs increases as expected, since the limited battery size can be mitigated by the large number of participating EVs. Due to the limited energy availability, EVs are better suited for short term services, rather than long term demand shaping.

- **Charger outlet**: The total power that can be handled by the charger outlet is limited. Very often a constant current constant voltage (CCCV) charging method is used, where first a constant charging current is applied until a certain voltage is reached. Afterwards, the voltage is kept constant while the charging current will decrease. While it is possible during the constant current mode to adjust the current and so control the power, this does not work during the constant voltage part. It might be necessary to treat the vehicles for part of the charging as uncontrollable loads. Also especially when the voltage of the battery is low it is recommended to use low charging currents [61]. In this thesis, we assume that the battery is connected through a converter with a maximum capacity that fits the charging capabilities of the battery and assume that it can be treated as a controllable power load. This sets a bound for the maximum controllable power and hence limits the support capabilities. This bound is mitigated by the large number of participating EVs. While we assume that most controlled charging will take place using a single phase outlet with a low charge rate such as a domestic charger outlet, it is possible that three-phase charging becomes more usual with a much higher power capability. Such a development would mitigate the limit even further. These assumptions are similar to the ones found in multiple other studies. For example, most studies assume that small single phase charger outlets with a capacity around 4 kW are used [25], [29], [41].


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- **Availability**: To be used as energy storage the EVs have to be connected to the grid. Due to their primary function as a transportation device, this is not given at all times. While a large number of EVs helps to compensate for the connection time, it is likely that the driving patterns are similar for a large percentage of the vehicles. For example, the use of EVs for commuting will result in similar patterns during weekdays. During those periods the EVs will not be available to support the distribution grid. Data collected about the driving patterns of vehicle owners are widely used to predict the connection times and the energy demand or capabilities posed by EVs [94].

While these limitations need to be considered, EVs are able to support the distribution grid considerably. Also, all these limitations can be mitigated with increasing penetration levels of EVs and progresses in technology, as increased charge rates or larger batteries.

1.2.2 Thermostatically Controlled Loads

Besides EVs, TCLs are identified as useful for load management purposes [50], [59], [70], [73], [114]. This category includes refrigerators, electric hot water heaters, electric heaters, and also air conditioners. Some of these loads are controlled using a thermostat, which gives the name. They possess two important characteristics:

- **Energy storage**: All of these loads store thermal energy in some form, either heat or cooling, which is the reason why they are often referred to as heat storage devices. Thanks to this energy storage property, they allow a large freedom in scheduling.

- **Availability**: They are available in large numbers in the form of air conditioners and refrigerators in domestic places and offices. [21] states that roughly 50% of the electricity consumption in the United States is due to such loads. This enables them to control a considerable amount of power.

These two characteristics make them ideal agents for load management purposes as their power consumption can be adapted without a large impact on the customer. In fact, it is required for the temperature to remain within acceptable bounds at all times.

Similar to EVs these loads also have some restrictions. These are:

- **Minimum energy consumption**: As TCLs are required to keep the temperature within certain bounds to avoid the dissatisfaction of customers, a minimum amount of energy has to be consumed within a certain period of time. While the load is highly flexible in regard to when in this period the power is consumed, there is an upper limit on delaying the consumption of power. The amount of energy that is required within a certain time period depends on the load itself, as well as the outside and inside temperatures.
• **Maximum energy consumption:** There is also a maximum energy that a TCL is allowed to consume within a certain time period to maintain the temperature within its desired range.

• **Maximum power consumption:** The TCL is able to handle a maximum power at all times that cannot be exceeded. This limits the difference in the aggregated demand that the rescheduling of such loads creates. This effect is mitigated however by the large numbers of these loads.

While these loads also impose some limitations, they are perfect candidates for load management purposes due to their large numbers and the limited impact on the customer. In general, the customer will not notice the rescheduling unless the temperature leaves the desired range.

It is important to note that many hot water heaters are already under a simple load management scheme, where the energy provider is able to switch on and off the water heaters using signals that are sent directly over the power line. This form of control is denoted ripple control and directly affects the controlled loads. It is normally performed in emergency situations to manage the peak load [57]. Similarly, there is a recent program in Ontario, Canada that allows automatic control over the thermostat settings of participating air conditioners [99].

### 1.2.3 Existing and Proposed Load Management Schemes

The above considerations show the importance and the advantages of load management. In this section, we summarise what types of load management schemes are already in use and future possibilities.

A variety of load management programs have already been in use for a long time. These are mostly concerned with large power consumers [57]. For example, in Paris back in 1928 a direct load control mechanism was used to control public lightning. Such types of direct load control continue such that nowadays various countries still exert control over hot water heaters [57]. Most other methods that are readily in use are controlled through the price of electricity. There is in this case often a special contract between an energy provider and a high power consumer, which encourages the consumer to shift the demand to off-peak times [57]. Sometimes these contracts also include agreements that the consumer reduces the demand upon request by the provider.

While these contracts are only limited to a few selected consumers, many countries also adopted pricing incentives for low energy consumers and residential consumers to govern the use of electricity. This simply includes variable electricity tariffs that depend on the time of day [3]. However, these load management schemes are only partially effective in reducing the peak load. Other tasks, such as load tracking or peak shaving according to the real-time aggregated demand cannot be achieved using these pricing incentives.

In the past decade, new methods for load management of small scale consumers has been widely studied [2], [21], [25], [29], [36], [37], [41], [50], [59], [67], [73], [74], [90], [99], [109], [118]. These
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new methods vary throughout the literature. We identified some characteristics that allow the comparison and classification of different load management schemes. Those characteristics are:

- **Location of the intelligence of the control**: Is the load management scheme based upon a central control unit that is in charge or is a distributed approach used?

- **Communication requirements**: What information has to be transmitted and how often?

- **Controller ability**: Has the controller full controllability over the power consumption of the load? What limitations exist?

- **Time scale**: What is the time scale on which the controller operates? Is it a day, hour, minute, or second based regulation?

- **Direct control**: Does the load management scheme directly control the power consumption or is it governed indirectly by electricity pricing?

- **Supported services**: Which services does the load management scheme allow the loads to perform?

- **Additional required information**: Does the load management scheme require any predictions such as predictions of the demand by uncontrollable agents?

While there is a vast literature concerned with the control mechanisms and communication infrastructure, we found only little work where the proposed algorithms have been realised in a real environment. In [16], a testbed has been developed that allows the testing of a vast variation of load management algorithms with different communication capabilities. In the following, we discuss the aforementioned characteristics of such algorithms.

1.2.3.1 Location of the Intelligence of the Control

The first important characteristic is where the intelligence of the algorithm is implemented. We distinguish three distinct methods of control:

- **Centralised control**: A central control unit collects data from the participating agents and other connected measuring devices, decides the scheduling of the agents, and sends instructions to all agents.

- **Decentralised control**: There exists no central unit that coordinates the agents. Rather the single agents adopt their own power consumption according to the information they have. The agents might communicate with each other or measure some quantities, such as the frequency, to react appropriate.

- **Distributed control**: There exists a central management unit which collects data from some measuring devices and informs the agents accordingly. The agents that receive the information from the central management unit, decide their power consumption individually. Each agent’s decision depends on its own needs, possibly some local measurements, and the
information that is received from the central management unit. Additionally, an agent might need to communicate with its neighbours to gain the necessary information to make an appropriate decision.

While those distinctions are commonly found, distributed algorithms are sometimes also included in decentralised or centralised control depending on their required communication exchange with the central unit.

We can find examples of load management algorithms for all three categories in the literature: [25], [29], [67], [73], [86], [109] suggest centralised control algorithms, [36], [37], [41], [50], [59], [74], [118] suggest distributed algorithms, and [5] suggests a decentralised algorithm that uses the frequency of the power grid as an indication of the grid state.

Centralised algorithms are sensitive to the failure of any component of the system, while distributed and decentralised algorithms are more robust towards failure of single components.

Also, centralised control schemes are generally more likely to require large amounts of data to be communicated among the agents and the central unit than distributed or decentralised algorithms. This is however not true in all cases and we will investigate the communication requirements of the algorithms separately.

Apart from such challenges, centralised controllers show the full potential of load management schemes. Using a centralised controller it is possible to find an optimal solution to the task, since all the required data is collected at a central unit, while decentralised or distributed algorithms are not always able to find the optimal solution. While it is often possible to find an optimal solution to the load management task, it might occur that in some cases due to restricted time, restricted communication, and computation power, the computations cannot be performed, such that a sub-optimal solution is preferred [8]. Hence, the performance might be similar to distributed or decentralised algorithms.

1.2.3.2 Communication Requirements

The next point is the communication requirements which are linked with the previous point. In general, distributed and decentralised algorithms reduce the communication requirements compared to a central controller. However, the communication requirements and the necessary data communicated varies widely. For example, we find distributed algorithms with relatively high communication requirements in [50], [74], [118], while the ones found in [36], [37], [41] have much lower demands. Also, one version of the centralised controller in [73] requires only little communication. However, in this case, the reduced amount of communication is accompanied by a performance decrease.

The importance of the amount of data communicated is due to several factors. In cases with large communication demands, there might be considerable delays occurring that influence the responsiveness to instructions of the algorithms. This is especially important for fast reacting services. Also, in some cases the information that has to be transmitted scales badly with the
number of participating agents, such that a high number of agents causes a high communication load. The communication network in this case has to be designed to handle this load, otherwise it might be pushed beyond its limit when more agents connect. It is preferable to design an algorithm where the communication load does not scale with the number of participating agents, but rather stays approximately constant. In that way, the network can be designed to be efficiently utilised without pushing it beyond its limit.

While the above technical issues may arise with large communication requirements it is possible to address them. Another critical point that arises when transferring data is data protection and privacy. For example, in the case of EVs transferring data such as the expected disconnection time or required energy, the commuting behaviour of the owner can be interpreted. This data, if allowed to be collected, needs to be protected to prevent misuse. A possible instance of misuse is that the expected disconnection time can be used by a thief to check when the persons are absent from their home.

For example the EV charging coordination algorithm proposed by [29] requires the owners to select a preferred charging time frame, that has to be communicated to the central controller which then performs the optimisation. Further, it requires the load profile of each vehicle and its connection and disconnection times. In turn, the central unit directs the vehicle when to charge. While the constant update of the charging requires communication, the overall load is relatively small. Note though that such an algorithm requires point to point communication such that the central unit informs each vehicle separately. Similarly, also [41] requires point to point communication, where the central unit sends a price signal and the connected EVs send their charging profiles repeatedly until the prices and profiles converge. Even though in [25] a centralised algorithm is suggested, the communication load is similar to the ones mentioned before, as the only required information to send is a required charging deadline from the vehicles, and the charging profiles from the central unit. As before this requires a point to point communication between the agents and the central unit.

The decentralised algorithm in [5] uses no communication among the agents at all, but utilises the frequency of the power grid directly as a control signal. This naturally requires that the agent is able to measure the frequency.

1.2.3.3 Controller Ability
The next important characteristic is the freedom the controller has over the charge rate, i.e. whether the charge rate is continuously adaptable or only a binary control can be imposed. The effects of such constraints are investigated in [74], where the authors find that the more freedom the controller has over the power consumption of the agents, the better it is able to govern the aggregated power consumption. However, more freedom in control comes with higher complexity in the infrastructure and the controller design, the advantages therefore have to be weighed carefully.

Often binary on-off scheduling is used for the coordination of loads [29], [50], [67], [71]. [71] is one of the first who suggested load management for heat storage devices. This work investigates the effects of turning off a population of refrigerators for a time period after which they return.
to normal operation. The recovery to normal operation in this setting is characterised by power oscillations. This study is useful to explain some effects that occur if heat storage systems are used for scheduling. On the other hand, EVs that use this method are modelled as constant power loads drawing a constant amount of active power. Especially, for EVs we find however studies assuming continuous variations of the charge rate [2], [25], [36], [41], [90], [109]. In this case, the power consumption is limited by the power outlet.

In our opinion, it is not possible to limit the control ability to one mode. It is likely that some loads will allow binary control while others may allow for continuous control. The reason for this is mainly because we assume that the shift towards a flexible load management scheme will happen gradually. Further, the additional cost of a continuously controllable agent compared to a binary controllable agent is an important factor which control ability will be available, as continuous control requires a more complicated hardware. Hence, the algorithms should be able to cope with both control types simultaneously. Most studies though focus on only one possible controller ability at a time [25], [29], [41], [67], [90].

1.2.3.4 Time Scale

The time scale of the controller is also an important factor. While some services require an hourly based adaptation, some require a second based control. For example, [29] uses a real-time controller based on 5 min slots. Similarly, the controllers used in [25], [90] work on 15 min time slots. On the other hand in [41] a non-real-time controller is suggested without specifying a specific time slot length. In their simulation studies, they assumed a time slot length of an hour. The time slot length that can be controlled depends on the loads and their capabilities as well as the communication network. We consider no specific time slot length in this work, since we believe that adaptation of the algorithm should be possible to allow for different time scales that range from several seconds up to hours.

1.2.3.5 Direct Control

Another important characteristic is how the control is achieved. With this we mean, whether the control is imposed via pricing, either with or without automatic control, or another way is chosen. Here, we distinguish three major versions [57], [114]:

- **Time of use:** The electricity price that the customer has to pay depends on the time of the day. Electricity use during hours with a generally higher demand are more expensive. The times, however, are predefined and fixed. Such schemes are already in use, for example in Australia the electricity provider AGL allows to select a time of use plan where three tariffs are possible: 1. Peak, 2. Shoulder, and 3. Off-Peak [3].

- **Real-time pricing:** Here the customers are directly charged the actual energy price as they participate in the market. In this way, customers that respond rapidly to price changes have an advantage. However, the response of the loads can often be automated such that the customer does not manually react to price changes.
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- **Direct load control:** Here the load’s power consumption is directly controlled without adjustment of a price. Naturally, participation might be encouraged by monetary incentives.

Note that in [57] various other methods are discussed, such as load curtailment programs, other pricing programs, or education programs. However, in the literature we found that the above three types are most commonly discussed and we will therefore focus on those.

Time of use pricing is already widely used. It is an indirect method of control which requires the interaction of the customer. Even though this method encourages the consumers to shift or reduce their electricity usage, the effectiveness in governing the demand is limited. In particular, it does not allow a real-time reaction by the agents. Hence, this method is not applicable for load management in the form we are interested in.

Multiple sources suggest the use of real-time pricing signals to govern the power consumption of the loads [36], [37], [41], [59], [118]. Often the price signal causes the agents to react automatically by optimising a specific cost defined by the algorithm. Firstly, this requires the customers to trust that the actions undertaken are in their best interest. Secondly, it shifts the burden partly towards them, since they are charged higher prices for the consumed power during peak demand. In cases where there is no automated reaction, the customer needs to know the instantaneous power consumption at all times to react to it, which might not be possible [21]. Another related method is to govern the demand by participation in a bidding market. In [99], for example, the utility pays the home owners a price for allowing control over the thermostat of their homes. This price is then increased until the required amount of owners are willing to participate. This also means that the reaction of the control is directly dependent on the reaction of the customers.

There are various load management schemes suggested that use direct control [2], [21], [25], [29], [50], [74], [90]. This method is more controversial as the customer’s needs may not be taken into account. However, in some cases constraints are introduced in the control which handle the needs of the customer. Another possibility is that the customers might easily opt out of the load management scheme for certain times of the day if the use of power is essential. Two examples of such constraints are found in [25], [29]. In [29], the owners of EVs select a priority group which is linked to the preferred charging time. While the highest priority whose time collides with the evening peak in the aggregated demand is the most expensive selection, the lower priority, which means that the charging occurs during the night when the demand is low, is the cheapest alternative. This method can increase the willingness of participation among the customers as they are sure to receive a financial incentive for allowing the postponing of the electricity use. In [25] the customer is allowed to define a deadline when the charging of the vehicle should be finished. This also takes care of the customer’s needs.

Often in regards to TCLs the method is to not directly steer the power consumption but adapt the temperature set point [19], [21], [86], [99]. This however directly impacts the user, as the desired temperature is changed. As long as the set point change is small, the effects on the customer are minimal and might go unnoticed, while larger deviations lead to customer dissatisfaction. Hence, these schemes should provide guarantees that the deviations remain minor.
with a maximum number of exceptions per year.

1.2.3.6 Supported Services

The next point is concerned with the supported services. While most sources concentrate on regulating and scheduling the power consumption of loads [25], [29], [36], [41], [50], [74], [90], some references go further to also allow the loads to inject power back into the power grid [2], [67], [119]. This is normally discussed if the controllable loads are vehicles since they can be treated as batteries. In this setting, that principle is termed V2G operations. For example in [2] two scenarios are investigated: either the EVs are only allowed to charge during the night and inject power during the day, or the EVs choose freely when to charge and inject power. Naturally, the second option allows for a more optimal control, since it increases the freedom of the controller.

Another possibility is reactive power compensation as suggested in [22] for EVs, where the grid is supported without impeding the state of charge. Those operations however require a more complicated control structure. The possible benefits are large, as it would be possible to increase the power factor to a higher level. Especially, for loads at industry sites with a bad power factor this might improve their demand and decrease distribution losses.

1.2.3.7 Additional Required Information

A large part of the algorithms for load management require some additional information, such as load demand predictions or driving patterns. In particular, the need for daily load profile predictions is very common [25], [41], [74]. Such predictions are already performed in the energy market [11]. The need for these forecasts and the robustness to errors in those predictions are important factors that influence the usability of the algorithm. For example [41] requires a prediction of the demand by the uncontrollable loads and the energy requirements of the EVs. In [25] the algorithm proposed uses in a first approach deterministic data, but in a second approach expands the algorithm to use stochastic data for accounting to prediction errors. They find that distribution losses caused by errors in the prediction are small. The reason thereof is that the time when the peak load occurs can be predicted well.

Especially for TCLs a model of the controlled agents is often required. For example in [73], in the version of the management scheme with small communication requirements, a broadcast signal is sent to all controlled loads, that uses estimates of the states of the load to compensate for the information loss compared to the algorithm with more communication.

1.3 Our Goals and our Contributions

In this project, we aim to develop a distributed load management scheme for electric vehicles and refrigerators. The limitation to those loads is reasonable since they fulfil the key requirements as mentioned before. Firstly, these loads are available in large numbers. Nearly every household is in possession of one or multiple refrigerators and there are several commercial refrigerators and cool rooms. Similarly, the number of EVs in use will most likely increase in the near future as discussed previously. In regard to the size of the fleet of conventional vehicles at the moment, i.e. about
12 million passenger vehicles in Australia [113], already a small penetration of EVs might have a positive effect on the grid if controlled. Further, it is assumed that with increasing penetration levels, the only option that does not require significant investments in the infrastructure of the existing grid is to control the charging of the EVs. Secondly, the power that is required by the agents to fulfil the end-use compared to the connection time is reasonable. With that we mean that the duty cycle of refrigerators is often relatively small, such that they do not require any power for long periods. Similarly, if the EVs charge whenever they are parked, their charging can be scheduled over a long period of time. Finally, the energy requirements for both loads are predictable. However, the load management scheme should be generic such that other load types can be integrated easily. Loads that are worth considering are air conditioners, hot water heaters, water pumps, washing machines, or water filtering systems.

We aim to develop a distributed load management scheme with a small communication load, which scales readily with a large number of participants. This is important in the future if increasing numbers of electrical loads are able to participate in load management tasks. Our load management scheme will therefore rely on broadcasting signals from a central management unit to the agents. An important effect is that it reduces the complications regarding data protection and privacy since the central management unit does not receive any information from the controllable loads. Naturally, there are still some predictions that we require. In fact, it is important for the energy provider to know the approximate power that the loads are able to control at any time and the predicted demand of all loads. This is required to keep the disruption by the load management service for the customer to a minimum. However, such predictions are more concerned with the long term need of electricity rather than the instantaneous demand. For example, there should be enough capacity to charge all connected EVs overnight, but it is not critical exactly when this capacity is available. Moreover, the control itself is performed in real-time and is not fixed the day ahead.

Further, the control ability of the agents is not necessarily homogeneous and it is most likely that, at least during some periods, there are loads which allow different controls. Therefore, it is important that the load management scheme is able to deal with simultaneous control of different loads. We aim to develop two algorithms: one for binary and one for continuously controllable loads. These two algorithms work in a similar way, such that it is possible to deploy agents with different control abilities simultaneously. This is an important difference to existing literature as discussed in Section 1.2.

One important difference to other literature is that we focus on the service provided to the customers within the limitations that are set by the provider of the service. While the customer’s wishes and needs may vary depending on the situation, especially if the task is to charge the customer’s EV, we believe that the fairness experienced among the customers is a central goal. Similarly to the wishes and needs, also the notion of fairness can change depending on specific situations. Hence, the proposed algorithms should be able to fulfil such fairness notions.

We were able to achieve these goals, as we will show in this thesis. We proposed a distributed load management scheme that uses a central management unit to govern the connected agents.
1. Introduction

The central management unit is able to broadcast signals to the connected agents on whether to increase or to decrease their power consumption. The management unit therefore compares the actual aggregated demand by all connected loads to a preselected desired value and broadcasts the signals depending on the outcome of this comparison. While the selection of this desired value is critical, we assume, in this thesis, that the value is selected. Through this simple load management scheme the aggregated demand by all loads is controlled to follow the preselected desired value.

The reaction of the agents is determined by their control abilities and whether they receive a broadcast signal. We propose two algorithms that define this reaction. These two algorithms are: a binary automaton (BA) algorithm for loads with a binary controllable power consumption that is based on stochastic decisions by the agents and an additive increase multiplicative decrease (AIMD) algorithm for loads with a continuously controllable power consumption which is commonly used in communication systems. This load management scheme is hence a form of direct control over the agents. There are no pricing signals involved and the owners of the loads are only able to influence the behaviour in a limited form, for example to choose the time frame after which a vehicle should be charged or to select the desired temperature within a refrigerator. Otherwise, the customer has no influence on the behaviour. Similar influence possibilities are for example found in [25], [29]. Due to the limited external influence by customers, we are able to predict the behaviour of the management scheme. In fact, we show mathematically some specific properties of the algorithms, like its convergence in certain cases. At the same time we give an in depth discussion of the behaviour that we do not analyse mathematically. In both cases, we illustrate our results and the behaviour of the algorithms through simulations. On the other hand, as the load management scheme is based upon simple principles, it can easily be implemented.

Besides governing the active power consumption of the controllable loads, we expanded the proposed AIMD algorithm for loads with a continuously controllable power consumption to allow reverse power flows. While some studies suggest this in the context of EV charging, see [2], [67], [119], most literature is concerned with governing the active power consumption. Additionally, to allow for reverse power flows, the AIMD algorithm is adapted to enable the control over the reactive power of the connected loads. We show in this thesis that there is a great potential to support the grid by locally balance the reactive power and so increase the power factor. While [56] investigates an EV charger with reactive power operation, there has been very little investigation in the control of agents with this capability, see [22].

Finally, the proposed load management scheme works in real-time by letting the aggregated demand follow a preselected value. So far we have not mentioned on what time scale this control takes place, i.e. how fast the control should react. Often for longer time periods it is not the instantaneous power consumption, but the average consumption during that period that should be controlled. The proposed load management scheme is very flexible and by adapting the time step size and some tuning parameters of the proposed algorithms, we believe that it is possible to allow the operation on various time scales. During tests we found that a time step length of one second, i.e. the management unit compares the actual demand and the preselected desired value every second and sends broadcast signals accordingly, allows the average power consumption for a period over five to fifteen minutes to follow the desired value. Such a time scale is often
1.4 Closely Related Work

Among the vast literature regarding load management of EVs and refrigerators we want to point out some of the works that are closely related to our suggested algorithm and in accordance with our goals.

In [50] the authors suggest the use of proactive and reactive congestion control for load management. Such algorithms are widely deployed in Transmission Control Protocol (TCP) congestion control and have shown to be reliable, highly flexible, and require only minimal communication. The work in [50] does not elaborate in detail on how to apply the algorithms on the load management problem. The authors more recently elaborated on the use of congestion control mechanism for distributed EV charging in [7]. They propose a framework that is identical to the one we are using. They argue that such a control, if the infrastructure is available, can govern the charge rate on a very fast time scale of a few milliseconds. The availability of the infrastructure is an important constraint here, as the communication delays and the time an agent takes to adjust its power consumption need to be short. One of the three types of possible control strategies that they suggest is to apply the algorithm that we are using for loads with continuously controllable power consumption. In fact, it is also a requirement for their charging control and they do not consider other control abilities. They do not show the implementation of such a control in detail.

Similarly, in [36] the authors suggest a method that is frequently used in communication networks to adapt the charge rate. The method that is used is called proportionally fair pricing, where a user is assigned a larger bandwidth if they pay more. This is also a highly flexible and reliable algorithm. Unlike the algorithm we suggest, the actual demand is governed by the price and not strictly controlled by an upper bound on the available power.

A proportionally fair control strategy is also suggested in [6], [8] where the problem of charging EVs is stated as an optimisation problem. Here, the authors rely on continuous congestion price signals upon which the EVs adjust their charge rate. As in the previously mentioned works, this algorithm is based on the ideas of congestion control in communication networks. It is a reliable, distributed algorithm that works also in real-time. Similar to our algorithm it relies only on broadcast signals. One major difference is however that this algorithm unlike the one we propose, relies on a continuous broadcast signal. Also, they only investigate cases where the EVs
have a continuously controllable power consumption without V2G or reactive power balancing. Further, we employ AIMD as congestion control, which differs from their implementation. In their studies, they consider multiple locations that can cause overload. While we performed only a brief simulation analysis of such cases and concentrated on local distribution networks with only one bottleneck present.

While the above works focus on the theoretical applications, [16] reports the development of a testbed for load management algorithms that are found in the existing literature. This testbed is able to implement a vast variety of load management tasks, including battery charging and refrigerator control. In this context, the authors of [16] chose to implement the algorithm that we proposed in [101] on their testbed, the algorithm is presented in Section 4.1 in this thesis. In this regard, the authors found that the algorithm operated as predicted.

In [62], [63] the algorithm that we proposed in [101] is adapted and applied on a simple low voltage distribution grid. The algorithm is adapted such that also voltage violations are considered rather than only controlling the aggregated demand. Finally, in [64] an enhanced algorithm is compared with an ideal centralised solution and a distributed price-feedback algorithm. Both algorithms achieve comparable results.

1.5 Thesis Organisation

In this section we outline the structure of this thesis more detailed.

Chapter 2 introduces the problem mathematically and describes the system. Therefore, we first state two constraints that are given by the distribution grid and the agents, respectively. Using these two constraints we can formulate the problem of load management, i.e. the problem of sharing a predefined amount of power fairly among the controllable agents, by stating sets of objectives with constraints. In total, we present three such sets. The first one is the basis and so the most simple of the three sets. The second adds the capability of reverse power flows, while the third set additionally allows reactive power balancing. Both the second and third set expand the basic set of objectives. In this regard, we also define several fairness notions, since we believe that fairness can be achieved in different ways and may also depend on the situation in which the customers are in. Afterwards, we introduce the general set-up of the system that is used for the control. The system allows for the integration of different algorithms that dictate the behaviour of the agents. In this thesis, we present in total two such algorithms, which are presented in Chapters 3 and 4, where the second algorithm is analysed mathematically in Chapter 5. The first of these algorithms requires the agents to have binary control over their power consumption and is so the simpler algorithm of the two. It is mathematically analysed in regards to its stability and behaviour. We illustrate our findings and claims throughout Chapter 3 with simulations performed using Matlab. The second algorithm requires continuously adjustable power consumption. It is far more flexible than the first one and allows for different fairness notions. We also expand this algorithm to allow both reverse and reactive power exchange. The analysis of the second algorithm in its basic form can be found in Chapter 5. We study therein the stability of the algorithm for which we make use of existing results in the literature and expand these to cover the differences between our implementation of
the algorithm and how it is commonly used in the literature. Further, we analyse a method how
the algorithm can be used to solve constrained convex optimisation problems and the behaviour
of the aggregated demand of agents that are controlled applying this algorithm. The effects of
the adaptations made to control also reverse and reactive power flows are investigated directly in
Chapter 4. Chapters 3 to 5 contain our main contributions. Due to convenience of the reader we
shifted some of the proofs to Appendix A.

Then, in Chapter 6 we investigate our proposed algorithms in a more realistic setting and
adapt them to allow for a more realistic use. In particular, we simulate the algorithms in a test
network using a combination of Matlab and the power grid simulation software OpenDSS. Further,
we examine scenarios where both algorithms are applied simultaneously, which we believe is an
advantage of our proposed load management scheme. These simulations illustrate the potential
of our proposed load management scheme in more realistic settings. However, we did not apply our
load management scheme in a real-world scenario.

There are some challenges that we did not address in this thesis. While some of these challenges
are specifically occurring with our proposed scheme, other load management schemes, such as the
ones discussed in Section 1.2.3 face the same or similar challenges. These challenges are discussed
in Chapter 7. Finally, we conclude this thesis in Chapter 8.

1.6 Published Papers

The content summarised in this thesis has been published in several works that are done in collabor-
ation with other researchers. Here, we introduce these publications and comment on which parts
of the thesis are based upon which publication. The contribution of the authors is also remarked
for each of the papers.

electric and plug-in hybrid vehicle charging policies,” International Journal of Control, vol.

In this paper, we suggested for the first time to use an AIMD algorithm, commonly found
in communication networks, for solving charging objectives for EVs. This is described in
this thesis in Sections 2.1 and 2.2. We introduced a total of three specific scenarios for the
charging of EVs in different scenarios. These are: a domestic scenario, a workplace scenario,
and a public charging station scenario. Both the domestic and the workplace scenarios found
in that paper are also reproduced in this thesis in Sections 2.2.1.1 and 2.2.1.2 and the used
implementation of the algorithm in Section 4.1. While the formulation in the paper differs
slightly from the one used in this thesis the actual ideas remain unchanged. In this paper,
only active power consumption is regarded and no reverse power flows from the vehicle to
the grid are considered. The simulations are performed using Matlab without taking the
structure of the grid or other uncontrollable loads into account.
1. **Introduction**

This work is a collaboration between my supervisors, R. Middleton and R. Shorten, a co-worker E. Crisostomi, and me. Prof. Crisostomi supported me in a supervisory role in addition to my supervisors. In detail, he answered questions, lead me on the right path by giving valued advice, and double checked my findings. The work in this paper has been done at the Hamilton Institute at the University of Maynooth in Ireland, where I started my studies as a PhD under the supervision of R. Middleton and R. Shorten. After my supervisor R. Middleton relocated to Newcastle, I was able to join and continue my studies at the University of Newcastle.


In this conference publication, we expanded the results from [101]. Three new scenarios have been added with varying objectives for EV charging: a shopping centre scenario, a restaurant scenario, and a parking area scenario. The shopping centre scenario can be found in Sections 2.2.1.3 and 4.1.2.3 in this thesis. As in [101] the simulations were performed using Matlab, without taking the grid structure or other uncontrollable loads into account. This is in accordance to our approach to concentrate on the varying needs and wishes of the customers.

Again, this work is in collaboration with my supervisors, R. Middleton and R. Shorten, and E. Crisostomi. While I did the implementation E. Crisostomi and my supervisors gave me advice and we had very helpful discussions. Also, this work was conducted during my time in Ireland.


In this short paper, we investigated a new approach how to look at the concept of reversed power flows while charging EVs. While most work concentrated on the positive effects on the grid and the environment, we investigated negative effects that the V2G concept can have if not properly coordinated. This means that if the needed power is simply selected equally among the participating agents, the environmental effect can be sub-optimal. Also, in some cases it is more effective to use an available power plant to deliver the energy rather than taking it from the EVs.

While this work cannot be found in the thesis directly, the use of V2G operation is studied in another form here, where the objectives for the coordination are concerned with the customer rather than the overall environmental impact.

This was a joint work with W. Griggs, E. Crisostomi and my supervisor R. Shorten. Both W. Griggs and E. Crisostomi supported me with discussions on the subject, valuable advice, and ideas. Especially, W. Griggs helped me with the implementation. Also, she is responsible for the extension regarding the financial aspects while I worked mostly on the environmental
1.6. Published Papers

aspects. This work was done shortly before my relocation to the University of Newcastle.


This paper elaborates the work in [103] and delivers the ideas in more detail and with a more realistic example. As before, this work cannot be found directly in this thesis. The work in this paper was started in Ireland, however I finished most of the work after my relocation to the University of Newcastle.


This publication is introducing the algorithm that we suggested for loads with a binary controlled power. In this paper, the loads no longer require a continuously adjustable power consumption, but only allow for switching on and off. Since the loads are only able to switch on and off, reactive power balancing or reverse power flows are not considered. Further, this work does not consider the distribution grid or any connected uncontrollable loads. This work is reported in Chapter 3.


This work investigates the algorithm suggested in [106] in a simple radial distribution network and compares it to both a centralised approach and uncoordinated approach. Here, the algorithm is further adapted to also take into account voltage deviations and the power consumption limitations in each phase. To do so the algorithm is slightly adapted to react also on these type of constraint violations. The simulations that are shown in Section 6.1 are based upon the model used in this paper.

The work presented in that paper is joint work with M. Liu, his supervisor S. McLoone, and my supervisors. The simulations, including the development of the network model, have been performed by M. Liu, while I was involved in discussions with M. Liu in solving the problems and the algorithm side. This work was performed during a three month long exchange during which I was working at IBM research under the supervision of R. Shorten and J. Yu.


While we concentrated before solely on the power coordination side of the load management task, we analyse in this work the communications requirements that are needed for the
realisation of the algorithm and developed methods to solve problems that may arise during the usage of the algorithm found in [101], [102]. Part of the considerations in this work can be found in Section 5.3.

This paper is the result of joint work between R. Khan, his supervisor J. Khan, my supervisors and myself. While I was working on the adaptation of the algorithm to the faced problems and the mathematical results, R. Khan was responsible for the communication side of the paper including the simulations.


This work is not related to my thesis and is the work of R. Khan. I was only involved through discussions. It presents a distributed algorithm developed by R. Khan for EV charging that is based on communication principles.


This paper expands the algorithm suggested in [101] with the capability for V2G and reactive power balancing, which is covered in Sections 4.2 and 4.3 of this thesis. In this paper, we still neglect the actual distribution grid and the demand by uncontrollable loads. Also, compared to the situation suggested in this thesis, the paper does not allow the agents to automatically switch from grid to vehicle (G2V) to V2G mode, rather it requires the management unit to detect whether the agents are required to draw or inject power, both active or reactive. This is a more constrictive assumption. As in a real application, it is difficult for the management unit to distinguish between reduced power consumption by loads or power injection by loads without collecting additional data.

Again this work is a collaboration between my supervisors, R. Middleton and R. Shorten, E. Crisostomi, and me. While I did the implementations, E. Crisostomi and my supervisor supported me with advice and discussions.


The content of this book chapter is a summary of the parts in Chapters 2 and 4 and Section 6.1 that are concerned with EV charging where the EVs have a continuously controllable power consumption.
1.6. Published Papers

It is based on the previous work with [101], [102], [108] and so was done in collaboration with E. Crisostomi and my supervisors. E. Crisostomi was involved in the editing, writing, and supported me in a supervisory role.


This paper presents how the AIMD algorithm can be utilised to solve a constrained convex optimisation problem. The main result is the convergence result of the adapted algorithm. The content of the paper can be found in this thesis in Section 5.2 with only minor changes. In the paper, the algorithm is illustrated on an optimisation problem that is not related to the load management problem investigated in this thesis. However, in this thesis we apply the method to the load management problem. This can be found in Sections 2.2.1.4 and 4.1.2.4.

This work has been done in collaboration with F. Wirth, J. Yu, M. Corless and my supervisor R. Shorten. The work was started during a three month long exchange at IBM research Ireland under the supervision of R. Shorten and J. Yuan Yu, but was finished after I returned to Australia.

12. F. Wirth, S. Stüdli, J. Yu, et al., Asynchronous algorithms for network utility maximisation with a single bit (i), Accepted to present at Control Conference (ECC), 2015 European, 2015

This paper is a conference version of the above paper, i.e. [117], where the proofs are omitted.

13. S. Stüdli, M. Corless, R. H. Middleton, et al., On the modified AIMD algorithm for distributed resource management with saturation of each user’s share, Submitted to Decision and Control (CDC), 2015 IEEE 54nd Annual Conference on, 2015

This work expands the convergence results of the AIMD algorithm for systems where the states are bounded. The work is presented in this thesis in Section 5.1.3 and Section 6.2.

This paper was done in close collaboration with M. Corless and my supervisors R. Middleton and R. Shorten. Note that the proof presented in this thesis differs from the version presented in the paper, which is reworked in close collaboration with M. Corless.
In this chapter, we first formulate three sets of objectives that represent the load management services that we presented in Chapter 1. These objectives are constrained by the needs of the distribution grid and the controller abilities of the controllable agents, which we discuss in Section 2.1. We then present these three sets of objectives. The first set describes the problem of regulating the active power flow from the distribution grid to the controllable loads. This set consists of the most basic objectives and can handle both of the controller abilities we discuss in this thesis. The other two sets of objectives expand this first set to additionally allow reverse power flows and to allow reactive power exchange with the distribution grid.

Subsequently, we describe the set-up of the system used that is the central part of our proposed load management scheme. The basic idea is that the agents react upon signals that are broadcast by a central management unit. The proposed system allows the utilisation of different algorithms by the agents, which strongly dictate the behaviour of the overall system. Two such algorithms are proposed in Chapters 3 and 4, respectively.

2.1 Constraints

In this section, we introduce two important constraints that allow us to describe the load management services mathematically. The first one is mainly invoked by the distribution grid, but also includes other factors. The second constraint is invoked by the single agent and its controller ability.

2.1.1 Power Constraint

In this thesis, we look at the two primary support services of load management which we identified in Section 1.2: “peak shaving” and “load tracking”. Both these tasks regulate the active power drawn by individual loads to support the distribution grid. While the aim of the two services differ, both can be represented as the task of sharing a limited resource, namely the available power, among several agents of which some may be uncontrollable. This leads to the first constraint.

The available power is specified by the energy provider as the desired maximum aggregated
demand of all connected loads. The selection of the available power is critical to maximise the advantages of load management for the distribution grid and the loads connected. For example, in [63] the authors adjust the available power according to the electricity price at the moment and its minimum value to encourage the shift away from times with a high electricity price. Note that times with a high electricity price are normally peak times. Figure 2.1 depicts the different parties and their link to the available power. The available power depends on various factors:

- **Region:** Since the distribution grid covers a large area with thousands of loads connected, it is important to select which loads should be affected by the limited availability of power. For example, such a limit can be introduced at a transformer station that connects a neighbourhood of domestic houses with the distribution grid. In this case only loads that are connected within this neighbourhood are affected, while loads that are connected to other parts of the distribution grid are unaffected.

- **Physical limitations:** All physical components of the distribution grid, such as transformers, power lines, etc., are limited by their physical constraints.

- **Power generation:** Naturally, the available power is influenced by the actual power that is generated. This also includes renewable power generation. However, in cases with distributed renewable power generation, it might make more sense to include those smaller power generators as uncontrollable loads with a negative power consumption.

- **Service provider:** The service provider is the company that provides the power. This can be directly the electricity provider or a company that delivers the service, for example a workplace or shopping centre that allows parked vehicles to charge. Those providers have their own desires and constraints that influence the shape of the available power. For example, the available power could be selected to minimise power losses or minimising the costs of the service.

- **Customer:** Naturally, the customer, i.e. the owner of the loads, has some expectations that need to be considered to guarantee customer satisfaction. For example, the temperature in a refrigerator should remain within certain bounds. The available power should be selected such that the controllable loads can still fulfil their task most of the time. Otherwise customer satisfaction will drop and customers might refuse to participate in the load management scheme.

- **Load:** The loads have individual physical constraints which need to be taken into account. These include mainly upper bounds on the power consumption and the availability of the loads. For example, a power outlet of a vehicle charger has an upper bound on the power it can draw or supply. By availability we mean that the load might not be connected for certain times. Two such examples are if a refrigerator is turned off for cleaning or a vehicle is used for driving.

All those factors should be taken into account when selecting the amount of available power. This critical task might require detailed analysis of the network, forecasts of the demand and
2.1. Constraints

Power Consumption

Available Power

Power Generation

Distribution Grid

Physical Limitations, Region

Demand

Service Provider

Power Delivery

Controllable Loads

Uncontrollable Loads

Customer

Requirements

D原标题

Figure 2.1: Simplified diagram showing the influences on and of the available power.

generation, and estimates of the availability and needs of controllable loads. Depending on the selection of the available power, the power may vary with time or remain constant throughout the day. In this thesis, we assume that such a selection has been performed and do not go into further details how the available power is selected.

While we concentrate our investigations on cases where there is a single power limit, it is possible to introduce multiple limitations at various locations and include also current or voltage limitations. This will be discussed shortly in Section 6.1.2 and Section 7.6.

We denote the amount of power available at time step \( k \) by \( \bar{P}(k) \). Further, the active power drawn by the controllable load \( i \) at time step \( k \) is denoted \( p_i(k) \). In total \( N(k) \) controllable agents are connected at time step \( k \). Generally, the number of connected agents varies with time. While in some cases all loads will be controllable, most applications will include uncontrollable loads.
The aggregated demand of these loads is denoted by \( \tilde{p}(k) \). Note that distributed power generation can be included by treating them as uncontrollable loads with a negative power consumption and include them in the term \( \tilde{p}(k) \). Then, the first constraint, denoted the power constraint, can be expressed by

\[
\sum_{i=1}^{N(k)} p_i(k) + \tilde{p}(k) \leq \bar{P}(k) \quad \forall k.
\]  

We assume that Equation (2.1) is a soft constraint, i.e. we allow minor transient excursions beyond \( \bar{P}(k) \). Such an assumption is for example in accordance with a thermal constraint at a line or transformer.

By inspecting Equation (2.1) it is clear that in some cases the constraint results in an infeasible problem depending on the selection of \( \bar{P}(k) \), the aggregated power consumption of uncontrollable loads \( \tilde{p}(k) \), and the aggregated power controllable by the agents. While proper selection of the available power should prevent cases like this, they might still occur due to unexpected circumstances. Then, even though the constraint is clearly violated regardless the action of the controllable loads, we are interested in a “best-effort” solution, where the controllable loads reduce their power consumption or increase their power injection as far as possible to lessen the violation.

### 2.1.2 Controller Ability of the Agents

The second constraint is the freedom that the controllers of the single agents possess in governing the power consumption. \cite{74} investigates how different restrictions on the control over the power consumption of EVs affect the ability to manipulate the aggregated power consumption by all loads. In detail, they investigate how the allowance of varying charge rates and allowing breaks during the charging improve the abilities to govern the aggregated demand. As expected, they find that with variable charge rates and allowed breaks during the charging the aggregated demand can be governed best.

In this thesis, we distinguish between two categories of loads depending on the controllability of the power consumption of the agent. Let the set \( S_i \) be the set from which the actual power consumption of agent \( i \) may be selected, i.e.

\[
p_i(k) \in S_i \quad \forall k.
\]  

We consider two categories of control:

- **Binary controlled power:** These loads can only be in one of two states, either “on” or “off”. While the load is on, it draws a constant power \( \bar{p}_i \), otherwise the power consumption is assumed to be zero, thus \( S_i = \{ \bar{p}_i, 0 \} \). Note that here we neglect the power required for the controller, as we assume it will be much smaller than the power consumption during the on state. Additional constraints in this category can be a minimal required pause after switching due to the physical capabilities of the appliance or to avoid damage. Nonetheless, the loads are allowed to interrupt their operation at any time.
• **Continuously controlled power:** The loads in this category allow continuous control over their power consumption. However, due to the physical constraints of the load, there exists an upper bound on the power consumption $p_i$. Further, loads in this category might be able to inject power into the power grid, which is equally bounded. Hence, in case a load does not support reverse power flows $S_i = [0, p_i]$ while otherwise $S_i = [-p_i, p_i]$. In the later case a negative power consumption, i.e. $p_i(k) < 0$, means that the load injects power into the grid. Note that we here assume that the maximum power that can be injected is equal to the maximum power that can be consumed. Naturally, it is possible to change these bounds such that they differ. This can be desired to bound the injected energy. In this thesis, we assume that they are identical as stated above.

Figure 2.2 shows what appliances we expect to find in which control category in the near future. In this thesis we only consider electric vehicles and domestic refrigerators. In case of EVs, the power capacity and the control capabilities of the outlet may vary depending on the charging situation. For example, a normal home outlet [60] and generally single phase or Level 1 charging allows only small charge rates [1], [24], [25], [29], [67], [75], [94]. There are public places that will possibly support higher charging rates such as three phase charging [1], [4], [87], which is not usually found at home. In some cases, EVs are enabled to inject power into the grid, such operation is denoted V2G [2], [38], [67], [95]. Not only does the charger’s power capacity vary, but so does its ability to exercise control over the charge rate. Since both agents with binary and continuously controlled power are commonly found in literature, we allow EVs to be in either of the two categories defined in Figure 2.2. When considering refrigerators, most households have one or more in use and commercial refrigerators are readily available. Even though commercial refrigerators also considerably contribute to the aggregated power consumption [77], we consider only domestic refrigerators in this thesis. The principles are similar though and future work could investigate the use of commercial refrigerators and other TCLs, such as air conditioners and hot water heaters, which account for a large part of the total energy consumption [114]. Since we only consider domestic refrigerators, we expect that their capabilities for control are rather limited, such that we limit their category to binary controllable agents, see Figure 2.2. The main reason for this is the small price of a single agent and the small power that such an agent draws. On the other hand their power capacity may vary due to variations in size, age, and their construction. Especially, commercial refrigerators might in the future also allow for continuous controlled power.

### 2.2 Basic Load Management Objectives

After stating the constraints in the previous section, i.e. the power constraint in Equation (2.1) and the agent’s limitation in Equation (2.2), we will now state the first set of objectives that describe the basic load management problem mathematically, where loads do not allow reverse power flows, i.e. the set $S_i$ is either $\{p_i, 0\}$ for loads with a binary controllable power or $[0, p_i]$ for loads with a continuously controllable power. This set consists of two objectives which build the basis of the load management problem. It governs the controllable agents such that they receive as much power as needed while the aggregated demand follows the available power. These basic
2. Problem and System Description

<table>
<thead>
<tr>
<th>Binary controlled power:</th>
<th>Continuous controlled power:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i(k) \in {0, \bar{p}_i}$</td>
<td>$p_i(k) \in [0, \bar{p}_i]$ or $p_i(k) \in [-\bar{p}_i, \bar{p}_i]$</td>
</tr>
</tbody>
</table>

Figure 2.2: Summary of the two different categories and appliances that are likely to be equipped with the controller capabilities of these categories.

The first objective we introduce is to govern the active power consumption of the agents such that the constraints in Equation (2.1) and Equation (2.2) are not violated, while maximising the power transferred to the agents. This can be represented by

$$O_1(k) = \max_{p_1(k), \ldots, p_N(k)} \sum_{i=1}^{N(k)} p_i(k)$$

s. t. $p_i(k) \in \mathcal{S}_i$ for all $i, k$  \hspace{1cm} (2.3)

$$\sum_{i=1}^{N(k)} p_i(k) + \bar{p}(k) \leq \bar{P}(k) \text{ for all } k,$$

where $N(k)$ denotes the number of agents participating at time step $k$, which can vary over time.

The first objective $O_1(k)$ outlines the basic problem and restrictions imposed. However, an important part has not yet been considered, namely how the available power should be shared among the agents. While it is clear that this should be in a “fair” manner for the customers, it is debatable what a “fair” share means. For example, the power can be shared proportionally fair according to some weights, but the selection of the weights is highly discussable. The weights may be selected equal for all agents, based on their required energy, or depend on the contract with the energy supplier. We believe that the notion of “fairness” is highly dependent on the situation and the appliance that is involved. We consider two appliances as mentioned before: EVs and refrigerators. Both impose different concepts of fairness in regard to the expectations of a customer. In general, let a cost function $f(p_1(k), \ldots, p_N(k))$ define a fairness criteria based on a specific scenario. A second objective $O_2(k)$ is defined, how to share the power among the agents.
2.2. Basic Load Management Objectives

in a fair manner, by

\[
O_2(k) = \min_{p_1(k), \ldots, p_N(k)} f(p_1(k), \ldots, p_N(k)) \\
\text{s. t. } p_i(k) \in S_i \text{ for all } i, k \\
\sum_{i=1}^{N(k)} p_i(k) = O_1(k) \text{ for all } k.
\] (2.4)

This allows us to define the load management problem mathematically. In a first step, the objective \(O_1(k)\) in Equation (2.3) is solved. If the solution allows additional degrees of freedom, the second objective is solved with a lower priority. We name such a problem a prioritised optimisation (PO).

In the following, we define specific scenarios, for which we define a notion of “fairness”. Note that there are other ways how to define a fair share in each of the scenarios and there are many more occasions and situations.

2.2.1 Fairness Notions

First, consider the EVs; The charging of an EV can occur in a multitude of places with different charger capabilities, since a vehicle is parked for more than 90% of the time \([22]\). Often, charging is done at home or at the workplace of the vehicle owner, see for example \([25]\), \([89]\), \([94]\). However, especially owners of EVs with smaller battery sizes might prefer additional charging at shops or on street, due to their shorter range \([9]\), \([94]\). While the situations and the charger capabilities change, so might the expectation of “fairness”. Hence, we define in this thesis four scenarios that deal with EV charging: Charge Rate Fairness (CRF), Required Energy Fairness (REF), Charge Time Fairness (CTF), and Global Optimum Fairness (GOF). Each of the four scenarios gives rise to a specific concept of fairness and defines a function \(f(p_1(k), \ldots, p_N(k))\) that is used in the second objective in Equation (2.4). While it might be advantageous for an EV to remain connected and continue participation in the load management scheme after finish charging, we assume that the EV disconnects as soon as it is fully charged. Additionally, the vehicle may be disconnected after a certain period by the owner who requires the vehicle. Both these disconnection events are captured in \(N(k)\), which at the same time also captures connecting vehicles. Here, we slightly misuse the notation, since in fact \(N(k)\) not only depends on the time step \(k\) but also on the power consumption of the EVs, their battery size, the time they are connected, as well as the owners decision to use the vehicle.

On the other hand refrigerators are likely to be deployed at all times in similar settings. Hence, we assume that the definition of “fairness” remains static and we consider in this thesis a single scenario for refrigerators.

2.2.1.1 Charge Rate Fairness Scenario for EVs

The most obvious and simple way how the limited power can be shared among the charging vehicles is to maintain an equal share of the power across the agents connected. Such a scenario might
be useful in a domestic charging setting, where it is considered unfair to give higher charge rates to some particular vehicles. We assume that there is enough energy available to fully charge all the vehicles even though they are not able to charge using their maximum charge rate. Hence, this scenario occurs most likely at places where the vehicles are connected overnight or for longer periods during the day.

We define the cost function \( f(p_1(k), \ldots, p_N(k)) \), that defines the fairness criterion, as

\[
 f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} |p_i(k) - p_j(k)|. 
\] (2.5)

This cost function encourages equal charge rates among the vehicles at all time steps \( k \). This is a very restrictive fairness condition, since it pushes the actual charge rates to be identical at each time instance. It is possible to relax the fairness criterion by encouraging that only the average charge rate is equal, which is sufficient in most situations. This average charge rate, denoted \( \rho_i(k) \), can, depending on the requirement, either be computed upon connection until the actual time step

\[
 \rho_i(k) = \frac{1}{k} \sum_{l=1}^{k} p_i(l) 
\] (2.6)

or over the past \( \tau \) time steps

\[
 \rho_i(k) = \frac{1}{\tau} \sum_{l=k-\tau+1}^{k} p_i(l). 
\] (2.7)

With this we can redefine the cost function to be

\[
 f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} |\rho_i(k) - \rho_j(k)|. 
\] (2.8)

### 2.2.1.2 Required Energy Fairness Scenario for EVs

The CRF scenario distributes the available power equally among the participating EVs. This might lead to a share where some EVs are not able to fully charge their batteries before they are needed by the owner again, while other vehicles finish charging long before they are needed. We therefore propose to prioritise the EVs according to how long they are going to be connected and their total energy requirement. Such a scenario can only take place at locations where the owners of the EVs are willing to collaborate with each other rather than compete. Further, we still assume that the vehicles are connected for longer periods of time such that most vehicles do not require charging with their maximum allowed rate. Hence, we believe that it could possibly take place at the parking area of an office building, where the employees are allowed to charge their vehicles throughout the day.

Assume that at time step \( k \) an EV requires a certain energy \( E_i(k) \) to fully charge its battery. Further, assume that the vehicle is going to be disconnected from the charging point after a specified time window \( \Psi_i(k) \). This window is for example given by the end of the working day.
2.2. Basic Load Management Objectives

or when the employee has to drive off to a meeting. In this remaining connection time $\Psi_i(k)$ the vehicle should manage to charge the remaining required energy $E_i(t)$. Hence, we define a desired power consumption

$$\hat{p}_i(k) = \min \left( \frac{E_i(k)}{\Psi_i(k)}, \overline{p}_i \right)$$  \hspace{1cm} (2.9)

which allows EV $i$ to finish charging within the time it is supposed to be connected. Note that the desired power consumption is bounded above by the maximum power consumption $\overline{p}_i$ allowed due to limitations of the charger outlet.

We prioritise the vehicles according to their desired power consumption such that vehicles with a high desired power consumption receive a higher power share than those with a lower desired power consumption. This can be achieved with the cost function

$$f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} \left| \frac{p_i(k)}{\hat{p}_i(k_0)} - \frac{p_j(k)}{\hat{p}_j(k_0)} \right|$$  \hspace{1cm} (2.10)

where $k_0$ denotes the time step at which the vehicle computes its desired power consumption. This can be either upon connection to the power grid or at certain reset time steps.

As for the CRF scenario, this objective is rather restrictive. Analogously, we can relax the objective by using the average charge rate $\rho_i(k)$ instead, defined in Equations (2.6) and (2.7), respectively. The cost function in this case is

$$f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} \left| \frac{\rho_i(k)}{\hat{p}_i(k_0)} - \frac{\rho_j(k)}{\hat{p}_j(k_0)} \right|.$$  \hspace{1cm} (2.11)

2.2.1.3 Charge Time Fairness Scenario for EVs

The two previous scenarios accommodate for situations where the EVs are connected for longer time periods. This is most likely the major charging mechanism for EVs in the near future. However, during the day there arise multiple situations where the vehicle is parked for shorter periods, for example parking spots at shopping centres, in the city centre, at highway restaurants, and various other places [14]. These locations might in the future more often provide the possibility to charge EVs for free as a complimentary service to attract more customers. We assume that with the accessibility of such charging stations and the advances in battery technology, such short term charging facilities do not have the charging of the vehicle as primary goal in mind, but deliver charging as a complimentary service, while the owner of the vehicle attends to the main task, for example shopping or dining. In these cases, the above “fairness” concepts that accommodate for longer charging times are not necessarily appropriate. The idea is therefore to reduce the share of the power gradually the longer an EV is connected. In that way, EVs which are connected for only short periods of time also receive a useful amount of energy.

First, we assign a priority to each vehicle which depends on the time that has elapsed since the vehicle’s connection to the charger. Upon connection the EV automatically is assigned the highest priority. After a predefined amount of time has elapsed, the priority of the EV is gradually lowered
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until the lowest priority is reached. The available power is shared during the charging among the vehicles according to their assigned priorities. Assume that the priority of each vehicle lies in the set \( \{0, 1, \ldots, L - 1\} \), where 0 is the lowest priority and \( L - 1 \) is the highest priority. We assume that the predefined time delay before the priority is reduced is a fixed value that is identical for all connected EVs. This delay is measured in time steps and is in the following denoted by \( \kappa \).

Further, the time step at which vehicle \( i \) connects to the distribution grid is denoted by \( T_i \). The priority that is assigned to EV \( i \) at time step \( k \), denoted \( \gamma_i(k) \), is then computed by

\[
\gamma_i(k) = \min \left( L - 1, \left\lfloor \frac{k - T_i}{\kappa} \right\rfloor \right), \tag{2.12}
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \).

The cost function depicting the fairness condition can be formulated using these priorities by

\[
f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1}^{N(k)} \left| \frac{p_i(k)}{\gamma_i(k)} - \frac{p_j(k)}{\gamma_j(k)} \right|. \tag{2.13}
\]

Similarly, as for the other scenarios, it is possible to relax the objective by using the average power consumption \( p_i(k) \) instead of \( p_i(k) \). However, as the expected connection time is short, it makes sense to use the finite time average over a short period as in Equation (2.7). Hence, the cost function becomes

\[
f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1}^{N(k)} \left| \frac{p_i(k)}{\gamma_i(k)} - \frac{p_j(k)}{\gamma_j(k)} \right|. \tag{2.14}
\]

2.2.1.4 Global Optimum Fairness Scenario for EVs

The previously described scenarios assume that the agents have their individual goal, while wishing a certain fairness. However, in some cases it might be desired to achieve a global goal rather than having individual goals. We believe that such scenarios can occur in a variety of situations, but in particular that it may be applied for fleets of EVs such as for company owned vehicles. Assume, that each EV has an individual cost assigned which may depend on the actual time, connection time, actual charge rate, desired charge rate and other internal parameters. The cost is chosen to be convex, continuously differentiable, and strictly increasing with the power consumption of the agents and is denoted by \( g_i(p_i(k)) \). The cost function that represents the fairness notion is then defined as the sum of these individual costs, i.e.

\[
f(p_1(k), \ldots, p_N(k)) = \sum_{i=1}^{N(k)} g_i(p_i). \tag{2.15}
\]

In this thesis, we choose to minimise the missing energy normalised by the battery capacity after a period of ten hours. Note that no future aspects nor any predictions are considered, such as the disconnection time of other agents, the connection of new agents, the varying demand by uncontrollable agents or any changes in the available power. Similarly, we do not consider the past such as how much energy the agent already received. In contrast, we use the energy required upon
connection to compute the missing energy normalised by the capacity of the battery. To penalise large amounts of energy missing quadratic cost terms are used. Let $E_i(T_i)$ be the required energy to fully charge vehicle $i$ when it starts its charging cycle and $\tilde{E}_i$ to be the total capacity of the battery. Further, let $\rho_i(k)$ be the average charge rate of the vehicle at time step $k$ as defined in Equation (2.6). Then, the individual cost of vehicle $i$ is defined as

$$g_i(p_i(k)) = \left( \frac{E_i(T_i) - 10 h \rho_i(k)}{\tilde{E}_i} \right)^2.$$  \hspace{1cm} (2.16)

Naturally, there is a wide range of other global goals that could be defined as a fairness notion such as the total cost of the energy delivered, the total environmental impact while charging, the customer satisfaction, etc.. Such other fairness notions might also take into account future developments of the demand by uncontrollable agents, the available power, or the connection and disconnection of agents. For example, that EVs reduce their power consumption if the demand is expected to fall later during their charging cycle to allow the charging of vehicles that will be disconnected during those times. In fact, every fairness notion including such factors that can be represented using convex, strictly increasing individual cost functions is allowed.

### 2.2.1.5 Refrigerator Scenario

We consider the simplest possible scenario for the refrigerators. This means that we require the inside temperature to lie in a certain range at all times. This reflects the fact that the customers most likely do not care when the refrigerator actually runs as long as the temperature is within the nominated range. Note that nowadays refrigerators most commonly are controlled by a thermostat which keeps the inside temperature oscillating between a lower and higher bound. We denote the temperature in the refrigerator by $\vartheta(k) \in [\vartheta_l, \vartheta_u]$. As long as the temperature is within the range the refrigerator participates in the load management scheme and receives an equal share of the available power. As soon as the temperature leaves the nominated range the refrigerator stops its participation until the temperature is driven to the desired interval. The cost function can be written as

$$f(p_1(k), \ldots, p_N(k)) = \sum_{i \in \mathcal{C}(k)} |\rho_i(k) - \rho_j(k)|$$  \hspace{1cm} (2.17)

where $\mathcal{C}(k)$ is the set of participating refrigerators at time step $k$, i.e.

$$\mathcal{C}(k) = \{ c \in \{1, 2, \ldots, N(k) \} | \vartheta_c(k) \in [\vartheta_l, \vartheta_u] \}.$$  \hspace{1cm} (2.18)

### 2.3 Reverse Power Flows Allowed

While reducing the active power consumption of loads is useful, the controllable power is limited. A possibility to increase the controllable power is to allow the agents to inject active power into the distribution grid. This is commonly referred to as V2G capabilities in regard to EV charging. Such V2G capabilities are investigated in recent studies, see for example [2], [38], [67], [95].

A controller that allows reverse power flows is more complex than one without this capability. If the controller allows reverse power flows, the additional equipment that is required to allow...
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continuous power control rather than binary power control is small. We therefore solely investigate
continuously controllable loads that allow reverse power flows, which are EVs in this thesis. The
inclusion of reverse power flows means that the set in which the power consumption may lie is
\[ S_i = [-\bar{p}_i, \bar{p}_i] \]. Here, positive values for \( p_i(k) \) mean that the agent is drawing power from the grid
and negative values indicate that the agent injects power into the grid. Further, \( \bar{p}_i \) represents the
physical constraint of the charger of EV \( i \).

While the inclusion of reverse power flows allows a larger amount of power to be controlled,
the available power \( \bar{P}(k) \) should still be selected carefully to balance the satisfaction of the vehicle
owners, the needs of the distribution grid, and the capabilities of the vehicles. This means that
cases where the agents are not able to provide enough power should be avoided, i.e. situations
in which Equation (2.1) remains violated regardless of the action of controllable agents. Due to
unforeseen circumstances, such cases might still occur. In such situations, we are interested in a
best effort solution as before, where the agents provide as much power as possible to lessen the
gap between the available power \( \bar{P}(k) \) and the total aggregated demand \( \sum_{i=1}^{N} p_i(k) + \bar{p}(k) \).

Besides the changed set \( S_i \), the load management services remain unchanged. The aggregated
demand is still constrained by the available power while transferring as much power to the agents
as possible. Hence, the first objective remains identical to the one in Equation (2.3) and is restated below

\[ O_1(k) = \max_{p_1(k), \ldots, p_N(k)} \sum_{i=1}^{N(k)} p_i(k) \]
\[ \text{s. t. } p_i(k) \in S_i \text{ for all } i, k \]  
\[ \sum_{i=1}^{N(k)} p_i(k) + \bar{p}(k) \leq \bar{P}(k) \text{ for all } k. \]  

(2.19)

In Section 2.2.1, we defined different fairness notions on how EVs should share the limited
power among themselves. When including reverse power flows, there are still various situations
which may lead to different definitions of fairness. We therefore define a PO as in Section 2.2.
The second objective defines how the power should be shared in case the first objective allows for
additional degrees of freedom. This second objective is identical to Equation (2.4) and is restated
here for the convenience of the reader

\[ O_2(k) = \min_{p_1(k), \ldots, p_N(k)} f(p_1(k), \ldots, p_N(k)) \]
\[ \text{s. t. } p_i(k) \in S_i \text{ for all } i, k \]
\[ \sum_{i=1}^{N(k)} p_i(k) = O_1(k) \text{ for all } k. \]  

(2.20)

2.3.1 Revised Fairness Notions for V2G Operations

In the following, we present different scenarios which define different definitions of fairness when
taking reverse power flows into account. The scenarios are mostly identical to the ones suggested
in Section 2.2.1 with small adaptations to accommodate for the reverse power flows.
2.3. Reverse Power Flows Allowed

Note that there are two different modes in which an agent can operate: consuming power and injecting power. Basically it is possible to define two different types of fairness notions depending on the mode the agent is operating in. However, we believe that the understanding of fairness is depending on the situation, rather than the mode the vehicle is operating in at that moment. According to that, we assume that the idea what is fair is identical, independent of the mode. Naturally, the actual definition of the cost function that captures the fairness has to be adapted to maintain the same fairness idea. Note also that there are far more possible scenarios than presented here.

2.3.1.1 Charge Rate Fairness Scenario

As discussed in Section 2.2.1.1 it may be fair to share the available power equally among the EVs. The fairness idea can be broadened logically to equalise the power consumption during G2V operation and equalise the power injection during V2G operation. Naturally, all the vehicles should be in the same mode, i.e. all are either drawing power from the distribution grid or injecting power into the distribution grid.

Similar to Section 2.2.1.1, we wish to equalise the averaged power consumption $\rho_i(k)$ as defined in Equations (2.6) and (2.7), respectively. Note that, unlike in Section 2.2.1.1, the EVs are able to inject power into the grid, which means that their power consumption can become negative. The cost function itself to capture the fairness criterion remains unchanged, i.e.

$$f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} |\rho_i(k) - \rho_j(k)|. \quad (2.21)$$

We assumed in the above implicitly that all participating EVs allow reverse power flows. In some cases, it might not be true and only a fraction of the participating EVs allow reverse power flows. In those cases, the set $S_i$ has to be adapted for the individual agents.

2.3.1.2 Required Energy Fairness Scenario

In Section 2.2.1.2, we argued that in various cases an equal share is not ideal, as some EVs might require more energy than others. Hence, we prioritise the EVs depending on their desired charge rate as in Equation (2.9). If the EVs consume power, the ones with a higher desired charge rate receive a higher share. Conversely if the EVs inject power, the ones with a higher desired charge rate have to inject a lower share, such that it is easier for these to recharge during the periods when power is available. The cost function during G2V operation can then be formulated as in Equation (2.10) in Section 2.2.1.2 by

$$f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} \left| \frac{p_i(k)}{\hat{p}_i(k_0)} - \frac{p_j(k)}{\hat{p}_j(k_0)} \right|. \quad (2.22)$$

During V2G operation the cost function is defined as

$$f(p_1(k), \ldots, p_N(k)) = \sum_{i,j=1 \atop i < j}^{N(k)} \left| p_i(k)\hat{p}_i(k_0) - p_j(k)\hat{p}_j(k_0) \right|. \quad (2.23)$$
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In the above, $k_0$ denotes the time step at which the vehicle computes its desired power consumption. This can either be upon connection to the power grid or at certain reset time steps during the charging period.

2.4 Reactive Power Balance

We have so far described the problem of regulating the active power consumption of controllable agents to support the power grid. In addition enabling the agents to consume or deliver reactive power allows them to participate in reactive power balancing services [22]. Reactive power control might be especially interesting in scenarios where a sufficiently large fleet of agents is connected to the grid in close proximity to consumers of reactive power, such as certain industries. The basic task of controlling the active power remains unchanged compared to what is presented in Sections 2.2 and 2.3. This means that also the constraint described in Section 2.1.1 remains unchanged, i.e. there is an available power defined that sets an upper bound on the aggregated active demand. The available power should be carefully selected, as described in Section 2.1, to accommodate the wishes of the suppliers and the demands of the customers. In addition to this task, a second goal is to achieve reactive power compensation within a certain area where the agents are connected. This means that the agents should as far as possible supply the reactive power that is required by other uncontrollable loads in that area.

As in Section 2.1.2, we assume that all agents that are capable of reactive power consumption, are able to control their power consumption continuously. This assumption is reasonable as the control of reactive power requires a more complex control structure, e.g. an active rectifier, which means that continuous control of the power consumption is enabled. Even though this structure would clearly allow for injection of active power, the owner of the controllable load might not be willing to participate in V2G operations. Naturally, the power outlet of the controllable load imposes some limitations on the amount of power that can be handled. This means that the active power lies in $S_i = [0, p_i]$ or $S_i = [-p_i, p_i]$. Additionally, to include the reactive power, an upper bound on the total apparent power is introduced, denoted $\bar{s}_i$ with $\bar{s}_i \geq p_i$. Then, the active and reactive power of the controllable agent has to fulfil

$$\sqrt{p_i(k)^2 + q_i(k)^2} \leq \bar{s}_i, \quad (2.24)$$

where $p_i(k)$ is the active power consumption of agent $i$ and $q_i(k)$ is the reactive power consumption.

Note that in most cases $\bar{s}_i = \bar{p}_i$ holds, because of the physical constraints of the charger. In some cases though it might be useful to limit the active power such that the apparent power bound is larger than the one for the active power. This makes sense to guarantee the availability of some reactive power.

The main objective remains to control the active power such that the agents receive as much power as possible without violating Equations (2.1) or (2.24). The distribution of the power among the agents should also happen in a fair way if possible. Hence, the first and second objective remain identical, as defined in Equations (2.3) and (2.4). In addition to govern the active power consumption, the agents should now compensate the reactive power consumption locally as far as
possible. So ideally
\[ \sum_{i=1}^{N} q_i(k) + \bar{q}(k) = 0 \quad \text{for all } k, \]  
(2.25)
where \( \bar{q}(k) \) denotes the aggregated reactive power consumption by uncontrollable loads in the area. However, often this might not be achievable depending on the agents’ limitations and the reactive power required. In those cases, we aim for a best effort solution, where the agents provide as much reactive power as possible. However, we assume that the reactive power compensation is less important than the active power transfer, such that a controllable agent should only deliver or draw reactive power if it does not interfere with its active power consumption or injection. Hence, we add a third objective \( O_3 \) with even lower priority, while the first and second objectives remain identical to Equations (2.3) and (2.4). The third objective is given by

\[
O_3(k) = \min_{q_1(k), \ldots, q_N(k)} \left| \sum_{i=1}^{N} q_i(k) + \bar{q}(k) \right|
\tag{2.26}
\]

s. t. \[ \sqrt{p_i(k)^2 + q_i(k)^2} \leq s_i. \]
(2.26)

Note that since the third objective solely regulates the reactive power the active power of each agent is set by the first and second objective alone.

In this thesis, we assume that the individual reactive power provided or consumed by each agent does not matter. However, in the future the owners of the agents might receive monetary compensations according to the reactive power that is actually exchanged. In this case, it becomes important to add a fairness criterion, similarly to the second objective \( O_2 \), for the reactive power exchange. This can easily be achieved by adding a fourth objective with lowest priority that defines the fairness notion for the reactive power. As for the active power we would need to define what fairness means. This can depend on the situation, the location, or the compensation provided to the owners for the reactive power exchange. Let \( g(q_1(k), \ldots, q_N(k)) \) be a function that indicates the fairness notion for the reactive power share. Note that this function could also depend on the active power consumption of the agents. The larger the value of the function the less fair is the share. Then, such a fourth objective \( O_4 \) could be stated as

\[
O_4(k) = \min_{q_1(k), \ldots, q_N(k)} g(q_1(k), \ldots, q_N(k))
\tag{2.27}
\]

s. t. \[ \sqrt{p_i(k)^2 + q_i(k)^2} \leq s_i \]
\[ \left| \sum_{i=1}^{N} q_i(k) + \bar{q}(k) \right| = O_3(k) \text{ for all } k. \]

In this thesis, we do not consider such an additional objective.

### 2.5 Management System

To fulfill the objectives introduced in Sections 2.2 to 2.4 many agents need to be coordinated in a widespread area. One of the main characteristics of any coordination is where the “intelligence”
2. Problem and System Description

of the controller lies. It is common to distinguish between three basic types of control:

Centralised control: A central control unit collects data from the participating agents and other connected measuring devices, decides the scheduling of the agents, and instructs all agents.

Decentralised control: There exists no central unit that coordinates the agents. Rather the agents adopt their power consumption according to the information they have. It can also include that the agents require to measure some quantities to react appropriately.

Distributed control: There exists a central management unit which collects data from some measuring devices and informs the agents accordingly. The agents receiving the information from the central management unit, decide their power consumption individually based on this information. The decision of each agent depends on its own needs, possibly some local measurements, and the information received from the central management unit. Additionally, an agent might need to communicate with its neighbours to gain the necessary information to make an appropriate decision.

We decide to propose a distributed load management scheme that relies on broadcast signals instead of centralised or decentralised control. A distributed load management scheme combines the advantages of both centralised and decentralised control, such as a small communication load, robustness against single component failures, adaptability, and lessens the concerns regarding data protection and privacy. Below we discuss the advantages and drawbacks of central and decentralised control in regard to a distributed control in detail.

A centralised controller has generally a high communication load which often scales badly with the system size. Since for load management purposes it is desirable that as many agents as possible participate to decrease the dependency on a single agent and minimise the impact on the customers, we expect a large number of agents to be controlled in the future [21]. This means that the communication load might be high, even if the single agent does not exchange much data with the central controller. Further, the controller has to handle frequently changing conditions, such as connection and disconnection of participating agents, changing expectations or needs of the customers, the changing power consumption of uncontrollable loads, and the variable available power. This means that the central controller might require frequent rescheduling of the power consumption by the controllable loads to achieve the optimal schedule [58]. This in turn might increase the communication load if the update requires frequent collection of data from the connected agents. In addition to the increased communication load, the collection of data at a central controller raises data protection and privacy issues. The information that the central controller might require can range from a single quantity to a multitude of variables regarding the state of the loads and the distribution grid, such as for example the temperature inside a refrigerator or the expected disconnection time of an EV. In general, the more information is available the better is the control algorithm able to schedule the power while taking into account both the needs of the distribution grid and the customers. While by limiting the needed information from the agents the concerns regarding data protection can be lessened, this also limits the ability of the controller to achieve an optimal schedule. Lastly, a centralised algorithm is less robust
against the failure of components, such that the failure of only few components might influence the control badly. When designing a central controller this has to be considered such that the effects can be limited. Besides the previously described disadvantages a central controller has also clear advantages. When using a central controller it is possible for the provider to adapt the behaviour of the controllable agents easily. For example, a change in the desired demand can easily be achieved by adjusting the central controller. Further, a central controller is able to find the most optimal schedule at any time, because all the necessary information is gathered by the central controller. However, in some cases the optimal schedule might be computationally challenging, such that it can not be applied in real-time. Not only the computational complexity but also the high information exchange can hinder the usage in a real-time setting, due to delays and packet loss.

The opposite of centralised control is decentralised control which requires generally less communication, in some cases even none, see for example [5] where refrigerators use the frequency to control whether they should turn on or off. Hence, there are no issues regarding data protection and privacy. Also, using a decentralised control increases the robustness of the system against failure of single agents compared to a centralised control, which is especially desirable in the load management case with a large number of participating agents. However, since there is no central management unit, it is hard for the provider to adapt the behaviour of the controllable agents without adapting the individual behaviour of each agent locally. For example, if the aggregated demand should be reduced, this has to be communicated to every single agent. Further, for decentralised control the agents might need to be equipped with measurement devices instead or along with communication devices. The cost and complexity of such devices if needed has to be considered when comparing the investment cost of different load management schemes. For example, in [5] the agents need to measure the frequency without the need for any communication.

The system underlying the proposed load management scheme is depicted in Figure 2.3. It includes three parts: a central management unit, the agents, and the distribution grid. These three parts are interlinked by exchanging information and power. The central management unit collects information from the power grid and broadcasts signals to the controllable agents. These agents then react individually upon the broadcast by adjusting their power consumption. The power is delivered by the power grid to both controllable and uncontrollable agents. Note that the management unit itself does not receive any information from the controllable agents which means that there are no issues regarding data protection or privacy. The system is described in detail below.

The central management unit is able to detect whether Equations (2.1) and (2.25) are violated by monitoring the total active and reactive power consumed in a region. This implies that the management unit is informed about the available active power $\overline{P}$. Then, the central management unit will broadcast to all controllable agents as soon as it detects that the power consumption should be reduced. This occurs for the active power whenever the aggregated demand exceeds the available power, i.e. Equation (2.1) is violated. We denote such an event an active capacity event (CE) and the signal that is broadcast by the central management unit an active CE signal. Similarly, if the reactive power consumption is larger than 0 a so called reactive CE occurs and
a reactive CE signal is broadcast. This indicates that all reactive power in the area has been compensated and additional consumption of reactive power would lead to over-compensation, see also Equation (2.25). These events can happen either for the active power, the reactive power, or for both simultaneously.

The agents themselves react upon the receipt of a CE signal or the lack thereof. Naturally, only agents with controller abilities that allow reactive power exchange react on reactive CEs. Hence, in other cases where such capabilities are exempt, we use CE short for active CE. The manner in which the agents react is defined by a specific algorithm that is executed by the agent. In this thesis, we propose two algorithms. The first is for loads with binary controllable power consumption and the second is for loads with continuously controllable power consumption and therefore more complex. These algorithms are presented in Chapters 3 and 4, respectively. While the first one only allows the most basic control of solely the active power consumption, the second allows for more complexity and is expanded to also allow reverse power flows and reactive power balancing. Both algorithms react on the same principle though. As long as no CE signal is received the aggregated power consumption is likely to increase, while the receipt of a CE signal reduces the power consumption. This holds for both active and reactive CE signals. Naturally, the agents therefore need to know what algorithm they should use. This can be either directly implemented in the appliances or communicated by the central management unit. The latter however requires the central management unit to know the abilities of the controller for each connected load. Hence, we assume that the first method is the case in this thesis.
Chapter 3

Binary Controllable Power Consumption

In Section 2.5, we described the load management scheme that solves the objectives as introduced in Sections 2.2 to 2.4. It is based on algorithms that dictate the behaviour of a single agent depending on CE signals that are broadcast by the central management unit. In this chapter, we present one such algorithm, the BA algorithm, that enables loads with binary controllable power consumption to participate in the load management scheme. Since the controller ability of the agents is restricted to binary controllable power consumption, this is the most basic algorithm. It solely adjusts the active power consumption without permitting reverse power flows. Hence, the BA algorithm only reacts on active CE signals and ignores any reactive CE signals that the agent receives. Further, we only consider the fairness notions presented in Sections 2.2.1.1 and 2.2.1.5, which correspond to an equal share of the available power or in this case equal long on-times. In Chapter 4, we propose an algorithm for loads with continuously controllable power consumption which is more advanced and can be expanded to allow reverse power flows and reactive power balancing. There, we also consider the other fairness notions. In Chapter 5, the non-expanded version of this second algorithm is analysed.

In the following, we first describe the BA algorithm and its operation. Then, we analyse the algorithm mathematically. In particular, we are interested in the interaction if multiple agents applying this algorithm participate in the load management scheme. We investigate properties of the system such as its stability and the achieved share of power. We therefore use two separate methods. The first treats every single agent individually, while the second method assumes a large number of agents and concentrates on their behaviour as a whole. The advantages of the first method are that we can give information about the behaviour of every single agent, such as its expected power consumption, as well as information about the complete system, such as the aggregated demand. However, this method scales badly with the size of the system such that it is infeasible for systems with large numbers of participants. The second method tackles this problem. However, the information about the single agents can no longer be obtained. Our results are illustrated using Matlab simulations.
3. Binary Controllable Power Consumption

3.1 Description of the BA Algorithm

The BA algorithm reacts upon the receipt of a CE signal. We identify two phases: the turn on (ON) phase, when no CE signal has been received, and the turn off (OFF) phase, when a CE signal has been received. Figure 3.1 shows a detailed flow chart of the BA algorithm.

The ON phase affects solely agents that are currently turned off, since it enables them to turn on as the name suggests. We define two variables: a probability $\nu_i(k) \in [0,1]$, and an integer $n_i$. At the beginning of the phase at time $k$ the agent decides in a stochastic manner whether to turn on. The probability of agent $i$ to turn on at time step $k$ is $\nu_i(k)$. If it chooses to stay off, the probability $\nu_i(k)$ is increased additively, i.e.

$$\nu_i(k+1) = \nu_i(k) + \frac{1}{n_i} \tag{3.1}$$

and

$$p_i(k+1) = p_i(k) = 0. \tag{3.2}$$

Otherwise, in case it decides to turn on the probability $\nu_i(k)$ is reset and the agent turns on, i.e.

$$\nu_i(k+1) = \frac{1}{n_i} \tag{3.3}$$

and

$$p_i(k+1) = p_i. \tag{3.4}$$

This means that after at most $n_i$ time steps without a CE the agent will turn on.

The OFF phase on the other hand affects agents that are currently turned on by enabling them to turn off. As for the ON phase, two variables are defined: a probability $\mu_i(k)$ and an integer $m_i$. In this phase, the agent attempts to turn off with probability $\mu_i(k)$. The probability is increased in case the agent does not turn off, i.e.

$$\mu_i(k+1) = \mu_i(k) + \frac{1}{m_i} \tag{3.5}$$

and

$$p_i(k+1) = p_i(k) = p_i. \tag{3.6}$$

In case the agent actually turns off the probability is reset, i.e.

$$\mu_i(k+1) = \frac{1}{m_i} \tag{3.7}$$

and

$$p_i(k+1) = 0. \tag{3.8}$$

Note that while the parameters $\nu_i(k)$ and $\mu_i(k)$ are time dependent, the parameters $n_i$ and $m_i$ are constants that define the behaviour of the agent. The larger those are chosen the longer the agent is expected to remain on and off, respectively.
3.1. Description of the BA Algorithm

Figure 3.1: Diagram depicting the BA algorithm executed by each agent.

Lemma 3.1. Assume an agent $i$ utilises the BA algorithm. Let $T_{\text{no CE}}$ be the number of time steps without a CE after switching off, which not necessarily have to be consecutive. Then, the expectation of $T_{\text{no CE}}$ is

\[
E[T_{\text{no CE}}] = (n_i - 1)! \sum_{k=1}^{n_i} \frac{k^2}{(n_i - k)!n_i^k}.
\]  

(3.9)

Also, the number of expected time steps with a CE, not necessarily consecutive, $E[T_{CE}]$ required
3. Binary Controllable Power Consumption

before turning off after being switched on on is

\[ E[T_{\text{CE}}] = (m_i - 1)! \sum_{k=1}^{m_i} \frac{k^2}{(m_i - k)! m_i^k}. \] (3.10)

**Proof.** Assume that an agent is turned off at time step \( k \). Then, its turning on probability is reset, such that \( \nu_i(k + 1) = \frac{1}{n_i} \). First, note that the probability to turn on after exactly \( \ell \) time steps without a CE, which do not necessarily occur consecutive, is given by

\[ \frac{\ell}{n_i} \prod_{k=1}^{\ell-1} \left(1 - \frac{k}{n_i}\right). \]

Then, the expected number of time steps \( E[T_{\text{no CE}}] \) is

\[ E[T_{\text{no CE}}] = \sum_{k=1}^{n_i} k \frac{k}{n_i} \prod_{\ell=1}^{k-1} \left(1 - \frac{\ell}{n_i}\right) = (n_i - 1)! \sum_{k=1}^{n_i} \frac{k^2}{(n_i - k)! n_i^k}. \] (3.11)

Analogously, the expected number of required CE, not necessarily consecutive, computes to

\[ E[T_{\text{CE}}] = \sum_{k=1}^{m_i} k \frac{k}{m_i} \prod_{\ell=1}^{k-1} \left(1 - \frac{\ell}{m_i}\right) = (m_i - 1)! \sum_{k=1}^{m_i} \frac{k^2}{(m_i - k)! m_i^k}. \] (3.12)

\[ \square \]

3.2 Analysis Using an Automata Game

The first method that we use to analyse the BA algorithm treats every single agent individually. We assume the following property for the available power and the aggregated demand from uncontrollable loads.

**Assumption 3.1.** The available power is constant and chosen such that

\[ \bar{P} - \bar{\rho} > 0 \] (3.13)

and

\[ \sum_{i=1}^{N} \bar{p}_i > \bar{P} - \bar{\rho}. \] (3.14)

The above assumption excludes two cases. The first case is where all the available power is already used by the uncontrollable loads. In this case, the controllable agents are never allowed to turn on. The second case is where all agents can draw power simultaneously without violating the power constraint in Equation (2.1). This means that the agents remain constantly on until they received the necessary energy.

First, we look at the behaviour of a single agent. We will use this analysis to find that the algorithm implements a fixed-structure automaton and that the overall system represents a game between multiple automata, as defined in [78]. We are then able to interpret the overall system as
3.2. Analysis Using an Automata Game

a Markov chain for which we show ergodicity. This means that the system converges to a steady
state distribution independent of the initial conditions. This is particularly relevant to examine
simulation results, since it is important to know that the simulation represents on average the
behaviour of the system and it does not depend on the initial states of the agents.

Looking at the single agent \( i \), it is clear from the description of the algorithm that when it is
off only the internal parameter \( \nu_i(k) \) is adapted and similarly if the agent is on only the internal
parameter \( \mu_i(k) \) is adapted. Further, \( \nu_i(k) \) and \( \mu_i(k) \) can only obtain a finite number of values,
namely \( n_i \) and \( m_i \), respectively. Therefore, we can define \( n_i + m_i \) states that the agent can occupy,
which are characterised by the value of \( p_i(k), \nu_i(k), \) and \( \mu_i(k) \). The transitions between the states
depend on whether a CE is received or not. In Figure 3.2, this behaviour is depicted in a transition
diagram for the two possible inputs from the environment: CE signal received or not. Note that
the states coloured in red mark the states where the agent is turned off, i.e. \( p_i(k) = 0 \), while green
marks the states when the agent is turned on, i.e. \( p_i(k) = p_i \). Hence, the behaviour of the agent is
identical to a fixed-structure automaton, as defined in [78], where the input is the CE signal, the
state set is the collection of \( n_i + m_i \) states, and the action is the power demanded by the agent
\( p_i(k) \).

Then, from this description, we find the following mathematical system description.

**Fact 3.2.** Let \( x^{(i)}(k) \) be the state vector for an agent \( i \) with entries \( x^{(i)}_l(k) \), such that

- \( x^{(i)}_l(k) \) is the probability that agent \( i \) is off and its internal parameters are \( \nu_i = \frac{l}{n_i} \) and
  \( \mu_i = \frac{1}{m_i} \) at time step \( k \) for \( l \in \{1, 2, \ldots, n_i\} \) and
- \( x^{(i)}_{n_i+l}(k) \) is the probability that agent \( i \) is on and its internal parameters are \( \nu_i = \frac{l}{n_i} \) and
  \( \mu_i = \frac{1}{m_i} \) at time step \( k \) for \( l \in \{1, 2, \ldots, m_i\} \).

Further, let

\[
A^{(i)}_{nc} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 - \frac{1}{n_i} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 - \frac{2}{n_i} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \frac{1}{n_i} & 0 & 0 & \cdots & \cdots & 0 \\
\frac{1}{n_i} & \frac{2}{n_i} & \cdots & \frac{n_i}{n_i} & 1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\] (3.15)

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Then, the reaction of the agent applying the algorithm described in Section 3.1 can be represented as a switched system

$$x(i)(k+1) = A(i)(k)x(i)(k)$$

(3.17)

where

$$A(i)(k) = \begin{cases} A_{nc}(i) & \text{if no CE signal is received at time step } k \\ A_c(i) & \text{if a CE signal is received at time step } k. \end{cases}$$

Next, we look at the interaction between $N$ agents if all of them utilise the BA algorithm. We find that this is a game between fixed-structure-automata, where the environment depends deterministically on the action of the agents. Using this we can find the below description of the complete system with $N$ agents participating as a Markov chain.

**Fact 3.3.** Let

$$z(0) = x(1)(0) \otimes x(2)(0) \otimes \cdots \otimes x(N)(0)$$

(3.18)

be the state vector of the complete system. Further, let $I_A$ be a diagonal matrix containing 1 if the state leads to a CE and 0 otherwise.

Then, the system can be represented by

$$z(k+1) = Hz(k),$$

(3.19)

where

$$H = \left( A_{nc}(1) \otimes \cdots \otimes A_{nc}(N) \right) (I - I_A) + \left( A_c(1) \otimes \cdots \otimes A_c(N) \right) I_A.$$  

(3.20)

The reasoning behind this representation can be found in Appendix A.1. Note that in Fact 3.3 we assume that at time step $k = 0$ the agents are independent. However, the system links the behaviour of the agents such that they can no longer be seen as independent for $k > 0$. It is possible to compute the single agents probability vector $x(i)(k)$ from the complete state vector $z(k)$ by adding up the proper entries of $z(k)$. 

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3.2. Analysis Using an Automata Game

Next, we establish some important results for the system that later allow us to show the ergodicity of the system in Fact 3.3. We show first that two specific states are reachable from any other state.

Lemma 3.4. Let

\[
t = \sum_{i=1}^{N-1} \left( n_i \prod_{j=i+1}^{N} (n_j + m_j) \right) + n_N + 1
\]  

and assume that Assumption 3.1 holds.

Then, state \( z_1 \), where all agents are turned off and their internal parameters are equal to \( \frac{1}{n_i} \) and \( \frac{1}{m_i} \), and the state \( z_t \), where all agents are turned on and their internal parameters are equal to \( \frac{1}{n_i} \) and \( \frac{1}{m_i} \), are reachable from any other state.

Proof. Every time an agent turns off it goes into its first state \( x_1^{(i)} \), similarly whenever an agent...
turns on it reaches its internal state $x_1^{(i)}$. Further, we know that the agent stays in its first state $x_1^{(i)}$ as long as CEs occur, otherwise it enters its state $x_2^{(i)}$ or turns on and enters its state $x_{n+1}^{(i)}$. Also, the agent stays in $x_{n+1}^{(i)}$ until a CE arises, upon which it switches to its state $x_{n+2}^{(i)}$ or turns off and switches back to $x_1^{(i)}$.

First, choose any state $s$ that leads to a CE. From this state, there is a non-zero probability that all agents that are on turn off. This state clearly does not lead to a CE since all the agents are turned off. Therefore, there exists a positive probability that all the agents turn on in the next time step. This leads to the state $z_t$, which leads to a CE, according to Assumption 3.1. From this state the probability to reach $z_1$ is positive, showing that both states $z_1$ and $z_t$ are reachable from any state leading to a CE.

Secondly, choose any state $s$ that does not lead to a CE. The probability that all remaining states turn on in the next time step is larger than 0. This leads to a state causing a CE according to Assumption 3.1, as all agents are turned on. Hence, the possibility exists that all agents turn off at the same time, leading to the state $z_1$. This state is in turn leading to a CE, such that the probability that all agents turn off is positive. This means that the state $z_t$ can occur. Thus, the states $z_1$ and $z_t$ are reachable from any state not leading to a CE and the fact is proven.

Next, a property of the Markov chain is derived.

**Lemma 3.5.** If Assumption 3.1 holds, then the Markov chain from Fact 3.3 contains only one essential class.

*Proof.* Assume there are two essential classes and let’s take one vertex from each class, vertex $i$ and $j$. Both of these vertices are essential and not communicating with each other, i.e. there is no path from vertex $i$ to vertex $j$. However from Lemma 3.4 and the definition of an essential vertex, it is clear that vertex $i$ communicates with state $z_1$. Further, vertex $j$ also communicates with state $z_1$, because of Lemma 3.4. This means that there exists a path from vertex $i$ to $j$ and from $j$ to $i$. This contradicts the assumption that they are from two different essential classes.

The next lemma shows an important property of paths within this essential class.

**Lemma 3.6.** If Assumption 3.1 holds. Then, every path through $z_1$ can be enlarged as often as wanted by two or three steps.

*Proof.* Let $t$ be defined as in Equation (3.21) and let

$$s = \sum_{i=1}^{N-1} \left( n_i + 1 \right) \prod_{j=i+1}^{N} (n_j + m_j) + nN + 1.$$  \hspace{1cm} (3.22)

From $z_0$ the possibility exists to reach state $z_1$. Then, the probability to return to state $z_1$ is positive, which corresponds to a cycle of length 2. From $z_1$ it is also possible to reach first $z_2$ and then return to $z_1$, which corresponds to a cycle of length 3, which proves the fact above.
3.2. Analysis Using an Automata Game

Next, we are able to show an important property of this essential class.

**Lemma 3.7.** The essential class associated with the system in Lemma 3.5 is primitive.

**Proof.** Generate a path from vertex $i$ to vertex $j$ through state $z_1$ for $i,j$ any essential vertex. It follows from Lemma 3.4 that such a path exists for every essential vertex. Then, choose the longest of those paths and set $r$ its length plus 2. Since from Lemma 3.6 all paths can be enlarged by 2 or 3 steps, it is possible to reach every essential vertex from any other in $r$ steps. Note that the generated $r$ is not necessarily the smallest step length to reach any other vertex.

Hartfiel [43] defines a stochastic matrix to be regular if it corresponds to precisely one essential class of vertices, and the sub-matrix corresponding to this essential class is primitive. In [43], the author is able to show the following theorem that we here copy for the convenience of the reader.

**Theorem 3.8 (Ergodic Theorem[43]).** If the stochastic matrix $A$ is regular, then

$$A^r \rightarrow I^T y$$

(3.23)

where $y$ is the stochastic eigenvector of $A$ belonging to the eigenvalue 1 and $I$ is a column vector of all ones.

Note that the above theorem uses the notation in [43], which means that $y$ is a row vector. This is in contrast to the notation we adapted for this thesis, where we consider column vectors, such that our matrices and vectors are transposed compared to [43]. For our case Theorem 3.8 means that, if the stochastic matrix $H$, defined in Equation (3.20), is regular, then

$$\lim_{k \rightarrow \infty} H^k = y1,$$

(3.24)

where $y$ is the stochastic right eigenvector of $H$ belonging to the eigenvalue 1. Using the above, we are now able to state the ergodicity of the system.

**Theorem 3.9.** If $N$ agents participate and Assumption 3.1 holds, then the Markov chain from Fact 3.3 is ergodic.

The proof of Theorem 3.9 follows directly from Theorem 3.8 and Lemmas 3.5 and 3.7.

Since the Markov chain is ergodic, the state $z(k)$ converges to a steady state $z^* = \theta y$, where $y$ is the Perron eigenvector corresponding to the eigenvalue 1 and $\theta$ is a scalar to normalise the vector. This vector can be computed numerically, for example by using Matlab, and contains useful information about the overall power consumption by the agents and their individual power consumption, which we show next.

From the power consumption of each agent it is simple to construct a matrix containing the total power consumption from each state in its diagonal:

$$P_{\text{tot}} = p_1V^{(1)} \oplus p_2V^{(2)} \oplus \ldots \oplus p_NV^{(N)},$$

(3.25)
where $\oplus$ stands for the Kronecker sum and $V^{(i)}$ is defined as
\[
V^{(i)} = \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix}.
\] (3.26)

Additionally, $z^*$ contains the steady state probability distribution of the states, which is identical to the expected fraction of time that a state occurs. Therefore, by summing over the entries of $z^*$ which corresponds to states with the same power consumption defined in $P_{\text{tot}}$, a histogram of the expected total power consumption distribution can be constructed.

Figure 3.3 shows such a diagram for five agents, all with a power consumption of 3.7 kW and a total power allowance of 10 kW. In a first test, all agents use the parameters $n = 5$ and $m = 5$ shown in dark and light blue. The second largest eigenvalue of $H$, which is a measure for the convergence rate, is in this case $0.624 + 0.438i$ with magnitude 0.763. As can be seen the distribution is symmetric such that the time the agents consume less power than is available is equal to the amount of time where they use more power. By changing the internal parameters, the distribution can be modified. For example, in a second test the agents use $n = 10$ and $m = 3$. In this example, the second largest eigenvalue computes to $0.661 + 0.409i$ with magnitude 0.777.

This test is also shown in Figure 3.3 as the red bars. As the bars are higher for lower power consumption values (0−7 kW), it is expected that the system stays longer in states with low power consumption than in the first test. Also, in both cases the analytically computed share of time accurately predicts the actual simulated behaviour, see the dark versus the light bars. The example shows clearly that it is possible to numerically predict the estimated power consumption. However, as can be seen in this example, the actual power consumption exceeds the available power for approximately half of the time in the first test where all agents use the parameters $n = 5$ and $m = 5$. In the second test the expected time where the available power is exceeded is reduced compared to the first test, however still accounts for a large ratio of the total time. Such a behaviour is not desired for a load management algorithm. The main reason for such long periods of overshoots are the chosen parameters $n$ and $m$. While for the illustration and the purpose of numerical analysis small numbers are advantageous, it is desirable to use much larger numbers for an actual implementation of the algorithm. Further, as the second test indicates if $n > m$ the algorithm pushes the power consumption to smaller values. Hence, to achieve a desirable load management scheme the parameters $n$ and $m$ should be chosen large with $n > m$. Naturally, even in that case the nature of the algorithm allows that an extreme overshoot can occur, i.e. that all agents are turned on at the same time, but the probability of such an event becomes smaller. To illustrate this, a simulation has been performed with the same initial set up as before. This time the parameters are set to $n = 1000$ and $m = 25$. In this test, we only simulated the behaviour and did not numerically compute the actual expected shares. Figure 3.4 shows the histogram for this third simulation test. The power consumption of the agents is now most of the time close to 7 kW which is below the available power. An overshoot still occurs for about 15% of the time. Increasing the parameters and allowing more agents to participate would reduce this time even further. The behaviour of the algorithm in a more realistic setting is shown in Section 6.1.1.1.

Not only the total power consumption, but also the share of the power among the agents is interesting. This means we compare the expected average time an agent is turned on and off,
3.3 Analysis for Large Scale Systems

In Section 3.2, the complete system was investigated, where we considered each agent separately. This has the advantage that the agents can be diverse and we also gain proper results with only a small number of agents. However, with increasing numbers of agents the complexity increases rapidly, which makes it hard to analyse. Therefore, we will analyse the system using a simplification.

Assumption 3.2. We assume that a sufficiently large number of agents with identical parameters \( n, m, \) and \( p \), participate and apply the BA algorithm, i.e. \( N \to \infty \).

Since \( N \to \infty \) we can assume that the collection of the agents react according to the expectation. We redefine the state vector \( x(k) \) with entries \( x_i(k) \) as:
3. Binary Controllable Power Consumption

Figure 3.4: Histogram of the power consumption for a simulation test with reasonably large parameters.

- \( x_l(k) \) is the fraction of agents that are turned off with \( \nu(k) = \frac{l}{n} \) and \( \mu(k) = \frac{1}{m} \) at time step \( k \) for \( l \in \{1, 2, \ldots, n\} \) and
- \( x_{n+l}(k) \) is the fraction of agents that are turned on where \( \nu(k) = \frac{1}{n} \) and \( \mu(k) = \frac{l}{m} \) at time step \( k \) for \( l \in \{1, 2, \ldots, m\} \).

Note that there are in total \( n + m \) states, where the sum of the first \( n \) states corresponds to the ratio of agents being off and the sum of the last \( m \) states corresponds to the ratio of agents being on. Further, it implies that \( x_i \leq 1 \) at all times.

The power constraint in Equation (2.1) causing a CE, can be formulated in the new state by

\[
\begin{bmatrix}
0 & 1^T
\end{bmatrix} \mathbf{x}(k) \leq \frac{\bar{P}(k) - \bar{p}(k)}{pN}.
\]

We assume that the available power is reasonably large, i.e. there is power for a certain fraction of the agents. Then, the available power can be written as \( \bar{P}(k) - \bar{p}(k) = \bar{P}(k)pN \) with \( \bar{P}(k) \in [0, 1] \). The interpretation of \( \bar{P}(k) \) is the fraction of agents for which there is enough power available. Then, the above power constraint can be rewritten as

\[
\begin{bmatrix}
0 & 1^T
\end{bmatrix} \mathbf{x}(k) \leq \bar{P}(k).
\]

Using the definition of the states we can find a relation between \( \mathbf{x}(k+1) \) and \( \mathbf{x}(k) \).
3.3. Analysis for Large Scale Systems

![Figure 3.5: The expected share of time an agent is on.](image)

**Lemma 3.10.** Let the matrices $A_{nc}$ and $A_c$ be as defined in Equations (3.15) and (3.16).

Then, the state $\mathbf{x}$ evolves according to the switched system

$$\mathbf{x}(k+1) = A(k)\mathbf{x}(k)$$

with

$$A(k) \in \{A_{nc}, A_c\}.$$  \hfill (3.28)

The proof can be found in Appendix A.2. Note the following characteristics of the matrices $A_{nc}$ and $A_c$:

1. Both matrices are non-negative, i.e. each element is non-negative.
2. The subspace $S = \{\mathbf{x} \in \mathbb{R}_+ | \sum x_i = c\}$ is closed under $A_{nc}$ and $A_c$ for any constant scalar $c$, i.e. if $\mathbf{x} \in S$ then $A_{nc}\mathbf{x} \in S$ and $A_c\mathbf{x} \in S$.
3. The matrices $A_{nc}$ and $A_c$ have a left eigenvector $1^T$ in common with the common eigenvalue 1. However, the right eigenvectors with the common eigenvalue 1 differ.

### 3.3.1 Local Stability

In the following, we state a local stability result for the switched system found in Lemma 3.10.

**Assumption 3.3.** For each constant value of $\hat{P}$ there exists at least one periodic state sequence $\mathbf{x}^\ast(k), \mathbf{x}^\ast(k+1), \ldots, \mathbf{x}^\ast(k+T)$ with an associated switching sequence $A(0), A(1), \ldots, A(T-1)$ such
that

$$x^*(k + T) = A(T - 1)A(T - 2) \cdots A(0)x^*(k)$$

and the switching criteria is fulfilled for each step, i.e.

$$\begin{bmatrix} 0 & I^T \end{bmatrix} x^*(k + l) \begin{cases} \leq \hat{P} & \text{if } A(l) = A_{nc} \\ > \hat{P} & \text{if } A(l) = A_c \end{cases}$$

holds for \(l \in \{0, 1, 2, \ldots, T - 1\}\).

We denote a periodic state sequence fulfilling the criteria in Assumption 3.3 a valid cycle of the system. Next, we find a condition under which two states experience the same switching behaviour. Superscripts are used to indicate such two different states, e.g. \(x^{(1)}(k)\) and \(x^{(2)}(k)\).

**Lemma 3.11.** The state \(x^{(1)}(k)\) experiences the same switching behaviour as the state \(x^{(2)}(k)\) if

$$\left\| x^{(1)}(k) - x^{(2)}(k) \right\|_1 < \epsilon$$

(3.29)

with \(\epsilon\) chosen such that

$$\epsilon \leq \left\| \hat{P} - \begin{bmatrix} 0 & I^T \end{bmatrix} x^{(2)}(k) \right\|.$$  

(3.30)

The proof can be found in Appendix A.3. The above result is important to relate two states to each other regarding their behaviour. Now, we show that if two states experience the same switching because they fulfil Equation (3.29) at time step \(k\), they also experience the same switching behaviour for all following time steps. Note that some states lead to the same switching behaviour at time step \(k\) without fulfilling Equation (3.29). However, these states do not necessarily follow the same switching for the following time steps.

**Lemma 3.12.** If state vector \(x^{(1)}(k)\) fulfils

$$\left\| x^{(1)}(k) - x^{(2)}(k) \right\|_1 < \epsilon$$

with \(\epsilon\) chosen such that

$$\epsilon = \left\| \hat{P} - \begin{bmatrix} 0 & I^T \end{bmatrix} x^{(2)}(k) \right\|.$$  

Then,

$$\left\| \left( x^{(1)}(k + 1) - x^{(2)}(k + 1) \right) \right\|_1 < \epsilon.$$  

**Proof.** Because of Lemma 3.11 the state \(x^{(1)}(k)\) and \(x^{(2)}(k)\) experience the same switching. Therefore, the following is true

$$\left\| x^{(1)}(k + 1) - x^{(2)}(k + 1) \right\|_1 = \left\| A(k) \left( x^{(1)}(k) - x^{(2)}(k) \right) \right\|_1 \leq \left\| A(k) \right\|_1 \left\| x^{(1)}(k) - x^{(2)}(k) \right\|_1.$$
Further, \( \|A(k)\|_1 = 1 \) as the \( l_1 \) norm is equal to the maximum absolute column sum according to the definition. Hence,
\[
\|x^{(1)}(k + 1) - x^{(2)}(k + 1)\|_1 \leq \|x^{(1)}(k) - x^{(2)}(k)\|_1 < \epsilon.
\]

Next, these properties of two state vectors are used to find a local convergence result for valid cycles, i.e. cycles that fulfill the conditions in Assumption 3.3.

**Theorem 3.13.** Take a valid cycle \( x^*(0), x^*(1), \ldots, x^*(T) \) with an associated switching sequence \( A(0), A(1), \ldots, A(T - 1) \). Choose
\[
\epsilon = \min_{l \in \{1, \ldots, T\}} \left| \tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^*(l) \right|.
\]

Then, each sequence with initial state such that
\[
\|x(0) - x^*(0)\|_1 < \epsilon \tag{3.31}
\]
converges to the valid cycle \( x^*(0), x^*(1), \ldots, x^*(T) \) in the sense that
\[
\|x(k) - x^*((k \mod T))\|_1 \to 0 \text{ for } k \to \infty.
\]

The proof of this theorem can be found in Appendix A.4. While Theorem 3.13 shows the local stability of a valid cycle the local region might still be very small, because the \( \epsilon \) in Equation (3.31) might be small. In particular, when at any point during the cycle the actual fraction of agents turned on is close to the available fraction of power \( \tilde{P} \), i.e. \( \epsilon \) reaches 0 when
\[
\tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^*(l) = 0 \tag{3.32}
\]
for any \( l \in \{1, \ldots, T\} \).

A second problem is that a valid cycle has to be found, while its existence or uniqueness is not given. These cycles can be very long and there might be in each case multiple valid cycles that are all locally stable.

For the above reasons, the results found in this section though promising, cannot be applied generally to analytically predict the behaviour of the system. While the algorithm can still be applied and shows good behaviour in simulations the lack of a theoretical analysis for large scale systems makes the tuning of the parameters \( n \) and \( m \) hard. While the turning on of agents is mainly influenced by the parameter \( n \), the second parameter \( m \) also impacts the time, an agent takes to turn on. This interconnection is due to the fact that during a OFF phase the already off agents do not respond in any way. Nevertheless, we found that a large \( n \) prolongs the period an agent is turned off. Vice versa, a large \( m \) prolongs the period for an agent to turn off. Hence, for most load management tasks it is good to choose \( n > m \). Further, larger parameters reduce the probability of a large overshoot. However, to tune the values simulation studies should be done. In that way it is also possible to find possible cycles that might occur in the system.

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In Chapter 3, we proposed an algorithm that enables loads with binary controllable power consumption to participate in the load management scheme. In accordance to the limited controller ability, the algorithm suggested is not capable of handling more complex tasks such as reverse power flows and reactive power balancing. However, we expect that in the future more agents will allow a continuously controllable power consumption. This means that the control can be more flexible. Since such controller capabilities require more expensive components, e.g. an active rectifier, we assume that the maximum power consumption of these agents is generally larger and the fairness notions become more important.

We suggest the application of the AIMD algorithm, originally introduced in [23], for enabling agents with continuously controllable power consumption to participate in the load management scheme. The reasons we propose this algorithm are the similarities between a power distribution network and the communication network, as in detail discussed in [51]. The two objectives introduced in Section 2.2 that describe the load management problem can be interpreted as a resource sharing problem, where several agents compete for the resource power. Note that the AIMD algorithm is also mentioned briefly in [50] for a type of reactive control of loads and further given as an example in [7]. This algorithm is widely deployed in TCP congestion control and has proven to be reliable.

While the ideas in [7], [50] are similar to ours, they are not implemented in detail. Especially, in [7] the AIMD algorithm is given as one of three possible scheduling mechanisms. Also, we concentrate in this chapter on the fairness that the customers experience, which is not investigated before.

As mentioned earlier, [36] employs an algorithm used in communications. The actual implementation though differs from ours. Firstly, we do not use pricing to govern the power consumption directly. Secondly, in [36] the aggregated demand is not necessarily an upper bound rather the willingness to pay of the single users influences the aggregated demand of the controllable loads.
Finally, while the algorithm to find the charge rate is similar, the algorithm in [36] does not contain a multiplicative decrease.

Nowadays, various extensions and generalisations exist to the basic AIMD algorithm, mainly because the basic AIMD does not operate ideally in all situations [30], [49], [72]. [40] has for example proposed a high-speed TCP, where the response function upon detection of a congestion event CE, is adapted such that it behaves in cases with low packet drops similar to multiple parallel AIMD flows where the number of flows increases with the window length. Several other researches have followed this line of work making adaptations to the basic AIMD to accommodate for such high-speed networks [30], [49], [72], [76], [110].

Further, extensive literature is concerned with the algorithm’s behaviour in regards to its fairness and stability [76]. While early mathematical analysis of AIMD-like systems is done using flow models [68], more recently a stochastic model is used for the analysis based on a switched system representation, see for example [97], [98]. This stochastic model allows the use of various analysis techniques such as contraction arguments, non-negative matrix analysis tools, and iterated function systems. It is used to show the fairness of the time average and its convergence, see [97], [98], [115]. Higher moments, including second moments, are studied in [93], [96]. Also, various extensions of the basic algorithm are studied using this stochastic model representations.

[91] investigates the AIMD algorithm, where the additive increase factor is no longer a constant but a non-linear, non-decreasing function of the time. Such an algorithm is called non-linear additive increase multiplicative decrease (NAIMD). [91] shows the existence of a unique fixed point to which the algorithm converges using a contraction argument. In [53], the additive increase phase is chosen dependent on the time and the state, where it is a linear function of time, but a non-decreasing, non-linear function of the state. Similarly, [26] proposes an AIMD-like algorithm for high-speed networks, where the additive factor is chosen stochastic from a given set, and shows that ergodicity holds. There exist analogous stability results for NAIMD where the multiplicative factor is no longer linear while the additive increase phase is kept linear, also denoted additive increase non-linear multiplicative decrease (AINMD). In [27], the decrease is an increasing function of the state such that a larger state leads to a larger decrease. Further, [54] investigates the case where the constraint function is generalised, while both the additive increase phase and the multiplicative decrease phase remain linear.

Additionally, in [48] a multiple bottleneck problem is investigated, where more than one constraint occurs and each constraint affects only a subset of the flows. The authors of [48] prove the ensemble average fairness of such a case in which AIMD is used.

Finally, in [116], [117] a relation between convex optimisation problems with constraints and AIMD algorithms is established. This allows the use of an AIMD algorithm to solve large scale convex optimisation problems in a distributed way without additional communication.

While the application of the AIMD algorithm to our problem requires some small adaptations, we are able to base the analysis of the algorithm regarding its stability and fairness on the existing
stochastic model as described in [97].

The flexibility of the AIMD algorithm allows us to tune the parameters to accommodate for the different fairness notions introduced in Section 2.2.1. Additionally, using minor adaptations of the basic AIMD, we can expand the algorithm to allow reverse power flows and reactive power balancing. These extensions still allow the different fairness notions in Sections 2.2.1 and 2.3.1 to be addressed.

In the following, we describe the AIMD algorithm as it is applied by the agents that do not allow reverse power flows or reactive power exchange. Afterwards, we expand the basic AIMD algorithm to first allow reverse power flows and then in a next step to enable reactive power balancing, see Sections 4.2 and 4.3. The basic AIMD algorithm is analysed mathematically in regards to its convergence and the aggregated demand in Chapter 5, in which the available results regarding convergence are expanded to adapt to our implementation presented in this chapter. For the two extensions to the algorithm we mostly use simulations to illustrate the behaviour of the adapted system. The main reason for using simulations in this case is that most of the analysis of the basic AIMD in Chapter 5 still holds for the expanded versions, because they are strongly dependent on the basic algorithm. Note that we do not consider the detailed dynamic response of the power grid in this chapter. Similarly, we do not examine cases where both loads with algorithms presented here and in Chapter 3 participate simultaneously in the load management scheme. Simulations of the algorithms in a more realistic setting, where the power grid is taken into account, are shown in Section 6.1. There, we also discuss additional aspects such as when simultaneously loads with different controller abilities are present.

Note that since this algorithm requires continuous control over the power consumption, we consider solely EVs as agents, and therefore use the terms EV and agent as synonyms within this chapter.

4.1 The Basic AIMD Algorithm

In this section, we describe the AIMD algorithm as it is applied for controlling the active power consumption of agents. Since it does not allow any control over the reactive power, the algorithm ignores any reactive CEs. Afterwards, we will show how the algorithm can be tuned to accommodate for the different scenarios introduced in Section 2.2.1. The tuning is based on properties of the AIMD algorithm, such as its convergence. These properties are analysed mathematically in Chapter 5. For illustration purposes we use in this chapter simulations performed in Matlab.

4.1.1 Description of the AIMD Algorithm

The AIMD algorithm consists of two distinct phases: 1. the multiplicative decrease (MD) phase and 2. the additive increase (AI) phase. Depending on whether a CE signal has been received or not the agent executes either the MD phase, if a signal has been received, or the AI phase otherwise. Since it does not allow any reactive power control it only reacts upon active CE signals. We know from Section 2.5 an active CE signal is sent if the total power consumption should be reduced.
4. Continuously Controllable Power Consumption

Hence, the MD phase takes care of the reduction by multiplying the actual power consumption of the agent, \( p_i(k) \), with one of two possible multiplicative factors, \( \beta^{(1)}_{i}(k) \) and \( \beta^{(2)}_{i}(k) \), with \( 0 < \beta^{(1)}_{i}(k) < \beta^{(2)}_{i}(k) \leq 1 \) for all \( k \). In case \( \beta^{(2)}_{i}(k) = 1 \), some of the agents do not reduce their power consumption. This results in a similar behaviour to the case where not all agents receive the actual CE signal. Which of the two factors is applied is selected in a stochastic manner. Let \( \lambda_{i}(k) \) be the probability with which \( \beta^{(1)}_{i}(k) \) is chosen at time step \( k \) in case a CE signal has been received. Note, that probabilities of 1 correspond to a deterministic version of the algorithm. Then, the MD phase can be described by

\[
p_i(k + 1) = \begin{cases} 
\beta^{(1)}_{i}(k)p_i(k) & \text{with probability } \lambda_{i}(k) \\
\beta^{(2)}_{i}(k)p_i(k) & \text{with probability } 1 - \lambda_{i}(k).
\end{cases}
\]  

As long as no CE signal is received the agent executes the AI phase which increases the power consumption of the agent additively. Each vehicle has its individual additive factor \( \alpha_{i}(k) \) and a common constant additive factor \( \overline{\alpha} \) equal for all agents. This is to allow for regulation of the increase rates by the central management unit. Those values are multiplied to gain the total additive factor with which the power consumption is increased. Note that by limiting the range of the individual additive factor \( \alpha_{i}(k) \) to \([0, 1]\) the increase of the aggregated power during one time step is bounded by \( \overline{\alpha} \) per participating agent. Naturally, the power consumption is still limited by the maximal power consumption \( \overline{p} \). Hence, the AI phase can be described by

\[
p_i(k + 1) = \min \left( p_i(k) + \alpha_{i}(k)\overline{\alpha}, \overline{p} \right).
\]  

Figure 4.1 shows the above procedure as it is executed by each vehicle at each time step as long as it is participating in the load management scheme. Possible reasons for stopping the participation are for example if the vehicle is fully charged or if the owner drives away.

The AIMD parameters introduced above \( \alpha_{i}(k), \beta^{(1)}_{i}(k), \beta^{(2)}_{i}(k), \) and \( \lambda_{i}(k) \) allow tuning of the algorithm such that it is possible to share the available power according to the different objectives that were defined in Section 2.2.1. We will next show how to tune those parameters for the various fairness notions proposed. To understand the reasoning behind the choices of the AIMD parameters, some properties of the AIMD algorithm, such as its convergence, are important. These properties along with some investigations concerning the aggregated demand are given in Chapter 5.
4.1. The Basic AIMD Algorithm

4.1.2 The AIMD Algorithm in Different Scenarios

In Chapter 5, we will find that the behaviour of the algorithm is largely defined by the AIMD parameters $\alpha_i$, $\beta^{(1)}_i$, $\beta^{(2)}_i$, and $\lambda_i$. Hence, proper tuning of these parameters allows to accommodate for different shares of power among the agents. We use this to implement the different fairness notions defined in Section 2.2.1. In the following sections, we illustrate how the parameters should be tuned and give some illustrative simulation results indicating the use of the algorithm.

4.1.2.1 Charge Rate Fairness Scenario

To achieve the objective in Section 2.2.1.1, the power consumption of the agents should be equal unless the rate is limited. From, a basic property of the AIMD algorithm discussed in the next chapter, i.e. Theorem 5.15 in Chapter 5, it is possible to find that the objective holds for the deterministic AIMD, if the AIMD parameters are chosen identical for all agents, i.e. $\alpha_i = \alpha$ and $\beta^{(1)}_i = \beta^{(1)}$ for all $i$. Assuming that this expands to the stochastic AIMD, see Claim 5.18 in Chapter 5, we find that also $\beta^{(2)}_i = \beta^{(2)}$ and $\lambda_i = \lambda$ should hold for all $i$.

It is important to note that for this scenario the agents are required to have a consensus on the parameters. This is possible by either allowing communication among them upon the connection to find consensus on the parameters, informing the agents of the parameters by the central management unit, or programming the charger directly. The first possibility requires the agents to send information, which means that they require additional communication capabilities. The second possibility requires the management unit to broadcast the parameters, which means that it needs to know them. Also, the communication load will increase since the parameters have to be resent periodically to accommodate newly connected agents. The third possibility has the
advantage that no additional communication occurs. However, an adaptation of the parameters requires that each controller is reprogrammed. In this thesis, we assume that upon connection the parameters are known and neglect how this is achieved.

For illustrative purposes, we use a Matlab simulation that does not take the distribution grid into account. Simulations where the distribution grid is taken into account can be found in Section 6.1. A total of 20 EVs participate. The available power is set to a constant 20 kW and no uncontrollable loads are connected. All EVs use identical AIMD parameters which are summarised in Table 4.1. The time they connect to the distribution grid is uniformly distributed in the first 6 hours of the simulation and each vehicle remains connected for up to ten hours or until it finished charging. The energy required by the vehicle is randomly drawn from a uniform distribution between 0 and 24 kW h. Further, the maximum charger outlet is set to 3.6 kW. Figure 4.2 shows the filtered power consumption of five randomly selected agents using a moving average filter with a window length of 600 time steps. Clearly the average charge rate of the EVs equalises and is able to adapt if EVs disconnect or connect.

| case   | $\alpha$ | $\bar{\alpha}$ | $\beta^{(1)}$ | $\beta^{(2)}$ | $\lambda$ | $E[\beta]$
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<td>0.7</td>
<td>0.99</td>
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<td>0.758</td>
</tr>
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<td>0.7</td>
<td>0.99</td>
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<tr>
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<td>0.7</td>
<td>0.99</td>
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<td>0.758</td>
</tr>
<tr>
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<td>0.7</td>
<td>0.99</td>
<td>adapted</td>
<td>-</td>
</tr>
</tbody>
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Table 4.1: The AIMD parameters used in the simulations for the different scenarios.

Figure 4.2: Evolution of the charge rates for five EVs applying the CRF scenario.
4.1.2.2 Required Energy Fairness Scenario

The second fairness notion is introduced as the REF scenario. Here, the power should be shared among the vehicles depending on their needs. Due to Equation (2.11) this means that the ratio

\[ \frac{\hat{p}_i}{\bar{p}_i} \]  

(4.3)

should be equal for all agents unless their rate is limited. Using the property that each agent’s share is proportional to

\[ \frac{\alpha_i}{1 - \beta^{(1)}_i}, \]  

(4.4)

as long as the charge rate does not reach the upper bound, which will be investigated in Theorem 5.15 in Chapter 5, it becomes clear that the desired share can be achieved with a deterministic AIMD algorithm by setting its parameters to

\[ \frac{\alpha_i}{1 - \beta^{(1)}_i} \propto \hat{p}_i. \]  

(4.5)

By expanding this to the stochastic case, i.e. Claim 5.18 as discussed in Chapter 5 holds, we find that

\[ \frac{\alpha_i}{1 - (\beta^{(1)}_i \lambda_i + \beta^{(2)}_i (1 - \lambda_i))} \propto \hat{p}_i. \]  

(4.6)

Note that it does not matter which AIMD parameters are adapted as long as the ratio fits. However, the adaptation of the different parameters influences the behaviour of the aggregated demand as discussed in Section 5.3. For this scenario we choose to adapt the additive AIMD parameter \( \alpha_i \) of each agent such that

\[ \alpha_i \propto \hat{p}_i, \]  

(4.7)

while the other AIMD parameters are chosen equal for all agents. To find the actual additive factor we scale the desired power consumption with a constant \( \varrho \) which is set identical for all agents. We choose the constant \( \varrho \) to be larger than the maximum allowed power consumption of a single agent that is connected, i.e.

\[ \varrho \geq \max_{i=1, \ldots, N} \bar{p}_i. \]  

(4.8)

This limits the additive factor to lie in the range \([0, 1]\). Even though this is not necessary, it is desirable as then the constant factor \( \overline{\alpha} \) which is identical for all agents bounds the increase per time step and agent. Hence, the additive factor is set to

\[ \alpha_i = \frac{\hat{p}_i}{\varrho}. \]  

(4.9)

As for Section 4.1.2.1, we assume that upon connection the other parameters and the constant \( \varrho \) are known. The agent computes its desired power consumption \( \hat{p}_i \) and their individual additive factor \( \alpha_i \) upon connection.

For illustration purposes, we use the same Matlab simulation as in Section 4.1.2.1 that does not take the distribution grid into account. Again, a total of 20 EVs participate, the available power is set to 20 kW, and no uncontrollable loads are connected. The settings of the EVs are chosen equal to the settings in Section 4.1.2.1, see Table 4.1 for a summary of the AIMD parameters. The only
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difference between the simulations is that now the additive factor $\alpha_i$ is chosen upon the connection of the vehicle according to Equation (4.9). Figure 4.3 shows the filtered power consumption of the same five agents selected in Section 4.1.2.1 using a moving average filter with a window length of 600 time steps, which corresponds to 10 min. Clearly the average charge rates of the EVs no longer equalise. Naturally, this is not the objective, but as stated in Section 2.2.1.2 the goal is to equalise the ratio between the average power consumption and the desired charge rate. In Figure 4.4, this ratio is shown. As clearly visible this ratio equalises throughout the simulation.

![Figure 4.3: Evolution of the filtered charge rates for five EVs applying the REF scenario.](image)

4.1.2.3 Charge Time Fairness Scenario

For the CTF scenario the requirement, as described in Section 2.2.1.3, is that the power consumption is proportional to the priority assigned to the agents. Hence, analogous to Section 4.1.2.2, we find that

$$\alpha_i \propto \gamma_i$$

(4.10)

should hold for the stochastic AIMD algorithm. As before, we choose again to adapt only the additive factor $\alpha_i$ of the agent individually and select the remaining AIMD parameters identical for all participating agents. This means that the additive factor should be proportional to the priority of the agent, i.e.

$$\alpha_i \propto \gamma_i.$$  

(4.11)

We scale the additive factor here with $L$, i.e. the number of available priorities such that

$$\alpha_i = \frac{\gamma_i}{L}.$$  

(4.12)

Then, the additive factor $\alpha_i$ lies in the interval $[0, 1]$. While it is not essential that the additive factor lies in this interval, it is a useful property, since then the common additive factor $\alpha$ is an upper bound for the increase per time step and agent as we will discuss in Section 5.3.
4.1. The Basic AIMD Algorithm

As before, we illustrate this behaviour using a Matlab simulation. The settings of the vehicles are identical to the case in Sections 4.1.2.1 and 4.1.2.2. The AIMD parameters are also chosen identical with the sole difference that the additive factor $\alpha_i(k)$ is adapted during the simulation according to Equation (4.12), see Table 4.1. Unlike in Section 4.1.2.2 the additive factor is updated throughout the simulation to take the changes in the priority into account. Figure 4.5 shows the ratio between the filtered power consumption and the priority of the five chosen vehicles before.

As can be seen the ratio is equalised. Note that the discrepancy at the end is caused by the upper bound on the power consumption of agent 1. Further the spikes visible are caused by updates in the priority and the time the algorithm requires to adjust its share.

4.1.2.4 GOF Scenario

The fairness notion for the GOF scenario tries to optimise a global cost. As discussed in Section 2.2.1.4, that means that each vehicle has an assigned cost $g_i(p_i(k))$ and the second objective is to minimise the sum of these individual costs. To fulfil such a scenario we will adapt the probability $\lambda_i$ of each agent continuously at each CE. We use the technique described in [116], [117] and in this thesis in Section 5.2 to fulfil the second objective of the load management problem. Note there are two subtle differences between the algorithm how it is used in Section 5.2 and our implementation here. The first difference is that the second multiplicative parameter $\beta^{(2)}$ is equal to 1. Secondly, we assume an upper bound on the individual power consumption that is not considered in Section 5.2. These two differences result in a slight change in the selection of the probability, such that each agent at a CE selects its probability to be

$$\lambda_i = \frac{\frac{\partial}{\partial p_i} g_i(p_i)|_{p_i}}{\rho_i} \cdot \frac{1 - \beta^{(2)}_i}{\beta^{(2)}_i - \beta^{(1)}_i}.$$  

(4.13)
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Figure 4.5: Evolution of the ratio between the filtered charge rates and the priorities for five EVs applying the CTF scenario.

The reasoning behind this slight adaptation is given in Appendix A.5. However, note that this is not a proof and we do not show that the adapted algorithm still converges analytically. However, in simulations the algorithm seems to operate as desired for the specific cost function discussed in Section 2.2.1.4, i.e. the expected total charging state. The derivative of the individual cost function is

$$\frac{\partial}{\partial \rho_i} g_i(p_i) \bigg|_{\rho_i} = \frac{2 \times 100 \times \rho_i - 20E_i(T_i)}{E^2}. \quad (4.14)$$

Note that for simplicity we omitted the units. With this we set the probability at the $k$-th CE to

$$\lambda_i(\tau_k) = \frac{2 \times 100 \times \rho_i(\tau_k) - 20E_i(T_i)}{E^2\rho_i(\tau_k)} - \frac{1 - \beta^{(2)}_i}{\beta^{(2)}_i - \beta^{(1)}_i} \quad (4.15)$$

and simulate this scenarios using the same setting as for the other scenarios including the AIMD parameters. However, this time the probability $\lambda_i(\tau_k)$ is adapted at each CE according to Equation (4.15), see Table 4.1.

Figure 4.6 shows the derivative of the cost functions of five agents, also selected previously for the other scenarios. The agents are able to find a consensus, which indicates that the objective is fulfilled. Here, we filtered the derivatives using a moving average filter with a window length of 3600 time steps which corresponds to an hour. When we apply a moving average filter with the same window size as for the other scenarios, i.e. 600 time steps, the derivatives show large deviations. This is because the long term average converges in the algorithm in [116], [117].

Note that as the main idea is based upon finding a consensus, fairness notions that are based on a consensus can also be fulfilled. Naturally, it has to be guaranteed that the probability will lie
4.1. The Basic AIMD Algorithm

Figure 4.6: Evolution of the derivatives of the cost for five EVs applying the GOF scenario.

within the reasonable range $[0, 1]$. Further, the functions should all be strictly increasing and the probability $\lambda_i(\rho_i)$ has to fulfil the three assumptions given in Assumption 5.9 and in [116], [117].
4. Continuously Controllable Power Consumption

4.2 Expansion to Handle Reverse Power Flows

In this section, we show how to extend the AIMD algorithm described in Section 4.1 for agents that allow reverse power flows without allowing reactive power control. Since only the active power should be controlled, the agents react only to active CE signals and ignore all reactive CE signals they receive. We first describe the expanded AIMD algorithm in detail which is based on the basic AIMD algorithm proposed in Section 4.1. Since, the expanded algorithm is very similar, we illustrate afterwards the effect the adaptations have on the behaviour of the algorithm in regards to the aggregated demand and the power consumption of a single agent. Most properties that will be discussed in Chapter 5 also hold for this algorithm.

4.2.1 Description of the Expanded AIMD Algorithm

To allow bi-directional power flow, the suggested algorithm operates in two modes: the G2V mode, in which the agent draws active power from the distribution grid, and the V2G mode, in which the agent injects active power into the distribution grid.

If the agent operates in the G2V mode its operation is nearly identical to the one described in Section 4.1, i.e. the algorithm consists of two phases: the AI phase and the MD phase. While the AI phase is executed according to Equation (4.2), the MD phase is altered to guarantee a lower bound on the decrease. In that way, it is possible to handle situations where the power consumption of an agent is close to zero, which occurs for example during transitions between G2V and V2G mode operations. Let $\beta$ be the minimum decrease that is chosen identically for all participating agents and let $d(x, y)$ be a function such that

$$d(x, y) = \begin{cases} 
\beta & \text{if } x < y \\
0 & \text{otherwise.}
\end{cases}$$

(4.16)

Then, the MD phase can be described by

$$p_i(k + 1) = \begin{cases} 
\beta^{(1)}_i(k)p_i(k) - d((1 - \beta^{(1)}_i(k))p_i(k), \beta) & \text{with probability } \lambda_i(k) \\
\beta^{(2)}_i(k)p_i(k) - d((1 - \beta^{(2)}_i(k))p_i(k), \beta) & \text{with probability } 1 - \lambda_i(k).
\end{cases}$$

(4.17)

Note that the only difference to Equation (4.1) is the inclusion of the minimal decrease.

If the agents operate in the V2G mode, the AIMD algorithm described above is inverted. This means that upon receiving a CE the agents increase their power injection additively, which corresponds to an actual decrease of the power consumption. Similarly, when no CE is received the agents decrease their power injection multiplicatively, which corresponds to an increase in power consumption. By analogy with the G2V mode, we denote the first AI phase and the latter MD phase.

The agents can automatically recognise at which point they need to change the operating mode, i.e. from G2V to V2G or vice versa. Therefore, let $\nu_i(k)$ indicate whether the $i$-th agent operates...
in G2V mode at time step $k$, i.e. $v_i(k) = 1$ if at time step $k$ agent $i$ is in G2V mode and $v_i(k) = 0$ otherwise.

The switch from G2V to V2G mode occurs after a CE if the actual power consumption is very small. This means the indicator is updated after each CE event according to

$$v_i(k + 1) = \begin{cases} 1 & \text{if } p_i(k + 1) > \epsilon \text{ and } v_i(k) = 1 \\ 0 & \text{if } p_i(k + 1) \leq \epsilon \text{ and } v_i(k) = 1, \end{cases}$$

where $\epsilon$ is a small positive scalar. The return from V2G mode to G2V mode occurs when no CE is received, and the indicator changes as

$$v_i(k + 1) = \begin{cases} 1 & \text{if } p_i(k + 1) > -\epsilon \text{ and } v_i(k) = 0 \\ 0 & \text{if } p_i(k + 1) \leq -\epsilon \text{ and } v_i(k) = 0. \end{cases}$$

Figure 4.7 illustrates the AIMD algorithm described above as it is executed by an agent.

### 4.2.2 Behaviour of the Modified AIMD System

We now illustrate the effect of the parameters newly introduced on the algorithm using simulations. Therefore, we simulated a total of five different settings of AIMD parameters. The additive factor $\alpha_i$ is selected randomly from a uniform distribution between 0.2 and 1 for each agent. In all five settings $\alpha_i$ is selected identical. The remaining AIMD parameters are chosen identical for all agents. The AIMD parameters used in the five simulation settings are summarised in Table 4.2.

There are no uncontrollable loads connected and the available power is chosen to vary step wise, where positive values mean that the agents should consume power and negative values mean the agents are required to inject power.

In Figure 4.8, the aggregated power consumption and injection is shown for the five scenarios. The selected period shows a step change in the available power from $-10$ kW to 10 kW, i.e. the agents are supposed to inject power in the first part and consume power in the second part. It shows that the minimum decrease $\beta$ helps the system to follow changes in the power consumption which require the agents to change their mode. Larger minimum decreases lead to a faster change. While a large minimum decrease enhances the following of such power mode changes, it also decreases the efficiency of the share as the typical saw tooth pattern is increased in cases where there is not a lot of power available for the controllable agents. The parameter $\epsilon$ has only a small influence in how fast the aggregated demand is able to follow such changes. In this regard, we would be able to remove this parameter without any major effect.

In Figure 4.9, we selected an agent that is supposed to inject a small amount of power during a period where the agents are required to inject power. In such a case the agent might toggle between consuming and injecting power frequently. The frequency of such changes can be controlled using the parameter $\epsilon$. While both too large or too small values result in frequent switches, a careful selected $\epsilon$ can decrease the number of such switches. This is indicated in Figure 4.9 by comparing the plot using the base parameters compared to no or a large $\epsilon$. Note that also the parameter $\beta$...
4. CONTINUOUSLY CONTROLLABLE POWER CONSUMPTION

Figure 4.7: Illustrative diagram of the AIMD algorithm with V2G capabilities.

- **G2V mode**
  - If the CE signal is received, the algorithm proceeds with the following updates:
    - \( p_i(k+1) = \beta_i(k)p_i(k) \)
    - \( v_i(k+1) = \begin{cases} 1 & \text{if } p_i(k+1) > \epsilon \\ 0 & \text{if } p_i(k+1) \leq \epsilon \end{cases} \)

- If the CE signal is not received, the algorithm proceeds with:
  - \( p_i(k+1) = p_i(k) + \alpha_i(k)\pi \)

- **V2G mode**
  - If the CE signal is received, the algorithm proceeds with:
    - \( p_i(k+1) = \beta_i(k)p_i(k) \)
    - \( v_i(k+1) = \begin{cases} 1 & \text{if } p_i(k+1) > -\epsilon \\ 0 & \text{if } p_i(k+1) \leq -\epsilon \end{cases} \)

- If the CE signal is not received, the algorithm proceeds with:
  - \( p_i(k+1) = p_i(k) - \alpha_i(k)\pi \)

Influence factors such as \( \beta \) influence the number of switches, such that a large \( \beta \) results in frequent and large switches, while no \( \beta \) results in no switches at all. Hence, this yields a trade-off for the parameter \( \beta \) to reduce the number of switches when supplying or providing small amounts of power, while maintaining a fast response as in Figure 4.8.

In Figure 4.10, the average power consumption of each connected agent is shown where each subplot shows one of the scenarios. As can be seen the minimum decrease strongly affects the influence of the additive factor. A large minimum decrease pushes the share of agents down that
already should have a lower share, while no minimum decrease leads to a share as it would be expected from the basic AIMD without V2G capabilities. The parameter $\epsilon$ has a minor influence on how the power is shared among the agents.

The above discussion shows that the selection of the newly introduced parameters is important to achieve a desired behaviour in regards to both the aggregated demand and the share each agent receives.

<table>
<thead>
<tr>
<th>case</th>
<th>$\alpha$</th>
<th>$\alpha$</th>
<th>$\beta(1)$</th>
<th>$\beta(2)$</th>
<th>$\lambda$</th>
<th>$\beta$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>base</td>
<td>-</td>
<td>0.1</td>
<td>0.7</td>
<td>0.99</td>
<td>0.8</td>
<td>0.15</td>
<td>0.1</td>
</tr>
<tr>
<td>no $\epsilon$</td>
<td>-</td>
<td>0.1</td>
<td>0.7</td>
<td>0.99</td>
<td>0.8</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>large $\epsilon$</td>
<td>-</td>
<td>0.1</td>
<td>0.8</td>
<td>0.99</td>
<td>0.8</td>
<td>0.15</td>
<td>0.3</td>
</tr>
<tr>
<td>no $\beta$</td>
<td>-</td>
<td>0.1</td>
<td>0.7</td>
<td>0.99</td>
<td>0.8</td>
<td>-</td>
<td>0.1</td>
</tr>
<tr>
<td>large $\beta$</td>
<td>-</td>
<td>0.1</td>
<td>0.8</td>
<td>0.99</td>
<td>0.8</td>
<td>0.4</td>
<td>0.1</td>
</tr>
<tr>
<td>CRF scenario</td>
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<td>0.1</td>
<td>0.7</td>
<td>0.99</td>
<td>0.8</td>
<td>0.15</td>
<td>0.1</td>
</tr>
<tr>
<td>REF scenario</td>
<td>adapted</td>
<td>0.1</td>
<td>0.7</td>
<td>0.99</td>
<td>0.8</td>
<td>0.15</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 4.2: The AIMD parameters used in the simulations of the AIMD algorithm with V2G capabilities.

Figure 4.8: Aggregated demand for five settings of different AIMD parameters.

4.2.3 Application in Different Scenarios

As mentioned in Section 2.3.1, there are different ways how to achieve a fair share among the agents. While we discussed in Section 4.1.2 how to tune the AIMD parameters in case of G2V operation, we now will do the same when allowing V2G capabilities.

4.2.3.1 CRF Scenario

The objective defined in Section 2.3.1.1 is to share the available power equally among the agents during G2V operation and supply equal amounts of power in V2G operation within the power
limitations of the agents. Analogously to Section 4.1.2.1, this objective can be achieved by selecting identical AIMD parameters for all agents. This means that we select $\alpha_i = \alpha$, $\beta^{(1)}_i = \beta^{(1)}$, $\beta^{(2)}_i = \beta^{(2)}$, and $\lambda_i = \lambda$ for all $i$. Note that the other AIMD parameters $\overline{\alpha}$, $\epsilon$, and $\beta$ are always chosen identical for all participating agents.

We now illustrate this behaviour during both G2V and V2G operation using a Matlab simulation. Therefore, we let the available power $\overline{P}(k)$ vary step-wise throughout the simulation period and take both positive and negative values. We assume that there are no uncontrollable loads connected, such that $\overline{p}(k) = 0$ for all $k$. In total 20 vehicles participate in the load management task. All EVs have a battery size of 24 kW h, have the same upper bound on the power consumption of $\overline{p}_i = 3.6$ kW, and all agents permit bidirectional power flows. In this setting, we assume that complete discharge of the battery is permitted in V2G mode. This can easily be prevented by limiting the amount of energy they are allowed to use. Further, they all disconnect after at most 10 h or as soon as they are fully charged. Once disconnected the vehicles are also unable to inject power into the grid. The AIMD parameters are chosen identical for each vehicle and set according to Table 4.2. The average power consumption of five randomly selected agents is shown in Figure 4.11. As can be seen the power consumption and injection is equalised. Note that agent 1 stops early in the first period when the agents are supposed to inject power. The reason thereof is that the battery is empty such that the agent can no longer provide energy.
4.2. Expansion to Handle Reverse Power Flows

Figure 4.10: The average power consumption/injection of the connected agents applying the AIMD algorithm with V2G capabilities.

4.2.3.2 REF Scenario

According to Section 2.3.1.2, a second fairness notion is introduced where the available power should be distributed depending on the needs of the vehicles, such that the ratio

\[
\frac{\rho_i(k)}{\tilde{p}_i(k_0)}
\]  

(4.18)
4. Continuously Controllable Power Consumption

![Graph showing power consumption over time for five agents]

Figure 4.11: Evolution of the average power consumption for five randomly selected agents applying the AIMD algorithm for the CRF scenario with V2G capabilities.

is equal for all agents during the G2V mode and

\[ \rho_i(k) \hat{p}_i(k_0) \]  

(4.19)

is equal during V2G operation unless their rate is limited.

Analogously to Section 4.1.2.2, we adapt the additive factor \( \alpha_i \) to achieve the above objective while the remaining AIMD parameters are selected identical for all agents. During G2V mode we adapt the factor according to

\[ \alpha_i = \frac{\hat{p}_i}{\varrho} \]  

(4.20)

where

\[ \varrho = \max_{i \in \{1,...,N\}} \bar{p}_i. \]  

(4.21)

Note that this is identical to the adaption found in Section 4.1.2.2. During V2G operation we adapt the additive factor such that

\[ \alpha_i = \frac{\varrho}{\bar{p}_i}. \]  

(4.22)

This behaviour is illustrated using the same simulation settings as for the CRF scenario. The used AIMD parameters are summarised in Table 4.2, where the additive parameter is adapted using Equations (4.20) and (4.22). In Figure 4.12, the terms in Equations (2.22) and (2.23) during G2V and V2G operation, respectively, are shown of five randomly selected agents.
4.3 Active and Reactive Power Management

As described in Section 2.4, the abilities of the agents can be extended to allow additionally reactive power balancing. This adds another level to the load management problem by enabling the loads to support reactive power balancing. We will suggest in this thesis a dual additive increase multiplicative decrease (DAIMD) algorithm that governs both the active and reactive power consumption of an agent. Hence, the agents are supposed to react on both active and reactive CE signals sent by the central management unit. Since the DAIMD algorithm is strongly based on the AIMD algorithm, it leads to a similar behaviour. In the following, we describe the DAIMD algorithm in detail. Afterwards, we illustrate briefly its behaviour in regard to the aggregated demand using Matlab simulations. Due to the strong similarities between the AIMD and the DAIMD algorithm most of the properties that will be discussed in Chapter 5 hold also for the DAIMD algorithm. Further, the adaptation to different fairness notions are done identically as in Sections 4.1.2 and 4.2.3 depending on whether the agents forbid reverse active power flows or not. Hence, the discussions in this section are kept brief.

4.3.1 Description of the DAIMD Algorithm

The DAIMD algorithm comprises an active power AIMD, which manages the active power, and a reactive power AIMD, which governs the reactive power exchange. Figure 4.13 shows a flow chart implementing such a DAIMD.

In a first step, the active power AIMD is executed, as described in Section 4.1 or Section 4.2 and illustrated in Figures 4.1 and 4.7, respectively. Afterwards, a second AIMD algorithm, the
active power AIMD algorithm

reactive power AIMD algorithm

\[ p_i(k + 1) = \max \left( -\bar{p}_i, \min \left( \bar{p}_i, p_i(k + 1) \right) \right) \]
\[ q_i(k + 1) = \max \left( -\sqrt{s_i^2 - p_i(k + 1)^2}, \min \left( \sqrt{s_i^2 - p_i(k + 1)^2}, q_i(k + 1) \right) \right) \]

end

Figure 4.13: Illustration of the DAIMD algorithm

The reactive power AIMD algorithm depends on reactive CEs, which are generated as described in Section 2.5. Similarly to the active power AIMD algorithm, the agents are able to draw or inject reactive power depending on the requirements of the power grid. Hence, the algorithm operates as well in two different modes: the G2V mode and the V2G mode. In the G2V mode the reactive power consumption additively increases if no reactive CE occurs, and multiplicatively decreases otherwise. As for the active AIMD this behaviour is swapped during V2G operation, i.e. the power injection increases additively if a CE has been received and multiplicatively decreases otherwise. Figure 4.14 illustrates this behaviour in detail.

To distinguish between the parameters used in the reactive power AIMD from the ones used in the active power AIMD, we denote the individual additive parameter \(a_i(k)\), the global additive scaling factor \(\pi\), the two multiplicative factors \(b^{(1)}_i(k)\) and \(b^{(2)}_i(k)\), the associated probability \(\xi_i(k)\), and the indicator \(\zeta_i(k)\).

Naturally, at all times, the power outlet imposes a maximum bound on the apparent power that can be exchanged between the agents and the distribution grid, such that Equation (2.24) holds. Regarding this bound, it is important to note that we first bound the active power consumption, and then we bound the reactive power consumption, see Figure 4.13. Thus, we give a higher priority to the active power exchange rather than to the reactive power exchange. This is deliberate and based on the assumption that the operation of the agents, i.e. the charging of the EVs, is more
important than satisfying some ancillary services for the grid and in accordance with the definition of the set of objectives in Section 2.4. If necessary the priorities can easily be reversed, giving the reactive power exchange first priority and active power exchange a lower priority.

### 4.3.2 The DAIMD Algorithm Illustrated

The algorithm is simulated in Matlab. A total number of 20 agents participate, their active AIMD parameters are set to $\alpha = 1$, $\overline{\alpha} = 0.05$, $\beta^{(1)} = 0.7$, $\beta^{(2)} = 0.99$, $\lambda = 0.8$, $\overline{\beta} = 0.15$, and $\epsilon = 0.1$ and their parameters for the reactive AIMD are chosen identical. The available power varies in steps, where positive values indicate that the agents should consume power and negative values indicate that the vehicles should inject power. In Figure 4.15, the active and reactive aggregated power consumed by the agents is depicted. As can be seen, the agents are able to follow the available power and to balance the reactive power by uncontrollable loads.

Note that the parameters can be tuned equivalent to Sections 4.1.2 and 4.2.3 to allow to accommodate for the same scenarios, see Sections 2.2.1 and 2.3.1.
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Figure 4.14: Reactive AIMD algorithm in detail.
4.3. Active and Reactive Power Management

Figure 4.15: Evolution of the aggregated demand by the agents.
Chapter 5

Analysis of the AIMD Algorithm

In this chapter we mathematically analyse the AIMD algorithm proposed in Section 4.1. Properties of the algorithm such as its convergence and the aggregated demand of the agents are analysed. Our analysis will be based on the stochastic model described in [97]. After giving a short summary of the convergence results known from the literature, we concentrate on the analysis of the effects caused by differences between our implementation and those found in the literature. We expand the stability results that are available to be in line with our implementation of the AIMD algorithm. In the situations where we are unable to expand the results analytically, we investigate the implications on the agents and the aggregated demand using both mathematical reasoning and simulations. Afterwards, we investigate how the AIMD algorithm can be applied to solve constrained convex optimisation problems in a distributed manner. Finally, we investigate how the aggregated demand by the controllable agents behaves dependent on the AIMD parameters and the number of agents participating.

5.1 Convergence of AIMD

For our problem the most interesting property is the algorithm’s convergence if a constant number of EVs participate. Two important studies in this regard are [97], [115] which analyse the average behaviour of the algorithm. These results can be applied directly to our problem under some simplifying assumptions. The first assumption allows a continuous implementation of the algorithm.

\textbf{Assumption 5.1.} The AI phase of the algorithm is assumed to be continuous, hence slowly increasing the power consumption, while the MD phase is assumed to be instantaneous.

Let \( \tau_k \) be the time at which the \( k \)-th CE has been received and \( p_i(\tau_k) \) denote the power consumption of EV \( i \) at the \( k \)-th CE directly after the decrease. Also, let \( \beta_i(\tau_k) \) be such that

\[
\beta_i(\tau_k) = \begin{cases} 
\beta_i^{(1)}(\tau_k) & \text{with probability } \lambda_i(\tau_k) \\
\beta_i^{(2)}(\tau_k) & \text{with probability } 1 - \lambda_i(\tau_k). 
\end{cases}
\]
Then, we find the following equations for time $t$, with $\tau_k < t \leq \tau_{k+1}$, during the AI phase
\[ p_i(t) = p_i(\tau_k^+) + \alpha_i(\tau_k) \pi(t - \tau_k) \] (5.2)

where
\[ p_i(\tau_k^+) = \beta_i(\tau_k) p_i(\tau_k) \] (5.3)
is the state directly after the MD phase. Since the AI phase is monotonic, the $k$-th CE is triggered, when
\[ \sum_{i=1}^{N} p_i(\tau_k) = \bar{P}(\tau_k) - \bar{p}(\tau_k). \] (5.4)

We also assume that some values of the system remain constant during the AI phase. In particular, we take the available power, the aggregated demand of uncontrollable loads, and the AIMD parameters as constant, i.e.

**Assumption 5.2.** The available power and the aggregated demand of the uncontrollable loads are considered to be constant, i.e. $\bar{P}(k) = \bar{P}$ and $\bar{p}(k) = \bar{p}$.

and

**Assumption 5.3.** The additive factor, the multiplicative factors, and the probability to use the first multiplicative factor are constant in time, i.e. $\alpha_i(k) = \alpha_i$, $\beta^{(1)}_i(\tau_k) = \beta^{(1)}_i$, $\beta^{(2)}_i(\tau_k) = \beta^{(2)}_i$, and $\lambda_i(\tau_k) = \lambda_i$.

Further,

**Assumption 5.4.** The power consumption of a single agent is not limited, i.e. $\bar{p}_i = \infty$.

Note, that even with Assumption 5.4 the actual power consumption of the agents is indirectly limited by the available power.

To investigate the convergence and fairness [97], [115] study the system at CEs directly before the multiplicative decrease, i.e. the power consumption at times $\tau_k$. Let $p(\tau_k)$ be the vector containing the actual power consumption of the agents at the $k$-th CE. Further, let $\alpha$ be the vector containing the individual additive factors, and $\beta(\tau_k)$ be the vector containing the individual multiplicative factors of the agents at the $k$-th CE, i.e.
\[ \alpha = \begin{bmatrix} \alpha_1 & \cdots & \alpha_N \end{bmatrix}^T \] (5.5)
and
\[ \beta(\tau_k) = \begin{bmatrix} \beta_1(\tau_k) & \cdots & \beta_N(\tau_k) \end{bmatrix}^T \] (5.6)

Then, we find as in [97] the following relation.
Lemma 5.1. Let $\mathcal{P}$ be the set such that

$$\mathcal{P} = \{ p \in \mathbb{R}_+^N | 1^T p = \bar{p} \}$$

and assume that Assumptions 5.1 to 5.4 are fulfilled.

Then, the basic AIMD from one CE at $\tau_k$ to the next at $\tau_{k+1}$ maps a vector $p(\tau_k) \in \mathcal{P}$ into a vector $p(\tau_{k+1}) \in \mathcal{P}$ such that

$$p(\tau_{k+1}) = A(\tau_k) p(\tau_k),$$

where the matrix $A(\tau_k)$ is

$$A(\tau_k) = \text{diag}(\beta(\tau_k)) + (1^T \alpha)^{-1} \alpha (1 - \beta(\tau_k))^T.$$

The proof of Lemma 5.1 is identical to the one found in [97] and can be found in Appendix A.6.

Note that the power consumption of agent $f$ from one CE at $\tau_k$ to the next at $\tau_{k+1}$ evolves according to

$$p_f(\tau_{k+1}) = \beta_f(\tau_k) p_f(\tau_k) + \alpha_f \sum_{i=1}^{N} \alpha_i \sum_{i=1}^{N} (1 - \beta_i(\tau_k)) p_i(\tau_k).$$

The matrix $A(\tau_k)$ in Equation (5.9) depends on the probabilistic outcome at the $k$-th CE of each agent’s multiplicative factor. Let $\mathcal{I}(\tau_k)$ be the set of agents $i$ which use the factor $\beta^{(1)}_i(\tau_k)$ at time $\tau_k$. There are a total of $r = 2^N$ possible unique sets, i.e. $\mathcal{I}(\tau_k) \in \{\mathcal{I}_1, \ldots, \mathcal{I}_r\}$. Hence, the matrix $A(\tau_k)$ is taken from a set $\mathcal{A} = \{A_{\mathcal{I}_1}, \ldots, A_{\mathcal{I}_r}\}$ with probabilities

$$\tilde{\lambda}_{A_{\mathcal{I}_k}} = \Pr [A(\tau_k) = A_{\mathcal{I}_k}] = \prod_{i \in \mathcal{I}_k} (\lambda_i) \prod_{j \notin \mathcal{I}_k} (1 - \lambda_j).$$

The matrices in the set $\mathcal{A}$ are all non-negative, stochastic matrices, which posses useful properties that are used to prove convergence in [97], [115]. Matrices from this set are referred to as AIMD matrices. We denote the matrix $A_{\mathcal{I}_k}$ to be the matrix where all agents select the factor $\beta^{(1)}_i$. AIMD matrices have the useful property that they leave the subspace defined by

$$\mathcal{V} = \{ x \in \mathbb{R}^N | 1^T x = 0 \}$$

invariant, since they are all column stochastic matrices. AIMD matrices possess a useful contractive property, as shown in [115].

Lemma 5.2. Let $A_{\mathcal{I}_k}|\mathcal{V}$ be the restriction of $A_{\mathcal{I}_k}$ to the invariant subspace $\mathcal{V}$. Then, for all $A_{\mathcal{I}_k} \in \mathcal{A}$ it follows that

$$\|A_{\mathcal{I}_k}|\mathcal{V}\|_1 \leq 1.$$

Further for $A_{\mathcal{I}_1}$, there exists a constant $c \in (0, 1)$ such that

$$\|A_{\mathcal{I}_1}|\mathcal{V}\|_1 = c < 1.$$
5. Analysis of the AIMD Algorithm

**Theorem 5.3** ([97]). Consider the stochastic system defined in Lemma 5.1. Then, the expectation of \( \Pi(\tau_k) = A(\tau_k)A(\tau_{k-1}) \cdots A(\tau_0) \) is given by

\[
E[\Pi(\tau_k)] = E[A^{k+1} = (\sum_{\ell=1}^{r} \bar{\lambda} A_{\xi_{\ell}} A_{\xi_{\ell-1}} \cdots A_{\xi_0})]
\]

and the asymptotic behaviour of \( E[\Pi(\tau_k)] \) satisfies

\[
\lim_{k \to \infty} E[\Pi(\tau_k)] = x^* I^T
\]

where

\[
x^* = C \left[ \beta_1^{(1)} \alpha_1 \cdots \beta_N^{(1)} \alpha_1 \right]^T
\]

and the scalar factor \( C \in \mathbb{R} \) is such that \( I^T x^* = 1 \).

The proof of this theorem can be found in [97] and is repeated with minor changes to adapt to our system in Appendix A.7.

**Theorem 5.4** ([115]). Consider the average of the power consumption over time, evolving according to Lemma 5.1

\[
\rho_k = \frac{1}{k+1} \sum_{\ell=0}^{k} p(\tau_\ell) = \frac{1}{k+1} \sum_{\ell=0}^{k} \Pi(\tau_\ell) p(\tau_0).
\]

Then, the expected value of the time average is

\[
E[\rho_k] = \frac{1}{k+1} (I + E[A(\tau_0)]^1 + E[A(\tau_0)]^2 + \cdots + E[A(\tau_0)]^{k-1}) p(\tau_0)
\]

and the limit for \( k \to \infty \) is

\[
\lim_{k \to \infty} E[\rho_k] = x^*_p I^T p(\tau_0) = p^*
\]

where \( x^*_p \) is defined as in Theorem 5.3.

The proof of this theorem follows directly from Theorem 5.3 as stated in [115]. The above theorem shows the almost sure convergence of the algorithm to a point

\[
p^* = C \left[ \beta_1^{(1)} \alpha_1 \cdots \beta_N^{(1)} \alpha_1 \right]^T
\]

where the factor \( C \) is chosen such that \( I^T x^* = \bar{P} - \bar{p} \). Since this also implies a convergence result in probability, we find the following property. Its proof can be found in Appendix A.8.

**Lemma 5.5.** Consider the Markov chain as in Lemma 5.1. Then, for every \( \varepsilon, \delta > 0 \) there exists a \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \)

\[
\Pr \left[ \left\| \frac{1}{k+1} \sum_{\ell=0}^{k} \Pi(\tau_\ell) - x^*_p I^T \right\|_1 > \delta \right] < \varepsilon.
\]

In particular, there exists a \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \) and all \( p(\tau_0) \in \mathcal{P} \) we have

\[
\Pr \left[ \left\| \frac{1}{k+1} \sum_{\ell=0}^{k} p(\tau_\ell) - p^* \right\|_1 > \delta \right] < \varepsilon.
\]
5.1. Convergence of AIMD

The above analysis only applies if Assumptions 5.1 to 5.4 hold, which are not applicable for our system under consideration. In the following sections we show the subtle differences that occur when these assumptions are relaxed. Therefore, we expand the convergence results analytically. To illustrate our findings we use simulations performed in Matlab. In cases where this is not possible we analyse the impact both mathematically and using simulations.

5.1.1 Discrete Time AIMD

For the analysis above we assumed that the AIMD operates continuously, see Assumption 5.1. This is in contrast to the description of our algorithm and how we assume that the algorithm will be implemented. Naturally, the influence of the discretisation becomes smaller the shorter time steps are chosen. However, there are some important implications using discrete time steps that we want to investigate here using simulations. Afterwards, we will expand the convergence results for such discrete implementations.

The most immediate impact is that at a CE Equation (5.4) does not hold anymore, rather

\[
\sum_{i=1}^{N} p_i(\tau_k) \geq \bar{P}(\tau_k) - \bar{p}(\tau_k) \quad (5.24)
\]

and

\[
\sum_{i=1}^{N} p_i(\tau_k) < \bar{P}(\tau_k) - \bar{p}(\tau_k) + \sum_{i=1}^{N} \alpha_i. \quad (5.25)
\]

Note that this means that the actual constraint in Equation (2.1) is exceeded for short periods of time. However, by placing a security margin such that a CE is caused before the aggregated power reaches the available power this can be prevented. In this thesis, we assume that no problem is caused by the violation of Equation (2.1) as long as the period remains short and the overshoot small, i.e. the constraint Equation (2.1) is seen as a soft constraint. Naturally, smaller time steps lead to a behaviour that is closer to a continuous time situation. In the continuous time version the aggregated demand is exactly equal to the power that is available for the agents. We illustrate these considerations by simulating the discrete time, deterministic AIMD with two different time step lengths. The first time, the time step the length is 1 s, the second time it is 100 µs. In Figure 5.1 the aggregated demand by the agents is shown. As can be seen in the case where the longer time step length is used the aggregated demand overshoots the available power, while in the case with the shorter time step length the overshoot does not occur. Note that Figure 5.1 shows an excerpt from the complete simulation and that the overshoot occurs well after the start of the simulation.

In the following we will expand the convergence result of the discrete algorithm. Let Assumptions 5.2 to 5.4 hold. Further, we assume

Assumption 5.5. \( \lambda_i(\tau_k) = 1 \) for all \( i, \tau_k \).

Assumption 5.5 means that the AIMD algorithm is deterministic. Let \( \alpha, \beta \) be defined as in
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Equations (5.5) and (5.6), which are two constant vectors due to Assumptions 5.3 and 5.5. Then, we find the following system description for the discrete time AIMD.

Lemma 5.6. Assume that Assumptions 5.2 to 5.5 hold. Then, the discrete AIMD from one CE at $\tau_k$ to the next at $\tau_{k+1}$ can be modelled by

$$ p(\tau_{k+1}) = Ap(\tau_k) + \left(1^T\alpha\right)^{-1}(\zeta_2 - \zeta_1)\alpha, $$

(5.26)

where the matrix $A$ is

$$ A = \text{diag}(\beta) + \left(1^T\alpha\right)^{-1}\alpha(1 - \beta)^T $$

(5.27)

and where $\zeta_1$ and $\zeta_2$ are two scalars such that

$$ 0 \leq \zeta_1 < 1^T\alpha $$

(5.28)

and

$$ 0 \leq \zeta_2 < 1^T\alpha. $$

(5.29)

Further,

$$ \sum_{i=1}^{N} p_i(\tau_{k+1}) = \bar{P} - \bar{p} + \zeta_2 $$

(5.30)

and

$$ \sum_{i=1}^{N} p_i(\tau_k) = \bar{P} - \bar{p} + \zeta_1. $$

(5.31)

The proof of Lemma 5.6 can be found in Appendix A.9.

Figure 5.1: Aggregated demand for the discrete time, deterministic AIMD with different time step lengths.

![Figure 5.1: Aggregated demand for the discrete time, deterministic AIMD with different time step lengths.](image)
5.1. Convergence of AIMD

In the above we implicitly assumed that after at most one CE the power constraint is no longer violated. This means that the additive factors $\alpha$ are reasonably small compared to $\bar{P} - \bar{p}$ and the multiplicative factors $\beta$. Lemma 5.6 shows that the discrete time AIMD behaves like the continuous AIMD with a bounded disturbance $\zeta_2 - \zeta_1$. Using Lyapunov theory we are now able to show that the discrete AIMD converges to a region around an equilibrium point. First we state a result for a general disturbed system.

**Lemma 5.7.** Consider the system

$$p(\tau_{k+1}) = Ap(\tau_k) + \alpha(\gamma_2 - \gamma_1)$$

(5.32)

where $\gamma_1 \leq \gamma_1 \leq \bar{\gamma}_1$

(5.33)

$\gamma_2 \leq \gamma_2 \leq \bar{\gamma}_2$.

(5.34)

Further assume that

$$1^T p(\tau_{k+1}) = \bar{P} - \bar{p} + \gamma_2$$

(5.35)

and

$$1^T p(\tau_k) = \bar{P} - \bar{p} + \gamma_1.$$  

(5.36)

Then,

$$\|p(\tau_{k+1}) - p^*\|_1 \leq \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1 + \left( \max_i \left( \beta^{(1)}_i \right) + 1 \right) \|\gamma_1\| + \|\gamma_2 - \gamma_1\|,$$  

(5.37)

where $p^*$ is the Perron eigenvector of $A$ such that

$$1^T p^* = \bar{P} - \bar{p}.$$  

(5.38)

The proof of Lemma 5.7 is moved to Appendix A.10. Next, the following limits for the discrete AIMD are found.

**Lemma 5.8.** We consider the system from Lemma 5.6. Let Assumptions 5.2 to 5.5 hold. Then,

$$\lim_{k \to \infty} \|p(\tau_k) - p^*\|_1 \leq \frac{2 + \max_i \left( \beta^{(1)}_i \right) 1^T \alpha \pi}{1 - \max_i \left( \beta^{(1)}_i \right)}.$$  

(5.39)

**Proof.** The system in Lemma 5.6 can be rewritten in the form of the system in Lemma 5.7 where

$$\gamma_1 = \gamma_2 = 0$$

(5.40)

$$\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma} 1^T \alpha.$$  

(5.41)

Then, applying Lemma 5.7 we find that

$$\|p(\tau_{k+1}) - p^*\|_1 \leq \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1 + \left( \max_i \left( \beta^{(1)}_i \right) + 1 \right) \pi 1^T \alpha + \pi 1^T \alpha.$$  

(5.42)

Finally, this yields

$$\lim_{k \to \infty} \|p(\tau_k) - p^*\|_1 \leq \frac{2 + \max_i \left( \beta^{(1)}_i \right) 1^T \alpha \pi}{1 - \max_i \left( \beta^{(1)}_i \right)}.$$  

(5.43)
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Figure 5.2 illustrates these findings. As can be seen after a few CEs the error $||p(\tau_k) - p^*||_1$ drops below the upper bound found in Lemma 5.8. In this example, $\pi$ is set to 0.001. The individual additive and multiplicative factors are drawn randomly at the beginning of the simulation from uniform distributions, such that the additive factors lie between 0.5 and 1, and the multiplicative factors lie between 0.5 and 0.8. The maximum multiplicative decrease parameter is 0.798. Figure 5.2 shows not only that our bound is met in this particular case but also that the bound seems to be very conservative. Such a behaviour is expected since our result is based on a Lyapunov argument. However, the above shows only a particular case and the parameters might have an influence on the behaviour.

First, we investigate the dependency on the maximum multiplicative parameter, while the additive factors remain identical. Therefore, we simulate the system repeatedly where we choose the same additive factors as in the previous example while the maximum multiplicative factor is increased gradually from 0.1 up to 0.9. The other multiplicative factors are chosen randomly from a uniform distribution to lie between 0.05 and the maximum multiplicative factor. To show that those are not the most favourable cases for each value of the multiplicative factor the simulation is repeated ten times where the multiplicative factors are randomly generated anew. Then, the average of these ten simulations is computed. Note that also the agent which has the maximum multiplicative factor assigned is changed. The results of these simulation tests are shown in Figure 5.3. Note that in cases where there is no bar plotted the simulation ended before the number of CEs occurred. These simulation tests indicate that independent of the multiplicative factors the upper bound is very conservative. It also seems that the larger the multiplicative factor is chosen the slower is the convergence. This fits very well with the analytic result.

Figure 5.2: Evolution of the error $||p(\tau_k) - p^*||_1$ for the discrete time AIMD algorithm in one example with $\max_i(\beta(i)) = 0.798$. 

Figure 5.3: Evolution of the error $||p(\tau_k) - p^*||_1$.
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Figure 5.3: Mean error normalised to the upper bound for the discrete time AIMD algorithm for increasing values of $\max_i (\beta^{(1)}_i)$.
Secondly, we repeat the above simulation test, when the additive factors are adapted while the multiplicative factors remain identical for all simulations and are chosen as in the first example, i.e. they are selected between 0.5 and 0.8 with the maximum multiplicative decrease parameter being 0.798. The maximum additive factor $\bar{\alpha}$ increases from 0.005 to 0.05 in steps of 0.005. At each step ten simulations are repeated where each time the additive factors $\alpha_i$ are newly selected. These are drawn from a uniform distribution between 0.5 and 1. The mean error which is normalised to the upper bound for each simulation is plotted in Figure 5.4. As before the simulations indicate a very conservative bound where the error is after only a few CEs well within the bound. Since the analytic result is based on a Lyapunov argument, the conservativeness of the result is expected. The result can though be used as a guideline to select the parameters. Further, they guarantee a worst case behaviour.

![Graph showing mean error normalised to the upper bound for the discrete time AIMD algorithm for increasing values of $\bar{\alpha}$](image)

**Figure 5.4**: Mean error normalised to the upper bound for the discrete time AIMD algorithm for increasing values of $\bar{\alpha}$.

### 5.1.2 Varying Available Power and Aggregated Power Consumption of Uncontrollable Loads

While the convergence analysis explicitly assumes that both the available power $\bar{P}(k)$ and the aggregated power consumption of uncontrollable loads $\bar{p}(k)$ is constant, this is rarely the case in the load management problem. Even though in some cases the available power $\bar{P}(k)$ might be chosen constant, the consumption of the uncontrollable loads is often varying. In some cases, we can bound the variations of the available power, i.e.

**Assumption 5.6.** Let $\bar{P}, \bar{p}, \psi, \bar{\psi}$ be scalars. Then,

$$\bar{P}(k) = \bar{P} - \bar{p} + \psi(k),$$  \hspace{1cm} (5.44)
with
\[
\psi \leq \psi(k) \leq \bar{\psi}.
\] (5.45)

Note that Assumption 5.6 allows the available power and the aggregated demand of the uncontrollable loads to vary within the two bounds \(\psi\) and \(\bar{\psi}\). In most cases the bounds on the variation depend on the available power \(\bar{P}(k)\) and the expected demand variations caused by the uncontrollable loads \(\tilde{p}(k)\). However, we assume that for the period of the load management those bounds are constant, as seen in Assumption 5.6. The principle is depicted in Figure 5.5. Note that the relations of the curves do not depict proper size relation. We implicitly assume that these variations are small compared to \(\bar{P} - \tilde{p}\) such that after at most one CE the aggregated demand by the controllable agents is below \(\bar{P} - \tilde{p} + \psi\).

For this case the convergence analysis no longer holds as the system definition in Lemma 5.1 depends on the equality of the aggregated power consumption of all agents at CEs.

![Figure 5.5: Principle of the AIMD algorithm with varying available power and the aggregated power of uncontrollable loads.](image)

However, a useful system representation from one CE to the next can be found.

**Lemma 5.9.** Consider the AIMD algorithm and let Assumptions 5.1 and 5.3 to 5.6 hold. Then, the evolution of the power consumption from one CE at \(\tau_k\) to the next at \(\tau_{k+1}\) can be described by
\[
p(\tau_{k+1}) = Ap(\tau_k) + (I^T \alpha)^{-1} \alpha (\zeta - \psi(k))
\] (5.46)
where \(A\) is the AIMD matrix in Equation (5.9) and \(\zeta\) is a scalar such that
\[
\bar{\psi} \leq \zeta \leq \bar{\psi}.
\] (5.47)

The proof of Lemma 5.9 can be found in Appendix A.11. Then, we are able to show the following Lemma.

**Lemma 5.10.** Consider the system in Lemma 5.9 and let Assumptions 5.1 and 5.3 to 5.6 hold.
Further, let $p^*$ be the Perron eigenvector of $A$ such that

$$I^T p^* = \bar{P} - \bar{p}. \tag{5.48}$$

Then,

$$\lim_{k \to \infty} \| p(\tau_k) - p^* \|_1 \leq \frac{(1 + \max_i (\beta^{(1)}_i)) \max \left( \frac{\| \bar{\psi} \|}{\| \bar{\psi} \|}, \frac{\| \bar{\psi} \|}{\| \bar{\psi} \|} \right) + \bar{\psi} - \bar{\psi}}{1 - \max_i (\beta^{(1)}_i)}. \tag{5.49}$$

**Proof.** We apply Lemma 5.7, where

$$\gamma_1 = \gamma_2 = \psi \tag{5.50}$$

$$\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\psi}. \tag{5.51}$$

Then, we find

$$\| p(\tau_{k+1}) - p^* \|_1 \leq \max_i \left( \beta^{(1)}_i \right) \| p(\tau_k) - p^* \|_1 + \left( \max_i \left( \beta^{(1)}_i \right) + 1 \right) |\gamma_1| + |\gamma_2 - \gamma_1| \tag{5.52}$$

Using this we find the claim in Lemma 5.10. \square

This behaviour is illustrated in Figure 5.6. As can be seen after a few CEs the error

$$\| p(\tau_k) - p^* \|_1 \tag{5.53}$$

drops below the upper bound found in Lemma 5.10. Figure 5.6 shows an example simulation using the same values for the additive increase and multiplicative decrease factors as in the example for the discrete time system in Section 5.1.1.

### 5.1.3 Upper Bounds on the Power Consumption of each Agent

The convergence result so far assumed that the power consumption of an agent is limited by the actual available power and does not take into account that the outlet may itself impose a constraint, see Assumption 5.4. In this section we relax this assumption, while we keep Assumptions 5.1 to 5.3.

We first investigate the special case with $\lambda_i = 1$ for all $i$, i.e. Assumption 5.5 holds.

Note that Assumption 5.5 means that the AIMD algorithm is deterministic and all agents reduce their power consumption at every CE using the first multiplicative AIMD parameter $\beta^{(1)}_i$. Additionally, we introduce an upper limit on the power consumption of each agent which replaces Assumption 5.4.

**Assumption 5.7.** The individual agent’s power consumption $p_i(t)$ is limited by an upper bound $\bar{p}_i$, i.e.

$$p_i(t) \leq \bar{p}_i \quad \text{for all } i. \tag{5.54}$$

Since the power consumption of each agent is limited, at one point during the AI phase the power consumption might saturate and not continue to increase for some agents. To guarantee
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that a CE occurs at some point we assume that there is less power available than the maximum power that the participating agents are able to consume, i.e.

**Assumption 5.8.** \( \hat{P}, \tilde{P}, \) and the maximum power consumption of the agents \( \bar{p}_i \) are such that

\[
\sum_{i=1}^{N} \bar{p}_i \geq \hat{P} - \tilde{P} \tag{5.55}
\]

holds.

Let \( \mathbf{p}(\tau_k) \) be the vector containing the actual power consumption of the agents at the \( k \)-th CE and let \( \mathbf{x}(t) \) be the vector containing a fictional power consumption at time step \( t \), with \( \tau_k < t \leq \tau_{k+1} \), if we assume that the AI phase would not limit the actual power consumption, i.e.

\[
\mathbf{x}(t) = \mathbf{p}(\tau_k^+) + \alpha(\tau_k)(t - \tau_k), \tag{5.56}
\]

while the actual power consumption of the agents is bounded element-wise such that

\[
p_i(t) = \min(\bar{p}_i, x_i(t)). \tag{5.57}
\]

Further, let \( \mu(\tau_k) \) be the difference between the actual power consumption \( \mathbf{p}(\tau_k) \), that is saturated, and the fictional power consumption \( \mathbf{x}(\tau_k) \), i.e.

\[
\mathbf{x}(\tau_k) = \mathbf{p}(\tau_k) + \mu(\tau_k). \tag{5.58}
\]

Using the definition of the fictional power consumption \( x_i(\tau_k) \) and the difference \( \mu_i(\tau_k) \), we find the following fact.
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**Fact 5.11.** For every agent $i$ with $p_i(\tau_k) < \ov{p}_i$ it follows that

$$\mu_i(\tau_k) = 0$$

and

$$x_i(\tau_k) = p_i(\tau_k).$$

In this setting there are two slightly different variations of how the AIMD with saturated power consumption can be implemented. The first version, multiplicatively decreases directly the actual power consumption of the agent $p_i(\tau_k)$ at a CE. In this case, for time $t$ with $\tau_k < t \leq \tau_{k+1}$ the evolution of the power consumption can be written as

$$x_i(t) = p_i(t) + \mu_i(t) = \beta^{(1)}_i p_i(\tau_k) + \alpha_i (t - \tau_k).$$  \hfill (5.59)

The second version instead decreases the fictional power consumption $x_i(\tau_k)$ using the same multiplicative factor at a CE. Hence, at time $t$ with $\tau_k < t \leq \tau_{k+1}$ the power consumption is

$$x_i(t) = p_i(t) + \mu_i(t) = \beta^{(1)}_i x_i(\tau_k) + \alpha_i (t - \tau_k).$$  \hfill (5.60)

We will here investigate the first version and expand the convergence results. Therefore, we study the system at CEs.

**Lemma 5.12.** Let $\mathcal{P}$ be the set such that

$$\mathcal{P} = \{ p \in \mathbb{R}_+^N | I^T \ov{p} = \ov{P} - \ov{p} \text{ and } p_i \leq \ov{p}_i \}$$

and let $\mathcal{M}_p$ be a set that is associated with a vector $p \in \mathcal{P}$ such that

$$\mathcal{M}_p = \{ \mu \in \mathbb{R}_+^N | \mu_i = 0 \text{ whenever } p_i < \ov{p}_i \}.$$  \hfill (5.62)

Then, the AIMD algorithm, as defined in Equation (5.59), from one CE at $\tau_k$ to the next at $\tau_{k+1}$ maps a vector $p(\tau_k) \in \mathcal{P}$ into a vector $p(\tau_{k+1}) \in \mathcal{P}$ such that

$$p(\tau_{k+1}) = A p(\tau_k) + \left( (I^T \alpha)^{-1} \alpha I^T - I \right) \mu(\tau_{k+1}),$$

where the matrix $A$ is an AIMD matrix, i.e.

$$A = \text{diag}(\beta^{(1)}) + (I^T \alpha)^{-1} \alpha (I - \beta^{(1)})^T$$

and $\mu(\tau_{k+1}) \in \mathcal{M}_p(\tau_{k+1})$ is chosen appropriately.

**Proof.** The proof is analogous to the one found in [97]. Let $\Delta \tau_k$ be the time between the $k$-th and the $(k+1)$-th CE, then using Equation (5.59) we find the evolution from one CE to the next to be

$$p(\tau_{k+1}) + \mu(\tau_{k+1}) = \text{diag}(\beta) p(\tau_k) + \alpha \Delta \tau_k.$$  \hfill (5.65)
Using Equations (5.4) and (5.65) we can find $\Delta \tau_k$ by

$$1^T p(\tau_{k+1}) + 1^T \mu(\tau_{k+1}) = 1^T \text{diag}(\beta)p(\tau_k) + 1^T \alpha \Delta \tau_k$$

$$\Delta \tau_k = (1^T \alpha)^{-1} \left( (1^T - \beta^T)p(\tau_k) + 1^T \mu(\tau_{k+1}) \right). \quad (5.66)$$

When inserting Equation (5.66) into Equation (5.65) we find the claim in Lemma 5.12. \hfill \Box

Note that the power consumption of agent $f$ from one CE at $\tau_k$ to the next at $\tau_{k+1}$ evolves according to

$$p_f(\tau_{k+1}) = \beta^{(1)f} p_f(\tau_k) + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N (1 - \beta^{(1)i}) p_i(\tau_k) + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N \mu_i(\tau_{k+1}) - \mu_f(\tau_{k+1}) \quad (5.67)$$

For the system described in Lemma 5.12 we can find an important property of a fixed point.

**Lemma 5.13.** Let $p^*$ and $\mu^*$ be a fixed point of the system in Lemma 5.12, then

$$\frac{(1 - \beta^{(1)f}) p_f^* + \mu_f^*}{\alpha_f} = \frac{(1 - \beta^{(1)i}) p_i^* + \mu_i^*}{\alpha_i} \quad (5.68)$$

holds for any $f, i$ in $\{1, \ldots, N\}$.

**Proof.** Using Equation (5.67), we find that for any fixed point

$$p_f^* = \beta^{(1)f} p_f^* + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N (1 - \beta^{(1)i}) p_i^* + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N \mu_i^* - \mu_f^*$$

has to hold. Rearranging leads to

$$\frac{p_f^* - \beta^{(1)f} p_f^* + \mu_f^*}{\alpha_f} = \frac{1}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N (1 - \beta^{(1)i}) p_i^* + \frac{1}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N \mu_i^*. \quad (5.69)$$

As can be seen the right hand side of Equation (5.69) is independent of $f$, i.e. identical for $f = 1, \ldots, N$. Hence, the claim in Lemma 5.13 holds. \hfill \Box

We now establish an important property of the set $\mathcal{P}$ as defined in Lemma 5.12, which is later used to prove the existence of a unique fixed point and the convergence of the system.

**Lemma 5.14.** Assume $\rho^{(a)} \in \mathcal{P}$ and $\rho^{(b)} \in \mathcal{P}$ are two vectors such that $\rho^{(a)} \neq \rho^{(b)}$ and let $\mu^{(a)} \in \mathcal{M}_{\rho^{(a)}}$ and $\mu^{(b)} \in \mathcal{M}_{\rho^{(b)}}$, where $\mathcal{P}$ and $\mathcal{M}$ are defined as in Lemma 5.12. Then, the following holds

1. there exists an index $i$ such that $\rho^{(a)}_i > \rho^{(b)}_i$
2. there exists an index \( j \) such that \( \rho_j^{(a)} < \rho_j^{(b)} \)
3. \( \forall i \) if \( \rho_i^{(a)} > \rho_i^{(b)} \) then \( \mu_i^{(a)} \geq \mu_i^{(b)} = 0 \)

Proof. The first and second item follow directly from the definition of the set \( P \). If \( \rho_i^{(a)} > \rho_i^{(b)} \) we know that \( \rho_i^{(b)} \) is not saturated and therefore \( \mu_i^{(b)} \) is equal to 0. As \( \mu_i^{(a)} \geq 0 \), item 3 holds.

\[ p_i^* = \min \left( C \frac{\alpha_i}{1 - \beta^{(1)}}, \bar{p}_i \right), \text{ for all } i, \tag{5.70} \]

and difference

\[ \mu_i^* = C \alpha_i - \left(1 - \beta^{(1)}\right)p_i^*. \tag{5.71} \]

In the above \( C \) is a constant such that

\[ \mathbf{1}^T \mathbf{p}^* = \bar{P} - \bar{p} \tag{5.72} \]

holds.

The proof of Theorem 5.15 can be found in Appendix A.12. To find the constant \( C \) one can use an iterative method. We set

\[ C(0) = \frac{\bar{P} - \bar{p}}{\sum_{i=1}^{N} \frac{\alpha_i}{1 - \beta^{(1)}}}. \]

Then, we use the following recursion for calculating the constant \( C \). The power consumption is updated using

\[ p_i^*(k + 1) = \min \left( C(k) \frac{\alpha_i}{1 - \beta^{(1)}}, \bar{p}_i \right). \]

Then, the difference between the available power and the consumed power is

\[ d(k + 1) = \bar{P} - \bar{p} - \sum_{i=1}^{N} p_i^*(k + 1). \]

Using this we find the constant \( C(k + 1) \)

\[ C(k + 1) = C(k) + \frac{d(k + 1)}{\sum_{i \in \mathcal{G}} \frac{\alpha_i}{1 - \beta^{(1)}}}, \]

where the set \( \mathcal{G} \) is the set of all agents that are not yet saturated, i.e.

\[ \mathcal{G} = \{ i \in \{1, 2, \ldots, N\} | p_i^*(k + 1) < \bar{p}_i \}. \]

The recursion is stopped if the difference \( d(k) \) reaches a sufficiently small number.

Note, that in case no agent saturates during the AI phase, we find \( C = C(0) \) which leads to the fixed point being equal to the one found for the AIMD algorithm without an upper bound.

Next we are able to state a convergence result for the system in Lemma 5.12.
Theorem 5.16. The system in Lemma 5.12 converges to the unique fixed point found in Theorem 5.15 such that

$$\|p(\tau_k) - p^*\|_1 \leq \max_i \left(\beta^{(1)}_i\right)^k \|p(\tau_0) - p^*\|_1.$$  

(5.73)

The proof can be found in Appendix A.13. To verify these findings, a simulation has been performed for four agents with random values for $\alpha$ and $\beta^{(1)}$. Over the simulation period the available power increases in steps from 4 kW to 8 kW. Figure 5.7 shows the actual power consumption of those four agents and the fixed points calculated according to Equation (5.70). The upper bounds of the four agents are chosen to be 1 kW, 1.5 kW, 2.5 kW, and 3 kW, respectively.

![Figure 5.7: Simulation of the saturated AIMD with four connected agents.](image)

Note that Assumption 5.8 excludes some situations, when the available power is selected very large. Then, the agents might not be able to increase their rate further, since they are already saturated at their maximum power $\overline{p}_i$. This behaviour is illustrated in Figure 5.8. A total of 20 agents are simulated with identical AIMD parameters. Their maximum power consumption on the other hand is different for each vehicle. In the first part of the extract of the simulation the available power is large, such that all agents are able to consume their maximum power. In the second part the available power is reduced. Now, only agent 2 is limited by its maximum power consumption. The figure shows for comparison the power consumption of the same three agents if there is no upper bound in place. Note that in that case the power consumption of the agents is identical, as their AIMD parameters are identical.

Next we derive a relation between a quadratic optimisation problem and the system defined in Lemma 5.12. Its proof can be found in Appendix A.14.
5. Analysis of the AIMD Algorithm

![Graph showing power demand over time for three randomly selected agents with and without upper bounds.]

Figure 5.8: Demand by three randomly selected agents applying the AIMD algorithm with and without an upper bound on their charge rate.

**Lemma 5.17.** The algorithm in Lemma 5.12 solves the following constrained quadratic optimisation problem

$$
\arg\min_p \sum_{i=1}^{N} \frac{1 - \beta^{(1)}_i p_i^2}{2\alpha_i} \quad \text{s.t.} \quad p_i \leq \bar{p}_i \forall i
$$

$$
\sum_{i=1}^{N} p_i = \bar{P} - \tilde{p}.
$$

(5.74)

The above convergence analysis holds only for the deterministic case, i.e. Assumption 5.5 holds. We claim that similar results as in Theorems 5.15 and 5.16 hold for the stochastic case.

**Claim 5.18.** The stochastic AIMD algorithm with upper bounds approaches on average a unique point defined by

$$
p_i^* = \min \left( C \frac{\alpha_i}{1 - b_i} \bar{p}_i \right) \quad \text{for all } i,
$$

(5.75)

with

$$
b_i = \lambda_i \beta^{(1)}_i + (1 - \lambda_i) \beta^{(2)}_i.
$$

(5.76)

In the above $C$ is a constant such that

$$
1^T p^* = \bar{P} - \tilde{p}
$$

(5.77)
While we do not prove the above claim, we illustrate on simulations that the claim holds in some cases. Figure 5.9 shows two simulations of such a stochastic algorithm where the available power is adjusted. A total of four agents participate with identical upper bounds as in the previous simulation. The additive factors $\alpha_i$, the first multiplicative factors $\beta_i^{(1)}$, and the probabilities $\lambda_i$ are chosen randomly. The second multiplicative factor is set to 0.99.

![Figure 5.9: Filtered demand of four agents with an upper bound on their charge rate applying a stochastic AIDM algorithm.](image-url)
5. Analysis of the AIMD Algorithm

5.2 Optimisation with the AIMD Algorithm

In Section 5.1.3 we showed a connection between a constrained optimisation problem with quadratic cost terms and the AIMD algorithm in a deterministic setting with upper bounds. In this section we will show that in fact the AIMD algorithm can be used to solve constrained convex optimisation problems [116], [117]. While this was used in Section 4.1.2.4 to fulfil the GOF scenario, we here investigate a general optimisation problem of the form

$$\min_{p_1, \ldots, p_N} \sum_{i=1}^{N} g_i(p_i)$$

s. t. $p_i \geq 0$ for all $i$

$$\sum_{i=1}^{N} p_i = \bar{P} - \bar{p}. \quad (5.78)$$

We assume that the individual cost terms $g_i(p_i)$ are strictly convex and continuously differentiable. Note that such an optimisation is achieved for Equation (2.4), if we assume that the agents are able to draw more power than the constraint in Equation (2.1) allows, such that

$$O_1(k) = \bar{P} - \bar{p}. \quad (5.79)$$

Also note that we assume that the power consumption of an individual agent is not bounded, that $\bar{P} - \bar{p}$ is constant, and that the algorithm is continuously implemented, i.e. Assumptions 5.1, 5.2 and 5.4 hold. The basic idea is that the optimal point $\mathbf{p}^*$ is characterised by the Karush-Kuhn-Tucker (KKT) conditions. These are characterised for the above case by:

$$\sum_{i=1}^{N} p_i = \bar{P} - \bar{p} \quad \text{(5.80)}$$

$$\frac{\partial}{\partial p_i} g_i(p_i) - \mu_i = \nu \quad i = \{1, \ldots, N\} \quad \text{(5.81)}$$

$$\mu_i \geq 0 \quad i = \{1, \ldots, N\} \quad \text{(5.82)}$$

$$\mu_i p_i = 0, \text{ and} \quad \text{(5.83)}$$

$$p_i \geq 0 \quad i = \{1, \ldots, N\}. \quad \text{(5.84)}$$

Due to the compactness of the feasible space and the strict convexity of the individual cost terms such an optimal point exists and is unique. Note that we assume here that the optimum is reached when the power consumption of the agents is larger than 0. Note that in that case the Lagrange multiplier $\mu_i$ is identical to 0 for all $i$. Then, the AIMD algorithm is adapted such that the probability $\lambda_i$ is adapted at each CE depending on the average power consumption $\rho_i$ of agent $i$, while all other AIMD parameters, i.e. $\alpha_i, \pi_i, \beta^{(1)}$, and $\beta^{(2)}$, are identical for all agents. Hence, the probability $\lambda_i$ is a function of $\rho_i$ such that

$$\lambda_i(\rho_i) = \Gamma \frac{\partial}{\partial p_i} g_i(p_i) \bigg|_{\rho_i}, \quad \text{(5.85)}$$

where $\frac{\partial}{\partial p_i} g_i(p_i) \bigg|_{\rho_i}$ denotes the partial derivative of the individual cost function evaluated at the average power consumption $\rho_i$ and $\Gamma$ is a constant positive scalar identical for all agents. The
average power consumption $\rho$ can be either computed upon connection

$$\rho_i(\tau_k) = \frac{1}{k + 1} \sum_{l=0}^{k} p_i(\tau_l)$$

(5.86)

or over a finite number $T$ of past CEs

$$\rho_i(\tau_k, T) = \frac{1}{T} \sum_{l=k-T+1}^{k} p_i(\tau_l).$$

(5.87)

Note that unlike Equations (2.6) and (2.7), we only average over the power consumption at CEs.

Note that since the function in Equation (5.85) is a probability, it is implicitly assumed that there exists a $\Gamma$ such that the probability remains in the range $[0, 1]$. This naturally restricts the allowed individual cost function terms. It is however possible to also treat more general functions. One important case is if the probability will go to infinity as $\rho_i$ goes to 0. In this case it is possible to saturate the probability at 1. If a small enough $\Gamma$ is chosen this will still find the optimal point.

5.2.1 Convergence Analysis

In the following, we analyse the convergence of the adapted AIMD algorithm. Therefore, we assume that Assumptions 5.1, 5.2 and 5.4 hold. Further, let $\beta^{(2)} = 1$ and $\bar{\pi} = 1$. As mentioned previously, $\rho$ can be either defined as a finite average as in Equation (5.87) or as long term average as in Equation (5.86). The analysis of the convergence uses different methods for the two approaches. In the first case, the problem can be reformulated as a homogeneous Markov chain with state-dependent probabilities, sometimes also called an iterated function system (IFS). In this setting existing results ensure the existence of an attractive invariant measure and ergodicity results follow [15], [31], [98]. In this case however the optimum point, i.e. the point that solves the optimisation problem, is not reached, though as the window length $T$ increases the result approaches the optimum. The more interesting case is the second one, where the long term average is used. This case gives rise to a non-homogeneous Markov chain for which the powerful methods that exist for the first case are not available. The method to show convergence relies in this case on a detailed analysis of the system dynamics using Lyapunov functions.

Further we assume the following for the analysis

**Assumption 5.9.** The probabilities $\lambda_i$ of the AIMD algorithm as defined in Section 5.1 are functions such that

$$\lambda_i : [0, \bar{P} - \tilde{p}] \to [0, 1], \quad i = 1, \ldots, N,$$

(5.88)

and satisfy

1. $\lambda_i$ is continuous, $i = 1, \ldots, N$;
2. $r \mapsto r \lambda_i(r)$ is strictly increasing on $[0, \bar{P} - \tilde{p}], i = 1, \ldots, N$;
3. There exists a constant $\lambda_{\min}$ such that $\lambda_i(r) \geq \lambda_{\min} > 0$ for all $r \in [0, \bar{P} - \tilde{p}], i = 1, \ldots, N$;
Note that the first two points hold if the probability is chosen as in Equation (5.85), due to the convexity and continuously differentiable individual costs.

Investigating the system from one CE to the next and using the definitions in Section 5.1, we find that the system can be described as in Lemma 5.1 by the switched system from Equations (5.8) and (5.9), i.e.

\[
p(\tau_{k+1}) = A(\tau_k)p(\tau_k) \tag{5.89}
\]

with \( A(\tau_k) \in \mathcal{A} \). Let, as in Section 5.1, \( I(\tau_k) \) be the set of agents \( i \) which use the factor \( \beta_i^{(1)} \) at time \( \tau_k \). Note that there are in total \( r = 2^N \) such sets. Further, let \( A_{I_1} \) denote the matrix where all agents decrease with factor \( \beta_i^{(1)} \). The probability \( \lambda_{A_{I_1}} \) depends on the probabilities as defined in Equation (5.85) and so the history of the sample path, i.e. the actual realisations of the AIMD matrices. We assume as before that the agents react independently, such that the probability \( \lambda_{A_{I_1}} \) can still be computed using Equation (5.11).

### 5.2.1.1 Finite Average

When using Equation (5.87) to compute the average power consumption that is used in Equation (5.85), the probability \( \lambda_{A_{I_1}} \) needs to be interpreted along sample paths. This means for each realisation of the Markov chain in Equation (5.8), the probabilities at time step \( \tau_k \) are a function of the average over the time interval \([\tau_{k-T+1}, \ldots, \tau_k]\) for the given realisation. By augmenting the state this system can be modelled as a Markov chain with state-dependent probabilities. We define the new variable

\[
z(\tau_k) = \left[ p(\tau_k) \quad \frac{1}{2} (p(\tau_k) + p(\tau_{k-1})) \quad \ldots \quad \frac{1}{T} (p(\tau_k) + \ldots + p(\tau_{k-T+1})) \right]^T. \tag{5.90}
\]

Note that \( z(\tau_k) \) lies in the set

\[
\mathcal{P} = \left\{ z \in \mathbb{R}_+^{(1+1)N} \left| \sum_{i=1+tN}^{(1+1)N} z_i = \bar{P} - \bar{p} \right. \text{ for } t = 0, T-1 \right\}. \tag{5.91}
\]

Then, the evolution from one CE to the next of \( z(\tau_k) \) is described by the Markov chain

\[
z(\tau_{k+1}) = \begin{bmatrix}
A(\tau_k) & 0 & \ldots & 0 \\
\frac{1}{2} (A(\tau_k) + \mathbf{I}) & 0 \\
\frac{1}{3} A(\tau_k) & \frac{2}{3} \mathbf{I} & 0 & \vdots \\
\vdots & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 0 \\
\frac{1}{T} A(\tau_k) & 0 & \ldots & 0 & \frac{T-1}{T} \mathbf{I} & 0 \\
\end{bmatrix} z(\tau_k) = A_T(\tau_k) z(\tau_k). \tag{5.92}
\]

Each matrix in \( \mathcal{A} \) defines a matrix \( A_T \), where \( A_{T,j} \) is computed by inserting \( A_j \) in Equation (5.92) for \( A(\tau_k) \). The set of possible matrices \( A_T \) occurring in Equation (5.92) is defined in this way, denoted in the following by \( \mathcal{A}_T \). The probabilities that a specific matrix \( A_{T,j} \) is selected at time step \( \tau_k \) can be computed using the \( T \)-th component vector of \( z(\tau_k) \) analogue to Equation (5.11). We denote in the following the \( i \)-th component vector of \( z(\tau_k) \) by \( z_i(\tau_k) \in \mathcal{P} \). The
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The following norm on \( \mathbb{R}^{TN} \) simplifies the analysis of the Markov chain considerably, as it reveals the contractive properties of the Markov chain. We define

\[
\| z \|_{H_T} = \max_{i=1,\ldots,T} ||z_i||_1,
\]

where \( z = [z_1^T \ldots z_T^T]^T, z_i \in \mathbb{R}^N, i = 1,\ldots,T. \) Next, we state that the linear system with matrices \( A_{T,j} \) where these matrices are chosen randomly possesses contractive properties with respect to the above norm.

**Lemma 5.19.** The following statements hold:

(i) For all \( A_{T,j} \in \mathcal{A}_T \) the matrix norm induced by \( \| \cdot \|_{H_T} \) satisfies

\[
\| A_{T,j} \|_{H_T} \leq 1.
\]

(ii) The subspace

\[
W = \{ z \in \mathbb{R}^{TN} | f^T z_i = 0, \forall i = 1,\ldots,T \}
\]

is invariant under all \( A_{T,j} \in \mathcal{A}_T. \)

(iii) For all \( z \in W, A_{T,j} \in \mathcal{A}_T \) it holds that

\[
\| A_{T,j} z \|_{H_T} = \| z \|_{H_T} \Rightarrow A_j z_1 = z_1.
\]

In particular, we have

\[
\| A_{T,1} | W \|_{H_T} \leq \frac{c + T - 1}{T} < 1,
\]

where \( c < 1 \) is the constant given by Lemma 5.2.

The proof of the above properties can be found in Appendix A.15. These properties are very important for proving the existence of a unique invariant and attractive measure for the Markov chain. Before proving this we need an assumption on the probability functions \( \lambda_i \) that guarantees strong contractivity on average.

**Theorem 5.20.** Assume that the probability functions \( \lambda_i \) satisfy Assumption 5.9 and are Lipschitz continuous. Then, for all \( T \geq 1 \), there exists a unique invariant and attractive measure \( \pi^T \) on \( \mathcal{P}^T. \) Furthermore, for all \( z(\tau_0) \in \mathcal{P}^T, \) we have that almost surely

\[
\lim_{k \to \infty} \frac{1}{k+1} \sum_{\ell=0}^k z(\tau_\ell) = \int_{\mathcal{P}^T} z \ d\pi^T(z) = E[\pi^T].
\]

**Proof.** It is easy to show that the sufficient conditions provided in [15] are satisfied. In particular, these conditions can be met by requiring that

\[
\sup_{z,w \in \mathcal{P}} \sum_{j=1}^{2^N} \Pr[A_T = A_{T,j} | z] \frac{||A_{T,j}(z - w)||_{H_T}}{||z - w||_{H_T}} < 1.
\]
Note that \( z - w \in W \). Hence, due to item (iii) we find
\[
||A_{T,1}(z - w)||_{H_T} \leq \frac{(c + T - 1)}{T} ||z - w||_{H_T} < ||z - w||_{H_T}.
\]
(5.99)

Then, item (i) in Lemma 5.19 implies that the sum does not exceed 1. Also, Assumption 5.9 ensures that for each \( z \in P \) the probability \( \Pr[A_T = A_{T,1} | z] \) is larger equal to \( \lambda_{\text{min}}^N > 0 \). Thus, the probability that matrix \( A_{T,1} \) occurs is bounded away from zero. Hence, the supremum in Equation (5.98) is bounded away from 1.

By Theorem 2.1 in [15] the existence of an attractive invariant measure follows. Uniqueness is then a consequence of attractivity. The ergodic property in Equation (5.97) now follows from [31]. \( \square \)

The above result shows that the AIMD system in which the probabilities are adapted according to Equation (5.85) using the finite average in Equation (5.87) is indeed converging in a strong sense. In particular, long term averages converge almost surely. While the limit is not exactly the desired KKT point that fulfills the conditions in Equation (5.80), simulations suggest that the limit gets closer as \( T \) increases. Also for large \( T \), with high probability along a sample path, the average of the windows of size \( T \) is close to the desired KKT point.

5.2.1.2 Long Term Average

Next we investigate the situation where the probabilities are adapted using the long-term average in Equation (5.86). Note that the resulting system in Equation (5.89) does not define a Markov chain on \( P \), since the probabilities do not depend on the current state \( p(\tau_k) \) but rather on the complete history of a sample path. To obtain a formulation as a Markov chain, the state is augmented to include the average. We introduce the variable
\[
z(\tau_k) = \begin{bmatrix} p(\tau_k) \\ \rho(\tau_k) \end{bmatrix}
\]
(5.100)
where \( \rho(\tau_k) \) is defined as in Equation (5.86). From the definition of \( \rho(\tau_k) \) we find that
\[
\rho(\tau_{k+1}) = \frac{1}{k+2} p(\tau_{k+1}) + \frac{k+1}{k+2} \rho(\tau_k).
\]
(5.101)

Hence, the AIMD algorithm from one CE to the next can be described in the new state according to
\[
z(\tau_{k+1}) = \tilde{A}(k) z(\tau_k),
\]
(5.102)
where
\[
\tilde{A}(\tau_k) = \begin{bmatrix} A(\tau_k) & 0 \\ \frac{1}{k+2} A(\tau_k) & \frac{k+1}{k+2} \mathbf{1} \end{bmatrix}.
\]
(5.103)

Note that \( \tilde{A}(\tau_k) \) is dependent on the selected matrix \( A(\tau_k) \in A \) and \( k \) itself. Given \( A \in A \), we introduce
\[
\tilde{A}_A(\tau_k) = \begin{bmatrix} A & \frac{1}{k+2} A \\ \frac{1}{k+2} A & \frac{k+1}{k+2} \mathbf{1} \end{bmatrix}.
\]
(5.104)
Further, the set
\[ \tilde{A} = \{ \tilde{A}_A(\tau_k) | A \in \mathcal{A}, k \in \mathbb{N} \} \]  
(5.105)
is defined. This set contains all possible matrices appearing in Equation (5.102). Then, the probability to select a specific matrix \( \tilde{A}_A(\tau_k) \) given the state \( z(\tau_k) \) can be computed using the second component of \( z(\tau_k) \), i.e. \( \rho(\tau_k) \) according to Equation (5.11). We denote
\[ \Pr [ \tilde{A}(\tau_k) = \tilde{A}_A(\tau_k) | z(\tau_k) = y ] = \gamma_A(y) \]  
(5.106)
for \( y \in \mathcal{P} \).

This defines a non-homogeneous Markov chain with place dependent probabilities. Note that the non-homogeneity comes from the time-varying nature of the matrices \( \tilde{A}_A(\tau_k) \), whereas the functions \( \gamma_A(\cdot) \) describing the place-dependent probabilities do not depend on time.

Similar as for the finite average case, we intent to obtain contractive properties of the Markov chain in Equation (5.102). A special norm on \( \mathbb{R}^{2N} \) is defined analogous to Equation (5.93), i.e. for \( x \in \mathbb{R}^{2N} \) we define the norm
\[ ||x||_{H_2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \max \left( ||x_1||_1, ||x_2||_1 \right), \]  
(5.107)
where \( x_1, x_2 \in \mathbb{R}^N \) are the two components of \( x \). The matrix norm induced by this norm on \( \mathbb{R}^{2n \times 2n} \) is also denoted by \( ||\cdot||_{H_2} \). Then, the following useful properties for the matrices in \( \tilde{A} \) can be found.

**Lemma 5.21.** The following statements hold:

(i) For all \( \tilde{A}_A \in \tilde{A} \)
\[ ||\tilde{A}_A||_{H_2} \leq 1. \]  
(5.108)

(ii) The subspace
\[ W = \{ x \in \mathbb{R}^{2N} | 1^T x_1 = 1^T x_2 = 0 \} \]  
(5.109)
is invariant under all \( \tilde{A}_A \in \tilde{A} \).

**Proof.** For any \( x \) and \( \tilde{A}_A \in \tilde{A} \) it follows that
\[ ||\tilde{A}_A x||_{H_2} = \max \left( ||A x_1||_1, \frac{1}{k+2} ||A x_1 + \frac{k+1}{k+2} x_2||_1 \right) \leq \max \left( ||x_1||_1, \frac{1}{k+2} ||x_1||_1 + \frac{k+1}{k+2} ||x_2||_1 \right) \leq ||x||_{H_2}. \]  
(5.110)

Hence, the induced matrix norm is smaller equal to 1. For the second part note that \( 1^T A = 1^T \) for all \( A \in \mathcal{A} \). Therefore, for the first component of \( \tilde{A}_A x \) it follows that
\[ 1^T A x_1 = 1^T x_1 = 0. \]  
(5.111)
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Similarly, for the second component, we have
\[
1^T \left( \frac{1}{k+2} A x_1 + \frac{k+1}{k+2} x_2 \right) = \frac{1}{k+2} 1^T x_1 + \frac{k+1}{k+2} 1^T x_2 = 0. \tag{5.112}
\]

Hence, Lemma 5.21 holds.

To obtain a convergence result we require some intermediate steps. To understand the reasoning for them, we first briefly outline the method how we are going to show almost sure convergence of this adapted AIMD algorithm.

The key intuition is that in the long run the dynamics of the long term average \( \rho \) become slow. In other words, for large \( T \) and relatively short intervals of length \( m \) of the form \( \{T + 1, \ldots, T + m\} \), \( \rho \) is almost a constant, where \( m \) is to be understood to be small compared to \( T \). The reason for this is the relation
\[
\rho(\tau_{T+m}) = \frac{T}{T+m} \rho(\tau_T) + \frac{m}{T+m} \left( \frac{1}{m} \sum_{\ell=1}^{m} p(\tau_{T+\ell}) \right), \tag{5.113}
\]
which holds along any sample path. Note that the second term tends to 0 if \( T \) is large.

If \( \rho \) is almost a constant on a certain interval, then the probabilities for choosing the matrices \( A_j \) are almost constant, and we can approximate the dynamics using the results on AIMD with constant probabilities. This means that Lemma 5.5 becomes relevant. This result means in our case that, provided that \( m \) is large enough, the average over the next \( m \) steps is close to the expectation of the AIMD Markov chain with constant probabilities. This holds for all initial states \( p(\tau_T) \) and with a high probability.

While this basic intuition turns out to be true, we need to resolve the fact that the ergodic limit of the “fixed-probability system” depends on \( T \). Specifically, \( m \) and \( T \) depend on each other, such that it is important to understand this relationship.

To resolve this, we use the following interpretation of Equation (5.113). For \( p \in \mathcal{P} \), we denote by \( P(p) \) the expectation of the invariant measure of the AIMD algorithm with the fixed probabilities \( \lambda_i(p_i), i = 1, \ldots, N \), which is defined by Theorems 5.3 and 5.4 as
\[
P(p) = p^* = C \left[ \frac{\alpha_1}{\lambda_1(p_1)(1-\beta_1^{[1]})} \ldots \frac{\alpha_N}{\lambda_N(p_N)(1-\beta_N^{[N]})} \right]^T. \tag{5.114}
\]
Note that \( C \) is a constant to assure that \( P(p) \in \mathcal{P} \).

We then rewrite Equation (5.113) as
\[
\rho(\tau_{T+m}) = \frac{T}{T+m} \rho(\tau_T) + \frac{m}{T+m} \left( P(\rho(\tau_T)) + \Delta(\tau_T) \right), \tag{5.115}
\]
where we interpret \( \Delta(\tau_T) \) as a suitable perturbation term, that aggregates the effect that the probabilities are not precisely constant on \( \{T + 1, \ldots, T + m\} \), and further the effect that we are not at the expectation but only close to it.
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To understand the dynamics in Equation (5.115) we study the system

\[
\rho(\tau_{T+m}) = \frac{T}{T+m} \rho(\tau_T) + \frac{m}{T+m} P(\rho(\tau_T))
\]

(5.116)

and interpret the system in Equation (5.115) as a perturbed version thereof. First, in the next subsection, the system in Equation (5.116) and its perturbed version in Equation (5.115) is investigated. Then, we use the found results to establish almost sure convergence.

Analysis of the Systems in Equations (5.115) and (5.116) In this section a deterministic system defined by successive convex combinations of a point in \( P \) with the expectation of this point \( p^* \) as defined in Equation (5.114) is studied. Note that the definition in Equation (5.114) is based on the result in Theorem 5.4. Since we assume for the adapted AIMD that the parameters \( \alpha_i \) and \( \beta^{(1)}_i \) are constant and identical for all agents, and \( \beta^{(2)}_i = 1 \) for all agents, the expectation of the invariant measure \( P(p) = p^* \), see Equation (5.114), describes a function that maps a vector \( x \in P \) to a vector in \( P \) such that

\[
P(x) = C \left[ \frac{1}{\lambda_1} \cdots \frac{1}{\lambda_N} \right]^T
\]

(5.117)

where \( C \) is a constant such that \( P(x) \) lies in \( P \) and is computed by

\[
C = \frac{P - \bar{p}}{\sum_{i=1}^N \frac{1}{\lambda_i}}.
\]

(5.118)

Let \( \text{ri} \ P \) denote the relative interior of \( P \), i.e. \( \text{ri} \ P = \{ x \in P | x_i > 0 \text{ for } i = 1, \ldots, N \} \). Further, we denote the convex hull of a set \( X \) with \( \text{conv} X \), which can be defined as the smallest convex set containing \( X \). Then, \( P(P) \subset \text{ri} \ P \) is compact due to the third point in Assumption 5.9. This means that we can chose a constant \( \delta^- > 0 \) such that

\[
P(P) + B_1(0, \delta^-) \subset \text{conv} P(P) + B_1(0, 2\delta^-) \subset \text{ri} \ P,
\]

(5.119)

where \( B_1(0, \delta) \) denotes the closed ball of radius \( \delta \) around 0 with respect to the \( l_1 \) norm. Above and in the following both scaling and sum operations of sets are in the sense of Minkowski sums. The factor 2 in the above is chosen arbitrarily. It is solely required that this factor exceeds 1. Further, we define the constant

\[
\delta^+ = \max_{y \in \text{ri} \ P} (\text{dist}_1 (y, P(P)) ),
\]

(5.120)

where \( \text{dist}_1 (x, X) = \min(||x - z||_1 ; z \in X) \) is the distance of a point \( x \) to a set \( X \) with respect to the \( l_1 \) norm. In the following we investigate successive convex combinations of the state \( x \in P \) and \( P(x) \). Hence, for a sequence \( \{ \varepsilon_k \}_{k \in \mathbb{N}} \subset (0, 1) \), the system

\[
x(\tau_{k+1}) = (1 - \varepsilon_k)x(\tau_k) + \varepsilon_k P(x(\tau_k))
\]

(5.121)

is considered. We note the following simple properties of this iteration concerning a fixed point.

**Lemma 5.22.** Assume that Assumption 5.9 holds and let \( N \geq 2 \). Then, \( P(\cdot) \) has the following properties.

1. \( P \) has a fixed point \( x^* \in P \) such that \( P(x^*) = x^* \).
5. Analysis of the AIMD Algorithm

2. The fixed point \( x^* \) is unique and has the following property

\[
\gamma_F = x_i^* \lambda_i(x_i^*) = x_j^* \lambda_j(x_j^*) \quad \forall i, j \in \{1, \ldots, N\}.
\]

(5.122)

3. For every \( \varepsilon \in (0, 1] \) the fixed point \( x^* \) defined in item 2 is the unique fixed point of

\[
x \rightarrow (1 - \varepsilon)x + \varepsilon P(x).
\]

(5.123)

4. For every \( x_0 \in P \) and every sequence \( \{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, 1) \) the solution of Equation (5.121) \( x(\tau_k) \) satisfies \( x_i(\tau_k) > 0 \) for all \( i \) and \( k \geq 1 \).

The proof of the above lemma can be found in Appendix A.16. To simplify the notation, we introduce for \( \varepsilon \in [0, 1] \) the map \( R_\varepsilon : P \to P \) by

\[
R_\varepsilon(x) := (1 - \varepsilon)x + \varepsilon P(x).
\]

(5.124)

Because of Lemma 5.22, the fixed point \( x^* \) of \( P \) is also the unique fixed point of \( R_\varepsilon \) for all \( \varepsilon \in (0, 1] \).

In our analysis of the dynamics we require two types of contractive properties of the map \( R_\varepsilon \) in combination with robustness results. We will also consider set-valued maps of the form

\[
\Psi^\delta_\varepsilon(x) = R_\varepsilon(x) + \varepsilon B_1(0, \delta) = (1 - \varepsilon)x + \varepsilon (P(x) + B_1(0, \delta)),
\]

(5.125)

where \( 0 < \delta < \delta^* \). Note that by definition of \( \delta^* \) this ensures that \( \Psi^\delta_\varepsilon(x) \subset \text{ri} P \).

Next, we analyse properties of the map \( \Psi^\delta_\varepsilon \) by studying individual elements in its image. Let \( P_{\text{co}} \) be the convex set such that

\[
P_{\text{co}}(\delta) = \text{conv} P(P) + B_1(0, \delta).
\]

(5.126)

**Lemma 5.23.** Let \( x \in P \) and \( \delta^* \) as defined in Equation (5.119). Then for all \( 0 < \varepsilon \leq 1 \):

1. For all \( 0 < \delta < \delta^* \) and \( \Delta \in \mathbb{R}^N \) with \( I^T \Delta = 0 \) and \( \|\Delta\|_1 \leq \delta \) we have

\[
\text{dist}_1(R_\varepsilon(x) + \varepsilon \Delta, P_{\text{co}}(\delta)) \leq (1 - \varepsilon)\text{dist}_1(x, P_{\text{co}}(\delta)).
\]

(5.127)

2. With \( \delta^+ \) as defined in Equation (5.120), for all \( 0 < \delta < \delta^* \) and all \( y \in P \), we have

\[
\text{dist}_1((1 - \varepsilon)x + \varepsilon y, P_{\text{co}}(\delta)) \leq (1 - \varepsilon)\text{dist}_1(x, P_{\text{co}}(\delta)) + \varepsilon \delta^+.
\]

(5.128)

3. For every \( 0 < \delta < \delta^* \) there exists a \( C_\delta > 0 \) such that for all \( 0 < \varepsilon < 1 \) and all \( 0 < \delta < \delta^* \) we have the following implication: If \( x \in P \) satisfies \( \text{dist}_1(x, P_{\text{co}}(\delta)) > \delta \) and \( y \in P \), then

\[
\text{dist}_1((1 - \varepsilon)x + \varepsilon y, P_{\text{co}}(\delta)) \leq (1 + C_\delta \varepsilon)\text{dist}_1(x, P_{\text{co}}(\delta)).
\]

(5.129)
5.2. Optimisation with the AIMD Algorithm

Proof. To show item 1, we define $z \in P_{co}(\delta)$ such that

$$||x - z||_1 = \text{dist}_1 (x, P_{co}(\delta)).$$

(5.130)

Then, by convexity $(1 - \varepsilon)z + \varepsilon(P(x) + \Delta) \in P_{co}(\delta)$ and so

$$\text{dist}_1 (R_c(x) + \varepsilon \Delta, P_{co}(\delta)) \leq ||(R_c(x) + \varepsilon \Delta) - ((1 - \varepsilon)z + \varepsilon(P(x) + \Delta))||_1 = (1 - \varepsilon) ||x - z||_1.$$

(5.131)

Note that for any convex set $C$, every convex combination remains in $C$, i.e. $C = (1 - \varepsilon)C + \varepsilon C$. Then, using the definition of $\delta^+$, we find that

$$\text{dist}_1 ((1 - \varepsilon)x + \varepsilon y, P_{co}(\delta)) = (1 - \varepsilon)\text{dist}_1 (x, P_{co}(\delta)) + \varepsilon \text{dist}_1 (y, P_{co}(\delta))$$

(5.132)

and item 2 holds.

Finally, with the assumption $\text{dist}_1 (x, P_{co}(\delta)) > \bar{\delta}$ it follows that

$$\delta^+ \leq \frac{\text{dist}_1 (x, P_{co}(\delta))}{\delta}.$$

(5.133)

So item 3 follows from Equation (5.128) with an appropriate choice of $C_\delta > 0$. \qed

The above lemma describes three important features of the iteration

$$x(\tau_{k+1}) \in \Psi^\delta_k(x(\tau_k)).$$

(5.134)

Firstly, the iteration converges with rate $(1 - \varepsilon)$ to the set $P_{co}$, due to item 1 in Lemma 5.23. Secondly, by item 2 in Lemma 5.23 the increase of the distance to the convex set can be bounded, if the iteration is perturbed so that there is a convex combination with some $y \in \mathcal{P}$. Finally, the error induced by such a perturbation $y$ can be linearly bounded in $\varepsilon$, provided that we are sufficiently far way from $P_{co}$.

Next, we establish similar properties close to the fixed point $x^*$. We use for this the Hilbert metric on $ri\mathcal{P}$ which is given by

$$\text{dist}_H (x, y) = \max_i \log \left( \frac{x_i}{y_i} \right) - \min_j \log \left( \frac{x_j}{y_j} \right)$$

(5.135)

for two vectors $x, y \in ri\mathcal{P}$ and let

$$e^{d_H} (x, y) = \frac{\max_i \frac{x_i}{y_i}}{\min_j \frac{x_j}{y_j}}$$

(5.136)

for vectors $x, y \in ri\mathcal{P}$. Note that the latter is closely linked to the Hilbert metric. In particular, for $||x_k - y||_1 \to 0$ with increasing $k$ holds if and only if $\text{dist}_H (x_k, y) \to 0$ which is equivalent to $e^{d_H} (x_k, y) \to 1$. Further, note that $\text{dist}_H (x, y) < \text{dist}_H (x, z)$ holds if and only if $e^{d_H} (x, y) < e^{d_H} (x, z)$. The ball around $x$ with radius $\delta$ with respect to the Hilbert metric is then denoted as $B_H (x, \delta)$. As before we assume implicitly that the ball intersects $\mathcal{P}$. 113
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**Theorem 5.24.** Let $\mathbf{x}^* \in \mathcal{P}$ be the unique fixed point of $P$, as described in Lemma 5.22. For every $\eta > 0$, there is $1 > \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds

$$
\text{dist}_H (\mathbf{x}, \mathbf{x}^*) \geq \eta \quad \Rightarrow \quad \text{dist}_H (R_\varepsilon (\mathbf{x}), \mathbf{x}^*) < \text{dist}_H (\mathbf{x}, \mathbf{x}^*). \quad (5.137)
$$

The proof can be found in Appendix A.17. Note that from the above result we find the following for which the proof is stated in Appendix A.18.

**Corollary 5.25.** Let $\mathbf{x}^* \in \mathcal{P}$ be the unique fixed point of $P$, as described in Lemma 5.22. For every $\eta > 0$, there is $1 > \varepsilon_0 > 0$ and a constant $C_\eta > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have

$$
\text{dist}_H (\mathbf{x}, \mathbf{x}^*) \geq \eta \quad \Rightarrow \quad e^{d_H} (R_\varepsilon (\mathbf{x}), \mathbf{x}^*) < (1 - C_\eta \varepsilon) e^{d_H} (\mathbf{x}, \mathbf{x}^*), \quad (5.138)
$$
or in terms of the Hilbert metric

$$
\text{dist}_H (\mathbf{x}, \mathbf{x}^*) \geq \eta \quad \Rightarrow \quad \text{dist}_H (R_\varepsilon (\mathbf{x}), \mathbf{x}^*) < \text{dist}_H (\mathbf{x}, \mathbf{x}^*) + \log(1 - C_\eta \varepsilon) < \text{dist}_H (\mathbf{x}, \mathbf{x}^*) - C_\eta \varepsilon. \quad (5.139)
$$

Next we obtain two robustness results. The first concerns the perturbed AIMD system in 5.134, while the second yields a bound on the worst case behaviour of convex combinations with arbitrary points in $\mathcal{P}$.

**Lemma 5.26.** Let $\mathbf{x}^* \in \mathcal{P}$ be the unique fixed point of $P$, as described in Lemma 5.22. Consider $\delta^- > 0$ as defined in Equation (5.119). There exists a constant $K > 0$ such that for all $0 < \delta < \delta^-$, $\varepsilon \in (0, 1)$, all $x \in P_{\infty} (\delta) = \text{conv} P(\mathcal{P}) + B_1 (0, \delta)$, and all $\Delta \in \mathbb{R}^N$ with $\mathbf{1}^T \Delta = 0$ and $||\Delta||_1 \leq \delta$ we have

$$
e^{d_H} (R_\varepsilon (\mathbf{x}) + \varepsilon \Delta, \mathbf{x}^*) - e^{d_H} (R_\varepsilon (\mathbf{x}), \mathbf{x}^*) \leq K \varepsilon \delta. \quad (5.140)
$$

**Proof.** The assumption on $\delta$ yields that $\text{conv} P(\mathcal{P}) + B_1 (0, 2\delta) \subset \text{ri} \mathcal{P}$, see Equation (5.119). By definition we have

$$
e^{d_H} (R_\varepsilon (\mathbf{x}) + \varepsilon \Delta, \mathbf{x}^*) - e^{d_H} (R_\varepsilon (\mathbf{x}), \mathbf{x}^*) = \max_i \left( \frac{R_\varepsilon (\mathbf{x}) + \varepsilon \Delta_i}{\varepsilon \delta_j} \right) - \max_j \left( \frac{R_\varepsilon (\mathbf{x}) + \varepsilon \Delta_j}{\varepsilon \delta_i} \right) \quad (5.141)
$$

Assume that the indices $i$ and $j$ are chosen such that the maximum and the minimum is attained for the perturbed term, respectively. Note further that since $||\Delta||_1 \leq \delta$, it follows that $-\delta \leq \Delta_i \leq \delta$ for all $i$. Then, Equation (5.141) can be simplified to

$$
e^{d_H} (R_\varepsilon (\mathbf{x}) + \varepsilon \Delta, \mathbf{x}^*) - e^{d_H} (R_\varepsilon (\mathbf{x}), \mathbf{x}^*) \leq \frac{((R_\varepsilon (\mathbf{x})_i + \varepsilon \delta) R_\varepsilon (\mathbf{x})_j - R_\varepsilon (\mathbf{x})_j (R_\varepsilon (\mathbf{x})_i - \varepsilon \delta)) x^*_j}{((R_\varepsilon (\mathbf{x})_j - \varepsilon \delta) R_\varepsilon (\mathbf{x})_j) x^*_j} = \varepsilon \delta \left( \frac{R_\varepsilon (\mathbf{x})_j + R_\varepsilon (\mathbf{x})_i}{R_\varepsilon (\mathbf{x})_j - \varepsilon \delta} \right) x^*_j. \quad (5.142)
$$

To complete the proof, we need to show that the factor of $\varepsilon \delta$ in the expression on the right can be uniformly bounded for all $\mathbf{x} \in P_{\infty} (\delta)$. By assumption, $P_{\infty} (\delta)$ is a compact subset of $\text{ri} \mathcal{P}$,
so that all entries of $x$ and $R_{\epsilon}(x)$ are bounded away from 0. Let $e_i$, be the $i$-th canonical basis vector for $\mathbb{R}^N$. The terms $R_{\epsilon}(x)_j - \epsilon \delta$ are bounded away from 0, because for arbitrary indices $j' \neq j$ we have $R_{\epsilon}(x) - \epsilon \delta e_j + \epsilon \delta e_{j'} \in \text{conv} \ P(\mathcal{P}) + \overline{B}_1(0, 2\delta) \subset \mathcal{P}$. Thus the factor of $\epsilon \delta$ in the final expression may be bounded by a constant, as the denominator is bounded away from 0. This constant only depends on $\delta^*$. Hence, Lemma 5.26 holds. 

From the above we find also.

**Corollary 5.27.** Let $x^* \in \mathcal{P}$ be the unique fixed point of $P$, as described in Lemma 5.22. For a given $\eta > 0$, let $1 > \epsilon_0 > 0$ and $C_\eta > 0$ be the constants from Corollary 5.25 such that Equations (5.138) and (5.139) hold. Let

$$\delta^* = \min \left( \frac{C_\eta \epsilon_0}{2K}, \delta^- \right),$$

(5.144)

where $e^\cdot$ denotes the exponential function.

Then for every $0 < \delta < \delta^*$, all $0 < \epsilon < \epsilon_0$ and all $x \in \text{conv} \ P(\mathcal{P}) + B_1(0, \delta)$ and all $\Delta \in \mathbb{R}^N$ with $I^T \Delta = 0$ and $||\Delta||_1 \leq \delta$ we have $R_{\epsilon}(x) + \epsilon \Delta \in P_{\epsilon_0}(\delta)$ and

$$\text{dist}_H(x, x^*) \geq \eta \quad \Rightarrow \quad e^{d_H}(R_{\epsilon}(x) + \epsilon \Delta, x^*) < (1 - \frac{C_\eta \epsilon}{2})e^{d_H}(x, x^*),$$

(5.145)

or equivalently,

$$\text{dist}_H(x, x^*) \geq \eta \quad \Rightarrow \quad \text{dist}_H(R_{\epsilon}(x) + \epsilon \Delta, x^*) < \text{dist}_H(x, x^*) + \log \left( 1 - \frac{C_\eta \epsilon}{2} \right) < \text{dist}_H(x, x^*) - \frac{C_\eta \epsilon}{2}. \quad (5.146)$$

**Proof.** The first claim $R_{\epsilon}(x) + \epsilon \Delta \in P_{\epsilon_0}(\delta)$ is due to convexity. Under the assumptions we may apply Corollary 5.25. Hence, $\text{dist}_H(x, x^*) \geq \eta$ implies that

$$e^{d_H}(R_{\epsilon}(x), x^*) < (1 - C_\epsilon \epsilon)e^{d_H}(x, x^*).$$

(5.147)

Then, applying the above and Lemma 5.26 we find

$$e^{d_H}(R_{\epsilon}(x) + \epsilon \Delta, x^*) \leq e^{d_H}(R_{\epsilon}(x), x^*) + K \epsilon \delta \leq (1 - \frac{C_\eta \epsilon}{2})e^{d_H}(x, x^*) + \epsilon \left( -\frac{C_\eta \epsilon}{2} e^{d_H}(x, x^*) + K \delta \right)$$

$$\leq (1 - \frac{C_\eta \epsilon}{2})e^{d_H}(x, x^*) + \epsilon \left( -\frac{C_\eta \epsilon}{2} e^{\eta} + K \delta \right).$$

Due to the assumption $0 < \delta < \delta^*$ the last term on the right hand side is negative. Hence, Equation (5.145) holds. Equation (5.146) can be obtained by taking the logarithm on both sides and applying the logarithm inequality that $\log x < x - 1$ for $x > 0$. 

Now we can find the second robustness result that bounds the worst case behaviour of convex combinations with arbitrary points in $\mathcal{P}$. 

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Lemma 5.28. Let $x^* \in \mathcal{P}$ be the unique fixed point of $P$, as described in Lemma 5.22. There exists a constant $C > 0$ such that for all $x, y \in \mathcal{P}$ where all entries of $x$ are positive, i.e. $x_i > 0$ for all $i$, and for all $\varepsilon \in [0, 1)$ it holds that

$$\text{dist}_H((1 - \varepsilon)x + \varepsilon y, x^*) \leq \text{dist}_H(x, x^*) + \log \left(1 + C \frac{\varepsilon}{1 - \varepsilon}\right).$$  \hspace{1cm} (5.148)

In particular, for any $0 < \varepsilon_0 < 1$ there is a constant $C_0$ such that for all $x, y \in \mathcal{P}$ where $x_i > 0$ for all $i$ and for all $\varepsilon \in [0, \varepsilon_0)$ it holds that

$$\text{dist}_H((1 - \varepsilon)x + \varepsilon y, x^*) \leq \text{dist}_H(x, x^*) + C_0 \varepsilon.$$  \hspace{1cm} (5.149)

Proof. Let $x, y \in \mathcal{P}$ be arbitrary with $x_i \geq 0$ for all $i$. Then, we find that

$$e^{d_H((1 - \varepsilon)x + \varepsilon y, x^*)} = \frac{\max_i \left((1 - \varepsilon)x_i + \varepsilon y_i\right)}{\min_j \left((1 - \varepsilon)x_j + \varepsilon y_j\right)} \leq \frac{\max_i \left((1 - \varepsilon)x_i + \varepsilon(\bar{P} - \bar{p})\right)}{\min_j \left((1 - \varepsilon)x_j\right)}$$

$$\leq \frac{(1 - \varepsilon) \max_i \left(\frac{x_i}{x^*_i}\right) + \varepsilon(\bar{P} - \bar{p}) \max_i \left(\frac{1}{x^*_i}\right)}{(1 - \varepsilon) \min_j \left(\frac{x_j}{x^*_j}\right)}$$

$$\leq e^{d_H(x, x^*)} + \varepsilon(\bar{P} - \bar{p}) \frac{\max_i \left(\frac{1}{x^*_i}\right)}{1 - \varepsilon} \frac{\max_i \left(\frac{x_i}{x^*_i}\right)}{\min_j \left(\frac{x_j}{x^*_j}\right)}.$$  \hspace{1cm} (5.150)

Note that $x_i^* \geq x^*_{\min}$ and $\max_i (x_i/x_i^*) \geq 1$. Then, we find

$$e^{d_H((1 - \varepsilon)x + \varepsilon y, x^*)} \leq e^{d_H(x, x^*)} \left(1 + \frac{\varepsilon(\bar{P} - \bar{p})}{1 - \varepsilon} \frac{1}{x^*_\min}\right).$$

Equation (5.148) follows by taking the logarithm and defining $C$ appropriately. Then, Equation (5.149) follows as $1/(1 - \varepsilon)$ is bounded on an interval of the form $[0, \varepsilon_0]$ for $\varepsilon_0 < 1$. \hfill \Box

Almost Sure Convergence

In this section, we will show the almost sure convergence of the adapted AIMD algorithm.

Theorem 5.29. Let the functions $\lambda_i$ defined in Equation (5.88) satisfy Assumption 5.9 and let $p^*$ denote the fixed point guaranteed by Lemma 5.22. Consider the non-homogeneous Markov chain in Equation (5.102). For any initial condition

$$z(\tau_0) = \begin{bmatrix} p(\tau_0) \\ \rho(\tau_0) \end{bmatrix}^T \in \mathcal{P}^2$$  \hspace{1cm} (5.150)

we have that the second component of $z(\tau_k)$ satisfies

$$\lim_{k \to \infty} \rho(\tau_k) = p^* \quad \text{almost surely.}$$
5.2. Optimisation with the AIMD Algorithm

The above theorem shows the convergence of the algorithm. Further, the equilibrium point is actually the solution to the optimisation problem posed earlier. Hence, the AIMD algorithm with appropriately adaptation of the probabilities allows to solve constrained convex optimisation problems. The proof of the theorem can be found in Appendix A.19. It makes use of the two following properties concerning series of random variables.

Lemma 5.30. Let \( X_k \) be a sequence of independent, identically distributed, real-valued random variables, satisfying \( \Pr[X_k = a] = p_1 \) and \( \Pr[X_k = b] = p_2 \), where \( a < 0 < b, p_1 + p_2 = 1 \), and \( E[X_k] = r < 0 \). Let \( \{\varepsilon_k\}_{k \in \mathbb{N}} \) be a sequence of positive real numbers, that is square summable, but not summable. Then,

\[
\sum_{k=1}^{L} \varepsilon_k X_k \to -\infty \quad \text{almost surely as } L \to \infty. \tag{5.151}
\]

Furthermore,

\[
\lim_{\ell \to \infty} \sup_{L} \sum_{k=\ell}^{\ell+L} \varepsilon_k X_k = 0 \quad \text{almost surely} \tag{5.152}
\]

Proof. As \( \text{Var}[\varepsilon_k X_k] = \varepsilon_k^2 \text{Var}[X_k] \), the variance of the variables in the series from Equation (5.151) is square summable, and so by [17, Theorem 22.6] the series

\[
\sum_{k=1}^{\infty} \varepsilon_k (X_k - E[X_k])
\]

converges almost surely to a finite value. Due to the assumption that \( r < 0 \), the series

\[
\sum_{k=1}^{\infty} \varepsilon_k E[X_k] = r \sum_{k=1}^{\infty} \varepsilon_k \tag{5.153}
\]

diverges to \(-\infty\) and Equation (5.151) follows. For the second part, i.e. Equation (5.152), consider

\[
\sum_{k=\ell}^{\ell+L} \varepsilon_k X_k \leq \sum_{k=\ell}^{\ell+L} \varepsilon_k (X_k - E[X_k]).
\]

Again by [17, Theorem 22.6] the partial sums on the right are almost surely partial sums of a convergent series. Then, the Cauchy criterion says that there are only finitely many \( \ell \in \mathbb{N} \) such that the sum exceeds a given \( C > 0 \). This shows that Equation (5.152) holds as smaller equal to 0. The equality follows from the case \( L = 1 \).

The second property extends the continuity result in Lemma 5.5 to the family of Markov chains with fixed probability \( z(\tau_0) \in \mathcal{P} \). In the following result we use the notation \( \Pr_{\lambda(z(\tau_0))} \) to indicate a probability statement for the Markov chain in Lemma 5.1 with fixed probability \( \lambda = \lambda(z(\tau_0)) \). Its proof can be found in Appendix A.20.

Lemma 5.31. Assume that Assumption 5.9 holds. Consider the family of Markov chains such as in Lemma 5.1 with fixed probability \( \lambda = \lambda(y) \), parameterised by \( y \in \mathcal{P} \). Then, for all \( \delta > 0 \) and all \( \theta \in (0, 1] \) there exists a \( m \in \mathbb{N} \) such that for all \( y \in \mathcal{P} \)

\[
\Pr_{\lambda(y)} \left( \left\| \frac{1}{m+1} \sum_{\ell=0}^{m} \Pi(\tau_\ell) - \frac{P(y)}{p - \bar{p}} T^T \right\|_1 > \delta \right) < \theta. \tag{5.154}
\]
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5.3 Behaviour of the Aggregated Demand

While the above analysis concentrates on the single agents and their average share, the evolution of the total power consumption is important for the distribution grid. Therefore, we investigate three features of the algorithm and how they depend on the AIMD parameters: \( \alpha \), \( \alpha_i \), \( \beta^{(1)}_i \), \( \beta^{(2)}_i \), and \( \lambda_i \). These features are:

- the total increase during the AI phase per time step,
- the total decrease at a CE, and
- the time between two consecutive CEs.

Those three features allow to specify the behaviour of the algorithm in terms of the overall power consumption. For simplicity, we assume that Assumption 5.2 holds, i.e. the power consumption by uncontrollable agents \( \tilde{p}(k) \) and the available active power \( \bar{P}(k) \) are constant.

5.3.1 The Total Increase during the AI Phase

The increase during the AI phase at each time step is defined by the sum of the additive factors of all controllable agents. Let \( A_\alpha(k) \) denoted the total increase at time step \( k \), then

\[
A_\alpha(k) = \pi \sum_{i=1}^{N} \alpha_i(k).
\]

Hence, the increase at each time step depends solely on the additive parameters of the individual agents. This means, the larger the individual factors \( \alpha_i \) are, the larger is the increase at each time step. Also, the number of connected agents influences the increase in the way that more connected agents lead to a higher increase per time step. Further, by adapting the value of the global additive factor \( \pi \) it is possible to influence the increase. This value could for example be transmitted with a CE without increasing the communication load.

To illustrate the dependence of the additive parameters on the increase, we conduct a Matlab simulation with 20 participating agents for a total duration of 1500 time steps. For simplicity, we assume that Assumptions 5.5, 5.7 and 5.8 hold. The first test is used as a reference and denoted in the following “base case”. The second test uses a larger additive factor to capture the above described dependency on the additive factor. We denote this case the “additive case”. The AIMD parameters used are summarised in Table 5.1. An extract of the resulting aggregated power is shown in Figure 5.10, where the total power consumption for the two cases is compared. As easily can be seen the increase per time step is larger for the additive case. Additionally, the additive case leads to more CEs. The reason for this will become clear in the discussion of the third feature, the time between CEs. The overshoot that is visible in Figure 5.10 is due to the discrete implementation of the algorithm. As can be seen for the additive case a repetitive cycle over two CEs occurs consisting of a small and large overshoot, while the base case has a repetitive cycle over one CE. Both cases are within the bounds found in Equation (5.25) in Section 5.1.1. The behaviour that occurs in particular is dependent on the available power, and the parameters of
the agents. There seems to be multiple cases where cyclic behaviour shows up over multiple CEs. While a cycle length of one seems to be the most common, we observed also cycles over three and more CEs.

![Graph showing cyclic behaviour](image)

Figure 5.10: Influence of the additive AIMD parameter on the aggregated power consumption.

<table>
<thead>
<tr>
<th>case</th>
<th>$\alpha$</th>
<th>$\pi$</th>
<th>$\beta^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>base</td>
<td>1</td>
<td>0.1</td>
<td>0.7</td>
</tr>
<tr>
<td>additive</td>
<td>1</td>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>multiplicative</td>
<td>1</td>
<td>0.1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 5.1: The AIMD parameters used in the simulations of the deterministic, discrete time AIMD.

### 5.3.2 Decrease during the MD Phase

The second feature is the decrease occurring at a CE, which is defined as the sum of the individual decrease by each agent. Let $D_\beta(\tau_k)$ denote the total decrease at the $k$-th CE, then

$$D_\beta(\tau_k) = \sum_{i=1}^{N} (1 - \beta_i(\tau_k)) p_i(\tau_k), \quad (5.156)$$

where $\beta_i(\tau_k)$ is as defined in Equation (5.1). Hence, the actual decrease depends not only on the individual AIMD parameters of the agents, but also on their actual power consumption and the stochastic decision at time step $\tau_k$.

In the special case of a deterministic MD phase i.e. Assumption 5.5 holds, the decrease can be simplified to

$$D_\beta(\tau_k) = \sum_{i=1}^{N} (1 - \beta^{(1)}_i(\tau_k)) p_i(\tau_k). \quad (5.157)$$
By further constraining the multiplicative factor to be equal for all agents, we can simplify the decrease to
\[ D_\beta(\tau_k) = (1 - \beta(1)(\tau_k)) \sum_{i=1}^{N} p_i(\tau_k). \]  
(5.158)

From this it is clear that a larger multiplicative factor leads to a smaller decrease. To illustrate this behaviour we simulated a third case, here denoted “multiplicative” case, where the multiplicative factor is decreased compared to the base case. The parameters used are summarised in Table 5.1. As can be seen in Figure 5.11 the decrease is larger with the lower factor \( \beta(1) \) as expected. The larger decrease also leads to a decrease in CEs. This is easily explained when looking at the next feature, the time between CEs. First, the influence of the probability on the decrease is investigated.

![Figure 5.11: Influence of the multiplicative AIMD parameter on the aggregated power demand in the deterministic case.](image)

When Assumption 5.5 does not hold, the influence becomes more complicated. We expect however, that a smaller probability or smaller parameter \( \beta(1) \) leads to a larger decrease, due to inspection of Equation (5.156). This implies that the aggregated power tends to be smaller. Further, we expect that if the parameters are chosen such that the expected multiplicative factor \( E[\beta(\tau_k)] \) in the stochastic version is equal to the multiplicative factor \( \beta(1)(\tau_k) \) in the deterministic version that then the average aggregated power consumption is equal. These expectations are illustrated by simulating three stochastic cases with varying parameters. The used AIMD parameters are summarised in Table 5.2. Figure 5.12 shows the aggregated power consumption filtered using a moving average filter with a window size of 30 time steps. As can be seen the aggregated power is smaller with the lower expected multiplicative factor \( E[\beta(\tau_k)] \).
5.3. Behaviour of the Aggregated Demand

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<th>$\beta^{(2)}$</th>
<th>$\lambda$</th>
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<td>0.5</td>
<td>0.99</td>
<td>0.59</td>
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</tbody>
</table>

Table 5.2: The AIMD parameters used in the simulations of the stochastic, discrete time AIMD.

![Figure 5.12](image-url)

Figure 5.12: Influence of the multiplicative AIMD parameters and the probability on the aggregated power consumption in the stochastic case.

5.3.3 Time between Two Consecutive CEs

The last feature is the time between two consecutive CEs. Assume Assumption 5.1 holds and that the additive factors are constant from one CE to the next. Let $\Delta \tau_k$ be the time between the $k$-th and $(k+1)$-th event. It is clear that then $\Delta \tau_k$ is the fraction between the total decrease $D_\beta(\tau_k)$ and the total increase $A_\alpha(\tau_k)$, i.e.

$$\Delta \tau_k = \frac{D_\beta(\tau_k)}{A_\alpha(\tau_k)}$$

(5.159)

Using Equations (5.155) and (5.157) we find

$$\Delta \tau_k = \frac{\sum_{i=1}^N (1 - \beta_i(\tau_k)) p_i(\tau_k)}{\sum_{i=1}^N \alpha_i(\tau_k)}.$$  

(5.160)

Hence, a larger decrease leads to a longer time between the CEs, while a larger increase leads to a shorter time between the CEs. This explains the behaviour visible in Figures 5.10 and 5.11 where the amount of CEs increased and decreased due to the change of the parameters.
In this chapter, we investigate the proposed load management scheme in more realistic settings. We base the investigation mainly on simulations of the distribution grid using the power simulation tool OpenDSS in combination with Matlab. These investigations cover both the basic case where one type of agent is simulated in a simple radial test network as well as more complex settings, where multiple types are connected simultaneously. Further, we investigate more complex structures where more than one central management unit is present. Lastly, we include a lower bound on the power consumption for continuously controllable loads. While such a lower bound on first sight does not contribute to the main topic of this chapter, we believe that it is an important step towards a more realistic load management scheme. The lower bound on the power consumption may be of great practical relevance since we are then able to guarantee a minimum customer satisfaction. For example, such a bound can guarantee that a controllable EV is charged within a specified time.

6.1 Network Simulations

In this section, we illustrate the potential of the algorithms in simulations using Matlab and OpenDSS interlinked. Matlab is used to compute the power consumption of each controllable agent according to the algorithms described in Chapters 3 and 4. On the other hand OpenDSS, a power simulation tool developed by EPRI [33], is used to simulate the power distribution network including its loads.

Note that we do not consider the communication network over which the signals are transmitted. An initial analysis of the communication load can be found in [105]. In this thesis, we assume that there are no delays and all agents react synchronously.

The sampling time is set to one second. We assume that this gives the agents enough time to react to any received CE signals, such that at the next measurement the actions of the agents are observed.
6. Towards a More Realistic Load Management Scheme

6.1.1 IEEE 37 Bus Test Network

First, we test our algorithms on a revised version of the distribution power system based on the IEEE 37 bus test feeder found among the OpenDSS examples [32]. This is depicted in Figure 6.1. Note Figure 6.1 only shows the interconnections between the loads, buses, and the transformer, and is not meant to depict the real dimensions.

Figure 6.1: IEEE 37 bus test network.

Overall there are 25 uncontrollable three-phase loads, indicated by triangles in Figure 6.1, connected to different buses. Such loads follow a pre-specified load pattern over a day. This pattern is fixed and does not change. The algorithms however have no prior knowledge of the demand by the uncontrollable loads.

Further, we assume that the central management unit which sends the broadcast signals to the connected controllable agents is located at the transformer connecting the loads with the external
grid. The transformer is depicted in Figure 6.1 as a rectangular block. The voltage levels of the transformer are adapted to 10 kV at the supplier side and 400 V at the distribution side using a Wye connection. The 400 V distribution with three phases is often used in Europe and Australia. Such a distribution is different from the system commonly used in North America where the voltage level is 120 – 240 V mainly using a split-phase method. Both methods of distribution are different and so different problems might arise when the algorithm is applied. One such problem prominent is, for example, load imbalances in the three phases, where a single phase carries a lot more or less power than the other two phases. In this thesis we however neglect this particular problem. The management unit measures both the active and reactive power levels at this transformer. The available active power is fixed to 190 kW if EVs are connected and to 220 kW if refrigerators are connected. We chose the first value such that it is a bit lower than the maximum power demand by uncontrollable loads. This allows us to see the potential of including V2G options for the controllable agents. Since it is the maximum power consumption, during most of the simulation period the power drawn by uncontrollable loads is below the limit. Additionally, the algorithm controls the reactive power flow to obtain an ideal reactive power balance at the transformer of 0 VAR.

In the following, we simulate the different algorithms one by one in this setting and compare the results. The results obtained using the proposed algorithms are also compared with:

1. the case where no controllable agents at all are connected, to evaluate the possibly different utilisation of the uncontrollable loads and
2. the case where controllable agents operate uncontrolled, i.e. EVs charge with the maximum charge rate until they are fully charged and refrigerators operate using a thermostatic control.

We will first investigate the case when all loads have a binary controllable power consumption. Afterwards, we allow the connection of continuously controllable agents instead. Finally, the case where both categories of loads are present simultaneously is investigated.

6.1.1.1 Binary Controlled Power Consumption

In this section, we investigate the behaviour of the algorithm for binary controlled power, described in Chapter 3. While initially only EVs or refrigerators connect to the grid, we also simulate a situation where both loads are simultaneously connected. These situations are also compared with the two basic cases. The simulated situations are:

**No agents** No controllable agent is connected. The uncontrollable agents are connected as described previously, see Figure 6.1.

**Uncontrolled EVs** All controllable agents are EVs. The EVs do however react like uncontrollable loads, i.e. they consume the maximum allowed power \( \bar{p}_i \) upon connection until fully charged.

**EVs (BA)** All controllable agents are EVs and apply the BA algorithm.
6. Towards a More Realistic Load Management Scheme

**Uncontrolled refrigerators** All controllable agents are refrigerators, but their power consumption is not controlled, i.e. they apply a thermostatic control.

**Refrigerators** All controllable agents are refrigerators and apply the BA algorithm.

**Uncontrolled EVs and refrigerators** Both EVs and refrigerators are connected, however none of the loads is controlled.

**EVs (BA) and refrigerators** EVs and refrigerators are connected, where both loads apply the BA algorithm.

In the above situations, 20 EVs and/or 60 refrigerators can connect to the distribution grid. For all situations the simulation settings are identical such that the different situations can easily be compared. As discussed in Chapter 3, the selection of the parameters for the BA algorithms is complex and we were not able to define an analytic method to define those parameters. While it might be advantageous to change the parameters depending on the structure of the distribution grid, as well as the number and type of connected agents, both EVs and refrigerators, we refine from doing so in this thesis. This means that also the EVs and refrigerators are using the same parameters for the BA algorithm independent of the connection of other loads.

The locations where the EVs are connected are predefined. The connections are all single phase with a maximum power capacity of 3.7 kW. In Figure 6.2 the connection of an EV is indicated by an ellipse, where the different colours indicate the phase. For illustrative purposes, we make the assumption that the peak load due to the uncontrollable loads overlaps the charging periods of the EVs. While this assumption is clearly not always true, since most EVs can be recharged at night [29], it still allows us to investigate a worst-case setting. Also, note that there are some studies that predict domestic charging to occur partly during the evening load peak, directly after work. For example, in [25] three charging periods are identified: during daytime, during the night, and during the evening. Similarly, [89] assumes that without control the charging starts around 6pm and identifies such a setting as worst case. Hence, the EVs connect between hours 8 and 12 of the simulation, with a uniform probability. Note that due to this worst case setting and the relatively low chosen limit on the transformer, there is not enough power available to fully charge all EVs. In practice this should be avoided as far as possible or be limited to occur only rarely. However, these settings show the potential benefit of the load management scheme. The EVs require an energy between 15 kWh and 20 kWh to fully charge their batteries which is assigned randomly using a uniform distribution. Their battery size is set to 24 kWh. Further, the EVs disconnect from the distribution grid as soon as they are fully charged, i.e. received the required energy. Additionally, an EV disconnects after a total charging time of 9 h, independently of its charging state. In case the EV uses the BA algorithm its parameters are \( n = 5000 \) and \( m = 1500 \). The selection of the parameter here is some what random in the sense that the parameters are chosen manually to approximate desired behaviour. Since, we wish to gain good tracking with as little overshoot as possible, the parameter \( n \) is chosen larger than \( m \). Both parameters are chosen fairly large which means that the power consumption is expected to be close to the available power. Note that the parameters are not optimised in any way, such that they lead most probable to suboptimal behaviour. Also, an optimal selection of the parameters might take into account additionally the...
number and type of connected agents as well as the structure of the distribution grid. However, since we were not able to give any useful analytical methods how to select these parameters, we take here constant parameters for all simulated situations.

The refrigerators' connection points are also predefined and illustrated in Figure 6.2 as circles. Their maximum power capacity is selected randomly between 1 kW and 4 kW. Similarly, the temperature change occurring if off and on is selected randomly between $0.001 \, \degree C \, s^{-1}$ and $0.009 \, \degree C \, s^{-1}$, and $-0.07 \, \degree C \, s^{-1}$ and $-0.03 \, \degree C \, s^{-1}$, respectively. The allowed temperature inside the refrigerators can vary from $0 \, \degree C$ to $5 \, \degree C$. All these values are fixed throughout the simulation. In case the refrigerator leaves the allowed temperature band it turns on and off respectively, until its temperature is within the band. If a refrigerator applies the BA algorithm its parameters are $n = 11000$ and $m = 1500$. As in the case of an EV these parameters are selected manually. The very large $n$ parameter, especially compared to the case of EVs, reflects the fact that for long periods of time a refrigerator is not required to consume power without an effect on the end user. Hence, after a cooling period it can stay longer off. This allows other appliances to utilise the power during these periods. In contrast, EVs prefer to consume as much power as possible, such that their incentive is to turn on as soon as possible. So it might be beneficial to select the parameters according to the duty cycle of a refrigerator during its normal thermostatic cycle, i.e., the ratio of the time it is required to be on in regard to the length of the complete cycle. Since such considerations are however very complex due to the interactions between the parameters, the available power, and other agents, we here selected the parameters constant and identical for all agents.

In Figure 6.3, the evolution of the active power at the transformer for the situations with only EVs connected is shown. The power consumption is filtered using a moving average filter with a window size of 600 time steps, which corresponds to 10 min. The BA algorithm reduces the peak demand that occurs when no control is used to a value comparable without the EVs. At the same time, the peak is prolonged since EVs charge after the demand of the uncontrollable loads drops.

In Figure 6.4, the evolution of the active power at the transformer for the situations where refrigerators are connected is shown. The power consumption is again filtered using a moving average filter with a window size of 600 time steps, which corresponds to 10 min. Note the refrigerators are not able to support the distribution grid for longer periods, since it is required that they remain within the nominated temperature range. Also, the load by uncontrollable refrigerators is around the desired available power. Note that after hour 13 of the simulation in the case where the refrigerators are controlled the decrease of the aggregated demand is delayed. The reason for this becomes clear if we compare the temperature inside the refrigerators. In Figure 6.5, the temperature of two randomly selected refrigerators is depicted. While without control the temperature is around 2.5 $\degree C$, the BA algorithm increases the temperature during the period where the aggregated demand is above the available power, while during the other periods the temperature is pushed to lower values.

When the EVs are connected and apply the BA algorithm in Figure 6.4 the aggregated demand stays at the same level compared to the case where no EVs are connected as intended. Similar to before, the peak is prolonged, while the EVs finish their charging.
6. Towards a More Realistic Load Management Scheme

6.1.1.2 Continuously Controlled Loads

Previously, all controllable agents applied the BA algorithm and allowed solely binary control over their power consumption. In this section, we assume that all controllable agents are EVs that allow for continuous control over their charge rate. Their settings including the location where they connect, are chosen identical as previously, see also Figure 6.2. We simulate a total of six cases, where the agents deploy a different scenario:

1. CRF scenario using the AIMD algorithm for active power and G2V only.
2. CRF scenario using the DAIMD algorithm for active and reactive power and with V2G.
3. REF scenario using the AIMD algorithm for active power and G2V only.

Figure 6.2: IEEE 37 bus test network with the locations of connected controllable loads.
6.1. Network Simulations

Figure 6.3: The aggregated power consumption at the transformer of the IEEE 37 bus test network where the controllable agents are EVs.

4. REF scenario using the DAIMD algorithm for active and reactive power and with V2G.

5. CTF scenario using the AIMD algorithm for active power and G2V only.

6. GOF scenario using the AIMD algorithm for active power and G2V only.

The AIMD parameters are selected identical for all scenarios unless the scenario requires the adaptation of a parameter, i.e. the additive factor $\alpha_i$ is adapted for the REF and the CTF scenarios while the probability $\lambda_i$ is adapted for the GOF scenario. The parameters for the AIMD controlling the active power are: $\alpha = 1$, $\sigma = 0.1$, $\beta^{(1)} = 0.7$, $\beta^{(2)} = 0.99$, $\lambda = 0.8$, $\beta = 0.15$, and $\epsilon = 0.1$. In case the reactive power is controlled the reactive power AIMD uses the same parameters as the active power AIMD.

Figure 6.6 compares the aggregated active and reactive power for the different scenarios simulated. As can be seen there is no significant difference between the different scenarios. However, there are differences observed depending on the abilities of the algorithm.

We compare the situations previously introduced, i.e. no agents, uncontrolled EVs, EVs (BA) and two new situations:

**EVs (AIMD)** The connected controllable agents are EVs, which apply the AIMD algorithm in the CRF scenario.

**EVs (AIMD,V2G)** The connected controllable agents are EVs, which apply the DAIMD algorithm for active and reactive power and with V2G capabilities in the CRF scenario.
Figure 6.4: The aggregated power consumption at the transformer of the IEEE 37 bus test network where the controllable agents are refrigerators and/or EVs.

Figure 6.7 shows the aggregated active and reactive power consumption at the transformer for these situations. The power consumption has been filtered using a moving average filter with a window length of 600 time steps, which equals to 10 min. It can be clearly seen in Figure 6.7a that the EVs increase the peak load significantly without any control. Using the AIMD algorithm with only G2V capabilities achieves a reduction of the active power consumption equally to the case without any connected EVs. However, the duration of the peak is prolonged, as the EVs charge as soon as the demand drops below the imposed limit. The effects are similar as to the EVs (BA) situation. If V2G capabilities are included the controllable agents are able to push the power consumption below the imposed limit. Comparing the reactive power demand in Figure 6.7b, note that compared to the case with no EVs connected the reactive power increases. This is due to the larger peak demand and the losses caused by it in the lines. Hence, using the basic AIMD algorithm reduces the reactive power back to the level with no agents connected. On the other hand, when allowing the agents to participate in reactive power compensation, the reactive power is pushed close to the ideal level of 0.

6.1.1.3 Simultaneous Connection of Different Load Types

Previously, the connected loads either applied the BA or the AIMD algorithm. Often both categories of loads are present simultaneously which we investigate in this section. We therefore investigate the two situations:

**EVs (AIMD) and refrigerators** EVs and refrigerators are connected. While the refrigerators
apply the BA algorithm, the EVs apply the AIMD algorithm.

**EVs (AIMD,V2G) and refrigerators** EVs and refrigerators are connected. While the refrigerators apply the BA algorithm, the EVs apply the DAIMD algorithm with both V2G and reactive power capabilities.

The two situations above are compared with the previously introduced situations: no agents, uncontrolled EVs and refrigerators, and EVs (BA) and refrigerators. Figure 6.8 shows the aggregated demand for these situations. All situations lead to a similar result. Note that when allowing V2G and reactive power capabilities the aggregated active demand can be reduced to the desired available power. Similarly, in that case the reactive power can be reduced to approximately 0 for a large part of the simulation.

### 6.1.2 Network with Two Transformers

In Section 6.1.1, the algorithms have been simulated on the IEEE 37 bus test feeder, which is a radial grid structure with a single transformer connected. At this transformer the central management unit is placed. However, in the future the algorithm might be applied in more complex structures where more than one central management unit is present. We investigate such a scenario briefly in this section. Therefore, the IEEE 37 bus test feeder is adapted such that two transformers are connected. This is depicted in Figure 6.9. At each transformer there is a management unit present that is able to send CE signals to the connected loads. Both management units measure the active power at the transformers and ignore the reactive power. The first transformer has an...
available power of 180 kW assigned, while the second transformer’s available power is set to 40 kW. In this test the reactive power is ignored and the transformers have to share the required reactive load among them. There are multiple methods how to control such paralleled transformers [47]. Mostly, these control actions are not influenced by the active power. While it is possible that the load management scheme if operating in reactive power balancing mode can badly interact with controls that are currently in place, a detailed analysis is required. By selecting the bound on the reactive power depending on the setting at other parallel transformers it might be possible to support the control actions that are already in use.

Overall, there are 25 uncontrollable three-phase loads, indicated by triangles in Figure 6.9, connected to different buses. These loads remain identical to the setting in Section 6.1.1.

Up to 20 EVs are able to connect to the distribution grid that all allow continuous control over their charge rate. The locations, where these EVs connect are illustrated in Figure 6.9 where each EV is indicated by an ellipse. The connection point is single phase with a maximum power capacity of 3.7 kW. The different colours indicate the connection to the different phases of the power grid. As previously in Section 6.1.1, the EVs connect between hours 8 and 12 of the simulation, require between 15 kW h and 20 kW h to fully charge, and remain connected for 9 h.

In the basic form all EVs would react to any CE that is sent by either the first or the second transformer. While this will result in a fair share and guarantees that at both transformers Equation (2.1) is not violated, this most likely will not lead to the most efficient power transfer to the EVs, i.e. it might be possible to transfer more power to the connected agents without violating Equation (2.1). The reason for this speculation is that not all loads have an identical impact on the aggregated demand at the transformer level due to their connection.

Hence, we propose here that the agents react differently on CE depending on the source, i.e. the central management unit that sends it. Therefore, the agents are assigned to different groups, manually selected, that are based on the distance to the transformers. We investigate here a case with three and one with two such groups. The areas of the groups are depicted in Figure 6.9. The reaction is then adapted by adjusting the multiplicative factor $\beta^{(1)}_i$ of the agents depending on the group and the source of the CE. Let $s^{(j)}_i$ with $j = 1, 2$ be a coupling factor between agent $i$ and transformer $j$ with $0 \leq s^{(j)}_i \leq 1$ and $s^{(1)}_i + s^{(2)}_i = 1$. Then, the multiplicative factor is adapted at the $k$-th CE such that

$$\beta^{(1)}_i(\tau_k) = \begin{cases} 1 - s^{(1)}_i \gamma & \text{if transformer 1 causes the CE} \\ 1 - s^{(2)}_i \gamma & \text{otherwise} \end{cases}$$

where $\gamma$ is a constant such that the multiplicative factor remains within selected levels. For example, $\gamma = 0.5$, which we choose in the simulations, means that the multiplicative factor lies in the range $[0.5, 1]$. The remaining AIMD parameters are set to $\alpha = 1$, $\alpha = 0.1$, $\beta^{(2)} = 0.99$, $\lambda = 0.8$. The coupling factors are selected manually according to Table 6.1 for different simulation settings. Note that equal coupling is identical to the case with no coupling present.

In Figure 6.10, the aggregated active power at the two transformers is shown for the different
coupling factors. As expected, using the coupling the power transfer can be increased during some periods. In other periods the coupling leads to a disadvantage or overshoot, this indicates that the coupling factors and the groups are not optimally chosen.

There are multiple open problems in regard to such more complex structures that we did not investigate here. Firstly, the modified algorithm requires an analytical investigation of its stability. This includes also an analysis of possible continuous oscillations that could occur. Such undesired interactions can occur due the reactions of the load towards the transformers. Secondly, a systematic method how to choose the groups and the coupling factors has to be developed. This second point is closely related to the previous one, since it might be possible to reduce undesired effects by proper selection of such groups and the associated parameters. On the other hand the above simulation study shows that there is the potential to apply such an algorithm in complex grid structures and that with minor modifications the behaviour in such power grids can be enhanced. Similar methods can be applied for more complex structures with more than two transformers or if additional management units are present to control other parts, such as voltage deviations at critical locations. Also while in this simulation study only the AIMD algorithm has been investigated it might be possible to achieve similar results when the BA algorithm is used. Additionally, the effects can be similar in cases where the agents have capabilities that allow reverse power flows and/or reactive power exchange with the grid. As mentioned before, in the case of allowing reactive power exchange the interaction with the transformers is very important and the controls that are in use if transformers are connected in parallel to avoid undesired interactions.

<table>
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Table 6.1: Coupling factors of the selected groups for different scenarios.
6. Towards a More Realistic Load Management Scheme

Figure 6.6: The aggregated power consumption at the transformer of the IEEE 37 bus test network where the controllable agents apply the AIMD algorithm in different scenarios.
Figure 6.7: Aggregated power consumption at the transformer of the IEEE 37 bus test network where the controllable agents apply the AIMD algorithm.
6. Towards a More Realistic Load Management Scheme

Figure 6.8: Aggregated power consumption at the transformer of the IEEE 37 bus test network where simultaneously some agents apply the BA and others the AIMD algorithm.

![Active power](image1)
![Reactive power](image2)
Figure 6.9: Adapted IEEE 37 bus test network with a second transformer.
Figure 6.10: The aggregated active power consumption at the transformer 1 and 2 where the agents apply the AIMD algorithm with different coupling factors.
6.2 Lower Bound on the Power Consumption for Continuously Controllable Loads

When charging EVs often the problem arises that while customers agree to reduce the charge rate of their vehicles, a guarantee that the vehicles are charged to a useful level at the end of the charging interval is desired. By reducing the chance of receiving insufficient energy, customers might be more willing to participate in the load management scheme. Such a guarantee can be achieved by using a minimum power consumption for each EV.

Here, we replace the upper bound on the power consumption of each agent discussed in Section 5.1.3 with a lower and upper bound, i.e. Assumption 5.7 is replaced with

**Assumption 6.1.** Each agent has a lower bound $p_i$ and an upper bound $\bar{p}_i$ assigned with

$$p_i < \bar{p}_i,$$  \hspace{1cm} (6.1)

such that

$$p_i \leq p_i(k) \leq \bar{p}_i$$  \hspace{1cm} (6.2)

for all $i = 1, \ldots, N$ and all $k$.

Let Assumptions 5.1 to 5.3 and 5.5 hold. Due to the introduction of this minimum power consumption, it might happen that there is not enough power to satisfy the minimum power requirements of each agent. This situation should be avoided by all means to avoid any inconvenience for the customer by selecting the available power accordingly. In worst case the distribution network might require expansion to be able to accommodate the needs of the customers. Hence, we introduce the following assumption.

**Assumption 6.2.** There is more power available for the controllable agents than their aggregate minimum requirement, i.e.

$$\sum_{i=1}^{N} p_i < \bar{P} - \bar{p}.$$  

At the same time Assumption 5.8 should hold, to avoid the case where all agents can charge with their maximum allowed rate without causing a CE. While the upper bound on the power consumption mainly affected the AI phase, the lower bound affects the MD phase. Hence, both phases are influenced, where the upper bound impacts the AI phase and the lower bound impacts the MD phase.

Analogously to Section 5.1.3, let $p(\tau_k)$ be the vector containing the actual power consumption of the agents at the $k$-th CE and let $x(t)$ be the vector containing a fictional power consumption at time step $t$ if the power has no upper limit imposed, i.e.

$$x(t) = p(\tau_k^+) + \alpha_i(\tau_k)(t - \tau_k),$$  \hspace{1cm} (6.3)
6. Towards a More Realistic Load Management Scheme

while the actual power consumption of the agents is bounded element-wise such that

\[ p_i(t) = \min(x_i(t), \bar{p}_i). \quad (6.4) \]

Also, let \( \mu(\tau_k) \) be the difference between the actual power consumption \( p(\tau_k) \) and the fictional power consumption \( x(\tau_k) \), i.e.

\[ x(\tau_k) = p(\tau_k) + \mu(\tau_k). \quad (6.5) \]

Note that also Fact 5.11 holds in this case.

During the MD phase the power decrease is limited by the minimum allowed power consumption such that the power consumption does not fall below the minimal bound. This means that at a CE the agent adjusts its charge rate according to

\[ p_i(\tau_k^+) = \max \left( \beta^{(1)}_i p_i(\tau_k), \bar{p}_i \right). \quad (6.6) \]

It is important to note that in the above the power consumption \( p_i(\tau_k) \) is the actual power consumption and hence bounded by \( \bar{p}_i \). Using this the system with upper and lower bounds can be described by

\[ p_i(t) + \mu_i(t) = \max \left( \beta^{(1)}_i p_i(\tau_k), \bar{p}_i \right) + \alpha_i (t - \tau_k) \quad (6.7) \]

for agent \( i \) at time \( t \) with \( \tau_k < t \leq \tau_{k+1} \).

As in Section 5.1.3 we investigate the system from one CE to the next.

Lemma 6.1. Let \( \mathcal{P} \) be the set such that

\[ \mathcal{P} = \{ p \in \mathbb{R}_+^N \mid I^T p = \bar{p} - \bar{p} \text{ and } p_i \leq p_i \leq \bar{p}_i \} \quad (6.8) \]

and let \( \mathcal{M}_p \) be a set that is associated with a vector \( p \in \mathcal{P} \) such that

\[ \mathcal{M}_p = \{ \mu \in \mathbb{R}_+^N \mid \mu_i = 0 \text{ whenever } p_i < \bar{p}_i \} \quad (6.9) \]

Then, the AIMD algorithm as defined in Equation (6.7) from one CE at \( \tau_k \) to the next at \( \tau_{k+1} \) maps a vector \( p(\tau_k) \in \mathcal{P} \) into a vector \( p(\tau_{k+1}) \in \mathcal{P} \) according to

\[
p_f(\tau_{k+1}) = \max \left( \beta^{(1)}_f p_f(\tau_k), \bar{p}_f \right) + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N p_i(\tau_k) - \max \left( \beta^{(1)}_i p_i(\tau_k), \bar{p}_i \right) + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N \mu_i(\tau_{k+1}) - \mu_f(\tau_{k+1}) \quad (6.10)\]

for \( f \in \{1, 2, \ldots, N\} \).

Proof. Let the time \( \Delta T_k \) be the time between the \( k \)-th and \( (k+1) \)-th CE. We then sum Equation (6.7) over all participating agents and solve for \( \Delta T_k \) which gives

\[ \Delta T_k = \frac{1}{\sum_{i=1}^N \alpha_i} \left( \sum_{i=1}^N (p_i(\tau_{k+1}) + \mu_i(\tau_{k+1})) - \sum_{i=1}^N \left( \max \left( \beta^{(1)}_i p_i(\tau_k), \bar{p}_i \right) \right) \right). \quad (6.11) \]
6.2. Lower Bound on the Power Consumption for Continuously Controllable Loads

Using the fact that
\[ \sum_{i=1}^{N} p_i(\tau_{k+1}) = \sum_{i=1}^{N} p_i(\tau_k). \] (6.12)
allows us to find \( \Delta T_k \), i.e.
\[ \Delta T_k = \frac{1}{\sum_{i=1}^{N} \alpha_i} \left( \sum_{i=1}^{N} \left( p_i(\tau_k) + \mu_i(\tau_{k+1}) - \max \left( \beta^{(1)}_i p_i(\tau_k), \mu_i \right) \right) \right). \] (6.13)
Inserting \( \Delta T_k \) back into Equation (6.7) leads to Equation (6.10).

Next, we state some important properties of the set \( \mathcal{P} \) and the associated sets \( \mathcal{M}_p \).

**Lemma 6.2.** Assume \( \rho^{(a)} \in \mathcal{P} \) and \( \rho^{(b)} \in \mathcal{P} \) are two vectors such that \( \rho^{(a)} \neq \rho^{(b)} \) and let \( \mu^{(a)} \in \mathcal{M}_{\rho^{(a)}} \) and \( \mu^{(b)} \in \mathcal{M}_{\rho^{(b)}} \), where \( \mathcal{P} \) and \( \mathcal{M}_p \) are defined as in Lemma 6.1. Then, the following holds

1. there exists an index \( i \) such that \( \rho_i^{(a)} > \rho_i^{(b)} \)
2. there exists an index \( j \neq i \) such that \( \rho_j^{(a)} < \rho_j^{(b)} \)
3. \( \forall i \) if \( \rho_i^{(a)} > \rho_i^{(b)} \) then
   a) \( \mu_i^{(a)} \geq \mu_i^{(b)} = 0 \)
   b) \( \min \{(1 - \beta^{(1)}_i) \rho_i^{(a)}, \rho_i^{(a)} - p_i\} \geq \min \{(1 - \beta^{(1)}_i) \rho_i^{(b)}, \rho_i^{(b)} - p_i\} \)
   c) \( \max \left( \beta^{(1)}_i \rho_i^{(a)}, p_i \right) \geq \max \left( \beta^{(1)}_i \rho_i^{(b)}, p_i \right) \)
   d) \( \max \left( \beta^{(1)}_i \rho_i^{(a)}, p_i \right) - \max \left( \beta^{(1)}_i \rho_i^{(b)}, p_i \right) \leq \beta^{(1)}_i (\rho_i^{(a)} - \rho_i^{(b)}) \)

The proof is similar to the proof for Lemma 5.14 and can be found in detail in Appendix A.21.

Finally, we show an important relationship that holds for any fixed point of the system defined in Lemma 6.1.

**Lemma 6.3.** Let \( p^* \) be a fixed-point of the system described in Lemma 6.1, then
\[ \frac{p_i^* - \max \left( \beta^{(1)}_i p^*, p_i \right) + \mu_i^*}{\alpha_i} = \frac{p_j^* - \max \left( \beta^{(1)}_i p^*, p_j \right) + \mu_j^*}{\alpha_j} \]
for all \( i, j \).

**Proof.** Due to the definition of a fixed point
\[ p_i^* = \max \left( \beta^{(1)}_i p_{f_i}^*, p_i \right) + \frac{\alpha f}{\sum_{i=1}^{N} \alpha_i} \sum_{i=1}^{N} p_i^* - \max \left( \beta^{(1)}_i p_i^*, p_i \right) + \frac{\alpha f}{\sum_{i=1}^{N} \alpha_i} \sum_{i=1}^{N} \mu_i^* - \mu_f^* \] (6.14)
holds. By subtracting
\[ \max \left( \beta^{(1)}_i p_{f_i}^*, p_i \right) - \mu_f^* \]

on both sides and divide afterwards both sides by $\alpha_f$, Equation (6.14) becomes

$$\frac{p_f^* - \max \left( \beta^{(1)} f p_f^*, p_f \right) + \mu_f^*}{\alpha_f} = \frac{1}{\sum_{i=1}^{N} \alpha_i} \sum_{i=1}^{N} p_i^* - \max \left( \beta^{(1)} i p_i^*, p_i \right) + \frac{1}{\sum_{i=1}^{N} \alpha_i} \sum_{i=1}^{N} \mu_i^*. \quad (6.15)$$

Note that the right hand side of Equation (6.15) is independent of $f$. This means that the left hand side is identical for all $f \in \{1, 2, \ldots, N\}$ and the claim holds.

We establish the following theorem concerned with the existence of a unique fixed point.

**Theorem 6.4.** The system defined in Lemma 6.1 has a unique fixed point defined by

$$p_i^* = \min \left( \max \left( C\alpha_i + p_i, C \frac{\alpha_i}{(1 - \beta^{(1)} i)} \right), \bar{p}_i \right) \quad (6.16)$$

and

$$\mu_i^* = C\alpha_i - p_i^* + \max \left( \beta^{(1)} i p_i^*, p_i \right), \quad (6.17)$$

where $C$ is a constant to guarantee that

$$\sum_{i=1}^{N} p_i^* = \bar{P} - \bar{p}. \quad (6.18)$$

Its proof can be found in Appendix A.22. The next step is to show the convergence of the algorithm.

**Theorem 6.5.** The system defined in Lemma 6.1 converges to the unique fixed point defined in Theorem 6.4 such that

$$\|p(\tau_k) - p^*\|_1 \leq \max_i \left( \beta^{(1)} i \right)^k \|p(\tau_0) - p^*\|_1 \quad (6.19)$$

The proof can be found in Appendix A.23. To illustrate our findings, a simulation has been performed where four agents with random values for $\alpha$ and $\beta$ participate in the load management scheme. The maximum and minimum power consumption are selected randomly for each agent. Over the simulation period the available power increases stepwise from 3.5 kW to 8 kW. Figure 6.11 shows the actual power consumption of those four agents and the fixed points calculated according to Theorem 6.4.
Figure 6.11: Simulation of the AIMD algorithm with four agents that saturate with an upper and lower bound.
Chapter 7

Challenges

There are a multitude of challenges that we have not considered in this thesis. In this chapter, we will briefly outline these challenges. Additionally, we discuss some extensions that could be worth investigating in future settings.

7.1 Fairness Notions

Since we also considered the perspective of the customer, we introduced fairness notions that depend on the situation the customers are in. We believe that such fairness notions might encourage customers to participate in the load management scheme. The likelihood of participation can depend on how frequently the load management scheme has a large impact on the customers. For example, customers might prefer continuous control with a minor impact, such as we suggest in this thesis, over an infrequent control with a major impact on them. In this regard, it might be further encouraging that the control is achieved in a fair manner, i.e. everyone is equally contributing to support the distribution grid. In this thesis, we suggested a total of four notions for the charging of EVs and one for the control of refrigerators. Naturally, there are other possible notions of fairness that we did not consider. Further, we did not consider how such fairness notions should be selected. The selection process is however very important and might have a strong effect on the likelihood of participation. More people might be willing to allow control over their appliances or accept smaller monetary incentives, if the process is addressing their needs and wishes in a fair way. While this is only speculation, studies should investigate the effect of different fairness notions towards the willingness of participation and customer satisfaction. The selection process of the fairness notion, hence, requires an in depth analysis of the wishes and demands of the customers that are going to be connected as well as their situations. Additionally, the provider of the service might also want to include some demands, for example to encourage a certain behaviour by the EV owners. There are most likely multiple conflicting views in what is fairness for the individual persons such that a consensus has to be found that reflects the different interests.

Once the fairness notion for a region has been selected, the parameters of the algorithms must be tuned to fulfil the objectives. While the tuning of the parameters covers a multitude of possible fairness notions, there might be some specific fairness notions that are not possible to obtain
by mere adaptation of the parameters. For such specific fairness notions, the algorithm could eventually be adapted to achieve the objectives.

For the loads that apply the AIMD algorithm, i.e. loads with a continuously controllable power consumption, the AIMD parameters can be tuned for a wide variety of objectives. In this thesis, this is demonstrated by tuning the algorithm for four fairness notions concerning EVs: 1. CRF, 2. REF, 3. CTF, and 4. GOF. This is possible, because the AIMD algorithm achieves a share that is proportional to $p^*$ as found in Chapter 5. Hence, the adaptation of the parameters $\alpha_i$, $\beta^{(1)}_i$, $\beta^{(2)}_i$, and $\lambda_i$ allows fulfilment of a variety of different objectives. The fourth fairness notion makes use of the results found in Section 5.2, in which a method is given on how the AIMD algorithm is able to solve a convex optimisation problem under some conditions without requiring additional communication [116], [117]. This is done by adapting the AIMD parameter $\lambda_i$ depending on an individual cost function. Hence, if the objective can be represented as a convex optimisation problem such as in our fourth scenario that fulfils the conditions in Section 5.2, the algorithm is able to cope with it. Another method to allow the algorithm to solve a convex optimisation problem is presented in [101]. There, we allowed communication among the vehicles. Both ideas are based upon finding consensus among agents. This naturally also means that any fairness notion that is based on a consensus can be realised using the AIMD algorithm. While we here give a good insight in how the parameters can be tuned, the tuning has to be done specifically for every fairness notion.

On the other hand for loads applying the BA algorithm, i.e. loads with a binary controllable power consumption, the selection of the tuning parameters is not very well defined. While it is possible to numerically compute each agent’s expected individual power consumption for small scale systems, larger scale systems are too complex to compute. Note that here large scale systems can mean that there are either many participants or that the individual parameters of the agents are selected to be large, see Chapter 3. We have not been able to find an analytic method to compute the expected power consumption for such large scale systems. Further investigations are required on how these parameters influence the expected share of the agents and also on how to properly tune them. Unless such future studies are performed and guidelines to facilitate the parameter selection are established, the BA algorithm is most likely used solely for allowing equal on times among the participating agents. While this is a major drawback of the BA algorithm, there are two factors that diminish these drawbacks. Firstly, in the future we expect the number of loads that allow continuous adjustment of their power consumption to increase significantly. Secondly, we expect that the loads that are limited to binary control of their power consumption generally are cheaper loads with a low power consumption, which therefore may not benefit from other fairness notions.

7.2 The Available Power

The proposed load management scheme is based on the principle that the aggregated power consumption of controllable and uncontrollable agents should follow a specific upper bound that is set by the provider of the service: the available power $P$. From this it follows directly that the available power has to be selected carefully to incorporate the wishes and needs of the parties in-
7.3 Failure of Components

In a system where a large number of agents are involved, such as our load management scheme, some components will eventually fail. The failure occurs either at the single agent or the central management unit, which have a highly different impact on the system.

The impact on the aggregated power is minimal when the control of a single agent fails, since it is only one agent out of a large number. There are two possible scenarios how the control of an agent fails: 1: the agent consumes more power than allowed, or 2: the agent consumes less power than allowed.

In the first case, the agent increases its power consumption above the value that is set by the algorithm. This occurs if the agent is incapable to react on CE signals or if the agent increases its power consumption faster than anticipated. Such a defect can for example, occur if the communication path between the management unit and the agent is disturbed such that not all CE signals are received. This means that the aggregated power consumption of the agents is generally higher than anticipated, which results in that the management unit sends out more CEs. In turn, the power consumption of the other agents is therefore slightly reduced compared to the case where all agents operate normally. The reduction is minimal though as long as the majority of the agents respond properly to CEs. The defective agent does not experience any inconvenience, because it receives more energy than under normal operation, which should be more than is needed. However, the owner might be affected financially since the agent consumes more power during peak times which often have a higher tariff assigned. The possible financial consequences for the owner depend therefore on the agreement between the customer and the provider of the service.

In the second case, the agent reduces its power consumption more often or by a larger amount than anticipated by the central management unit or increases its power consumption slower than intended by the algorithm. One reason for this can be that the agent reacts to more CE signals than are sent by the management unit. Another more likely reason is that the agent’s parameters are not correct. This defect allows other agents to increase their power consumption, because there are slightly less CEs. Again, as only one agent is affected the impact on the other agents...
7. Challenges

is minimal. The impact on the owners of the agent can in this case though be severe since their appliance may receive too little energy. That can not only be inconvenient for the customers, but also inflict damage or extra costs. Examples thereof are food that has gone bad, because the refrigerator did not cool properly, or the need of a taxi, since the EV did not receive enough energy. Note that at all times there is a slight risk that an appliance does not operate as it should also without the participation in the load management scheme. Also, the first case is a more likely fault than this latter case.

Naturally, the provider requires methods to find defective agents such that they can be repaired. This requires that the provider receives information about the agents. However, this can inconvenience the customers or lead to data protection issues. Additionally, the provider prefers that the fault detection is fast, reliable, and cost efficient. To detect any defect, the provider can for example, collect data about the power consumption of the agents. However, the customers might not be willing to share such information. Especially, since we aimed to develop the algorithm such that the management unit does not require any information from the agents, the collection of such data for the purpose of fault detection contradicts this goal. Naturally, the customers could be trained to detect any defects themselves. This can be inconvenient for the customers, because it requires their time and effort. Similarly, performing regular tests at the customer’s location might not be accepted by the customers and are expensive to perform. It might be possible to use predictions about the behaviour of the aggregated demand to detect faults. The problem using such techniques is that prediction errors might be of a similar magnitude, if not larger, compared to the effect of a single defective agent. Similarly, it might be possible to detect faults by investigating the aggregated demand of the customer, that is recorded using smart meter technology. How the aggregated demand reacts at CEs might indicate any defects. Naturally, it has to be taken into account that also other appliances are included in the measurement that are not controllable. The above illustrates that the problem of fault detection is not trivial and requires additional consideration.

Another problem occurs if the central management unit is defect. In this case, it either sends out too many or too few CE signals. In the first case the actual demand will be smaller than the available power, while in the second case the aggregated demand overshoots the available power. The severeness of both cases depends on how the available power is selected. The consequences can include over- and under-voltage, higher transmission losses, or even damage to transformers or lines. Since the management unit belongs to the provider it should not be difficult to detect any defects within a short time and react accordingly. Similarly, it might be possible that the owner is able to stop or interrupt the participation in the load management scheme at any time and then later reactivate the participation. In this way, the customer would be able to alleviate such defects. However, the interruption of the participation might be discouraged by higher electricity tariffs or other financial penalties. Such an interrupt function would also allow the customers to react to local faults of their controllable agents or to receive the maximum power in emergency situations, for example where an EV has to be charged immediately to be available as soon as possible.
7.4 Security

Due to the importance of electricity any actual implementation of a load management scheme has to consider the security and stability in regard to sabotage, warfare and terrorism. Methods are required to achieve maximum security, which requires an in depth analysis of all possible security issues as well as the effects on the distribution grid and the customers.

One of the most obvious cases of sabotage is “cheating” customers. This means that single customers interfere with the operation of the algorithm for their own advantage. In general, this means that they try to increase the power consumption of their appliances. Most of the time, the effect on the distribution grid is small as long as the number of cheating customers is small compared to the total number of participants. Nonetheless, the fairness among the agents is compromised, since one agent receives more power than fairly assigned. The effects that arise when large groups of customers cheat requires more consideration. Also, some methods disrupt the algorithm in more problematic ways, even if only a small number of customers are involved.

One method to cheat is for example the interruption of the communication path between the central management unit and the agent, such that the load does not receive any CE signals. In this case, the power consumption of the agent is at the allowed maximum independent of the CE signals sent by the management unit. This means that the agent reacts like an uncontrollable load. The effects are similar as if a single agent fails. If the number of cheating customers becomes too large, the management unit looses control over a substantive amount of power and hence might not be able to prevent the violation of the constraint in Equation (2.1). Depending on the selection of $\bar{P}$, this can have severe effects on the stability of the power grid operation and cause voltage deviations and damage to lines or transformers.

Another method for customers to cheat is by changing the tuning parameters of the algorithm. By changing these parameters the customer can achieve that the appliance can consume more power by turning on faster and for longer periods or by increasing its consumption faster without decreasing it. This means that the additive factor $\alpha_i$ and multiplicative factors $\beta^{(1)}_i$ and $\beta^{(2)}_i$ should be increased and the probability $\lambda_i$ decreased for loads applying the AIMD algorithm, while for loads applying the BA algorithm the number of steps to turn on $n_i$ should be decreased and the number of steps to turn off $m_i$ should be increased. In worst case, the agent behaves like an uncontrollable load, while less drastic changes to the parameters lead mostly to a different share. In both cases, the fairness among the agents is disturbed. Nonetheless, as long as there are only a few agents cheating, the effects on the distribution grid are minimal. However, if a large part of the customers adapt these parameters, the effects on the entire system are very hard to predict. For example, the number of CEs most likely increases as the available power is reached faster, this in turn evokes a more frequent reduction in the power consumption. Hence, counteracting the initial goal of the customer to increase the agent’s power consumption. On the other hand, too frequent CEs might overload the communication system, which then disturbs the entire system. Further, as seen in Section 5.1.1, the tuning parameters influence the expected overshoot, which might cause severe problems to the distribution grid.
Finally, an increase in a customer’s own allowed power consumption can be achieved by sending false CEs to other connected agents. This means that the agents of other customers reduce their power consumption while the agents of the sender continue to increase their power consumption. This leads to a smaller aggregated power consumption than anticipated by the management unit. Further, such additional transmissions might overload the communication network, leading to wide spread disruptions. Possible effects thereof can include voltage deviations and overloaded lines and transformers.

There might be other unforeseen possibilities on how single customers can cheat. All these methods should be closely investigated to find the immediate effects as well as their worst case impact on the entire system. Further studies are needed on methods how cheating can be detected and prevented. These problems can be related to the problems arising for the detection of defective agents. One possibility is to use the aggregate demand of the customer, that is recorded using a smart meter, to investigate the participation.

While cheating customers is one of the most likely forms of sabotage, other forms of sabotage as well as terrorism and warfare have to be considered. The latter forms of sabotage are more dangerous, since they very likely will be aimed at disturbing large parts of the distribution grid. Such interruptions can cause less severe problems such as higher transmission losses or less efficient energy usage, but can also lead to severe voltage deviations, overloaded lines and transformers, and in worst case to black outs. In general, such disruptions have a higher impact on the stability of the distribution grid compared to cheating customers, because larger areas are affected. Due to these severe consequences, in depth studies are required to understand the effects caused by this and to prevent severe disruptions. For example, to prevent some form of sabotage it might be possible to use encryption methods for the sending of CEs by the central management unit. In this way, the transmission of false CEs could be prevented. However, this requires a secure set up of the encryption.

### 7.5 Communication Aspects

Since the communication link between the management unit and the agents is a crucial part of the proposed load management scheme, the communication aspects can readily influence the capabilities of the algorithm. Further, it is a likely target of sabotage, as discussed previously. Therefore, the security and robustness are vital parts of the design to prevent such events. There is nowadays a multitude of possible communication methods; from signals sent over the power lines to wireless communication links. All these technologies have their advantages and disadvantages, such that the selection of the communication system becomes an important task in the design of the actual load management scheme.

In [105], some of the communication aspects have been briefly considered. In this paper, the communication side of the AIMD algorithm was investigated in a wide area wireless communications network. We showed that the communication load is highly dependent on the number of participating agents as well as their internal parameters. Apart from that, a dwell time has been introduced to account for the delays in the communication and the reaction of the agents. This
dwell time directly affects the ability of the algorithm to follow the available power. This partly happens because while the management unit is affected by the dwell time, the agents still increase their actual power consumption. This leads to a lower limit to which an agent is able to reduce its power consumption. This lower limit depends on the dwell time and the agent’s internal parameters. In the paper, we suggested to broadcast the global additive factor $\pi$ and the multiplicative factor $\beta$ at a CE to resolve this problem. Another method would be to choose the length of a time step such that it is longer than the expected delay and synchronise the operation of the agents and the management unit. For the BA algorithm no such studies have been performed. Similar problems may however arise. We would expect that the agents are more likely to turn on than off, shifting the actual consumed power to larger values than expected without a dwell time.

In general, if the communication network causes delays that are much shorter than the used time step length, we expect no problems. The length of the time steps on the other hand depends on how fast the algorithm should react. While in this thesis we normally assumed time step lengths of one second, other lengths can be used for faster or slower services. We found that the average power consumption over a 10 min to 15 min period is relatively good controlled using the one second long time steps, which is seen in the simulations in Section 6.1. The communication method should then be selected according to the maximum allowed delays in the system. Such considerations are discussed in a more generalised form with regard to smart grids in [21].

### 7.6 Power Grid Structures

Some grid structures may have an adverse effect on the operation of the load management scheme. The grid structure includes the actual interconnection between the loads as well as the distribution methods. While we have not experienced any of such adverse effects during the simulation tests on a radial low voltage test network with three-phase distribution, there is no guarantee that they do not exist. For example, while a three-phase distribution is common in Australia, there are some remote connections that use a single-wire earth return (SWER) connection. It is unlikely that such loads would be controlled using the proposed algorithm though, since the load is remote, while the algorithm is expected to be used at locations with many loads connected. Similar in North America often a split-phase system is used for the distribution. In this thesis we have not investigated such a network. If the algorithm should be applied in these networks it is important to investigate whether problems arise.

Also the distribution grid’s interconnections, e.g. the length of connecting power lines, the exact interconnection of the loads, influences the power and voltage at each point. This means that while the algorithms perform satisfactorily on one specific structure, this might not be the case for distribution grids with other structures and elements that by themselves are controlled, such as some distributed energy generators. For example, assume that the CE causing location is selected to be at a transformer to prevent it from overloading. Then, while the transformer does not overload, other elements, such as other transformers or lines, can still be overloaded. Also, there are multiple different grid structures that might be used in the future, which we have not considered, such as mesh structures, micro grids, or grids with large amounts of distributed energy generation. These more complex grid structures can impact the algorithms in undesired ways. For example,
7. Challenges

A distribution grid that is fed by multiple transformers, where at one transformer a management unit is controlling the power, influences also the power and voltage levels at the other transformers. Similarly, high amounts of distributed energy generation and the methods used to control their effect can interact with the load management scheme proposed here, and lead to oscillations both of the demand by the agents as well as the power provided by the distributed energy generators. Such interconnections need to be studied extensively for each occurring situation, before the algorithm can be applied in the specific situation. Another problem that we have neglected is an imbalance between the three phases. While imbalanced phases might also occur in cases where there is no load management scheme present it might be that the proposed algorithm increases the imbalance by reducing the power demand of one phase more than the other phases due to the connections of the loads. It is important to investigate such adverse effects in detail. Finally, the distribution grids we used for the simulations are relatively small, while the load management scheme might be used for large scale distribution grids. To assess the operation of the load management scheme in such grid structures, additional simulation studies are needed.

It is possible that additional management units are present, which are all able to send CE signals to the connected agents. In diverse grid structures additional management units can help to control multiple critical locations and aspects of the power grid, that would not be possible otherwise. While some of these management units still operate using an available power, others might control other quantities like voltage deviations, frequency deviations, temperatures, or currents. It is critical where such management units are located and which measures they use. Also, the available power at each management unit needs to be selected properly. Similarly, management units that operate on other measures have to be designed to send a CE signal at desired moments. For example, in [62]–[64] additional management units have been suggested that operate on voltage deviations and showed promising results in simulation studies. Further studies should investigate where such management units should be placed, what measures these should use, and when to send CE signals.

As soon as two or more management units are able to send CE signals, these can interfere with each other. In this thesis, we briefly investigated the impact on the overall system if two management units are selected. Here, both units operate on the available power but are located at two different transformers. We showed that during some periods of time mostly at one of the two units the active power is at the limit while at the other the available power is not yet reached. This indicates that it is possible to increase the total transferred power to the controllable agents without violating any of the constraints set by the management unit. In fact, we allowed the agents to adapt their tuning parameters related to the decrease $\beta^{(1)}$, $\beta^{(2)}$, and $\lambda$ dependent on the source of the CE. In this way, they are able to react with a larger decrease to one of the sources and a smaller decrease to the other. Naturally, such a reaction has a negative effect on the fairness among the agents, as some agents are more affected by the constraints of one source than the other. Our investigation however lacks the needed depth and was done to show the possible effects and potential of the load management scheme. In our simulations, we assigned the agents to regions that are arbitrarily chosen. These regions then defined the reaction of the agents to each of the two sources. It is very important though to choose such regions effectively to achieve a trade off
7.7. Expanding the Possibilities

between the violation of the fairness among the agents and the actual increase in total delivered power to the controllable agents. Further, how the agents react in such regions to each CE source needs to be optimised to achieve the maximum efficiency. The selection of the regions and their reaction depends highly on the structure of the distribution grid. It might even be possible to allow each agent a specific reaction on a received CE signal depending on its impact on the source of the CE instead of grouping several agents together.

A so far not mentioned problem that becomes very important both in case of controlling the reactive power as well as in case of multiple management units, is that the control of the agent does not cause unintended oscillations or a positive feedback behaviour. For example, when using reactive power control with multiple management units, at the beginning all agents use their maximum active power, however a management unit detects a voltage drop and hence sends out a signal to reduce the active power. Since now the agents are no longer using their full capacity they are able to provide additionally reactive power which stabilises the voltage. This allows the agents to increase their active power, while reducing their reactive power, which then again will cause the initial voltage drop. While the proposed algorithm is based on a probing up until a constraint is hit, we assumed that the reactive and active power limits are independent of each other. This is not the case and so there can be stronger oscillations than expected. The second problem that was mentioned is a positive feedback behaviour. With that we mean that the reaction to a CE might in turn provoke a CE at another location and so forth. Such effects, while unlikely to occur in a simple radial feeder network with one management unit, become more likely with increasingly complex grid structures and an increasing number of management units.

7.7 Expanding the Possibilities

While the above mentioned challenges have to be addressed before the suggested scheme can be applied on an actual power grid, we showed in this thesis that there is a potential for this flexible load management scheme. In the future, the system could be expanded to allow other services as well as other types of loads or simply improve the control. We would like to mention one of these possible services that caught our interest: to balance the power among the three phases. By balancing the phases under and over voltage issues on the power grid can be reduced. This service could be realised by allowing the management unit to measure the power consumption in each phase separately and send CE signals separately for each phase. Then, the available power at each phase, as selected by the provider as usual, could be adapted at each time step according to the demand of the other loads. Naturally, such adaptations have an impact on the fairness. Also, the modification has to be carefully selected to guarantee the most efficient use of the available resource. A major difficulty is also to prevent unintended feedback among the phases, i.e. that the reduction of the available power in one phase does not immediately cause a reduction in the next phase etc..

Regarding the possible agents that are able to be controlled, we would like to point out that an important category of allowed controller capabilities has not been considered in this thesis. These are loads that allow the controller to choose from multiple predefined power levels instead. Such control abilities might especially in the future be interesting as a transition between binary
and continuously controllable power consumption. Allowing such agents the participation in the proposed load management scheme requires the development of a third algorithm. There are many possibilities how such an algorithm could be realised.
In this thesis, we developed a distributed load management scheme to control two types of loads: electric vehicles (EVs) and refrigerators. These loads, or agents, are chosen due to their characteristics that make them suitable for the load management scheme which is discussed in Section 1.2.1. The load management scheme is based on broadcast signals that are sent to the controllable agents whenever the aggregated demand by both controllable and uncontrollable agents exceeds the available power $\bar{P}$. Some of the controllable agents reduce their power consumption upon the receipt of such broadcast signals, denoted capacity event (CE) signals. As long as no CE signal is received the aggregated demand of the controllable agents increases. Hence, this method allows the aggregated power of both controllable and uncontrollable agents to follow a specific curve: the available power $\bar{P}$. Note that in this thesis we concentrated on how to achieve the control and do not discuss how the available power should be selected. The principle of the proposed load management scheme is fairly simple, so that the implementation of such a system should be straightforward. Further, since the load management scheme relies solely on broadcast signals without any data collection from single agents, the communication load is relatively small and there are no concerns regarding data protection or privacy. This also allows the load management scheme to be scalable up to large numbers of controllable loads.

We investigated two distinct grid services: active power regulation and reactive power balancing. The first governs the active power consumption of the loads. This includes the case where the loads reduce or increase their own active power consumption and the case where the loads are also able to inject power into the distribution grid. The second service also adds the balancing of the reactive power consumption in a region to the active power regulation. In particular, the second service allows the agents to support the distribution grid without affecting the end-use operation. We believe this service shows in particular a great potential in industrial areas due to the high reactive power required in these areas.

The loads employ two different control abilities which are binary controllable power consumption, or continuously controllable power consumption. In this regard, we suggest two different methods on how the agents should react depending on the receipt of the CE signals. Since both methods are based on the same load management principle, it is possible that in a region both
methods are simultaneously present, i.e. some loads apply the first method while others apply the second. To the best of our knowledge, this is a difference to other work, where normally only one control ability is investigated. Nonetheless, we believe that the ability to simultaneously control different load types is an important contribution to facilitate the control of various appliances at the same time.

For loads with a continuously controllable power consumption, we apply the additive increase multiplicative decrease (AIMD) algorithm, which is commonly used in congestion control for communication networks. This algorithm allows us to focus on the service to the customer rather than primarily on the needs of the distribution grid. In this context, we believe that the customers should receive a fair share of the available power and reduce their power consumption by a fair amount, respectively. However, the notion of what fairness is may depend on the situation that the customer is in. Here, we suggest four basic notions of fairness for charging EVs and show how these could be fulfilled using the AIMD algorithm. Note that we do not claim that these are the only possible fairness notions. Rather they illustrate the flexibility of the proposed algorithm. This new perspective is in our opinion an important contribution to load management, since it has the potential to encourage customers with single agents and with fleets of agents to participate in the load management scheme. The load management scheme proposed in this thesis continuously controls the power consumption of the agents, which means that the single customer will frequently experience a slight reduction in the received power. While large reductions are rare due to the large number of participants. Also, as we focus on the wishes of the customers, the reduction is fair, i.e. all customers carry the burden to reduce the aggregated demand fairly among them. We think that such a control might be more broadly accepted than infrequent reductions that have a large impact on the customers without any fairness assurances. Further, it will contribute to achieve a trade off between benefits for the power grid and the fairness experienced by the customers. We also suggested a simple binary automaton (BA) algorithm based on stochastic decisions by the agents, when the load allows binary control over its power consumption. This algorithm allows for equal on times among the agents. It might also be possible to achieve other fairness notions by proper tuning of the internal parameters of the algorithm. We were however not able to provide analytic methods for selecting these parameters.

Both algorithms: the AIMD and the BA algorithm, have been analysed mathematically in regards to their stability, fairness among the participating agents, and behaviour of the aggregated demand, where we concentrated mostly on the AIMD algorithm. For the AIMD algorithm, we could use a vast amount of previous research that investigates the behaviour of the algorithm. However, there are some slight differences between the algorithm we apply here and the AIMD as it is applied in communications networks. These differences have been discussed in detail in Chapter 5 and we were able to expand some of the stability results analytically to cover our case, i.e. a discrete implementation of the system, varying available power, and the inclusion of upper and lower bounds for each agent’s power consumption. These results in turn might be useful for the application of the AIMD algorithm in other areas. Finally, a method is shown how the AIMD algorithm can be used for solving constrained optimisation problems in Section 5.2. We used this method in this thesis to fulfil one of the fairness notions that we proposed. However, the algorithm
has the potential to be applied in a variety of other fields where a large number of agents require to solve a constrained convex optimisation problem or the agents require to find a consensus with only limited communication capabilities. For the BA algorithm, we were also able to give an analytical stability result. Further, we showed methods how the behaviour of small scale systems can be predicted numerically. However, this analysis is not feasible for large scale systems. We worked on one method how to deal with such large scale systems and found a local stability argument. The problem with this result is that it is very restrictive and limiting in the form of the system structure. Nevertheless, additional analysis might lead to better analytic results. Additional to the mathematical analysis we simulated the system in a realistic power network to illustrate the possible benefits. This is presented in Section 6.1.

While the above characteristics are clear advantages of our proposed load management scheme, there are various open challenges that have not been addressed in detail in this thesis, including interference by customers and external sources, security issues and specific grid structures. These challenges would have to be solved before widespread adoption of the suggested load management scheme is possible.
Appendix A

Reasoning and Proofs

A.1 The BA Algorithm as a Markov Chain

First, we consider the simple case where there is only one agent. We assume that whenever this agent is on, a CE occurs. Its internal parameters are $n$ and $m$. Then the complete system can be in one of $n + m$ states, i.e. all the states the single agent can be in. Let $z(k)$ be the vector containing the probabilities that the system is in a specific state. In this system with only one participating agent we have

$$ z(k) = x(k). \quad (A.1) $$

Let $I_A$ be a diagonal matrix of size $(n + m) \times (n + m)$ containing 1 if the agent is turned on and 0 otherwise, i.e.

$$ I_A = \begin{bmatrix} 1 & 0 \\ 0 & I_m \end{bmatrix}. \quad (A.2) $$

Hence, $I_A$ indicates which states lead to a CE and which don’t. From that the transition matrix can be constructed by

$$ H = A_{nc}(I_{n+m} - I_A) + A_cI_A. \quad (A.3) $$

The Markov Chain associated with the system with only one agent participating is then

$$ z(k+1) = Hz(k) = (A_{nc}(I_{n+m} - I_A) + A_cI_A)z(k). \quad (A.4) $$

Next, consider the system where $N$ agents participate. From [78], we know that such a system can be described as a Markov Chain. Its states are all combinations of the states the single automata can be in. This means there is a total of $\prod_{i=1}^N (n_i + m_i)$ states. Let $z(0)$ again be the vector where each entry is the probability that the system is in one of these states at time step $k = 0$. This vector can be constructed by

$$ z(0) = x^{(1)}(0) \otimes x^{(2)}(0) \otimes \cdots \otimes x^{(N)}(0) \quad (A.5) $$

if we assume that the agents are independent at the beginning. Note that due to the interaction with the system the probabilities will not remain independent.
A. Reasoning and Proofs

Each state $z_i$ has a defined power consumption and hence defines whether the agents receive a CE signal or not depending on the available power $\bar{P}$ and the demand by the uncontrollable agents $\bar{p}$. For instance, consider the case where a total of three agents participate. Agent 1 and 3 are turned off with $\nu_i = \frac{1}{n_i}$ and $\mu = \frac{m_i}{m}$, but agent 2 is turned on with the same internal parameters, then the total consumed power is $\sum p_i(k) = p_2$. This state is associated with $z_{n_2(n_3+m_3)+1}$.

Let $I_A$ be again a diagonal matrix, containing 1 if the state leads to a CE and 0 otherwise, then the column stochastic transition matrix of the complete system with $N$ agents is

$$H = \left( A_{nc}^{(1)} \otimes \cdots \otimes A_{nc}^{(N)} \right) \left( I - I_A \right) + \left( A_c^{(1)} \otimes \cdots \otimes A_c^{(N)} \right) I_A. \quad (A.6)$$

where $\otimes$ denotes the Kronecker product.

A.2 Proof of Lemma 3.10

If we assume there is no CE at time step $k$ then only agents that are currently off react. Which means that all agents that are on remain in the same state. Further, all agents that turn on switch to the state $x_{n+1}$. Therefore, the ratio in all other on states keeps constant, i.e.

$$x_{n+l}(k+1) = x_{n+l}(k) \quad \text{for} \quad l \in \{2, 3, 4, \ldots, m\}.$$  

Similarly, all agents currently in state $x_1$ definitely leave this state either turning on and ending up in state $x_{n+1}$, where all the agents that turn on switch to, or remaining off and switching to the next state $x_2$. Hence,

$$x_1(k+1) = 0.$$

Using the law of large numbers and the fact that the agents react independently of each other the expected ratio of agents in state $x_1$ remaining off is assumed to be equal to the actual fraction of agents remaining off. Hence, the fraction of agents switching to state $x_2$ is

$$\left( 1 - \frac{1}{n} \right) x_1.$$

Additionally, all agents currently in $x_2$ leave that state. Hence,

$$x_2(k+1) = \left( 1 - \frac{1}{n} \right) x_1(k).$$

Continuing the argument for the following states, we get

$$x_l(k+1) = \left( 1 - \frac{1}{n} \right) x_{l-1}(k) \quad \text{for} \quad l \in \{2, 3, \ldots, n\}.$$  

All agents that are expected to turn on switch to state $x_{n+1}$. Additionally, all agents in state $x_{n+1}$ remain in this state. Hence, the fraction is evolving according to

$$x_{n+1}(k+1) = x_{n+1}(k) + \sum_{l=1}^{n} \frac{l}{n} x_l.$$
A.3. Proof of Lemma 3.11

Summarising the above leads to the following recursive system

\[
\begin{aligned}
x_1(k+1) &= 0 \\
x_l(k+1) &= \left(1 - \frac{l}{n}\right)x_{l-1}(k) \quad \text{for } l \in \{2, 3, \ldots, n\} \\
x_{n+1}(k+1) &= x_{n+1}(k) + \sum_{l=1}^{n} \frac{l}{n}x_l \\
x_{n+l}(k+1) &= x_{n+l}(k) \quad \text{for } l \in \{2, 3, \ldots, m\}.
\end{aligned}
\]

Following the same argument, we find the recursive equations at a CE to be

\[
\begin{aligned}
x_1(k+1) &= x_1(k) + \sum_{l=1}^{m} \frac{l}{m}x_{n+l} \\
x_l(k+1) &= x_l(k) \quad \text{for } l \in \{2, 3, \ldots, n\} \\
x_{n+1}(k+1) &= 0 \\
x_{n+l}(k+1) &= \left(1 - \frac{l}{m}\right)x_{n+l-1}(k) \quad \text{for } l \in \{2, 3, \ldots, m\}.
\end{aligned}
\]

These equations yield the switched system

\[
x(k+1) = A(k)x(k) \quad \text{with } A(k) \in \{A_{nc}, A_c\} \quad (A.7)
\]

where the switching law depends on the state and the matrices \(A_{nc}\) and \(A_c\) are defined as in Equations (3.15) and (3.16), respectively.

A.3. Proof of Lemma 3.11

First note that

\[
\begin{aligned}
\left|\begin{bmatrix} 0 & 1^T \end{bmatrix} (x^{(1)}(k) - x^{(2)}(k))\right| &= \left| \sum_{i=n+1}^{n+m} x_i^{(1)}(k) - x_i^{(2)}(k) \right| \\
&\leq \sum_{i=n+1}^{n+m} \left| x_i^{(1)}(k) - x_i^{(2)}(k) \right| \\
&\leq \left| x^{(1)}(k) - x^{(2)}(k) \right|_1. \quad (A.8)
\end{aligned}
\]

If state \(x^{(2)}(k)\) leads to a CE, we have

\[
\bar{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) < 0
\]

which means that

\[
\left| \bar{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) \right| = \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) - \bar{P}.
\]

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Also, we know that
\[
\begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(1)}(k) = \begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(2)}(k) + x^{(1)}(k) - x^{(2)}(k) \right) = \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) + \begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(1)}(k) - x^{(2)}(k) \right).
\] (A.9)

So as long as
\[
\begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) + \begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(1)}(k) - x^{(2)}(k) \right) > \tilde{P}
\] (A.10)
holds, both $x^{(1)}(k)$ and $x^{(2)}(k)$ experience a CE. Choose $\epsilon$ such that
\[
\epsilon \leq \left| \tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) \right|,
\] (A.11)
then
\[
\begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(1)}(k) - x^{(2)}(k) \right) > \tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}
\] (A.12)
holds if
\[
\left| \begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(1)}(k) - x^{(2)}(k) \right) \right| < \epsilon.
\] (A.13)

Note that Equation (A.13) is more restrictive than the condition in Equation (A.12) Because of Equation (A.8) we find that the claim in Lemma 3.11 holds in this case.

Secondly, we investigate the case where $x^{(2)}(k)$ does not lead to a CE. Then, the following holds
\[
\tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) \geq 0
\]
and we get
\[
\left| \tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k) \right| = \tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k).
\]

Since Equation (A.13) holds, we find that $x^{(1)}(k)$ does not experience a CE as long as
\[
\begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(1)}(k) - x^{(2)}(k) \right) < \tilde{P} - \begin{bmatrix} 0 & 1^T \end{bmatrix} x^{(2)}(k)
\] (A.14)
holds. With $\epsilon$ chosen as in Equation (A.11) and the above we find that the condition in Equation (A.14) holds if
\[
\left| \begin{bmatrix} 0 & 1^T \end{bmatrix} \left( x^{(1)}(k) - x^{(2)}(k) \right) \right| < \epsilon.
\] (A.15)

Hence, the state $x^{(1)}(k)$ and $x^{(2)}(k)$ experience the same switching if the condition in Equation (3.29) holds.

### A.4 Proof of Theorem 3.13

We first show that the sequence starting with $x(0)$ is stable in the sense that it will remain close to the valid cycle $x^*$. Then in a second step we show that it actually approaches the valid cycle.

Due to Equation (3.31) and Lemma 3.11 $x(0)$ and $x^*(0)$ experience the same switching. Then, because of Lemma 3.12
\[
||x(1) - x^*(1)||_1 \leq \epsilon.
\]

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Again, as ε is defined as the minimum, the two states experience the same switching. The argument can be repeated endlessly. This shows that the sequence experiences identical switching and that the distance to the valid cycle remains bounded by ε.

In the next step, we show that the distance to the valid cycle reduces whenever a switch from $A_{nc}$ to $A_c$ occurs. A switch from $A_{nc}$ to $A_c$ means that first $A_{nc}$ is applied at time step $k$ followed by $A_c$ at time step $k + 1$. We define for simplicity

$$d(k) = x(k) - x^*((k \mod T)).$$

(A.16)

Then,

$$||d(k + 2)||_1 = ||A_cA_{nc}(x(k) - x^*((k \mod T)))||_1 = ||A_cA_{nc}d(k)||_1.$$\[\text{Using the definition of } A_{nc} \text{ and } A_c \text{ we find}\]

$$||d(k + 2)||_1 = \left|\left| \frac{1}{m} \sum_{l=1}^{n} \frac{l}{n} d_l(k) + \sum_{r=1}^{m} \frac{r}{m} d_{n+r}(k) \right|\right|_1 = \frac{1}{m} \sum_{l=1}^{n} \frac{l}{n} d_l(k) + \left|\left| \sum_{r=1}^{m} \frac{r}{m} d_{n+r}(k) \right|\right|_1 + \sum_{l=1}^{n-1} \left(1 - \frac{l}{n}\right) d_l(k) + \sum_{l=2}^{m-1} \left(1 - \frac{l}{m}\right) d_{n+l}(k).$$

Also, note that

$$1^T d(k) = 0$$

(A.17)

for all $k$, as

$$1^T x(k) = 1^T x^*((k \mod T)) = 1.$$ (A.18)

This means that $d(k)$ is either equal to 0 or contains both positive and negative entries. Hence,

$$||d(k + 2)||_1 < \frac{1}{nm} \sum_{l=1}^{n} \frac{l}{n} |d_l(k)| + \sum_{l=1}^{n-1} \left(1 - \frac{l}{n}\right) |d_l(k)| + \sum_{l=2}^{m-1} \left(1 - \frac{l}{m}\right) |d_{n+l}(k)| + \left|\left| \sum_{r=1}^{m} \frac{r}{m} d_{n+r}(k) \right|\right|_1 + \sum_{l=2}^{m-1} \left(1 - \frac{l}{m}\right) |d_{n+l}(k)|.$$
We now apply the triangle inequality and rearrange the terms such that

\[ ||d(k+2)||_1 < \frac{1}{nm} \sum_{l=1}^{n} l ||d_l(k)|| + \sum_{l=1}^{m} \frac{l}{m} ||d_{n+l}(k)|| + \sum_{l=1}^{n-1} \left( 1 - \frac{1}{n} \right) ||d_l(k)|| \]

\[ + \left( 1 - \frac{1}{m} \right) \sum_{l=1}^{n} \frac{l}{n} ||d_l(k)|| \]

\[ + \sum_{l=2}^{m} \left( 1 - \frac{1}{m} \right) ||d_{n+l}(k)|| \]

\[ = \sum_{l=1}^{n} ||d_l(k)|| + \frac{1}{m} ||d_{n+1}(k)|| + \sum_{l=2}^{m} ||d_{n+l}(k)|| \]

\[ = \sum_{l=1}^{n} ||d_l(k)|| + ||d_{n+1}(k)|| + \sum_{l=2}^{m} ||d_{n+l}(k)|| = ||d(k)||_1 \]

Note that similar steps lead to a decrease when switching from \( A_c \) to \( A_{nc} \).

Note that after at most \( n \) non-CEs in a row, a CE is generated. Analogously, there are at most \( m \) CEs in a row. This means that every valid cycle contains at least one switching from \( A_{nc} \) to \( A_c \). Hence, the distance

\[ ||x(k) - x^*((k \mod T))||_1 \]

decreases over each cycle.

### A.5 Reasoning behind Section 4.1.2.4

In Section 5.2, the AIMD algorithm is applied to solve a convex optimisation problem with constraints. In Section 4.1.2.4, we argued that the same technique can be used for our implementation of an AIMD with small adjustments. Here, we justify these claims. Note that this is however no proof of the convergence of the algorithm.

We consider the second objective in Equation (2.4). We assume that the agents are able to draw more power than the constraint in Equation (2.1) allows. This means that

\[ O_1(k) = \bar{P} - \bar{p}. \]  \hspace{1cm} (A.19)

Otherwise, the second objective would not be considered as there is no freedom to schedule the agents, i.e. each agent would consume its maximum allowed power. When Equation (A.19) holds, we find that one of the Karush-Kuhn-Tucker (KKT) condition of the second objective is

\[ \frac{\partial}{\partial p_i} g_i(p_i) + \gamma_i - \nu = 0, \]  \hspace{1cm} (A.20)

where \( \gamma_i \) and \( \nu \) are Lagrange multipliers caused by the upper bound and the constraint in Equation (A.19), respectively. Hence, we find a similar condition as in [116], [117]. Inspecting three remaining KKT conditions:

\[ \gamma_i \geq 0, \]  \hspace{1cm} (A.21)

\[ p_i \leq \bar{p}_i \]  \hspace{1cm} (A.22)
and
\[ \gamma_i(p_i - \overline{p}_i) = 0 \quad \text{(A.23)} \]
for all \( i \), we find that the consensus of the derivative as long as the upper bound of the power is not reached is an indication for the optimum. Unlike in \[116\], \[117\] we assume in this thesis that \( \beta^{(2)}_i \) is not equal to 1. Let us assume that Claim 5.18 holds. This means that the average power consumption of an agent converges to
\[ p_i^* = \min \left( C \frac{\alpha_i}{1 - b_i}, \overline{p}_i \right), \quad \text{(A.24)} \]
where \( C \) is a constant to assure
\[ \mathbf{1}^T p^* = \bar{P} - \bar{p}. \quad \text{(A.25)} \]
Let \( \mu_i \in \mathcal{M}_{p_i^*} \) be as introduced in Section 5.1.3, Then, Equation (A.24) can be rearranged to
\[ p_i^* + \mu_i^* = C \frac{\alpha_i}{1 - b_i}. \quad \text{(A.26)} \]
Using the definition of \( b_i \) in Equation (5.76) and rearranging the above is equivalent to
\[ \left( \frac{1 - \beta^{(2)}_i}{\beta^{(2)}_i - \beta^{(1)}_i} + \lambda_i \right) p_i^* + \frac{1 - b_i}{\beta^{(2)}_i - \beta^{(1)}_i} \mu_i^* = C \frac{\alpha_i}{\beta^{(2)}_i - \beta^{(1)}_i}, \quad \text{(A.27)} \]
where we assume that the agents use identical AIMD parameters \( \alpha, \beta^{(1)}_i, \) and \( \beta^{(2)}_i \). Comparison with the KKT condition in Equation (A.20) show that the condition holds if
\[ \frac{\partial}{\partial p_i^*} g_i(p_i^*) = p_i^* \left( \frac{1 - \beta^{(2)}_i}{\beta^{(2)}_i - \beta^{(1)}_i} + \lambda_i \right), \quad \text{(A.28)} \]
\[ \gamma_i = \frac{1 - b_i}{\beta^{(2)}_i - \beta^{(1)}_i} \mu_i^*, \quad \text{(A.29)} \]
\[ \nu = C \frac{\alpha_i}{\beta^{(2)}_i - \beta^{(1)}_i}, \quad \text{(A.30)} \]
This means that \( \lambda_i \) should be chosen as proposed in Equation (4.13). In \[116\], \[117\] and most likely in our case the probability \( \lambda_i \) has to fulfil three assumptions:

1. \( \lambda_i(\rho_i) \) is continuous
2. \( \rho_i \lambda_i(\rho_i) \) is strictly increasing on \([0, \overline{p}_i]\)
3. \( \lambda_i(\rho_i) \) is bounded above 0.

Naturally, the probability should be selected to lie in the range \([0, 1]\). The second assumption holds for any convex cost function, hence we only consider convex objectives. The first and third assumption require the objective to be selected accordingly. Note that it is possible to introduce a positive scaling factor \( \Gamma \) to accommodate for these constraints such that the probability can be selected according to
\[ \lambda_i = \Gamma \frac{\beta^{(2)}_i g_i(p_i)}{\rho_i} - \frac{1 - \beta^{(2)}_i}{\beta^{(2)}_i - \beta^{(1)}_i}, \quad \text{(A.31)} \]
Notice that the scaling factor \( \Gamma \) is a system wide constant and as such has to be communicated to the participating agents. It remains to check whether Equations (A.21) to (A.23) hold. Since
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\( \mu_1^* \geq 0, \beta^{(2)} > \beta^{(1)} \), and \( b_i < 1 \) Equation (A.21) holds at all times. Equation (A.22) holds by the definition of \( p^*_i \) in Equation (A.24). Finally, \( \mu^*_i = 0 \) if \( p^*_i \leq \overline{p}_i \), which means that Equation (A.23) holds.

A.6 Proof of Lemma 5.1

Analogously to [97], we investigate the state at CEs. Using Equations (5.2) to (5.4) the time between two consecutive CEs, i.e. the time between the \( k \)-th and \( (k+1) \)-th CE \( \Delta \tau_{k+1} \) can be computed by

\[
\Delta \tau_{k+1} = \frac{\sum_i (1 - \beta_i(\tau_k))p_i(\tau_k)}{\sum_i \alpha_i \alpha}.
\]  

(A.32)

Hence, by inserting \( \Delta \tau_{k+1} \) into Equations (5.2) and (5.3), the evolution from one CE to the next can be written as

\[
p_i(\tau_{k+1}) = \beta_i(\tau_k)p_i(\tau_k) + \alpha_i \frac{\sum_i (1 - \beta_i(\tau_k))p_i(\tau_k)}{\sum_i \alpha_i}.
\]  

(A.33)

Then, the complete system can be written in matrix form as

\[
p(\tau_{k+1}) = A(\tau_k)p(\tau_k) = A(\tau_k)A(\tau_{k-1}) \cdots A(0)p(0),
\]  

(A.34)

where \( A(\tau_k) \) is

\[
A(\tau_k) = \text{diag}(\beta(\tau_k)) + (1^T \alpha)^{-1} \alpha(1 - \beta(\tau_k))^T.
\]  

(A.35)

A.7 Proof of Theorem 5.3

This proof is restated from [97], [115] with minor changes to adapt to our system.

From the discussion in Section 5.1, we know that \( A(\tau_k) \) is taken from a set \( \mathcal{A} \), which has a total of \( r \) entries. Let \( \mathcal{I}_\ell \) be defined as in Section 5.1. The probability to choose a specific matrix is given in Equation (5.11). It is clear that in case Assumptions 5.2 and 5.3 hold this probability is independent of the CE event, i.e.

\[
p_{A_{\mathcal{I}_\ell}} = \text{Pr}[A(\tau_k)] = \text{Pr}[A(\tau_0)].
\]  

(A.36)

This means that the expected value of \( A(\tau_k) \) is also independent of the CE and hence

\[
E[A(\tau_k)] = E[A(\tau_0)] = \sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} A_{\mathcal{I}_\ell}.
\]  

(A.37)

Then, using again the independence of the variables \( A(\tau_k), \ldots, A(\tau_0) \) shows that

\[
E[\Pi(\tau_k)] = E[A]^T = \left( \sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} A_{\mathcal{I}_\ell} \right)^{k+1}.
\]  

(A.38)

Next, we want to find the asymptotic behaviour. First remark that from Equation (5.9), we find

\[
\sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} A_{\mathcal{I}_\ell} = \sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} \text{diag}(\beta_{\mathcal{I}_\ell}) + \sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} (1^T \alpha)^{-1} \alpha(1 - \beta_{\mathcal{I}_\ell})^T.
\]  

(A.39)
where $\beta_{\mathcal{I}_\ell}$ is the vector with the $i$-th element equal $\beta^{(1)}_i$ if $i \in \mathcal{I}_\ell$ and $\beta^{(2)}_i$ otherwise. We first investigate $\sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} \text{diag}(\beta_{\mathcal{I}_\ell})$. It is clear that the resulting matrix is a diagonal matrix where its $i$-th diagonal entry is

$$\lambda_i \beta^{(1)}_i + (1 - \lambda_i) \beta^{(2)}_i. \tag{A.40}$$

Then, considering the second part $\sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} (1^T \alpha)^{-1} \alpha (1 - \beta_{\mathcal{I}_\ell})^T$, we find that only $p_{A_{\mathcal{I}_\ell}}, \beta_{\mathcal{I}_\ell}$ depends on $\ell$, such that the other variables can be taken out of the sum. We find further that

$$\sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} (1 - \beta_{\mathcal{I}_\ell})^T \tag{A.41}$$

is a row vector where its $i$-th entry is equal to

$$1 - \left( \lambda_i \beta^{(1)}_i + (1 - \lambda_i) \beta^{(2)}_i \right). \tag{A.42}$$

Hence,

$$\sum_{\ell=1}^r p_{A_{\mathcal{I}_\ell}} A_{\mathcal{I}_\ell} = \text{diag}(b(\tau_k)) + (1^T \alpha)^{-1} \alpha (1 - b(\tau_k))^T, \tag{A.43}$$

where $b = \lambda_i \beta^{(1)}_i + (1 - \lambda_i) \beta^{(2)}_i$. This means that the expected value of $A(\tau_k)$ has the same structure as an AIMD matrix with multiplicative factors $b$. Hence, using [97], $E[A(\tau_k)]$ is a positive, column stochastic matrix, with a Perron eigenvector given by

$$x^*_p = \left[ \frac{\alpha_1}{1 - (\lambda_1 \beta^{(1)}_1 + (1 - \lambda_1) \beta^{(2)}_1)} \cdots \frac{\alpha_N}{1 - (\lambda_N \beta^{(1)}_N + (1 - \lambda_N) \beta^{(2)}_N)} \right]^T. \tag{A.44}$$

From the above Theorem 5.3 follows directly.

### A.8 Proof of Lemma 5.5

As $\mathcal{P}$ contains the canonical basis vectors $e_i$, and the norm on $\mathbb{R}^{N \times N}$ induced by $\|\cdot\|_1$ is the maximum absolute column sum, it follows that for all $\frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) \in \mathbb{R}^{N \times N}$

$$\left\| \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) \right\|_1 = \max_{i=1,\ldots,N} \left\| \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) e_i \right\|_1 = \max_{p \in \mathcal{P}} \frac{1}{p - \bar{p}} \left\| \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) p \right\|_1. \tag{A.45}$$

Fix $\varepsilon, \delta > 0$. Due to the convergence in probability and the fact that

$$\rho_k - \rho^* = \left( \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) - x^*_p 1^T \right) p(\tau_0), \tag{A.46}$$

we may choose $k_0(e_i)$ such that for all $k \geq k_0(e_i)$ we have

$$\Pr \left[ \left\| \left( \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) - x^*_p 1^T \right) e_i \right\|_1 > \delta \right] < \frac{\varepsilon}{N} \quad i = 1, \ldots, N. \tag{A.47}$$

By choosing $k_0 = \max_{i=1,\ldots,N} k_0(e_i)$ we obtain for all $k \geq k_0$ that

$$\Pr \left[ \left\| \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) - x^*_p 1^T \right\|_1 > \delta \right] = \Pr \left[ \max_{i=1,\ldots,N} \left\| \frac{1}{k+1} \sum_{\ell=0}^k \Pi(\tau_\ell) - x^*_p 1^T \right) e_i \right\|_1 > \delta \right] < \varepsilon, \tag{A.48}$$

where we used the standard estimate $\Pr [\cup_{i=1}^n W_i] \leq \sum_{i=1}^n \Pr[W_i]$ for events $W_i$. Hence, the claim in Equation (5.23) is an immediate consequence of A.45.
A.9 Proof of Lemma 5.6

Due to Equations (5.24) and (5.25), we can define \( \zeta_1 \) and \( \zeta_2 \) with

\[
0 \leq \zeta_1 < 1^T \alpha \tag{A.49}
\]

and

\[
0 \leq \zeta_2 < 1^T \alpha \tag{A.50}
\]

such that

\[
\sum_{i=1}^{N} p_i(\tau_k) = \bar{P} - \bar{p} + \zeta_2 \tag{A.51}
\]

and

\[
\sum_{i=1}^{N} p_i(\tau_k) = \bar{P} - \bar{p} + \zeta_1 \tag{A.52}
\]

hold.

Let \( \Delta k \) be a non-negative scalar. Then, Equations (4.1) and (4.2) allow us to find the evolution of the power consumption from one CE to the next to be

\[
p_i(\tau_{k+1}) = \beta(1)^i p_i(\tau_k) + \alpha_i \Delta k.	ag{A.53}
\]

Using Equations (5.30), (5.31) and (A.53) we find \( \Delta k \) to be

\[
\Delta k = \frac{1}{\alpha} \sum_{i=1}^{N} \alpha_i \sum_{i=1}^{N} \left( p_i(\tau_{k+1}) - \beta(1)^i p_i(\tau_k) \right)
\]

or

\[
\Delta k = \frac{1}{\alpha} \sum_{i=1}^{N} \alpha_i \left( \sum_{i=1}^{N} \left( p_i(\tau_k) - \beta(1)^i p_i(\tau_k) \right) + \zeta_2 - \zeta_1 \right).	ag{A.54}
\]

By inserting Equation (A.54) in Equation (A.53) we find the power consumption of each agent to be

\[
p_i(\tau_{k+1}) = \beta(1)^i p_i(\tau_k) + \frac{\alpha_i}{\sum_{i=1}^{N} \alpha_i} \left( \sum_{i=1}^{N} \left( p_i(\tau_k) - \beta(1)^i p_i(\tau_k) \right) + \zeta_2 - \zeta_1 \right).	ag{A.55}
\]

In matrix form this becomes

\[
p(\tau_{k+1}) = Ap(\tau_k) + (1^T \alpha)^{-1} \alpha (\zeta_2 - \zeta_1).
\]

A.10 Proof of Lemma 5.7

We consider the system

\[
p(\tau_{k+1}) = Ap(\tau_k) + \alpha (\gamma_2 - \gamma_1) \tag{A.56}
\]

\[
\gamma_1 \leq \gamma_1 \leq \bar{\gamma}_1 \tag{A.57}
\]

\[
\gamma_2 \leq \gamma_2 \leq \bar{\gamma}_2 \tag{A.58}
\]
A.10. Proof of Lemma 5.7

with
\[ 1^T p(\tau_{k+1}) = \hat{P} - \hat{p} + \gamma_2 \] (A.59)

and
\[ 1^T p(\tau_k) = \hat{P} - \hat{p} + \gamma_1. \] (A.60)

Let \( p^* \) be the Perron eigenvector of \( A \) such that
\[ 1^T p^* = \hat{P} - \hat{p}. \] (A.61)

Then, we consider the norm
\[ ||Ap(\tau_k) - p^*||_1. \] (A.62)

Using the definition of \( A \) and applying the triangle inequality yields
\[
||Ap(\tau_k) - p^*||_1 = \sum_{i=1}^{N} |\beta^{(1)}(p_i(\tau_k) - p^*_i) + \sum_{\ell=1}^{\ell} (1 - \beta^{(1)})(p_\ell(\tau_k) - p^*_\ell)|
\]
\[
\leq \sum_{i=1}^{N} \beta^{(1)} |p_i(\tau_k) - p^*_i| + \sum_{\ell=1}^{\ell} (1 - \beta^{(1)})(p_\ell(\tau_k) - p^*_\ell) \] (A.63)
\[
= \sum_{i=1}^{N} \beta^{(1)} |p_i(\tau_k) - p^*_i| + \sum_{\ell=1}^{\ell} (1 - \beta^{(1)})(p_\ell(\tau_k) - p^*_\ell)
\]

We define the two sets
\[ S^+ = \{ i \in \{1, \ldots, N \} | p_i(\tau_k) - p^*_i \geq 0 \} \] (A.64)

and
\[ S^- = \{1, \ldots, N \} \setminus S^+. \] (A.65)

Then, using these definitions and the triangle inequality, the bound on the norm can be rewritten as
\[
||Ap(\tau_k) - p^*||_1 \leq \sum_{i \in S^+} \beta^{(1)} |p_i(\tau_k) - p^*_i| + \sum_{i \in S^-} \beta^{(1)} |p_i(\tau_k) - p^*_i|
\]
\[
+ \sum_{i \in S^+} (-\beta^{(1)})(p_i(\tau_k) - p^*_i) + \sum_{i \in S^-} (-\beta^{(1)})(p_i(\tau_k) - p^*_i) + \sum_{i=1}^{N} p_i(\tau_k) - p^*_i \] (A.66)
\[
\leq \sum_{i \in S^+} \beta^{(1)} |p_i(\tau_k) - p^*_i| + \sum_{i \in S^-} \beta^{(1)} |p_i(\tau_k) - p^*_i| + \sum_{i=1}^{N} p_i(\tau_k) - p^*_i
\]
\[
+ \sum_{i \in S^+} (-\beta^{(1)})(p_i(\tau_k) - p^*_i) + \sum_{i \in S^-} (-\beta^{(1)})(p_i(\tau_k) - p^*_i)
\]

We now distinguish two cases

1. \( |\sum_{i \in S^+} (-\beta^{(1)})(p_i(\tau_k) - p^*_i)| \geq |\sum_{i \in S^-} (-\beta^{(1)})(p_i(\tau_k) - p^*_i)| \) and
2. \[ |\sum_{i \in S^+} (-\beta^{(1)} i)(p_i(\tau_k) - p_i^*)| < |\sum_{i \in S^-} (-\beta^{(1)} i)(p_i(\tau_k) - p_i^*)|. \]

In the first case, the norm can be bounded by

\[ \|Ap(\tau_k) - p^*\|_1 \leq \sum_{i \in S^+} |\beta^{(1)} i| |p_i(\tau_k) - p_i^*| + \sum_{i \in S^-} |\beta^{(1)} i| |p_i(\tau_k) - p_i^*| + \left| \sum_{i=1}^N p_i(\tau_k) - p_i^* \right| \]

(A.67)

Hence,

\[ \|Ap(\tau_k) - p^*\|_1 \leq 2 \sum_{i \in S^+} |\beta^{(1)} i| |p_i(\tau_k) - p_i^*| + \left| \sum_{i=1}^N p_i(\tau_k) - p_i^* \right| \]

(A.68)

Note that due to Equations (5.36) and (A.61), the following holds

\[ \sum_{i \in S^+} p_i(\tau_k) - p_i^* = \gamma_1 - \sum_{i \in S^-} p_i(\tau_k) - p_i^*. \]

(A.69)

Inserting Equation (A.69) into Equation (A.68) yields

\[ \|Ap(\tau_k) - p^*\|_1 \leq \max_i |\beta^{(1)} i| \left( \sum_{i \in S^+} (p_i(\tau_k) - p_i^*) + \sum_{i \in S^-} (p_i(\tau_k) - p_i^*) \right) + \left| \sum_{i=1}^N p_i(\tau_k) - p_i^* \right| \]

(A.70)

Using Equations (5.36) and (A.61) leads to the bound

\[ \|Ap(\tau_k) - p^*\|_1 \leq \max_i |\beta^{(1)} i| \left( \sum_{i \in S^+} |p_i(\tau_k) - p_i^*| + \sum_{i \in S^-} |p_i(\tau_k) - p_i^*| \right) + \max_i |\beta^{(1)} i| \gamma_1 + |\gamma_1| \]

(A.71)

Considering the second case, we similarly find that

\[ \|Ap(\tau_k) - p^*\|_1 \leq \sum_{i \in S^+} |\beta^{(1)} i| |p_i(\tau_k) - p_i^*| + \sum_{i \in S^-} |\beta^{(1)} i| |p_i(\tau_k) - p_i^*| + \left| \sum_{i=1}^N p_i(\tau_k) - p_i^* \right| \]

(A.72)
which yields

\[ ||Ac(p(\tau_k) - p^*)||_1 \leq 2 \sum_{i \in S^-} \beta(1)_i |p_i(\tau_k) - p_i^*| + \sum_{i=1}^N |p_i(\tau_k) - p_i^*| \]

\[ \leq 2 \max_i (\beta(1)_i) \sum_{i \in S^-} |p_i(\tau_k) - p_i^*| + \sum_{i=1}^N |p_i(\tau_k) - p_i^*| \]  \hspace{1cm} (A.73)

Using Equations (5.36) and (A.61) we find

\[ ||Ac(p(\tau_k) - p^*)||_1 \leq \max_i (\beta(1)_i) \left( \sum_{i \in S^-} |p_i(\tau_k) - p_i^*| \right) + \sum_{i=1}^N |p_i(\tau_k) - p_i^*| \]

\[ = \max_i (\beta(1)_i) \left( \sum_{i \in S^+} |p_i(\tau_k) - p_i^*| + \gamma_1 + \sum_{i \in S^-} |p_i(\tau_k) - p_i^*| \right) + \sum_{i=1}^N |p_i(\tau_k) - p_i^*| \]

\[ = \max_i (\beta(1)_i) ||Ac(p(\tau_k) - p^*)||_1 - \max_i (\beta(1)_i) \gamma_1 + \gamma_1 \]  \hspace{1cm} (A.74)

Hence, we have

\[ ||Ac(p(\tau_k) - p^*)||_1 \leq \max_i (\beta(1)_i) ||Ac(p(\tau_k) - p^*)||_1 + \left( \max_i (\beta(1)_i) + 1 \right) |\gamma_1|. \]  \hspace{1cm} (A.75)

We now consider the norm

\[ ||Ac(p(\tau_{k+1}) - p^*)||_1. \]  \hspace{1cm} (A.76)

Using the triangular inequality and Equations (5.35), (5.36), (A.61) and (A.75), we find

\[ ||Ac(p(\tau_{k+1}) - p^*)||_1 = \left( ||Ac(p(\tau_k) - p^*) + (1^T \alpha)^{-1} \alpha (\gamma_2 - \gamma_1) ||_1 \right) \]

\[ \leq ||Ac(p(\tau_k) - p^*)||_1 + \left( (1^T \alpha)^{-1} \alpha (\gamma_2 - \gamma_1) \right) ||_1 \]

\[ \leq \max_i (\beta(1)_i) ||Ac(p(\tau_k) - p^*)||_1 + \left( \max_i (\beta(1)_i) + 1 \right) |\gamma_1| + \left( (1^T \alpha)^{-1} \alpha (\gamma_2 - \gamma_1) \right) ||_1 \]

\[ = \max_i (\beta(1)_i) ||Ac(p(\tau_k) - p^*)||_1 + \left( \max_i (\beta(1)_i) + 1 \right) |\gamma_1| + \gamma_2 - \gamma_1 \]  \hspace{1cm} (A.77)

### A.11 Proof of Lemma 5.9

Assume at a given time step \( \tau_k \) the value of the variation in the available power \( \psi(\tau_k) \) is given. As the variation is bounded at each time step we can compute the minimum and maximum time to the next CE event from \( \tau_k \). Let \( \Delta T \) be the minimum inter CE time and \( \Delta \hat{T} \) be the maximum time. We know that the minimum time occurs if the variation in the power is constantly at its minimum \( \tilde{\psi} \). Hence, the minimum time can be computed by solving

\[ \tilde{P} - \tilde{p} + \tilde{\psi} = \sum_{i=1}^N \beta(1)_i p_i(\tau_k) + \alpha \tilde{\tau} \Delta \hat{T}. \]  \hspace{1cm} (A.78)
As \( \psi(\tau_k) \) is given the following holds

\[
\sum_{i=1}^{N} p_i(\tau_k) = \bar{P} - \bar{p} + \psi(\tau_k).
\] (A.79)

Inserting Equation (A.79) into Equation (A.78) yields

\[
\Delta T = \frac{1}{\sum_{i=1}^{N} \alpha_i} \left( \sum_{i=1}^{N} (1 - \beta^{(1)}_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right).
\] (A.80)

Similarly, the maximum inter CE time occurs if the variation is all the time at its maximum. Hence, solving

\[
\bar{P} - \bar{p} + \bar{\psi} = \sum_{i=1}^{N} \beta^{(1)}_i p_i(\tau_k) + \alpha_i \pi \Delta \bar{T}
\] (A.81)

finds \( \Delta \bar{T} \). By using Equation (A.79) we find

\[
\Delta \bar{T} = \frac{1}{\sum_{i=1}^{N} \alpha_i} \left( \sum_{i=1}^{N} (1 - \beta^{(1)}_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right).
\] (A.82)

From Equations (5.2) and (5.3) we see that the power consumption of each agent is linearly increasing with \( t - \tau_k \). Since \( \Delta T \) indicates the time between CEs, this means that the power consumption increases linearly with \( \Delta T \). Hence, inserting \( \Delta T \) and \( \Delta \bar{T} \) into Equations (5.2) and (5.3), respectively, delivers an upper and lower bound such that

\[
p_i(\tau_{k+1}) \geq \beta^{(1)}_i p_i(\tau_k) + \frac{\alpha_i}{\sum_{i=1}^{N} \alpha_i} \left( \sum_{i=1}^{N} (1 - \beta^{(1)}_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right)
\] (A.83)

and

\[
p_i(\tau_{k+1}) \leq \beta^{(1)}_i p_i(\tau_k) + \frac{\alpha_i}{\sum_{i=1}^{N} \alpha_i} \left( \sum_{i=1}^{N} (1 - \beta^{(1)}_i) p_i(\tau_k) - \psi(\tau_k) + \bar{\psi} \right).
\] (A.84)

There exists a scalar \( \zeta \) with \( \bar{\psi} \leq \zeta \leq \bar{\psi} \) such that

\[
p(\tau_{k+1}) = Ap(\tau_k) + (1^T \alpha)^{-1} \alpha (\zeta - \psi(k))
\] (A.85)

where \( A \) is the AIMD matrix defined in Equation (5.9).

\section*{A.12 Proof of Theorem 5.15}

The proof consists of two parts. First we show that the suggested fixed point is indeed a fixed point of the system. Then, it is shown that this fixed point is unique.

Let the set \( \mathcal{H} \) be

\[
\mathcal{H} = \left\{ h \in \{1, 2, \ldots, N\} \left| p_h^* = C \frac{\alpha h}{1 - \beta^{(1)}_h} \right. \right\}.
\]
This is the set of all indices for which the postulated fixed point is not saturated. Then, the right hand side of Equation (5.67) becomes

\[
\beta^{(1)} f p^*_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell=1}^N (1 - \beta^{(1)\ell}) p^{*\ell}_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell=1}^N \mu^{\ell}_f - \mu^*_f \\
= \beta^{(1)} f p^*_f - \mu^*_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell \in H} \left( (1 - \beta^{(1)\ell}) p^{*\ell}_f + \mu^{\ell}_f \right) + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{j \notin H} \left( (1 - \beta^{(1)j}) p^{*j}_f + \mu^{j}_f \right).
\]  

(A.86)

Inserting Equations (5.70) and (5.71) into Equation (A.86) and rearranging leads to

\[
\beta^{(1)} f p^*_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell=1}^N (1 - \beta^{(1)\ell}) p^{*\ell}_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell=1}^N \mu^{\ell}_f - \mu^*_f \\
= \beta^{(1)} f p^*_f - \mu^*_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell \in H} C \left( 1 - \beta^{(1)\ell} \right) \alpha^{\ell}_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{j \notin H} \left( (1 - \beta^{(1)j}) p^{*j}_f + C \alpha_j - \left( 1 - \beta^{(1)j} \right) p^{*j}_f \right) \\
= \beta^{(1)} f p^*_f - \mu^*_f + \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \sum_{\ell \in H} \left( \sum_{\ell \in H} \alpha^{\ell}_f + \sum_{j \notin H} \alpha_j \right) + C \frac{\alpha_f}{\sum_{\ell=1}^N \alpha_\ell} \left( \sum_{\ell \in H} \alpha^{\ell}_f + \sum_{j \notin H} \alpha_j \right) = \beta^{(1)} f p^*_f - \mu^*_f + C \alpha_f.
\]  

(A.87)

We distinguish two possible cases, one where \( p^{*}_f \) is unsaturated, and one where it is saturated. For both cases we insert Equations (5.70) and (5.71) into the right hand side found in Equation (A.87).

If the suggested fixed point \( p^{*}_f \) is not saturated, the right hand side of Equation (5.67) becomes

\[
C \frac{\beta^{(1)f}\alpha_f}{1 - \beta^{(1)f}} + C \frac{(1 - \beta^{(1)f}) \alpha_f}{1 - \beta^{(1)f}} = C \frac{\alpha_f}{1 - \beta^{(1)f}} \left( \beta^{(1)f} + 1 - \beta^{(1)f} \right) = C \frac{\alpha_f}{1 - \beta^{(1)f}} = p^*_f.
\]

In case \( p^{*}_f \) is saturated, the right hand side of Equation (5.67) becomes

\[
\beta^{(1)f} p - C \alpha_f + \left( 1 - \beta^{(1)f} \right) p_f + C \alpha_f = p_f = p^*_f.
\]

Hence, Equation (5.67) maps the suggested fixed point into itself, showing that it is indeed a fixed point of the system.

Next, we will show that this fixed point is also unique by contradiction, using a similar argument as in [117]. Let \( p^\circ \) and \( p^* \) be two fixed points such that \( p^\circ \neq p^* \). Without loss of generality, take \( p^\circ > p^*_f \) which means that \( \mu^\circ_f \geq \mu^*_f \) due to Lemma 5.14 and \( p^\circ_f < p^*_f \) which leads to \( \mu^\circ_j \leq \mu^*_j \).
A. Reasoning and Proofs

From Equation (5.68) we know that

\[
\frac{(1 - \beta^{(1)}_i)p_i^* + \mu_i^*}{\alpha_i} = \frac{(1 - \beta^{(1)}_j)p_j^* + \mu_j^*}{\alpha_j} \quad (A.88)
\]

holds for the two fixed points. However using Lemma 5.14 and Equation (A.89) we find

\[
\frac{(1 - \beta^{(1)}_i)p_i^* + \mu_i^*}{\alpha_i} \geq \frac{1}{\alpha_i} p_i^* + \frac{1}{\alpha_i} \mu_i^*
\]

\[
\geq \frac{1}{\alpha_i} p_i^* + \frac{1}{\alpha_i} \mu_i^*
\]

\[
= \frac{1}{\alpha_j} p_j^* + \frac{1}{\alpha_j} \mu_j^*
\]

\[
\geq \frac{1}{\alpha_j} p_j^* + \frac{1}{\alpha_j} \mu_j^*
\]

\[
> \frac{1}{\alpha_j} p_j^* + \frac{1}{\alpha_j} \mu_j^*,
\]

which contradicts Equation (A.88). This means that the suggested fixed point is unique.

A.13 Proof of Theorem 5.16

We consider the following norm

\[
\|p(\tau_{k+1}) - ((1^T \alpha)^{-1} \alpha 1 - I) \mu(\tau_{k+1}) - (p^* - ((1^T \alpha)^{-1} \alpha 1 - I) \mu^*)\|_1.
\]

Using the reverse triangle inequality we find

\[
\|p(\tau_{k+1}) - ((1^T \alpha)^{-1} \alpha 1 - I) \mu(\tau_{k+1}) - (p^* - ((1^T \alpha)^{-1} \alpha 1 - I) \mu^*)\|_1
\]

\[
= \sum_f \|p_f(\tau_{k+1}) - p_f^* - \left( \frac{\alpha_f}{\sum_l \alpha_l} \sum_l (\mu_l(\tau_{k+1}) - \mu_l^*) - (\mu_f(\tau_{k+1}) - \mu_f^*) \right)\|
\]

\[
= \sum_f \|p_f(\tau_{k+1}) - p_f^* - \frac{\alpha_f}{\sum_l \alpha_l} \sum_l (\mu_l(\tau_{k+1}) - \mu_l^*) + (\mu_f(\tau_{k+1}) - \mu_f^*)\| \quad (A.90)
\]

\[
\geq \sum_f \left( |p_f(\tau_{k+1}) - p_f^* + \mu_f(\tau_{k+1}) - \mu_f^*| - \left| \frac{\alpha_f}{\sum_l \alpha_l} \sum_l (\mu_l(\tau_{k+1}) - \mu_l^*) \right| \right).
\]

Using the fact that \(p_i(\tau_{k+1}) - p_i^*\) and \(\mu_i(\tau_{k+1}) - \mu_i^*\) have the same sign Equation (A.90) can be simplified to

\[
\|p(\tau_{k+1}) - ((1^T \alpha)^{-1} \alpha 1 - I) \mu(\tau_{k+1}) - (p^* - ((1^T \alpha)^{-1} \alpha 1 - I) \mu^*)\|_1
\]

\[
\geq \sum_f |p_f(\tau_{k+1}) - p_f^*| + \sum_f |\mu_f(\tau_{k+1}) - \mu_f^*| - \left| \sum_l (\mu_l(\tau_{k+1}) - \mu_l^*) \right| \quad (A.91)
\]

\[
= \sum_f |p_f(\tau_{k+1}) - p_f^*| + \sum_f |\mu_f(\tau_{k+1}) - \mu_f^*| - \sum_l (\mu_l(\tau_{k+1}) - \mu_l^*).
\]

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Finally, using the triangle inequality Equation (A.91) becomes
\[
\left\| p(\tau_{k+1}) - \left( (1^T \alpha)^{-1} \alpha 1 - 1 \right) \mu(\tau_{k+1}) - (p^* - ((1^T \alpha)^{-1} \alpha 1 - 1) \mu^*) \right\|_1 \\
\geq \sum_f |p_f(\tau_{k+1}) - p_f^*| + \sum_f |\mu_f(\tau_{k+1}) - \mu_f^*| - \sum_i |\mu_i(\tau_{k+1}) - \mu_i^*| \\
= \sum_f |p_f(\tau_{k+1}) - p_f^*| = \left\| p(\tau_{k+1}) - p^* \right\|_1.
\]

(A.92)

From Equation (5.63) we know that
\[
p(\tau_{k+1}) - \left( (1^T \alpha)^{-1} \alpha 1 - 1 \right) \mu(\tau_{k+1}) = Ap(\tau_k).
\]

Hence, we find
\[
\left\| p(\tau_{k+1}) - \left( (1^T \alpha)^{-1} \alpha 1 - 1 \right) \mu(\tau_{k+1}) - (p^* - ((1^T \alpha)^{-1} \alpha 1 - 1) \mu^*) \right\|_1 \\
= \left\| Ap(\tau_k) - Ap^* \right\|_1.
\]

(A.93)

Lastly, assume that \( p(\tau_k) \neq p^* \). Then, using the triangle inequality and Lemma 5.14 we find
\[
\left\| Ap(\tau_k) - Ap^* \right\|_1 \\
= \sum_{i=1}^N \left| \beta^{(1)}_{i,i} (p_i(\tau_k) - p_i^*) + \sum_{\ell=1}^N \frac{\alpha_{i,\ell}}{\alpha_{\ell}} \sum_{\ell=1}^N (1 - \beta^{(1)}_{\ell,i}) (p_\ell(\tau_k) - p_\ell^*) \right| \\
\leq \sum_{i=1}^N \left| \beta^{(1)}_{i,i} (p_i(\tau_k) - p_i^*) \right| + \sum_{i=1}^N \sum_{\ell=1}^N \frac{\alpha_{i,\ell}}{\alpha_{\ell}} \left| \sum_{\ell=1}^N (1 - \beta^{(1)}_{\ell,i}) (p_\ell(\tau_k) - p_\ell^*) \right| \\
= \sum_{i=1}^N \left| \beta^{(1)}_{i,i} (p_i(\tau_k) - p_i^*) \right| + \sum_{i=1}^N \sum_{\ell=1}^N \frac{\alpha_{i,\ell}}{\alpha_{\ell}} \left| \sum_{\ell=1}^N (1 - \beta^{(1)}_{\ell,i}) (p_\ell(\tau_k) - p_\ell^*) \right| \\
= \sum_{i=1}^N \left| \beta^{(1)}_{i,i} (p_i(\tau_k) - p_i^*) \right| + \sum_{i=1}^N \left| \sum_{\ell=1}^N (-\beta^{(1)}_{\ell,i}) (p_\ell(\tau_k) - p_\ell^*) \right|
\]

We define two sets
\[
S^+ = \{ i \in \{1, \ldots, N\} | p_i(\tau_k) - p_i^* \geq 0 \} \quad \text{and} \quad \text{(A.95)}
\]
\[
S^- = \{1, \ldots, N\} \setminus S^+. \quad \text{(A.96)}
\]

Then using those sets we find that the norm can further be bounded by
\[
\left\| Ap(\tau_k) - Ap^* \right\|_1 \leq \sum_{i \in S^+} \left| \beta^{(1)}_{i,i} (p_i(\tau_k) - p_i^*) \right| + \sum_{i \in S^-} \left| \beta^{(1)}_{i,i} (p_i(\tau_k) - p_i^*) \right| \\
+ \sum_{\ell \in S^+} (-\beta^{(1)}_{\ell,i}) (p_\ell(\tau_k) - p_\ell^*) + \sum_{\ell \in S^-} (-\beta^{(1)}_{\ell,i}) (p_\ell(\tau_k) - p_\ell^*)
\]

(A.97)

Note that due to the definitions of the two sets two cases can be distinguished:
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1. \( |\sum_{i \in S^+} (-\beta^{(1)}_i) (p_r(\tau_k) - p^*_i)| \geq |\sum_{i \in S^-} (-\beta^{(1)}_i) (p_r(\tau_k) - p^*_i)| \) and
2. \( |\sum_{i \in S^+} (-\beta^{(1)}_i) (p_r(\tau_k) - p^*_i)| < |\sum_{i \in S^-} (-\beta^{(1)}_i) (p_r(\tau_k) - p^*_i)| \).

In the first case, we find

\[
\|Ap(\tau_k) - Ap^*\|_1 \leq \sum_{i \in S^+} |\beta^{(1)}_i| |p_i(\tau_k) - p^*_i| + \sum_{i \in S^-} |\beta^{(1)}_i| |p_i(\tau_k) - p^*_i| \\
+ \sum_{i \in S^-} |\beta^{(1)}_i| |p_r(\tau_k) - p^*_i| - \sum_{i \in S^+} |\beta^{(1)}_i| |p_r(\tau_k) - p^*_i| \\
= 2 \sum_{i \in S^+} \beta^{(1)}_i |p_i(\tau_k) - p^*_i| \\
\leq 2 \max_i \left( \beta^{(1)}_i \right) \sum_{i \in S^+} |p_i(\tau_k) - p^*_i|.
\]

This can be rewritten using the definitions of the sets \( P, S^+, \) and \( S^- \) as

\[
\|Ap(\tau_k) - Ap^*\|_1 \leq \max_i \left( \beta^{(1)}_i \right) \left( \sum_{i \in S^+} |p_i(\tau_k) - p^*_i| + \sum_{i \in S^-} |p_i(\tau_k) - p^*_i| \right) \\
= \max_i \left( \beta^{(1)}_i \right) \sum_{i = 1}^N |p_i(\tau_k) - p^*_i| = \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1.
\]

Similarly, we find for the second case

\[
\|Ap(\tau_k) - Ap^*\|_1 \leq \sum_{i \in S^-} |\beta^{(1)}_i| |p_i(\tau_k) - p^*_i| + \sum_{i \in S^+} |\beta^{(1)}_i| |p_i(\tau_k) - p^*_i| \\
+ \sum_{i \in S^+} |\beta^{(1)}_i| |p_r(\tau_k) - p^*_i| - \sum_{i \in S^-} |\beta^{(1)}_i| |p_r(\tau_k) - p^*_i| \\
= 2 \sum_{i \in S^-} \beta^{(1)}_i |p_i(\tau_k) - p^*_i| \\
\leq 2 \max_i \left( \beta^{(1)}_i \right) \sum_{i \in S^-} |p_i(\tau_k) - p^*_i|.
\]

This leads to

\[
\|Ap(\tau_k) - Ap^*\|_1 \leq \max_i \left( \beta^{(1)}_i \right) \left( \sum_{i \in S^+} |p_i(\tau_k) - p^*_i| + \sum_{i \in S^-} |p_i(\tau_k) - p^*_i| \right) \\
= \max_i \left( \beta^{(1)}_i \right) \sum_{i = 1}^N |p_i(\tau_k) - p^*_i| = \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1.
\]

Combining Equations (A.92), (A.93), (A.99) and (A.101) it follows that

\[
\|p(\tau_{k+1}) - p^*\|_1 \leq \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1.
\]

Hence, by induction Equation (5.73) holds.
A.14 Proof of Lemma 5.17

To see the link between a quadratic optimisation problem and the algorithm in Lemma 5.12 we investigate the following general constrained optimisation problem

\[
\begin{align*}
\text{argmin}_p & \sum_{i=1}^{N} a_i p_i^2 \\
\text{s.t.} & \quad p_i \leq \bar{p}_i \forall i \\
& \quad \sum_{i=1}^{N} p_i = \bar{P} - \bar{\rho}, \\
\end{align*}
\]

where \( a_i \) are weighting factors of the cost-function. According to [18] we find the KKT conditions to be

\[
\begin{align*}
2a_i p_i + \lambda_i & + \nu = 0 \forall \; i, \\
p_i & \leq \bar{p}_i \forall \; i, \\
\sum_{i=1}^{N} p_i & = \bar{P} - \bar{\rho}, \\
\lambda_i & \geq 0 \forall \; i, \\
\lambda_i (p_i - \bar{p}_i) & = 0 \forall \; i.
\end{align*}
\]

When using the AIMD algorithm from Lemma 5.12, Equations (A.105) and (A.106) are fulfilled by the algorithm at a CE. From Lemma 5.13 it follows that for the fixed point of Lemma 5.12

\[
\frac{(1 - \beta^{(1)}) p_i^* + \mu^*_i}{\alpha_i}
\]

is constant. Similarly, Equation (A.104) means that

\[
2a_i p_i + \lambda_i
\]

is constant. Comparison shows that when selecting

\[
a_i = \frac{(1 - \beta^{(1)} p_i^* + \mu^*_i)}{2\alpha_i},
\]

the two Equations (A.109) and (A.110) are equal. Hence, we lastly check whether Equations (A.107) and (A.108) are fulfilled for the selection as above.

As \( \alpha_i > 0 \) and \( \mu_i^* \geq 0, \frac{\mu_i^*}{\alpha_i} > 0 \) and Equation (A.107) holds. Using the definition of \( M_{p^*} \) and \( P \) it follows immediately that also Equation (A.108) holds. Hence, the KKT conditions hold for the selection

\[
a_i = \frac{(1 - \beta^{(1)} p_i^* + \mu^*_i)}{2\alpha_i},
\]

which means that the AIMD algorithm solves the quadratic optimisation problem with weights

A.15 Proof of Lemma 5.19

To prove item (i), note that for a vector \( x \)

\[
\| A_{T,j} x \|_{H_T} = \max_{i=1,...,T} \left\| \frac{1}{i} A_j x_1 + i - 1 \right\|_{i, \text{L}_1}.
\]
A. Reasoning and Proofs

Since \( \|A_j\|_1 \leq 1 \) we find that this is always smaller than

\[
\|A_{T,j}x\|_{H_T} \leq \max_{i=1,\ldots,T} \left( \frac{1}{i} \|A_j\|_1 \|x_1\|_1 + \frac{i-1}{i} \|x_{i-1}\|_1 \right)
\leq \max_{i=1,\ldots,T} \|x_i\|_1.
\]

(A.111)

This shows that the induced norm is bounded by 1.

Next, we show item (ii). Remember that for any \( A_j \in \mathcal{A}, \ 1^T A_j = 1^T \) holds. Also, note that the component of \( A_{T,j} z \) are equal to

\[
\frac{1}{i} A_j z_1 + \frac{i-1}{i} I z_{i-1}
\]

with \( i = 1, \ldots, T \). For each component the definition of the subspace \( W \) yields that

\[
1^T \left( \frac{1}{i} A_j z_1 + \frac{i-1}{i} I z_{i-1} \right) = \frac{1}{i} 1^T A_j z_1 + \frac{i-1}{i} 1^T z_{i-1} = \frac{1}{i} 1^T z_1 + \frac{i-1}{i} 1^T z_{i-1} = 0.
\]

(A.113)

This shows that \( W \) is invariant under all \( A_{T,j} \in \mathcal{A}_T \). For the first part of item (iii), assume that we have indices \( s,t \) such that

\[
\frac{\|A_j z_1 + \frac{s-1}{s} I z_{s-1}\|_1}{\|A_j z_1 + \frac{i-1}{i} I z_{i-1}\|_1} = \max_i \left( \frac{\|A_j z_1 + \frac{i-1}{i} I z_{i-1}\|_1}{\|A_j z_1 + \frac{s-1}{s} I z_{s-1}\|_1} \right)
\]

(A.114)

and

\[
\|z_t\|_1 = \max_i (\|z_i\|_1).
\]

(A.115)

Then, using the triangle inequality we find that

\[
\frac{\|A_j z_1 + \frac{s-1}{s} I z_{s-1}\|_1}{\|A_j z_1 + \frac{i-1}{i} I z_{i-1}\|_1} \leq \frac{1}{s} \|A_j z_1\|_1 + \frac{s-1}{s} \|z_{s-1}\|_1
\]

\[
\leq \frac{1}{s} \|z_1\|_1 + \frac{s-1}{s} \|z_{s-1}\|_1
\]

\[
\leq \frac{1}{s} \|z_t\|_1 + \frac{s-1}{s} \|z_t\|_1 = \|z_t\|_1.
\]

(A.116)

Since we assume the equality of Equations (A.114) and (A.115), from the above follows that

\[
\|z_t\|_1 = \|z_{s-1}\|_1 = \|z_1\|_1 = \|A_j z_1\|_1.
\]

(A.117)

This shows the first part of item (iii), since \( 1^T z_1 = 0 \). Finally, because of Lemma 5.2,

\[
\|A_{T,1} z\|_{H_T} = \max_{i=1,\ldots,T} \left\{ \frac{1}{i} A z_1 + \frac{i-1}{i} z_{i-1} \right\} \leq \max_{i=1,\ldots,T} \left\{ \frac{c}{i} \|z_1\|_1 + \frac{i-1}{i} \|z_{i-1}\|_1 \right\}
\]

\[
\leq \max_{i=1,\ldots,T} \left\{ \frac{c}{i} + \frac{i-1}{i} \right\} \|z\|_{H_T},
\]

(A.118)

where \( c \) is the constant in Lemma 5.2. This shows the second part item (iii).
A.16 Proof of Lemma 5.22

Since $P : \mathcal{P} \to \mathcal{P}$ is continuous and $\mathcal{P}$ is compact and convex, the existence of a fixed point for $P$ follows from Brouwer’s fixed point theorem. Hence, item 1 holds.

Since $x^*$ is a fixed point the following holds
\[ x_i^* = C \frac{1}{\lambda_i(x_i^*)} \]  \hspace{1cm} (A.119)
for $i = \{1, \ldots, N\}$. Since $C$ is a constant given a point $x$, it follows that for all $i$ the fixed point fulfils
\[ x_i^* \lambda_i(x_i^*) = C. \]  \hspace{1cm} (A.120)

Next, we show the uniqueness of the fixed point $x^*$. Assume there are two fixed points $x^* \neq y^*$ of $P$ where $x^*, y^* \in \mathcal{P}$, then there exist indices $i, j$ such that $x_i^* > y_i^*$ and $x_j^* < y_j^*$. This is a property of the set $\mathcal{P}$, see Lemma 5.1. From before it is clear that
\[ x_i^* \lambda_i(x_i^*) = y_i^* \lambda_i(y_i^*) \]  \hspace{1cm} (A.121)
and
\[ x_j^* \lambda_j(x_j^*) < y_j^* \lambda_j(y_j^*). \]  \hspace{1cm} (A.122)

Combining Equations (A.121) and (A.122) we find that
\[ x_i^* \lambda_i(x_i^*) > y_i^* \lambda_i(y_i^*) = y_j^* \lambda_j(y_j^*) > x_j^* \lambda_j(x_j^*). \]  \hspace{1cm} (A.123)
This contradicts Equation (A.121) and item 2 holds.

A fixed point for the map in Equation (5.123) needs to fulfil
\[ x^* = (1 - \varepsilon)x^* + \varepsilon P(x^*). \]  \hspace{1cm} (A.124)
This only holds if $P(x^*) = x^*$. Since $x^*$ is the unique fixed point of $P$, $x^*$ is also the unique fixed point of Equation (5.123) and item 3 holds.

Finally, item 4 follows since every element of $P(x)$ is larger than 0 for all $x \in \mathcal{P}$ by the definition in Equation (5.121) and point three in Assumption 5.9.

A.17 Proof of Theorem 5.24

If $x \in \mathcal{P}$ where some entries of $x$ are zero, then $\text{dist}_H(x, x^*) = \infty$. Also, every entry in $R_c(x)$ is larger than 0 due to its definition. Hence, Equation (5.137) holds. If $x \in \mathcal{P}$ where all entries are larger than 0, i.e. $x \in \text{ri} \mathcal{P}$ for $i \in \{1, 2, \ldots, N\}$, we find due to the properties of $\mathcal{P}$ that
\[ x \in \text{ri} \mathcal{P} \text{ and } x \neq x^* \Rightarrow \min_j \left( \frac{x_j}{x_j^*} \right) < 1 < \max_i \left( \frac{x_i}{x_i^*} \right). \]  \hspace{1cm} (A.125)
Using the above we find that for a sequence \( \{x_k\} \subset \text{ri}\mathcal{P} \) the following terms are equivalent

\[
e^\mathcal{H}(x_k, x^*) \to 1 \tag{A.126}
\]
\[
\min_j \left( \frac{x_{kj}}{x^*_j} \right) \to 1 \tag{A.127}
\]
\[
\max_i \left( \frac{x_{ki}}{x^*_i} \right) \to 1. \tag{A.128}
\]

We distinguish two separate cases. In the first case, we assume that \( e^\mathcal{H}(R^\varepsilon(x_k), x^*) \) is obtained for a pair \((i, j)\) with \( x_i > x^*_i \) and \( x_j < x^*_j \), the second case assumes that \( e^\mathcal{H}(R^\varepsilon(x_k), x^*) \) is obtained for a pair \((i, j)\) where \( x_i \leq x^*_i \) or \( x_j \geq x^*_j \).

In the first case, we find using the constant \( \gamma_F \) defined in Equation (5.122) and the second point in Assumption 5.9 that \( x^*_j \lambda_j(x_j) < \gamma_F < x_i \lambda_i(x_i) \). Therefore, and by recalling the definition of \( P(x) \) and \( C \) in Equations (5.114) and (5.118), respectively, the \( i \)-th element of \( R^\varepsilon(x) \) is equal to

\[
R^\varepsilon(x)_i = (1 - \varepsilon)x_i + \varepsilon C \frac{1}{\lambda_i(x_i)} \left( (1 - \varepsilon) + \varepsilon C \frac{1}{\gamma_F} \right) x_i. \tag{A.129}
\]

Similarly, the \( j \)-th element of \( R^\varepsilon(x) \) becomes

\[
R^\varepsilon(x)_j = (1 - \varepsilon)x_j + \varepsilon C \frac{1}{\lambda_j(x_j)} \left( (1 - \varepsilon) + \varepsilon C \frac{1}{\gamma_F} \right) x_j. \tag{A.130}
\]

Combining (A.129) and (A.130), we find that

\[
\frac{R^\varepsilon(x)_i}{x^*_i} \frac{x^*_j}{x^*_j} < \frac{x_i x^*_j}{x^*_i x_j} \leq e^\mathcal{H}(x, x^*). \tag{A.131}
\]

In the second case where \( e^\mathcal{H}(R^\varepsilon(x_k), x^*) \) is obtained for a pair \((i, j)\) such that \( x_i \leq x^*_i \) or \( x_j \geq x^*_j \), it is sufficient to show that we can choose \( \varepsilon_0 > 0 \) small enough so that for \( 0 < \varepsilon < \varepsilon_0 \) this second case can only occur for \( x \in B_H(x^*, \eta) \), since in Equation (5.137) that case is excluded. So assume \( \eta > 0 \) is fixed. To show this we use the following three inequalities:

1. Due to Equations (A.126) to (A.128) there exists a constant \( r \in (0, 1) \) such that for all \( x \in \text{ri}\mathcal{P} \) with \( \text{dist}_H(x, x^*) \geq \eta \) we have

\[
\min_{j=1, \ldots, N} \frac{x_j}{x^*_j} \leq r < 1. \tag{A.132}
\]

2. Again by Equations (A.126) to (A.128), there exists a constant \( R > 1 \) such that for all \( x \in \text{ri}\mathcal{P} \) with \( \text{dist}_H(x, x^*) \geq \eta \) we have

\[
\max_{i=1, \ldots, N} \frac{x_i}{x^*_i} \geq R > 1. \tag{A.133}
\]
3. Due to the third point in Assumption 5.9, i.e. \( \lambda_i(x_i) \geq \lambda_{\min} \), there are two constants \( c_1, c_2 \) with

\[
    c_1 = (\bar{P} - \bar{\rho}) \frac{\lambda_{\min}}{N} \quad \text{and} \quad c_2 = (\bar{P} - \bar{\rho}) \min \left( 1, \frac{1}{N} \frac{1}{\lambda_{\min}} \right).
\]

(A.134)

In particular, for these two constants we find that for all \( x \in \mathcal{P} \) it follows that

\[
P_i(x) = C \frac{1}{\lambda_i(x_i)} \geq C \lambda_{\min} = \frac{\bar{P} - \bar{\rho}}{\sum_{j=1}^{N} \frac{1}{\lambda_j(x_j)}} \geq \frac{\bar{P} - \bar{\rho}}{N} \lambda_{\min} = c_1
\]

and

\[
P_i(x) = C \frac{1}{\lambda_i(x_i)} \leq C = \frac{\bar{P} - \bar{\rho}}{\sum_{j=1}^{N} \frac{1}{\lambda_j(x_j)}} \leq (\bar{P} - \bar{\rho}) \min \left( 1, \frac{1}{N} \frac{1}{\lambda_{\min}} \right) = c_2.
\]

(A.135)

(A.136)

Note that the last step also makes use of the fact that \( P(x) \leq \bar{P} - \bar{\rho} \) due to its definition.

For simplicity, let \( x_{i, \min}^* = \min_{i=1, \ldots, N} x_i^* > 0 \) and \( x_{i, \max}^* = \max_{i=1, \ldots, N} x_i^* > 0 \). Further, we define \( \varepsilon_1 > 0 \) by

\[
    \varepsilon_1 = \frac{1 - r}{1 - r + \frac{c_2}{x_{i, \min}^*} - \frac{c_1}{x_{i, \max}^*}}.
\]

(A.137)

For \( 0 < \varepsilon < \varepsilon_1 \), fix \( x \in \mathfrak{R}, \text{dist}_H(x, x^*) \geq \eta \) and an index \( j \) so that \( \frac{x_i^*}{x_j^*} \leq r \) according to Equation (A.132). Let \( \ell \in \{1, \ldots, N\} \) be any index for which \( x_i^* \geq x_j^* \). Then, we obtain using Equations (A.132), (A.133), (A.135) and (A.136) as well as the definition of \( x_{i, \min}^*, x_{i, \max}^* \) that

\[
    \frac{R_{i}(x)}{x_j^*} = \frac{(1 - \varepsilon)x_j^* + \varepsilon P_j(x)}{x_j^*} \leq (1 - \varepsilon)r + \varepsilon P_j(x) \leq (1 - \varepsilon)r + \varepsilon \frac{c_2}{x_{i, \min}^*}.
\]

(A.138)

Note that since \( \varepsilon < \varepsilon_1 \) it follows that

\[
    \varepsilon < \frac{1 - r}{1 - r + \frac{c_2}{x_{i, \min}^*} - \frac{c_1}{x_{i, \max}^*}}
\]

(A.139)

\[
    \left( 1 - r + \frac{c_2}{x_{i, \min}^*} - \frac{c_1}{x_{i, \max}^*} \right) \varepsilon < 1 - r
\]

\[
    r - r\varepsilon + \frac{c_2}{x_{i, \min}^*} < 1 - \varepsilon + \frac{c_1}{x_{i, \max}^*} \varepsilon.
\]

Then,

\[
    \frac{R_{i}(x)}{x_j^*} < (1 - \varepsilon) + \varepsilon \frac{c_1}{x_{i, \max}^*} \leq (1 - \varepsilon) x_i^* + \varepsilon P_i(x) \frac{x_i^*}{x_j^*}.
\]

(A.140)

This means that the assumptions that \( d_H(x, x^*) \geq \eta \) and \( 0 < \varepsilon < \varepsilon_1 \) imply that

\[
    \frac{R_{i}(x)}{x_j^*} = \min_{\ell=1, \ldots, n} \frac{R_{i}(x)}{x_{\ell}^*} \quad \Rightarrow \quad x_j < x_{j, i}.
\]

(A.141)

Similarly, we define \( \varepsilon_2 > 0 \) by

\[
    \varepsilon_2 = \frac{R - 1}{R - 1 + \frac{c_2}{x_{i, \min}^*} - \frac{c_1}{x_{i, \max}^*}}.
\]

(A.142)
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Then, for index $i$ with $\frac{x_i^*}{x_i^*} \geq R$ and $\ell \in \{1, \ldots, N\}$ be any index for which $x_\ell \leq x_\ell^*$, we find

$$\frac{R_\epsilon(x)_i}{x_i^*} = \frac{(1 - \epsilon)x_i + \epsilon P(x)}{x_i^*} \geq (1 - \epsilon)\frac{c_1}{x_{\max}} > (1 - \epsilon)\frac{c_2}{x_{\min}} \geq \frac{(1 - \epsilon)x_\ell + \epsilon P(x)}{x_\ell^*}, \quad (A.143)$$

where we used similar steps as before.

Hence, the conditions dist$_H(x, x^*) \geq \eta$ and $0 < \epsilon < \epsilon_2$ imply that

$$\frac{R_\epsilon(x)_i}{x_i^*} = \max_{\ell=1, \ldots, n} \frac{R_\epsilon(x)_\ell}{x_\ell^*} \quad \Rightarrow \quad x_i > x_i^*.$$  

(A.144)

Summarising, we obtain the following result from Equations (A.141) and (A.144): If $0 < \epsilon < \epsilon_0 = \min(\epsilon_1, \epsilon_2)$, then dist$_H(x, x^*) \geq \eta$ implies that $e^{d_H}(R_\epsilon(x), x^*)$ is obtained in a pair such that $x_i > x_i^*$ and $x_j < x_j^*$. Hence, the first case studied occurs and the condition in Equation (A.131) is satisfied, which proves Theorem 5.24.

### A.18 Proof of Corollary 5.25

Fix $\eta > 0$, let $\epsilon > 0$ be the constant. In Appendix A.17, we found that if $\epsilon$ is chosen small dist$_H(x, x^*) \geq \eta$, then $e^{d_H}(R_\epsilon(x), x^*)$ is obtained in a pair such that $x_i > x_i^*$ and $x_j < x_j^*$. In fact, we can choose $\epsilon$ smaller such that $e^{d_H}(R_\epsilon(x), x^*)$ will be obtained for a pair $x_i > Rx_i^*$ and $x_j < rx_j^*$ with suitable constants $R > 1$ and $r < 1$. $\epsilon$ is then constructed as in Appendix A.17 with the suitable constants $R$ and $r$, i.e. $\epsilon = \min(\epsilon_1, \epsilon_2)$ with

$$\epsilon_1 = \frac{1}{1 - r} - \frac{\epsilon}{1 - 1 + \frac{1}{x_{\min}^*} - \frac{1}{x_{\max}^*}}$$

and

$$\epsilon_2 = \frac{R - 1}{R - 1 + \frac{1}{x_{\min}^*} - \frac{1}{x_{\max}^*}}.$$  

(A.145)  

(A.146)

Using Assumption 5.9 and Lemma 5.22, there are constant $L_1 < \gamma F < L_2$ such that

$$x_j \lambda_j(x_j) < L_1 \quad \text{for all } j = 1, \ldots, N, \text{ and all } x_j \text{ such that } \frac{x_j}{x_j^*} \leq r,$$

$$x_i \lambda_i(x_i) > L_2 \quad \text{for all } i = 1, \ldots, N, \text{ and all } x_i \text{ such that } \frac{x_i}{x_i^*} \geq R.$$  

If dist$_H(x, x^*) \geq \eta$ and if $i, j$ are indices as mentioned before, we find following the steps of Equations (A.129) and (A.130) that

$$e^{d_H}(R_\epsilon(x), x^*) = \frac{R_\epsilon(x)_i}{R_\epsilon(x)_j} = \frac{(1 - \epsilon) + \frac{\sum_{\nu=1}^{\nu} \lambda_{\nu}(x_{\nu})}{\sum_{\nu=1}^{\nu} \lambda_{\nu}(x_{\nu})}}{1 - \frac{1}{L_1} - \frac{1}{L_2}} = (1 - \epsilon) e^{d_H}(x, x^*)$$

where

$$C_\eta = \min_{\epsilon \in [0, 1]} \left( \frac{\frac{1}{L_1} - \frac{1}{L_2}}{\sum_{\nu=1}^{\nu} \lambda_{\nu}(x_{\nu})} \right) > 0.$$  

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Note that $C_\eta$ depends on $\eta$ since $r, R$ determine possible values for $L_1, L_2$ and their choice is a function of $\eta$. To gain the expression in the Hilbert metric the logarithm is applied on both sides. The final bound comes from the basic logarithm inequality that $\log x < x - 1$ for $x > 0$.

A.19 Proof of Theorem 5.29

In this proof, we use the properties of the deterministic system discussed earlier in Corollary 5.25. We will show that the behaviour of $\rho(\tau_T)$ is well approximated by that system for sufficiently large $T$.

Assume that the constants $\delta^-, \delta^+$ from Equations (5.119) and (5.120) have been fixed. Let $\eta > 0$ be fixed. We aim to show that almost surely the sample path $\rho(\tau_T) \in B_1(x^*, \eta)$ for all $T$ large enough. Since $\eta > 0$ is arbitrary, this will show the above claim.

We choose the following variables sequentially:

1. For the constant $\eta$ pick $\varepsilon_0 > 0$ and $C_\eta > 0$ according to Corollary 5.25, such that Equation (5.139) is satisfied for all $0 < \varepsilon < \varepsilon_0$.

2. Let $C_0$ be the constant guaranteed by Lemma 5.28 satisfying Equation (5.149) for all $0 < \varepsilon < \varepsilon_0$.

3. Let $K > 0$ be the constant given by Lemma 5.26.

4. Choose $\delta^*, \tilde{\delta}$ according to Equation (5.144), i.e.

$$\delta^* = \min \left( \frac{C_\eta e^{\eta^2}}{2K}, \delta^- \right), \quad (A.147)$$

and

$$0 < \tilde{\delta} < \delta^*/3, \quad (A.148)$$

so that Corollary 5.27 and the third point of Lemma 5.23 are applicable. Let $C_{\tilde{\delta}} > 0$ be the constant guaranteed by Lemma 5.23.

5. Pick $\theta \in (0, 1)$ so that

$$-(1 - \theta) + \theta(1 + C_{\tilde{\delta}}) < 0 \quad \text{and} \quad (A.149)$$

$$-(1 - \theta)C_\eta + \theta C_0 < 0. \quad (A.150)$$

6. We use now Lemma 5.31 to determine the length of the short averaging period discussed earlier. Using this and the fact that

$$\frac{1}{k+1} \sum_{t=0}^{k} \rho(\tau_t) - p^* = \left( \sum_{t=0}^{k} \Pi(\tau_t) - x^*_p 1^T \right) y(\tau_0) \quad (A.151)$$

as in Lemma 5.5, pick $m \in \mathbb{N}$ such that for all $y \in \mathcal{P}$ the Markov chain as in Lemma 5.1 with fixed probability $\lambda = \lambda(y)$ satisfies for all $T \in \mathbb{N}$

$$\Pr_{\lambda(y)} \left( \left\| \frac{1}{m} \sum_{t=1}^{m} \rho(\tau_{T+t}) - P(y) \right\|_1 > \delta \right) < \frac{\theta}{2}. \quad (A.152)$$

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7. Pick $T_0 \in \mathbb{N}$ such that for all $T \geq T_0$, $m/(T+m) < \varepsilon_0$ holds and so that for the Markov chain as in Equation (5.102) with place dependent probabilities, the average of the first component fulfils

$$\Pr_{p(0)} \left( \left\| \frac{1}{m} \sum_{i=1}^{m} p(\tau_{T+j}) - P(\rho(T)) \right\|_1 > \delta \right) < \theta.$$  

(A.153)

This is possible since this inequality is a perturbed version of Equation (A.152). In fact, with increasing $T$ the variation of $\rho(T+j)$ with $0 \leq j \leq m$ becomes arbitrarily small. This means that the variation becomes small in the first $m$ steps after time $T$. More precisely, for $j = 1, \ldots, m$ we have by definition

$$\| \rho(\tau_T) - \rho(\tau_{T+j}) \|_1 \leq \frac{2m}{T}.$$  

Hence, since $T \to \infty$ the place dependent probabilities of the Markov chain in Equation (5.102) that are considered on the interval $\{T, \ldots, T+m\}$ converge to the fixed probabilities $\Lambda(\rho(\tau_T))$. The above follows then from Equation (A.152) by continuity of the probability functions $\lambda_i$.

With these selections we now let $T \geq T_0$, such that by construction $\varepsilon = m/(T+m) < \varepsilon_0$. We will study the evolution of the value $\rho(\tau_{T+km}) \to \rho(\tau_{T+(k+1)m})$ with $k \in \mathbb{N}$. This is given by

$$\rho(\tau_{T+(k+1)m}) = \frac{T + km + 1}{T + (k+1)m + 1} \rho(\tau_{T+km}) + \frac{m}{T + (k+1)m + 1} \left( \frac{1}{m} \sum_{i=1}^{m} p(\tau_{T+km+j}) \right).$$

As before we define $P_{co}(\delta) = \text{conv} \, P(\mathcal{P}) + B_1(0, \delta)$. Further, let $\kappa(k) = \tau_{T+km}$ and $\varepsilon_k = \frac{m}{T+(k+1)m}$, such that the previous equation can be expressed as

$$\rho(\kappa(k+1)) = (1 - \varepsilon_k) \rho(\kappa(k)) + \varepsilon_k \left( \frac{1}{m} \sum_{j=1}^{m} p(\kappa(k) + j) \right).$$  

(A.154)

Note that the above equation has the same structure as the discrete iteration analysed before in Section 5.2.1.2. Hence, due to the selections made, see item 7, the probability that $\rho(\kappa(k+1))$ is close to $R_{\varepsilon_k}(\rho(\kappa(k)))$ is high.

Further, note that the constants are chosen such that both Lemma 5.23 and Corollary 5.27 are applicable. Hence, we use the estimate gained in 5.23 to show that trajectories starting in $\mathcal{P}$ will reach the set $P_{co}(2\delta)$ in a finite number of steps. Afterwards, we show that for trajectories starting in the strict superset $P_{co}(3\delta)$ the estimates of Corollary 5.27 yield that we reach the set $B_H(p^*, \eta)$ almost surely, again in a finite number of steps. Finally, we are able to show almost sure convergence. These three steps are performed in detail below.

**Step 1:** More precisely, we will first show that

$$\sigma_1(\rho(\tau_T)) = \min \{ k \in \mathbb{N} ; \rho(\kappa(k)) \in P_{co}(2\delta) \}$$

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is almost surely finite, which is the first time the set is reached. Obviously, if \( \rho(T) \in P_{\infty}(2\delta) \) there is nothing to show. By the first item in Lemma 5.23 and the choice made in item 7, it holds that if \( \text{dist}_1(\rho(\kappa(k)), P_{\infty}(\delta)) > \delta \), then

\[
\Pr_{\rho(0)} \left( \text{dist}_1(\rho(\kappa(k+1)), P_{\infty}(\delta)) \leq (1 - \varepsilon)\text{dist}_1(\rho(\kappa(k)), P_{\infty}(\delta)) \right) \geq 1 - \theta.
\]

In the complementary event, which happens with probability of at most \( \theta \) we have again by Lemma 5.23 that

\[
\text{dist}_1(\rho(\kappa(k+1)), P_{\infty}(\delta)) \leq (1 + \varepsilon_4 C_3)\text{dist}_1(\rho(\kappa(k)), P_{\infty}(\delta)).
\]

Combining these two observations we see that for \( \kappa(k) < \sigma_1(\rho(T)) \) we have that

\[
\text{dist}_1(\rho(\kappa(k)), P_{\infty}(\delta)) \leq \left( \prod_{\ell=1}^{k} a_\ell \right) \text{dist}_1(\rho(T), P_{\infty}(\delta)),
\]

where \( a_\ell, \ell \in \mathbb{N} \) is a random variable that has the value \((1 - \varepsilon)\) with probability \( 1 - \theta \) and the value \((1 + \varepsilon_4 C_3)\) with probability \( \theta \). Note that the random variables \( a_\ell \) are independent, since the bounds obtained do not depend on the particular sample path of the Markov chain.

To be able to apply Lemma 5.30 we first note that for all \( \ell \) large enough we have \( \log(1 + \varepsilon_\ell C_3) < \varepsilon_\ell (1 + C_3) \). By the choice of \( \theta \) in item 6, we obtain for all \( \ell \) large enough that

\[
\mathbb{E} [\log a_\ell] \leq (-1 - \theta) + \theta (1 + C_3) \varepsilon_\ell.
\]

Hence, Lemma 5.30 implies that \( \sum_{\ell=1}^{k} \log a_\ell \to -\infty \), almost surely. Thus, almost surely it holds that

\[
\lim_{k \to \infty} \text{dist}_1(\rho(\kappa(k)), P_{\infty}(\delta)) = 0,
\]

provided that \( \rho(\kappa(k)) \notin P_{\infty}(2\delta) \) for all \( k \). This is of course impossible, and so almost surely \( \rho(\kappa(k)) \in P_{\infty}(2\delta) \) for a finite \( k \).

**Step 2:** Similarly, if \( \rho(\kappa(k)) \in P_{\infty}(3\delta) \), then by Corollary 5.27 and the choice made in item 7 it holds that

\[
\Pr_{\rho(0)} \left( \text{dist}_H(\rho(\kappa(k+1)), \rho^*) < \text{dist}_H(\rho(\kappa(k)), \rho^*) - \frac{C_\eta \varepsilon_k}{2} \right) \geq 1 - \theta.
\]

On the other hand with probability of at most \( \theta \) we have by Lemma 5.28 that

\[
\text{dist}_H(\rho(\kappa(k+1)), \rho^*) \leq \text{dist}_H(\rho(\kappa(k)), \rho^*) + C_0 \varepsilon_k.
\]

In a similar fashion to the first step, as long as \( \rho(\kappa(k)) \in P_{\infty}(3\delta) \) and \( \text{dist}_H(\rho(\kappa(k)), \rho^*) > \eta \), we have

\[
\text{dist}_H(\rho(\kappa(\sigma_1(\rho(\tau_T) + k))), \rho^*) \leq \text{dist}_H(\rho(\sigma_1(\rho(\tau_T) + k)), \rho^*) + \sum_{\ell=1}^{k} b_\ell,
\]

where \( b_\ell \) is a random variable that takes the value \(-\varepsilon_\ell C_\eta\) with probability \((1 - \theta)\) and the value \( \varepsilon_\ell C_0 \) with probability \( \theta \). As before, Lemma 5.30 ensures that \( \sum_\ell b_\ell \) diverges to \(-\infty \), almost surely.
Note that it is always possible to leave the set $P_{co}(3\delta)$ with a small probability. In this case, Step 1 can be applied again, so that we re-enter the set $P_{co}(2\delta)$, almost surely. Now, by Equation (A.155) the process of entering $P_{co}(2\delta)$ and subsequently leaving $P_{co}(3\delta)$ requires that for some partial sum we have

$$
\sum_{k=\ell}^{\ell+L} \log(a_k) \geq \bar{\delta}.
$$

By Lemma 5.30, with probability 1, this happens only a finite number of times. Consequently, almost surely a sample path will reach $B_H(p^*, \eta)$.

**Step 3:** Finally, to obtain almost sure convergence, we need to show that almost surely

$$
\rho(k(k)) \in B_H(p^*, \eta),
$$

for all $k$ large enough. To this end we repeat the choices made above in item 1 to item 7 for the value $\eta/2$. Thus we can conclude that almost surely a sample path enters $B_H(p^*, \eta/2)$. If we assume that the sample path leaves $B_H(p^*, \eta)$ at some later time, then again by Steps 1 and 2 it will almost surely re-enter $B_H(p^*, \eta/2)$. Hence, the question is whether it is possible that infinitely often the sample path exits the ball $B_H(p^*, \eta)$ given that it was previously within the ball $B_H(p^*, 3\eta/4)$. In view of Equation (A.158), this amounts to saying that

$$
\sum_{k=\ell}^{\ell+L} b_k > \frac{\eta}{4}
$$

for pairs $(\ell, L) \in \mathbb{N}^2$ with arbitrarily large $\ell$. By Lemma 5.30 this almost surely does not happen. This shows Equation (A.159). The proof is complete by noting that the small variations of $\rho$ on the intervals $\kappa(k), \ldots, \kappa(k+1)$ do not destroy stability. Indeed, if $\rho(k(k)) \in B_H(p^*, \eta/2)$ for all $k$ large enough, then also $\rho(k(k)+j) \in B_H(p^*, \eta)$ for $j = 1, \ldots, m$, provided $k$ is large enough.

### A.20 Proof of Lemma 5.31

Let

$$
\mathcal{S}(m) = \frac{1}{m+1} \sum_{\ell=0}^{k} \Pi(\tau_\ell).
$$

(A.160)

Fix $\varepsilon, \delta > 0$ and $\hat{y} \in \mathcal{P}$. Due to Lemma 5.5 there exists an $\hat{m}$ such that

$$
\text{Pr}_{\lambda(\hat{y})} \left( \left\| \mathcal{S}(\hat{m}) - \frac{P(\hat{y})}{P - \tilde{\rho}} 1^T \right\|_1 > \delta \right) < \varepsilon.
$$

(A.161)

Now the map $P(y)$ is continuous due to Assumption 5.9. Furthermore, the map

$$
y \rightarrow \text{Pr}_{\lambda(y)} \left( \left\| \mathcal{S}(\hat{m}) - \frac{P(y)}{P - \tilde{\rho}} 1^T \right\|_1 > \delta \right)
$$

(A.162)

is continuous. We find therefore that

$$
\text{Pr}_{\lambda(y)} \left( \left\| \mathcal{S}(\hat{m}) - \frac{P(y)}{P - \tilde{\rho}} 1^T \right\|_1 > 2\delta \right) < 2\varepsilon
$$

(A.163)

holds on a neighbourhood on $\hat{y}$. As $\mathcal{P}$ is compact, it is covered by a finite number of such neighbourhoods. With this argument, and as $\varepsilon, \delta$ are arbitrary, we see that there are finitely many

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\( \hat{m}_1, \ldots, \hat{m}_n \in \mathbb{N} \) such that for every \( y \in \mathcal{P} \) there is a \( \hat{m}_j \) such that Equation (A.161) holds with \( \hat{m} = \hat{m}_j \). Finally, we apply Chebyshev’s inequality. For \( y \in \mathcal{P} \) and \( k \in \mathbb{N} \) consider the real valued random variable

\[
D(k) = \left\| \mathcal{S}(k) - \frac{P(y)}{\bar{p} - \bar{p}} 1^T \right\|_1.
\]

Note that \( 0 \leq D(k) \leq 2 \), as \( \mathcal{S}(k) \) and \( \frac{P(y)}{\bar{p} - \bar{p}} 1^T \) are column stochastic. Thus, \( E[D(k)^2] \leq 4 \). Also, if Equation (A.161) holds, it follows that

\[
E[D(\hat{m})] \leq (1 - \varepsilon)\delta + 2\varepsilon.
\]

We note that this inequality is independent of a particular \( y \) and just depends on the fact that \( \hat{m} \) is chosen so that Equation (A.161) holds. Fix \( \delta, \theta > 0 \) and assume that \( \varepsilon, \delta \) are chosen such that \( (1 - \varepsilon)\delta + 2\varepsilon < \frac{\theta}{2} \). Then, it follows that

\[
E[X(\hat{m})] < \frac{\delta}{2}.
\]

Hence, for multiples of \( \hat{m} \), it follows that

\[
\mathcal{S}(\ell \hat{m}) \leq \frac{1}{\ell + 1} \sum_{i=1}^{\ell \hat{m}} \Pi(i) = \frac{1}{\ell \hat{m}} \sum_{i=0}^{\ell - 1} \sum_{j=0}^{\hat{m} - 1} \Pi(\nu \ell + j)
\]

\[
= \frac{1}{\ell \hat{m}} \sum_{i=0}^{\ell - 1} \sum_{j=0}^{\hat{m} - 1} A(\nu \ell + j) \ldots A(\nu \ell) \Pi(\nu \ell).
\]

Hence,

\[
D(\ell \hat{m}) \leq \frac{1}{\ell} \sum_{i=0}^{\ell - 1} \left\| \frac{1}{\hat{m}} \sum_{j=0}^{\hat{m} - 1} A(\nu \ell + j) \ldots A(\nu \ell) \Pi(\nu \ell) - \frac{P(y)}{\bar{p} - \bar{p}} 1^T \right\|_1
\]

\[
\leq \frac{1}{\ell} \sum_{i=0}^{\ell - 1} \left\| \frac{1}{\hat{m}} \sum_{j=0}^{\hat{m} - 1} A(\nu \ell + j) \ldots A(\nu \ell) - \frac{P(y)}{\bar{p} - \bar{p}} 1^T \right\|_1 ||\Pi(\nu \ell)||_1
\]

\[
\leq \frac{1}{\ell} \sum_{i=0}^{\ell - 1} \left\| \frac{1}{\hat{m}} \sum_{j=0}^{\hat{m} - 1} A(\nu \ell + j) \ldots A(\nu \ell) - \frac{P(y)}{\bar{p} - \bar{p}} 1^T \right\|_1.
\]

By the independence assumption on the \( A(j) \) we see that the final term is the average of \( \ell \) independent copies of \( D(\hat{m}) \), the variance of which is bounded by \( \frac{4}{\ell} \). Let \( \{D_{\nu}(\hat{m})\}_{\nu \in \mathbb{N}} \) be a sequence of independent copies of \( D(\hat{m}) \). It follows from Chebyshev’s inequality that for all \( \ell \geq \frac{4}{\varepsilon \sqrt{2}} \) we have

\[
\Pr_{A(y)} \left( \left\| \mathcal{S}(\ell \hat{m}) - \frac{P(y)}{\bar{p} - \bar{p}} 1^T \right\|_1 > \bar{\delta} \right) \leq \Pr_{A(y)} \left( \sum_{\nu=0}^{\ell - 1} D_{\nu}(\hat{m}) > E[D(\hat{m})] + \frac{\bar{\delta}}{2} \right) < \theta.
\]

As the previous argument only depends on the validity of Equation (A.161), it holds uniformly for all \( y \in \mathcal{P} \) for which the choice of \( \hat{m} \) guarantees that Equation (A.161) holds. So choosing \( m \) sufficiently large such that it is a common multiple of \( \hat{m}_1, \ldots, \hat{m}_n \), Lemma 5.31 holds.
A.21 Proof of Lemma 6.2

Remark that we assume \( \rho^{(a)} \neq \rho^{(b)} \), which means that there exists an index \( i \) such that \( \rho_i^{(a)} \neq \rho_i^{(b)} \). We assume w.l.g. that for this index \( i \) we have \( \rho_i^{(a)} > \rho_i^{(b)} \). As both \( \rho^{(a)} \) and \( \rho^{(b)} \) are members of \( \mathcal{P} \), we know that

\[
1^T \rho^{(a)} = 1^T \rho^{(b)} = \bar{P} - \bar{p}.
\]

(A.166)

Hence, if \( \rho_i^{(a)} > \rho_i^{(b)} \) there exists an index \( j \) with \( \rho_j^{(a)} < \rho_j^{(b)} \) to maintain the equality. This shows items 1 and 2.

For item 3a we use the fact that if \( \rho_i^{(a)} > \rho_i^{(b)} \), that means that \( \rho_i^{(b)} \leq \bar{p}_i \) due to the definition. This in turn means that \( \mu_i^{(a)} = 0 \) and therefore

\[
\mu_i^{(a)} \geq 0 = \mu_i^{(b)}.
\]

(A.167)

For the proof of item 3b we note that

\[
(1 - \beta(1)_i)\rho_i^{(a)} > (1 - \beta(1)_i)\rho_i^{(b)}
\]

and

\[
\rho_i^{(a)} - \bar{p}_i > \rho_i^{(b)} - \bar{p}_i,
\]

(A.168)

whenever \( \rho_i^{(a)} > \rho_i^{(b)} \) holds. We now distinguish two possible cases. In the first case, we assume that

\[
(1 - \beta(1)_i)\rho_i^{(b)} > \rho_i^{(b)} - \bar{p}_i.
\]

(A.169)

Using Equation (A.168) we find that

\[
(1 - \beta(1)_i)\rho_i^{(a)} > \rho_i^{(b)} - \bar{p}_i
\]

(A.170)

which together with Equation (A.169) shows that item 3b holds.

Conversely, in the second case we assume

\[
(1 - \beta(1)_i)\rho_i^{(b)} \leq \rho_i^{(b)} - \bar{p}_i.
\]

(A.171)

Using Equation (A.169) we find that

\[
\rho_i^{(a)} - \bar{p}_i > (1 - \beta(1)_i)\rho_i^{(b)}.
\]

(A.172)

This together with Equation (A.168) shows that item 3b holds as well in this case.

To show that item 3c holds, note that

\[
\beta(1)_i \rho_i^{(a)} > \beta(1)_i \rho_i^{(b)},
\]

whenever \( \rho_i^{(a)} > \rho_i^{(b)} \). In case

\[
\beta(1)_i \rho_i^{(a)} > \bar{p}_i
\]

(A.173)

item 3c follows directly. In turn, if

\[
\beta(1)_i \rho_i^{(a)} \leq \bar{p}_i
\]

(A.174)
we find
\[ \beta^{(1)}_{i} \rho^{(b)}_{i} < \beta^{(1)}_{i} \rho^{(a)}_{i} \leq p_i \]  \hspace{1cm} (A.177)
and item 3c follows.

Finally, to show item 3d. Remark that if \( \rho^{(a)}_{i} > \rho^{(b)}_{i} \) three cases need to be investigated. Firstly, \( \beta^{(1)}_{i} \rho^{(a)}_{i} > p_i \) and \( \beta^{(1)}_{i} \rho^{(b)}_{i} > p_i \). In this case, the claim in item 3d follows directly. Secondly, \( \beta^{(1)}_{i} \rho^{(a)}_{i} > p_i \) but \( \beta^{(1)}_{i} \rho^{(b)}_{i} \leq p_i \) then
\[
\max \left( \beta^{(1)}_{i} \rho^{(a)}_{i}, p_i \right) - \max \left( \beta^{(1)}_{i} \rho^{(b)}_{i}, p_i \right) = \beta^{(1)}_{i} \rho^{(a)}_{i} - p_i \leq \beta^{(1)}_{i} \rho^{(a)}_{i} - \beta^{(1)}_{i} \rho^{(b)}_{i}. \]  \hspace{1cm} (A.178)
Finally, in the third case \( \beta^{(1)}_{i} \rho^{(b)}_{i} < \beta^{(1)}_{i} \rho^{(a)}_{i} \leq p_i \). Then,
\[
\max \left( \beta^{(1)}_{i} \rho^{(a)}_{i}, p_i \right) - \max \left( \beta^{(1)}_{i} \rho^{(b)}_{i}, p_i \right) = 0 \]  \hspace{1cm} (A.179)
and the claim holds.

A.22 Proof of Theorem 6.4

The proof is split in two parts, where the first part shows the uniqueness of a fixed point and the second part shows that the suggested point in Equation (6.16) is indeed a fixed point.

Assume that there exist two fixed points \( p^*, p^* \in P \) with \( p^* \neq p^* \). Due to Lemma 6.2 we can assume w.l.o.g. that for an index \( i \) and \( j \), \( p^*_i > p^*_i \) and \( p^*_j < p^*_j \) hold, respectively. From Lemma 6.2 it follows then that
\[
\frac{p^*_i - \max \left( \beta^{(1)}_{i} p^*, p_i \right) + \mu^*}{\alpha_i} = \min \left( \frac{(1 - \beta^{(1)}_i) p^*_i, p^*_i - p_i} {\alpha_i} + \mu^*_i \right) \\
> \frac{\min \left( (1 - \beta^{(1)}_i) p^*_i, p^*_i - p_i \right) + \mu^*_i} {\alpha_i} \\
\geq \frac{\min \left( (1 - \beta^{(1)}_i) p^*_j, p^*_j - p_j \right) + \mu^*_j} {\alpha_j} \\
= \frac{\min \left( (1 - \beta^{(1)}_j) p^*_j, p^*_j - p_j \right) + \mu^*_j} {\alpha_j} \\
> \frac{p^*_j - \max \left( \beta^{(1)}_{j} p^*, p_j \right) + \mu^*_j} {\alpha_j} .
\]
This contradicts Lemma 6.3, so the fixed point is unique.

Next, we show that the suggested fixed point in Equations (6.16) and (6.17) is indeed a fixed point. We define three sets \( S, T, \) and \( R \) such that
\[
S = \left\{ s \in \{1, 2, \ldots, N\} \left| p^*_s = C_0 \frac{\alpha_s}{1 - \beta^{(1)}_s} \right. \right\} \hspace{1cm} (A.180)
\]
\[
T \setminus S = \left\{ t \in \{1, 2, \ldots, N\} \left| p^*_t = C_\alpha + p_t \right. \right\} \hspace{1cm} (A.181)
\]
\[
R \setminus S \setminus T = \left\{ r \in \{1, 2, \ldots, N\} \left| p^*_r = p_r \right. \right\} . \hspace{1cm} (A.182)
\]
A. Reasoning and Proofs

Note that $S \cup T \cup R = \{1, \ldots, N\}$.

By inserting the fixed point into Equation (6.10), the right hand side becomes

$$\max \left( \beta^{(1)} p^*_f, p_f \right) + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N \left( p_i^* - \max \left( \beta^{(1)} p_i^*, p_i \right) \right) + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \sum_{i=1}^N \mu_i^* - \mu_f^*$$

$$= \max \left( \beta^{(1)} p_f^*, p_f \right) - \mu_f^* + \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \left( \sum_{s \in S} C \frac{\alpha_s}{1 - \beta^{(1)}_s} - \max \left( \beta^{(1)}_s C \frac{\alpha_s}{1 - \beta^{(1)}_s}, p_s \right) \right)$$

$$+ \sum_{i \in T} C \alpha_t + p_t - \max \left( \beta^{(1)}_t \left( C \alpha_t + p_t \right), p_t \right)$$

$$+ \sum_{r \in R} p_r - \max \left( \beta^{(1)} r \left( p_r \right), p_r \right)$$

$$+ \frac{\alpha_f}{\sum_{i=1}^N \alpha_i} \left( \sum_{s \in S} \mu_s^* + \sum_{i \in T} \mu_i^* + \sum_{r \in R} \mu_r^* \right).$$

Note that for every $s \in S$ we know from the definition of the set $S$

$$C \frac{\alpha_s}{1 - \beta^{(1)}_s} \geq C \alpha_s + p_s. \quad \text{(A.183)}$$

This means that

$$\max \left( \beta^{(1)}_s C \frac{\alpha_s}{1 - \beta^{(1)}_s}, p_s \right) = \beta^{(1)}_s C \frac{\alpha_s}{1 - \beta^{(1)}_s}, \quad \text{(A.184)}$$

which follows directly from Equation (A.183). Also, from Equations (6.17) and (A.184) it follows that

$$\mu_s = 0. \quad \text{(A.185)}$$

Similarly, for every $t \in T$

$$C \alpha_t + p_t \geq C \frac{\alpha_t}{1 - \beta^{(1)}_t} \quad \text{(A.186)}$$

holds due to the definition of the set $T$. This implies that

$$\max \left( \beta^{(1)}_t \left( C \alpha_t + p_t \right), p_t \right) = p_t. \quad \text{(A.187)}$$

Using Equation (6.17) we easily find that

$$\mu_t = 0. \quad \text{(A.188)}$$

Using Equations (6.17), (A.184), (A.185), (A.187) and (A.188) we are able to simplify the right
hand side of Equation (6.10) further to
\[
\max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \beta_i \left( p_i - \max \left( \beta^{(1)} i p_i^* , \beta^{(1)} i p_i \right) \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \mu_i^* - \mu_f^* \\
= \max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - \mu_f^* + \frac{\alpha_f}{N} \sum_{i=1}^{N} \beta_i \left( \sum_{s \in \mathcal{S}} \alpha_s \frac{\beta_i}{1 - \beta^{(1)} s} - \beta^{(1)} s \left( \frac{\alpha_s}{1 - \beta^{(1)} s} \right) \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \mu_i^* - \mu_f^*
\]

By rearranging, we find the right hand side of Equation (6.10) to be equal to
\[
\max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \beta_i \left( p_i - \max \left( \beta^{(1)} i p_i^* , \beta^{(1)} i p_i \right) \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \mu_i^* - \mu_f^* \\
= \max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - \mu_f^* + \frac{\alpha_f}{N} \sum_{i=1}^{N} \beta_i \left( \sum_{s \in \mathcal{S}} \alpha_s + \sum_{t \in \mathcal{T}} \alpha_t + \sum_{r \in \mathcal{R}} \bar{p}_r - \max \left( \beta^{(1)} r \bar{p}_r , \beta^{(1)} r \bar{p}_r \right) + \mu_r^* \right)
\]

By inserting the fixed point equation for \( \mu_r^* \), see Equation (6.17), and rearranging we get
\[
\max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \beta_i \left( p_i - \max \left( \beta^{(1)} i p_i^* , \beta^{(1)} i p_i \right) \right) + \frac{\alpha_f}{N} \sum_{i=1}^{N} \mu_i^* - \mu_f^* \\
= \max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - \mu_f^* + \frac{\alpha_f}{N} \sum_{i=1}^{N} \beta_i \left( \sum_{s \in \mathcal{S}} \alpha_s + \sum_{t \in \mathcal{T}} \alpha_t + \sum_{r \in \mathcal{R}} \bar{p}_r - \max \left( \beta^{(1)} r \bar{p}_r , \beta^{(1)} r \bar{p}_r \right) + \mu_r^* \right)
\]

We distinguish three possible cases: \( f \in \mathcal{S} \), \( f \in \mathcal{T} \), and \( f \in \mathcal{R} \). In the first case, due to Equations (A.184) and (A.185) the right hand side of Equation (6.10) simplifies to
\[
\max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - \mu_f^* + C \alpha_f = \beta^{(1)} f C \frac{\alpha_f}{1 - \beta^{(1)} f} - 0 + C \alpha_f = C \frac{\alpha_f}{1 - \beta^{(1)} f} = p_f^*.
\]

In the second case, where \( f \in \mathcal{T} \), we apply Equations (A.187) and (A.188), such that the right hand side of Equation (6.10) becomes
\[
\max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - \mu_f^* + C \alpha_f = p_f^* - 0 + C \alpha_f = p_f^*.
\]

In the last case, i.e. \( f \in \mathcal{R} \), we use Equation (6.17) which yields
\[
\max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - \mu_f^* + C \alpha_f \\
= \max \left( \beta^{(1)} f p_f^* , \beta^{(1)} f p_f \right) - C \alpha_f + \bar{p}_f - \max \left( \beta^{(1)} \bar{p}_f , \beta^{(1)} \bar{p}_f \right) + C \alpha_f = \bar{p}_f = p_f^*.
\]
This means that the suggested point $p^*$ is indeed a fixed point.

A.23 Proof of Theorem 6.5

For the proof of Theorem 6.5 we investigate the following sum

$$\sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*)|.$$  \hspace{1cm} (A.191)

First, using the reverse triangle inequality and afterwards Lemma 6.2 we find

$$\sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*)| \geq \sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^*| + |\mu_i(\tau_{k+1}) - \mu_i^*| - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*)$$ \hspace{1cm} (A.192)

$$= \sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^*| + |\mu_i(\tau_{k+1}) - \mu_i^*| - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*)$$

Applying the triangle inequality yields

$$\sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*)| \geq \sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^*| + |\mu_i(\tau_{k+1}) - \mu_i^*| - \sum_{\ell=1}^{N} |\mu_\ell(\tau_{k+1}) - \mu_\ell^*|$$ \hspace{1cm} (A.193)

$$= \sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^*| = ||p(\tau_{k+1}) - p^*||_1.$$  

Starting again with Equation (A.191) and using Equation (6.10) we find that

$$\sum_{i=1}^{N} |p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*)|$$

$$= \sum_{i=1}^{N} \left| \max \left( \beta^{(1)} p_i(\tau_k), p_i^* \right) + \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} \left( p_\ell(\tau_k) - \max \left( \beta^{(1)} p_\ell(\tau_k), p_\ell^* \right) \right) \right.$$ \hspace{1cm} (A.194)

$$- \left( \max \left( \beta^{(1)} p_i^*, p_i^* \right) + \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} \left( p_\ell^* - \max \left( \beta^{(1)} p_\ell^*, p_\ell^* \right) \right) \right) \right|.$$
Rearrangement and the use of the triangle inequality yields

\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_{\ell}(\tau_{k+1}) - \mu_{\ell}^*) \right|
\]

\[
= \sum_{i=1}^{N} \left| \max \left( \beta^{(1)} i_{i}(\tau_k), \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) - \max \left( \beta^{(1)} i_{i}^*, \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) \right|
+ \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} \left| p_{\ell}(\tau_k) - \max \left( \beta^{(1)} \ell_{\ell}(\tau_k), \frac{p_{\ell}}{p_{\ell}^*}, \frac{p_{\ell}}{p_{\ell}^*} \right) - \left( p_{\ell}^* - \max \left( \beta^{(1)} \ell_{\ell}^*, \frac{p_{\ell}^*}{p_{\ell}^*}, \frac{p_{\ell}^*}{p_{\ell}^*} \right) \right) \right| \] (A.195)

\[
\leq \sum_{i=1}^{N} \left| \max \left( \beta^{(1)} i_{i}(\tau_k), \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) - \max \left( \beta^{(1)} i_{i}^*, \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) \right|
+ \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} \left| p_{\ell}(\tau_k) - \max \left( \beta^{(1)} \ell_{\ell}(\tau_k), \frac{p_{\ell}}{p_{\ell}^*}, \frac{p_{\ell}}{p_{\ell}^*} \right) - \left( p_{\ell}^* - \max \left( \beta^{(1)} \ell_{\ell}^*, \frac{p_{\ell}^*}{p_{\ell}^*}, \frac{p_{\ell}^*}{p_{\ell}^*} \right) \right) \right| .
\]

This yields

\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_{\ell}(\tau_{k+1}) - \mu_{\ell}^*) \right|
\]

\[
\leq \sum_{i=1}^{N} \left| \max \left( \beta^{(1)} i_{i}(\tau_k), \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) - \max \left( \beta^{(1)} i_{i}^*, \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) \right|
+ \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} \left| p_{\ell}(\tau_k) - \max \left( \beta^{(1)} \ell_{\ell}(\tau_k), \frac{p_{\ell}}{p_{\ell}^*}, \frac{p_{\ell}}{p_{\ell}^*} \right) - \left( p_{\ell}^* - \max \left( \beta^{(1)} \ell_{\ell}^*, \frac{p_{\ell}^*}{p_{\ell}^*}, \frac{p_{\ell}^*}{p_{\ell}^*} \right) \right) \right| .
\] (A.196)

Let $S^+$ and $S^-$ be two sets such that

\[
S^+ = \{ i \in \{1, \ldots, N\} | p_i(\tau_k) - p_i^* \geq 0 \} \quad \text{and} \quad (A.197)
\]

\[
S^- = \{ 1, \ldots, N \} \setminus S^+ . \quad (A.198)
\]

Due to Lemma 6.2 for the two sets

\[
i \in S^+ \Rightarrow \max \left( \beta^{(1)} i_{i}(\tau_k), \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) - \max \left( \beta^{(1)} i_{i}^*, \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) \geq 0 \quad (A.199)
\]

\[
i \in S^- \Rightarrow \max \left( \beta^{(1)} i_{i}(\tau_k), \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) - \max \left( \beta^{(1)} i_{i}^*, \frac{p_i}{p_i^*}, \frac{p_i}{p_i^*} \right) \leq 0. \quad (A.200)
\]

holds.
A. Reasoning and Proofs

Then, Equation (A.196) can be expressed as

\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_{\ell}(\tau_{k+1}) - \mu_{\ell}^*) \right|
\]

\[
\leq \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
+ \sum_{i \in S^-} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
+ \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
+ \sum_{i \in S^-} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|.
\]  

(A.201)

Similar as in Appendix A.13 we distinguish two cases

\[
\left| \sum_{i \in S^+} \left( \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right) \right| \geq
\]

\[
\left| \sum_{i \in S^-} \left( \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right) \right| \]  

(A.202)

and

\[
\left| \sum_{i \in S^+} \left( \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right) \right| <
\]

\[
\left| \sum_{i \in S^-} \left( \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right) \right| \]  

(A.203)

In the first case, Equation (A.208) can be simplified to

\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* - \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_{\ell}} \sum_{\ell=1}^{N} (\mu_{\ell}(\tau_{k+1}) - \mu_{\ell}^*) \right|
\]

\[
\leq \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
+ \sum_{i \in S^-} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
+ \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
- \sum_{i \in S^-} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|
\]

\[
\leq 2 \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i^* \right) \right|.
\]  

(A.204)
Using Lemma 6.2 and the fact that
\[
\sum_{i \in S^+} |p_i(\tau_k) - p_i^*| = \sum_{i \in S^-} |p_i(\tau_k) - p_i^*| \tag{A.205}
\]
we find
\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* \right| = \sum_{i=1}^{N} \left| \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_\ell} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*) \right|
\leq 2 \sum_{i \in S^+} \beta^{(1)}_i |p_i(\tau_k) - p_i^*| \leq 2 \max_i \left( \beta^{(1)}_i \right) \sum_{i \in S^+} |p_i(\tau_k) - p_i^*| \tag{A.206}
\]
\[
= \max_i \left( \beta^{(1)}_i \right) \sum_{i \in S^+} |p_i(\tau_k) - p_i^*| + \sum_{i \in S^-} |p_i(\tau_k) - p_i^*|
= \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1.
\]

For the second case we find similarly,
\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* \right| = \sum_{i=1}^{N} \left| \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_\ell} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*) \right|
\leq \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i \right) \right|
+ \sum_{i \in S^-} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i \right) \right|
\leq 2 \sum_{i \in S^+} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i \right) \right|
\leq 2 \sum_{i \in S^-} \left| \max \left( \beta^{(1)}_i p_i(\tau_k), p_i^* \right) - \max \left( \beta^{(1)}_i p_i^*, p_i \right) \right|.
\tag{A.207}
\]
Then, applying Lemma 6.2 and Equation (A.205) leads to
\[
\sum_{i=1}^{N} \left| p_i(\tau_{k+1}) - p_i^* + \mu_i(\tau_{k+1}) - \mu_i^* \right| = \sum_{i=1}^{N} \left| \frac{\alpha_i}{\sum_{\ell=1}^{N} \alpha_\ell} \sum_{\ell=1}^{N} (\mu_\ell(\tau_{k+1}) - \mu_\ell^*) \right|
\leq 2 \sum_{i \in S^-} \beta^{(1)}_i |p_i(\tau_k) - p_i^*| \leq 2 \max_i \left( \beta^{(1)}_i \right) \sum_{i \in S^-} |p_i(\tau_k) - p_i^*| \tag{A.208}
\]
\[
= \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1.
\]
Hence,
\[
\|p(\tau_{k+1}) - p^*\|_1 \leq \max_i \left( \beta^{(1)}_i \right) \|p(\tau_k) - p^*\|_1 \tag{A.209}
\]
and the claim in Theorem 6.5 holds.


[116] F. Wirth, S. Stüdi, J. Yu, M. Corless, and R. Shorten, Asynchronous algorithms for network utility maximisation with a single bit (i), Accepted to present at Control Conference (ECC), 2015 European, 2015.

