
Available from: http://dx.doi.org/10.1109/TAC.2012.2218151

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Sufficient conditions for generic feedback stabilisability of switching systems via Lie-algebraic solvability

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Abstract—We address the stabilisation of switching linear systems (SLSs) with control inputs under arbitrary switching. A sufficient condition for the stability of autonomous (without control inputs) SLSs is that the individual subsystems are stable and the Lie algebra generated by their evolution matrices is solvable. This sufficient condition for stability is known to be extremely restrictive and therefore of very limited applicability. Our main contribution is to show that, in contrast to the autonomous case, when control inputs are present the existence of feedback matrices for each subsystem so that the corresponding closed-loop matrices are stable admits a Lie-algebraic stability condition that can become a generic property, hence substantially improving the applicability of such Lie-algebraic techniques in some cases. Since the validity of this Lie-algebraic stability condition implies the existence of a common quadratic Lyapunov function (CQLF) for the SLS, our results yield an analytic sufficient condition for the generic existence of a control CQLF for the SLS.

I. INTRODUCTION

Switched systems are dynamical systems that combine a finite number of subsystems by means of a switching signal [1], [2]. In recent years, considerable research effort has been devoted to studying the stability and stabilisability of switched systems [1], [3]–[5]. In this paper, we focus on the case where each subsystem is linear and also on stability under “arbitrary switching”; where stability holds for every possible switching signal. We refer to the switched systems under consideration as switching linear systems (SLSs).

A SLS may either be autonomous or have control inputs. For autonomous SLSs, it is known that the uniform global exponential stability (UES, where ‘uniform’ means ‘over all switching signals’) is equivalent to the existence of a Lyapunov function common to all subsystems [6]. A computationally appealing stability condition, though more restrictive, is the existence of a common quadratic Lyapunov function (CQLF) [5], [4,2]. A CQLF may be efficiently numerically sought for by solving linear matrix inequalities [7]. An analytical stability condition, even more restrictive, states that a SLS admits a CQLF (and is hence UGES) if every individual subsystem is stable and the Lie algebra generated by their evolution matrices is solvable. The solvability of a matrix Lie algebra is equivalent to the existence of a single similarity transformation that transforms each matrix into upper triangular form. This Lie-algebraic stability condition is simple to check numerically and holds both for discrete-time SLSs [8], [9] and continuous-time SLSs [10], [11]. These Lie-algebraic stability conditions, although mathematically elegant and possibly computationally advantageous (cf. [12]), have had very limited applicability due to their restrictiveness.

The situation can be radically different for SLSs with control inputs, where feedback may be employed to stabilise the SLS. Indeed, the main contribution of the current paper is to establish that the existence of feedback matrices for each subsystem so that the closed-loop SLS satisfies the aforementioned Lie-algebraic stability condition can become a generic property, namely, a property that is valid for almost every set of system parameters. We give conditions

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Work partially supported by ANPCyT grant PICT 2010-0783, Argentina.

that ensure the genericity of this property and thus may enhance the applicability of such Lie-algebraic stabilisation techniques when control inputs are present. These conditions depend on the number of states, subsystems and control inputs of each subsystem. In order to be satisfied, the given conditions require each subsystem to have a “substantial” number of inputs, although possibly fewer inputs than states.

Feedback control design based on Lie-algebraic solvability has been previously pursued by the authors [13]–[16]. A central contribution in [13], [14] is an iterative design algorithm that searches for a set of stabilising feedback matrices that attain the target simultaneously triangularisable closed-loop structure via the application of a common eigenvector assignment (CEA) procedure and the reduction of state dimension at each iteration. The main theoretical result in [15], [14] establishes that the proposed algorithm will iterate successfully until the state dimension is reduced to 1 if and only if feedback matrices exist so that the corresponding closed-loop subsystem matrices are stable and simultaneously triangularisable, i.e. if and only if feedback matrices exist so that the closed-loop system satisfies the aforementioned Lie-algebraic stability condition. A numerical implementation for the proposed iterative design algorithm and the CEA procedure are also provided in [14], together with a key structural condition, which, when satisfied, guarantees a directly computable solution for the CEA procedure. If this structural condition is not satisfied, then the required quantities are sought by means of an optimisation problem.

In addition to its limited applicability, the aforementioned Lie-algebraic stability condition is also non-robust, in the sense that even if it is satisfied for a given autonomous SLS, it is almost surely not satisfied by SLSs with parameters arbitrarily close to the given one. The work in [15] then provides a robust result by relaxing, for single input systems, the simultaneous triangularisation requirement to approximate (in a specific sense) simultaneous triangularisation. The main theoretical contribution in [15] establishes that if a single-control-input SLS satisfying the aforementioned Lie-algebraic condition exists in a suitably small neighbourhood of the given SLS, then the proposed algorithm is guaranteed to find feedback matrices so that the corresponding closed-loop SLS admits a CQLF even if the Lie-algebraic condition is not met by the given system. (Agrachev et al. [17] have recently derived, for autonomous SLSs, robust stability conditions related to Lie-algebraic solvability and formulated directly in terms of Lie brackets.)

Our current main results build upon the key structural condition provided in [14]: if such structural condition is satisfied at every iteration of the algorithm, then the considered feedback control design via Lie-algebraic solvability problem may be not restrictive at all for systems with the given dimensions. In this regard, the main result in [16] is the identification of the situation that prevents the structural condition from being satisfied at every iteration of the algorithm.

In the present paper, we build upon the results of [16] by providing sufficient conditions for the structural condition to hold at every iteration of the algorithm for almost every set of system parameters with the given dimensions. We thus provide sufficient conditions for the genericity of the property of existence of feedback matrices so that the closed-loop subsystem matrices are stable and generate a solvable Lie algebra, a property which implies the existence of a CQLF for the closed-loop system. Consequently, a side contribution of the current paper is the derivation of an analytic condition that ensures the genericity of the property of existence of feedback matrices so that the corresponding closed-loop SLS admits a CQLF. Preliminary results on this topic have been previously presented by the authors in [18]. Even though our previous results [13]–[16], [18] focus on discrete-time SLSs, the current results are valid for both discrete- and continuous-time SLSs.
Notation. The index set \( \{1, 2, \ldots, N\} \) is denoted \( \mathbb{N} \). The kernel (null space) of a matrix or linear map \( A : \mathcal{X} \to \mathcal{Y} \) is denoted \( \ker A \) and its image (range), \( \text{im} A \). Given a subspace \( \mathcal{B} \subseteq \mathcal{Y} \), the subspace \( \{ v \in \mathcal{X} : Av \in \mathcal{B} \} \) is denoted \( (A)^{-1} \mathcal{B} \). For \( x \in \mathbb{C}^{n \times m} \), its transpose is denoted \( x^* \), its conjugate transpose \( x^\dagger \) and its Moore-Penrose generalised inverse \( x^+ \). If \( S, T \) are vector spaces, then \( S \subseteq T \) means that \( S \) is a subspace of \( T \). \( S \oplus T \) denotes the direct sum of \( S \) and \( T \) (which implies that \( S \cap T = 0 \)), and \( \text{dim} (S) \) denotes the dimension of \( S \). If \( I \) is a finite set, then \( \# I \) denotes the number of elements in \( I \).

II. PROBLEM FORMULATION

Consider the discrete- or continuous-time SLSs

\[
\begin{align*}
\dot{x}_{k+1} & = A_{\sigma(k)} x_k + B_{\sigma(k)} u_{\sigma(k)}, \\
\dot{x}(t) & = A_{\sigma(t)} x(t) + B_{\sigma(t)} u_{\sigma(t)}(t),
\end{align*}
\]

(1) (2)

where the switching function \( \sigma(\cdot) \) takes values in \( \mathbb{N} \), \( x \in \mathbb{R}^n \) for all \( i \in \mathbb{N}, \ u^i \in \mathbb{R}^m \), the matrices \( A_i \in \mathbb{R}^{n \times n} \) and \( B_i \in \mathbb{R}^{n \times m} \) are known, and \( B_i \) have full column rank. We are interested in state-feedback design by seeking feedback matrices that enforce a common eigenvector with corresponding stable eigenvalues.

III. PREVIOUS RESULTS

Control design that causes the closed-loop system to be stable by satisfying the conditions of Lemma 1 can be performed iteratively by seeking feedback matrices that assign a common eigenvector with stable corresponding eigenvalues at every iteration \([13], [14]\). Although the latter references deal exclusively with discrete-time SLSs, the only difference between the discrete- and continuous-time cases is the stability region considered (the open unit disk or the open left half-plane). The control design method of \([13], [14]\) is represented by iterative feedback design for the SLSs (1) and (2) based on the Lie-algebraic solvability condition

\[
\begin{align*}
u_{\sigma(k)} = K_{\sigma(k)} x_k, \quad & u_{\sigma(t)}(t) = K_{\sigma(t)} x(t),
\end{align*}
\]

(3)

so that the resulting closed-loop system

\[
\begin{align*}
x_{k+1} & = A_{\sigma(k)}^c x_k, \quad & \dot{x}(t) = A_{\sigma(t)}^c x(t),
\end{align*}
\]

(4)

\[
A_i^c = A_i + B_i K_i, \quad \text{for } i \in \mathbb{N},
\]

(5)

admit a CQLF and hence be stable under arbitrary switching. Note that at every instant, the control law \([5]\) requires knowledge of the "active" subsystem generated by \( \sigma(k) \) or \( \sigma(t) \).

As is well-known, ensuring that each \( A_i^c \) be stable is necessary but not sufficient to ensure the stability of the autonomous SLS \([4]\) under arbitrary switching. A sufficient condition is given by the following result \([10] \) Theorem 2, \([2] \) Theorem 6.18.

Lemma 1 (Lie-algebraic-solvability stability condition): If every \( A_i^c \) is stable and the Lie algebra generated by \( \{A_i^c : i \in \mathbb{N}\} \) is solvable, then \([4]\) admits a CQLF and hence is UGES.

In this paper, we specifically consider stabilising state feedback design for the SLSs (1) and (2) based on the Lie-algebraic-solvability condition of Lemma 1. We thus focus on the SLS class defined next.

Definition 1 (SLASF): A set \( Z = \{A_i \in \mathbb{R}^{n \times n} : i \in \mathbb{N}\} \) is said to be SLASF (Solvable Lie Algebra with Stability by Feedback) if there exist \( K_i \in \mathbb{R}^{m \times n} \) such that \( A_i^c \) are stable (Schur-stable for a discrete-time SLS; Hurwitz-stable for a continuous-time SLS) and generate a solvable Lie algebra.

In matrix terms, the fact that the Lie algebra generated by the matrices \( A_i^c \) is solvable is equivalent to the existence of an invertible matrix \( T \in \mathbb{C}^{n \times n} \) such that \( T^{-1} A_i^c T \) is upper triangular for all \( i \in \mathbb{N} \). Note that even if the matrices \( A_i^c \) have real entries, those of \( T \) may be complex \([19]\).
structural condition introduced in \cite{14}, which, when satisfied, ensures that such a vector exists and allows its computation in a straightforward and numerically efficient way.

We introduce some notion required to state this structural condition. Define \( m_i^\ell = \operatorname{rank}(B_i^\ell) = d(\im B_i^\ell) \), and factor \( B_i^\ell = b_i \ell v_i \), where \( v_i : \mathbb{R}^{m_i^\ell} \to \mathbb{R}^{m_i^\ell} \) has full row rank and \( b_i^\ell : \mathbb{R}^{m_i^\ell} \to \mathbb{R}^{n_i^\ell} \) has full column rank. We adopt the convention that \( b_i^\ell \) is an empty matrix if \( m_i^\ell = 0 \). Note that \( \operatorname{img} B_i^\ell = \operatorname{img} b_i^\ell \). Let \( A_i \) be the vector with components \( \lambda_i \), \( i \in \mathbb{N} \), i.e.

\[
A_i = [\lambda_1^i, \lambda_2^i, \ldots, \lambda_N^i],
\]

and build the matrix

\[
Q_i(A_i) = [R_i(A_i), -B_i],
\]

\[
R_i(A_i) = \begin{bmatrix}
\lambda_1^i I - A_1^i \\
\vdots \\
\lambda_N^i I - A_N^i
\end{bmatrix}, \quad \text{and } B_i = \operatorname{blkdiag}[b_1^i, \ldots, b_N^i],
\]

where \( \operatorname{blkdiag} \) denotes block diagonal concatenation.

**Lemma 2 (Structural condition \cite{14, 19}):** Let

\[
p_{\ell} = n_{\ell} + \sum_{i=1}^{N} m_i^\ell - N n_{\ell}.
\]

Then,

(a) A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues \( \lambda_i^\ell \) for \( i \in \mathbb{N} \) exists if and only if \( d(\ker Q_i(A_i)) > 0 \).

(b) If \( Q_i(A_i) v = 0 \) with \( v \neq 0 \) partitioned as

\[
w = [v', u_1', \ldots, u_N'],
\]

then \( v \neq 0 \), and

\[
(A_i + B_i F_i) v = \lambda_i v, \quad \text{for } i \in \mathbb{N},
\]

for every \( F_i^\ell \) satisfying \( r_i^\ell F_i^\ell v = u_i \). For each \( i \in \mathbb{N} \) one such \( F_i^\ell \) is \( F_i^\ell = (r_i^\ell)^\dag u_i v \).

(c) \( \operatorname{det}(Q_i(A_i)) \geq p_{\ell} \) for every choice of \( A_i \) as in (14).

Consequently, if \( p_{\ell} > 0 \), then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.

**Lemma 2** gives a structural condition, namely \( p_{\ell} > 0 \), for a feedback-assignable common eigenvector \( v \) to exist for every choice of corresponding eigenvalues \( \lambda_i \). This condition is structural because the quantities involved in the computation of \( p_{\ell} \) are only matrix ranks and dimensions. If the structural condition \( p_{\ell} > 0 \) is satisfied, a feedback-assignable common eigenvector \( v^\ell \) as required at iteration \( \ell \) of Algorithm ITF can be computed as follows:

1. Select the corresponding (stable) closed-loop eigenvalues \( \lambda_i^\ell \) for each subsystem \( i \in \mathbb{N} \) and build \( A_i \) as in (14).

2. Find a vector \( w \neq 0 \) with components partitioned as in (17) so that \( Q_i(A_i) w = 0 \) (namely, so that \( w \in \ker Q_i(A_i) \));

3. Select the first \( n_{\ell} \) components of \( w \) to construct the subvector \( v \) in (17). The feedback-assignable common eigenvector sought is finally computed as \( v^\ell = \frac{v}{\|v\|} \).

Feedback matrices to assign the eigenvector \( v^\ell \) with corresponding eigenvalues \( \lambda_i^\ell \) can be obtained as \( F_i^\ell = (r_i^\ell)^\dag u_i v \). Procedure CEA is thus summarised in Figure 2 for the case when the structural condition of Lemma 2 is satisfied.

Even if the SLS matrices \( A_i, B_i \) have real entries, those of the matrices \( A_i^\ell, B_i^\ell \) internal to Algorithm ITF can be complex at some iteration \( \ell \). This is so because the vector \( v^\ell \) returned by Procedure CEA (a feedback-assignable common eigenvector) can have complex components even if \( A_i^\ell, B_i^\ell \) have real entries, causing \( A_i^{\ell+1}, B_i^{\ell+1} \) to have complex entries. However, when the structural condition \( p_{\ell} > 0 \) is satisfied, the closed-loop eigenvalues \( \lambda_i^\ell \), i.e. the components of \( A_i \), can be arbitrarily selected. Hence, selecting \( \lambda_i^\ell \in \mathbb{R} \) will cause the vector \( v^\ell \) to be real. In the sequel, we assume that real eigenvalues will be selected and hence all matrices internal to Algorithm ITF will have real entries.

**C. The Structural Condition**

If the structural condition as given by Lemma 2, namely \( p_{\ell} > 0 \), holds at iteration \( \ell \) of Algorithm ITF, then Procedure CEA can compute a feedback-assignable common eigenvector and the corresponding feedback matrices, for every choice of corresponding (stable) closed-loop eigenvalues. In addition, if \( p_{\ell} > 0 \) the closed-loop eigenvalues \( \lambda_i^\ell \) for every \( i \in \mathbb{N} \) can be freely chosen. The quantity \( p_{\ell} \) depends on \( m_i^\ell \), the rank of \( B_i^\ell \). At the first iteration of Algorithm ITF, i.e. when \( \ell = 1 \), the internal matrices \( B_i^1 = B_i \), have \( n = n_i \) rows, \( m_i \) columns, and since by assumption they have full column rank, then \( m_i^1 = m_i \). At subsequent iterations, the matrices \( B_i^\ell \) have \( n_{\ell} = n - \ell + 1 \) rows and \( m_i \) columns. Since the matrix (10) is unitary by construction, then according to (11) and (15) we have

\[
m_{\ell+1} - 1 \leq m_i^{\ell+1} \leq m_i^\ell,
\]

and moreover, \( m_i^{\ell+1} \) depends on the vector \( v^\ell \) returned by Procedure CEA as

\[
m_i^{\ell+1} = \begin{cases} m_i^\ell & \text{if } v^\ell \notin \operatorname{img} B_i^\ell, \\ m_i^\ell - 1 & \text{if } v^\ell \in \operatorname{img} B_i^\ell. \end{cases}
\]

From (20), then \( m_i^{\ell+1} = m_i^\ell - 1 \) when \( m_i^\ell = n_{\ell} \), because \( v^\ell \in \mathbb{R}^{n_{\ell}} = \operatorname{img} B_i^\ell \). The following theorem and corollary follow from (6), (16), and (20).

**Theorem I (\cite{19}):** Consider Algorithm ITF at iteration \( \ell \) and \( p_{\ell} \) as in (16), with \( m_i^\ell = \operatorname{rank}(B_i^\ell) \). Then, \( p_{\ell+1} \geq p_{\ell} - 1 \), with equality if and only if

\[
v^\ell \in B_i^\ell, \quad \text{with } B_i^\ell = \bigcap_{i=1}^{N} B_i^\ell \text{ and } B_i^\ell \subseteq \operatorname{img} B_i^\ell.
\]

**Corollary I:** Let \( p_{\ell} > 0 \). Then,

(a) \( p_{q \ell} > 0 \) for \( q = \ell, \ldots, \ell + p_{\ell} - 1 \).

(b) \( p_{\ell+1} > 0 \) if \( v^\ell \notin \operatorname{img} B_k^\ell \) for some \( k \in \mathbb{N} \).

(c) \( p_{\ell+1} \neq 0 \) if and only if \( p_{\ell} = 1 \) and (21) holds.

**Corollary II** identifies the condition that prevents the inductivity of the structural condition \( p_{\ell} > 0 \) from iteration \( \ell \) to iteration \( \ell + 1 \) of the algorithm. The following section builds on these results.

**IV. MAIN RESULTS**

In this section, we derive conditions to ensure that the structural condition \( p_{\ell} > 0 \) will hold at every iteration of Algorithm ITF, i.e. for \( \ell = 1, \ldots, n \). Subsequently, we will establish that, for some state vector dimensions, \( n \), number of subsystems, \( N \), and number
of control inputs, $m_i$, for $i \in \mathbb{N}$, these conditions are valid for almost every set of system parameters —the entries of $A_i$ and $B_i$ for $i \in \mathbb{N}$.

In Section IV-A, we recall and extend the property of transversality of subspaces (see, e.g., Chapter 0 of [20]), which is required for the derivation of our main results. In Section IV-B, we derive conditions that ensure the validity of the structural condition at every iteration of Algorithm ITF. In Section IV-C, we analyse the conditions derived in Section IV-B and relate them to the genericity of the SLASF property (recall Definition 1). A brief numerical example is given in Section IV-D. Proofs are provided in the Appendix.

A. Transversality of Subspaces

Definition 2 (Transverse): Two subspaces $S, T$ of an ambient space $X$ are said to be transverse when the dimension of their intersection is minimal, given the dimensions of $S$ and $T$, i.e. when

$$d(S \cap T) = \max\{0, d(S) + d(T) - d(X)\}. \quad (22)$$

Equivalently, $S$ and $T$ are transverse when the dimension of their sum is maximal. We extend this definition to sets of subspaces as follows. Let $S = \{S_1, \ldots, S_N\}$ be a set of subspaces of an ambient space $X$. We say that $S$ is transverse when both the intersection of the subspaces in every subset of $S$ has minimal dimension and the sum of the subspaces in every subset of $S$ has maximal dimension.

It is well-known [20, Ch. 0] that transversality of two subspaces $S$ and $T$ is a generic property, i.e. it is satisfied by almost every $S$ and $T$ selected “randomly” among all subspaces of $X$. Also, it is evident that the extension of this property to sets of subspaces according to Definition 2 preserves genericity, in the sense that almost every set containing a finite number of subspaces taken “randomly” among all subspaces of $X$ will be transverse according to Definition 2.

We will require the following properties related to transversality.

Lemma 3: Let $S = \{S_1, \ldots, S_N\}$ be a set of subspaces of the ambient space $X$, and define

$$p \doteq d(X) + \sum_{i \in \mathcal{S}} d(S_i) - N d(X). \quad (23)$$

Then,

(a) $d(S_i \cap S_j) = d(S_i) + d(S_j) - d(S_i + S_j)$.
(b) If $S$ is transverse, then $d(\bigcap_{i \in \mathcal{S}} S_i) = \max\{0, p\}$.
(c) If $S$ is transverse and $p \geq 0$, then $d(S_i + S_j) = d(X)$ for all $i, j \in \mathcal{S}$ with $i \neq j$.
(d) Let $J = I \cup \{j\}$, with $J \subseteq \mathcal{S}$ and $\# J = \# I + 1$. Suppose that $p \geq 0$ and that $\{S_i : i \in I\}$ is transverse. Then, $\{S_i : i \in J\}$ is transverse if and only if $\bigcap_{i \in I} S_i + S_j = X$.

B. Validity of the Structural Condition at every Iteration

The derivations of this section require deep analysis of the condition (21). In the sequel, let $S_i^f$ denote the set of vectors $v \in B_i^f = \text{Im} B_i^f$ for which there exist a matrix $F_i^f$ and a stable scalar $\lambda$ so that

$$(A_i^f + B_i^f F_i^f)v = \lambda v. \quad (23)$$

By definition, $S_i^f$ is the set of feedback-assignable stable eigenvectors for the internal subsystem $(A_i^f, B_i^f)$ that are contained in $B_i^f$. Consequently, if $v_i^f$ is a stable feedback-assignable common eigenvector, then $v_i^f \in B_i^f$ if and only if $v_i^f \in S_i^f$. The following result is straightforward.

Lemma 4:

(a) The set $S_i^f$ is a subspace.
(b) $v \in S_i^f$ if and only if $v \in B_i^f$ and $A_i^f v \in B_i^f$.

define the following quantities:

$$\rho_i^f \doteq d(S_i^f), \quad q_i \doteq n_i + \sum_{\ell \in \mathcal{N}} \rho_i^\ell - N n_i, \quad (24)$$

$$S_i^f \doteq \bigcap_{\ell \in \mathcal{N}} S_i^\ell, \quad \rho_i^f \doteq d(S_i^f). \quad (25)$$

The core technical result of the paper is given below as Theorem 2. This result gives conditions under which the structural condition of Lemma 1 will hold at every iteration of Algorithm ITF, irrespective of the choice of closed-loop eigenvalues $\lambda_i^f$ performed in Procedure CEA.

Theorem 2: Let $\{S_i^f : i \in \mathcal{N}\}$ be transverse, $q_i \geq 0$, and $(A_i, B_i)$ be controllable for all $i \in \mathcal{N}$. Then,

(a) $p_i > 0$ for $\ell = 1, \ldots, n$.
(b) The set $Z = \{(A_i, B_i) : i \in \mathcal{N}\}$, which identifies the given SLS, is SLASF.

C. Genericity of the SLASF Property

Theorem 2 gives sufficient conditions under which a given SLS will be SLASF. We next show that for some state vector dimensions, $n$, number of subsystems, $N$, and number of control inputs, $m_i$, for each $i \in \mathcal{N}$, these conditions are satisfied for almost every set of matrices $A_i \in \mathbb{R}^{n_i \times n}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ with $i \in \mathcal{N}$.

The three conditions required by Theorem 3 are that the set of subspaces $\{S_i^f : i \in \mathcal{N}\}$ be transverse, that the quantity $q_i$ be nonnegative and that the pairs $(A_i, B_i)$ be controllable. It is well-known that controllability is a generic property [20] and hence we next focus on the first two conditions.

We show first that transversality of $\{S_i^f : i \in \mathcal{N}\}$ is generic in the space of parameters of the matrices $A_i, B_i$. From Lemma 5, it follows that $S_i^f = B_i^f \cap (A_i^f)^{-1} B_i^f$. Note that arbitrary choices for the entries of $B_i = B_i^f$ yield arbitrary $B_i^f = \text{Im} B_i^f$, although generically of dimension $m_i = m_i^f$. In addition, arbitrary choices for the entries of $A_i = A_i^f$ yield arbitrary $(A_i^f)^{-1} B_i^f$, also generically of dimension $m_i$. Therefore, the subspaces $B_i^f$ and $(A_i^f)^{-1} B_i^f$ will be transverse generically and from (22),

$$d(S_i^f) = d(B_i^f \cap (A_i^f)^{-1} B_i^f) = \max\{0, 2m_i - n\} \quad \text{generically.} \quad (26)$$

Due to the fact that arbitrary $S_i^f$ can be produced, then the set $\{S_i^f : i \in \mathcal{N}\}$ is transverse generically in the space of parameters of $A_i, B_i$.

Consider next the quantity $q_i$. From (24) and (25) follows that

$$q_i = n + \sum_{\ell \in \mathcal{N}} \max\{0, 2m_i - n\} - N n \quad \text{generically.} \quad (27)$$

the SLASF property holds for almost every set of system parameters $A_i \in \mathbb{R}^{n_i \times n}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ for all $i \in \mathcal{N}$.

Corollary 2: Under the same conditions for $n, N$ and $m_i$ as in Theorem 3 the property of existence of feedback matrices so that the closed-loop SLS admits a CQLF is generic.

We next analyse the condition (27). Note that non-trivial cases are those for which $n \geq 2, N \geq 2$ and $1 \leq m_i \leq n - 1$ for all $i \in \mathcal{N}$. Combining these conditions with (27) leads to the following:
In order for (27) to hold in a non-trivial case, then it is necessary that $N \leq n/2$ and $m_i > n/2$.

Under the condition $N \leq n/2$, then it is sufficient (but not necessary) that $m_i = n - 1$ for all $i \in N$ for (27) to hold.

**D. Numerical Example**

Consider a discrete-time system of the form (1), with $n = 4$, $N = 2$, $m_1 = m_2 = 3$.

$A_1 = \begin{bmatrix} 3.2 & 4 \cdot 10^{-4} & -3.4 & 3.6 \\ 4.1 & 2 \cdot 10^{-2} & 4.7 & 0.8 \\ -3.7 & 5 \cdot 10^{-4} & 4.6 & 4.2 \\ 4.1 & 4 \cdot 10^{-2} & -0.2 & 2.9 \end{bmatrix}$  $A_2 = \begin{bmatrix} 2.6 & 2 \cdot 10^{-1} & 3.2 & -0.6 \\ 0.4 \cdot 10^{-1} & 4.7 & 2 \cdot 10^{-1} & 1.2 \\ -1 \cdot 10^{-1} & 2 \cdot 10^{-1} & 1.8 & 2.7 \\ 1.6 & 4 \cdot 10^{-1} & 4 \cdot 5 & 3.0 \end{bmatrix}$

$B_1 = \begin{bmatrix} -2.2 & 4 \cdot 10^{-4} & 0.1 \\ 1.6 & 9 \cdot 10^{-3} & 3.9 \\ 3.4 \cdot 10^{-4} & 2 \cdot 10^{-4} & 6.2 \end{bmatrix}$  $B_2 = \begin{bmatrix} -0.3 & 0.1 \\ 1.6 & 10^{-4} \\ 1.6 & 4 \cdot 10^{-1} \\ -3.4 \cdot 2 \cdot 10^{-4} \end{bmatrix}$

Each of the entries of $A_1$, $A_2$, $B_1$, $B_2$ has been generated by rounding a random value uniformly distributed in the interval $[-5, 5]$. By direct computation, it can be verified that all the eigenvalues of both $A_1$ and $A_2$ are real and $B_1$, $B_2$ are controllable. According to (16), we have $p_1 = 2$ and from (23), $q_0 = 0$. According to Theorem 2, the given system is SLAFS and $p_j > 0$ at every iteration of Algorithm ITF, irrespective of the choice of eigenvalues performed in Procedure CEA. Choosing the stable eigenvalues $\lambda_i^T = 0$ for $i = 1, 2$ at every iteration $\ell$, Algorithm ITF yields

$K_1 = \begin{bmatrix} 6.0309 & -2.4801 & -0.7205 & -3.8911 \\ 1.3598 & 0.1776 & 0.3202 & -0.3436 \\ 0.5767 & -0.5322 & -0.704 & -1.0552 \end{bmatrix}$

$K_2 = \begin{bmatrix} 1.1774 & -0.119 & 0.4596 & 0.4449 \\ -0.640 & -1.0734 & -0.6828 & -0.6902 \\ 0.7242 & -0.6091 & 0.7428 & 0.5267 \end{bmatrix}$

It can be verified that all the eigenvalues of $A_i^{CL}$ [recall (5)] are zero (within rounding accuracy) and that $T^{-1} A_i^{CL} T$ is upper triangular for $i = 1, 2$, with

$T = \begin{bmatrix} 0.1570 & 0.9876 & 0 & 0 \\ -0.5668 & -0.0991 & -0.8189 & 0 \\ -0.3808 & -0.6005 & 0.2702 & 0.8822 \\ 0.7135 & 0.1134 & -0.5063 & 0.4709 \end{bmatrix}$

See [16] for more numerical examples on cases where genericity conditions hold at all or just some iterations of Algorithm ITF.

**V. Conclusion**

We have considered both continuous- and discrete-time SLSs with control inputs and under arbitrary switching. A stability result for SLSs with no control inputs states that the SLS is stable if the subsystem $A$ matrices are stable and generate a solvable Lie algebra. This stability result encounters very limited applicability due to its restrictive and non-robustness. However, we have established that when control inputs are present, the property of existence of feedback matrices so that the closed-loop SLS subsystem matrices are stable and generate a solvable Lie algebra can become generic, i.e. valid for almost every system of system parameters. We have derived sufficient conditions that ensure the genericity of this property. In order for these conditions to hold in non-trivial cases, the number of subsystems of the SLS has to be not greater than half the number of system states and every subsystem is required to have more control inputs than half the number of states.

Since the aforementioned Lie-algebraic stability condition implies the existence of a CQLF for the SLS, our results also provide an analytic sufficient condition for the genericity of the existence of feedback matrices so that the closed-loop SLS admits a CQLF.

**Appendix**

**Proof of Lemma 3**

The proof of (a) is a direct application of subspace algebra.
From (20), it follows that $c_{j,k} \neq 0$ for at least one pair of indices $(j,k)$ such that $k = \kappa^i_{k,j} - 1$, or otherwise the vectors in $D$ would be linearly dependent, a contradiction.

Let $\kappa = \max \{ \kappa^i_{k,j} : c_{j,k} \neq 0 \text{ with } k = \kappa^i_{k,j} - 1 \}$, and let $i$ be such that $c_{i,k} \neq 0$. From the basis $D$, construct another basis, $\mathcal{B}$, by replacing the basis vector $(A^i_{\kappa+1}v^i_{\kappa+1}t_i)w_{\kappa}$. Note that Span$(U_{t_i+1}^i : t \in D) = \mathbb{R}^{m_{\kappa+1}}$ and $U_{t_i+1}^i = U_{t_i+1}^i$. By (10) and (12) and the fact that $A^i_{\kappa+1}v^i_{\kappa+1}t_i = U_{t_i+1}^i + A^i_{\kappa+1}v^i_{\kappa+1}t_i$ and hence $u_{t_i+1}^i(A^i_{\kappa+1}v^i_{\kappa+1}t_i)$ is a basis for $\mathbb{R}^{m_{\kappa+1}}$ (recall that, by (6), $n_{\kappa+1} = n_{\kappa} - 1$). From (13), it follows that $B^i_{\kappa+1} = U_{t_i+1}^i B^i_{\kappa}$. We have that a basis for $B^i_{\kappa+1}$, is $E = \{ U_{t_i+1}^i t_j : j = 1, \ldots, m_i \}$ if $\bar{\kappa} > 1$ or $E = \{ U_{t_i+1}^i t_j : j = 1, \ldots, m_i, j \neq i \}$ if $\bar{\kappa} = 1$. The condition $\bar{\kappa} = 1$ hence happens if and only if $v^i_{\kappa+1} \in \mathcal{S}$.

The preceding derivations show that the controllability indices of $(A^i_{\kappa+1}, B_{i}^{\kappa+1})$ are given by $k^i_{\kappa+1} = k^i_{\kappa}$ for $j = 1, \ldots, m_i$ with $j \neq i$ and $k^i_{\kappa+1} = k^i_{\kappa} - 1$ whenever $\bar{\kappa} = 1 > \bar{\kappa}$ and hence $(A^i_{\kappa+1}, B_{i}^{\kappa+1})$ has one controllability index equal to one more than $(A^i_{\kappa}, B_i^\kappa)$. The fact that $S_{\kappa+1}^i \supset U_{t_i+1}^i S_i^\kappa$ follows from the latter consideration and the basis $E$.

Proof of Theorem 2. (a) First, we prove that the conditions

$$\{ S_i^\ell : i \in N \} \text{ transverse, } q_\ell \geq 0, \quad (A_i^\ell, B_i^\ell) \text{ controllable},$$

imply that $p_\ell > 0$. Since $S_i^\ell \subset B_i^\ell$, then $p_\ell \leq m_i$ and $q_\ell \leq p_\ell$. From controllability of $(A_i^\ell, B_i^\ell)$ and Lemma 2, then $p_\ell = m_i$ if and only if $m_i = n_i$. Hence, if $q_\ell = p_\ell$, then $p_\ell = n_i > 0$. Otherwise, $0 < p_\ell < p_\ell$.

Next, we establish the validity of (32) for $\ell = 1, \ldots, n$. Note that (32) hold at $\ell = 1$ by assumption and because $A_1^\ell = A_1$ and $B_1^\ell = B_1$. Next, suppose that (32) hold at some $1 \leq \ell < n$. By the argument in the previous paragraph, then $p_\ell > 0$, which ensures the existence and computation of $v^i_{\ell+1}$ such that $A^i_{\ell+1}v^i_{\ell+1}t_i$ with scalar $\lambda^i_{\ell+1}$ for all $i \in N$. Hence, $(A_1^\ell, B_1^\ell)$ is controllable by Lemma 6.

Also by Lemma 6, we have $S_{\kappa+1}^i \supset U_{t_i+1}^i S_i^\ell$ for all $i \in N$. Since $q_\ell \geq 0$, by Lemma 3(b) we have that $p_\ell = q_\ell$, and from Lemma 3(c) we have $d(S_i^\ell + S_i^\ell) = n_i$ for all $i, j \in N$ with $i \neq j$. It follows that $d(S_i^\ell + S_i^\ell) \geq d(U_{t_i+1}^i(S_i^\ell + S_i^\ell)) = n_\ell - 1 = n_\ell$ for all $i, j \in N$ with $i \neq j$. The latterfact establishes that the sum of the sets in every subset of $\{ S_i^\ell : i \in N \}$ has maximal dimension and also that $\{ S_i^\ell + S_i^\ell \}$ is transverse for all $i, j \in N$ with $i \neq j$.

Let $\ell$ be a subset of $\{ S_i^\ell : i \in N \}$. We proceed by induction on the number of subspaces in $T$. We have already established that $T$ is transverse if $#T = 2$. Suppose next that $T$ is transverse whenever $#T = 2, \ldots, \alpha$, with $\alpha \leq N - 1$. Let $T = \{ S_i^\ell : i \in I \}$, with $I \subset N$ and $#I = \alpha$, and let $R = T \cup \{ S_i^\ell \}$ so that $#R = \alpha + 1$.

By Lemma 6 and properties of maps and subspaces, we have

$$\bigcap_{i \in I} S_i^\ell + S_i^\ell \supset \bigcap_{i \in I} U_i^\ell + S_i^\ell + S_i^\ell \supset U_i^\ell + \bigcap_{i \in I} S_i^\ell + S_i^\ell.$$