Beer, G., Marussig, B. and Duenser, C. (2013) ***Géotechnique Letters* 3**, No. 3, 108–111, http://dx.doi.org/10.1680/geolett.13.00009

Isogeometric boundary element method for the simulation of underground excavations

G. BEER*†, B. MARUSSIG* and C. DUENSER*

Isogeometric methods have become the latest trend in numerical simulation. Their attractiveness stems from the fact that the description of the geometry can be taken directly from CAD programs, avoiding the need for mesh generation. Since NURBS (non-uniform rational B-splines) functions that exhibit desirable properties such as high continuity and efficient refinement algorithms are also used for the description of the unknowns, a higher quality of results can be expected. This paper presents some results from the research project 'Fast isogeometric boundary element methods for tunneling'. The aim of the project is to implement a fast and accurate method for the simulation of underground excavations without the need for the generation and subsequent refinement of a mesh.

KEYWORDS: excavation; numerical modelling; tunnels

INTRODUCTION

The finite-difference method (FDM), the finite-element method (FEM) and the boundary element method (BEM) have been successfully applied for the simulation of underground excavations. These methods require the generation of a volume (in FEM and FDM) or a boundary mesh (in BEM). The effort required for the generation of such meshes can be significant and mesh quality can have a significant impact on the accuracy of the results (for a review see, for example, the paper by Gioda & Swoboda (1999)). The mesh generation effort is reduced considerably if the BEM is used, since only the excavation boundary has to be discretised. The additional advantage of the BEM is that an infinite domain can be modelled without the need for mesh truncation.

Excavation geometries are often available as CAD models. Figure 1 shows an example of a more complex geometry in underground construction. If the geometry could be taken directly from such a model without the need for the generation of a mesh, this would mean a significant reduction in effort and an increase in accuracy, since no geometrical approximation error is introduced.

A new trend in numerical modelling is isogeometric analysis, which has emerged in the last decade. The main idea of the method is to use technologies from computational geometry (as used in CAD programs), such as non-uniform rational B-splines (NURBS) (Piegl & Tiller, 1997), for the analysis. Isogeometric FEMs were first proposed by Hughes *et al.* (2005), but the isogeometric concept has only recently been applied to the BEM for elasticity problems with mechanical engineering applications (e.g. Belibassakis *et al.*, 2009; Li & Qian, 2011; Scott *et al.*, 2013; Simpson *et al.*, 2012). The idea behind the term 'iso' is to use the same NURBS for the description of the geometry as well as for variation of the unknown.

The purpose of the research project 'Fast isogeometric boundary element methods for tunneling' is to apply this concept to problems in geotechnical engineering, in particular exterior problems as typified by analyses of underground excavations. To the best of the authors' knowledge, this is the first application of the isogeometric concept to exterior, rather than interior, problems. The method was implemented in our research code BEFE++. The capabilities of this code include consideration of non-linear and non-homogeneous domains, sequential excavation/construction and fast methods of solution.

The main differences to the above-mentioned publications on the isogeometric BEM are that the method is applied to infinite domains and different functions are used for the description of the geometry, the tractions and the displacements. This latter difference means that the method is no longer 'isogeometric', since this implies using the same basis functions for all, but could novelly be termed 'supergeometric'. The advantages of this supergeometric approach are that, when refining the solution, no new coordinates of control points have to be computed and computation of the Jacobian and the outward normal vector remains unchanged. Considerable computational savings would therefore be expected for large three-dimensional (3D) examples.

Figure 2 shows an example of a designed shape of a tunnel defined by circular arcs. It so happens that a circular arc can be exactly described by a second-order NURBS, so the description of half the tunnel with three NURBS is exact and need not be improved. Actually, it is even possible to unite these NURBS to a single NURBS curve that represents the whole boundary of the tunnel but the interpolatory points between the circular arcs will only ensure C^0 rather than C^1 continuity. The exact representation of the boundary means that the outward normal is also exactly represented and that the excavation tractions are also exact. However, the description of the unknown displacements may not be and thus a refinement in the definition of the NURBS function may be necessary; strategies for doing this are outlined later in this article.

THEORETICAL BACKGROUND

The theoretical background can only be discussed briefly here, but more details can be found elsewhere (Beer *et al.*,

Manuscript received 1 February 2013; first decision 19 March 2013; accepted 3 July 2013.
Published online at www.geotechniqueletters.com on 26 July 2013.
*Institute for Structural Analysis, Graz University of Technology, Austria
†Newcastle University, Australia

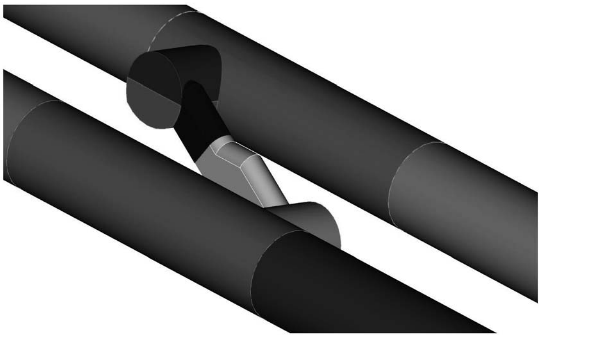

Fig. 1. Example of more complex excavation geometry (CAD model of Trans-Hudson Tunnel, courtesy of Halcrow Ltd)

in preparation). Using the theorem of Betti, the following integral equation is obtained (see Beer *et al.* (2008))

$$\mathbf{c}(P)\boldsymbol{u}(P) = \int_S \mathbf{U}(P,Q)\boldsymbol{t}(Q)\,\mathrm{d}S(Q) - \int_S \mathbf{T}(P,Q)\boldsymbol{u}(Q)\,\mathrm{d}S(Q) + \int_V \mathbf{E}(P,Q)\boldsymbol{\sigma}_0(Q)\,\mathrm{d}V \tag{1}$$

where P is the source point and Q is the field point on the boundary S. The coefficient $\mathbf{c}(P)$ is a free term related to the boundary geometry. $\boldsymbol{u}(Q)$ and $\boldsymbol{t}(Q)$ are the displacements and tractions on the boundary and $\mathbf{U}(P,Q)$ and $\mathbf{T}(P,Q)$ are matrices containing Kelvin's fundamental solutions (kernels) for the displacements and tractions respectively. $\boldsymbol{\sigma}_0(Q)$ are initial stresses arising from non-linear or non-homogeneous behaviour and $\mathbf{E}(P,Q)$ is a matrix containing fundamental solutions for the strains (see Beer *et al.* (2008)). Using the collocation method, the integral equation can be enforced on a number of points P_a, thus obtaining enough equations for solving a boundary value problem.

In order to solve the system of equations (1) numerically, the geometry $\boldsymbol{x}$, traction $\boldsymbol{t}$ and displacement $\boldsymbol{u}$ are described by the following interpolations

$$\boldsymbol{x} = \sum_{i=1}^{I} R_{i,p}\boldsymbol{B}_i \tag{2}$$

$$\boldsymbol{t} = \sum_{i=1}^{I_t} R_{i,p_t}\boldsymbol{q}_i \tag{3}$$

$$\boldsymbol{u} = \sum_{i=1}^{I_d} R_{i,p_d}\boldsymbol{d}_i \tag{4}$$

where I, I_t and I_d is the number of control points, $R_{i,p}$, R_{i,p_t} and R_{i,p_d} are NURBS basis functions (Piegl & Tiller, 1997), or T-splines (Scott *et al.* 2013), of order p, p_t and p_d for describing the geometry, the traction and the displacement (which are used instead of the conventional isoparametric functions), $\boldsymbol{B}_i$ are the coordinates of the control points and $\boldsymbol{q}_i$ and $\boldsymbol{d}_i$ are parameters for the tractions and displacements at the control points. The special feature of NURBS is that they are defined by a knot vector, control points and weights. The control points do not, in general, lie on the curve and therefore the physical values along the curve (tractions and displacements) are controlled by parameters rather than physical values at nodal points. Indeed, the concept of nodal points is no longer valid and therefore the coordinates of the collocation points also have to be explicitly computed (see, for example, Li & Qian (2011)).

The example presented later on uses for the exact definition of a quarter circle, quadratic NURBS basis functions based on an open knot vector $\Xi = \{0, 0, 0, 1, 1, 1\}$, where knots are repeated $p+1$ times at the beginning and the end. Due to the repeated knots, the first and last control point lie on the curve. The knot vector comprises $I+p+1$ components. Hence, only three control points are needed for the exact representation. As an example, Fig. 3 shows a comparison of the exact description of a circular arc with NURBS and the approximation given by isoparametric basis functions. The two types of basis functions are compared in Fig. 4. More details on the implementation can be found elsewhere (Beer *et al.*, in preparation).

REFINEMENT STRATEGIES

Different strategies can be used to refine the solution, the main ones being knot insertion, order elevation and *k*-refinement. Note that these strategies do not involve any mesh generation but rather a refinement of the function space describing the unknown, by increasing the number of basis functions or the order of the basis functions. For the sake of simplicity, these strategies are explained for a two-dimensional (2D) case only.

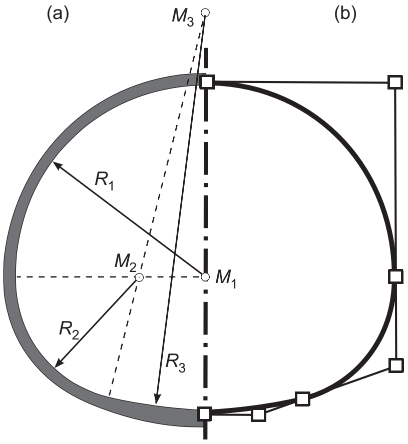

Fig. 2. Design cross-section of a tunnel defined by circular arcs (a) and exact representation with NURBS (b)

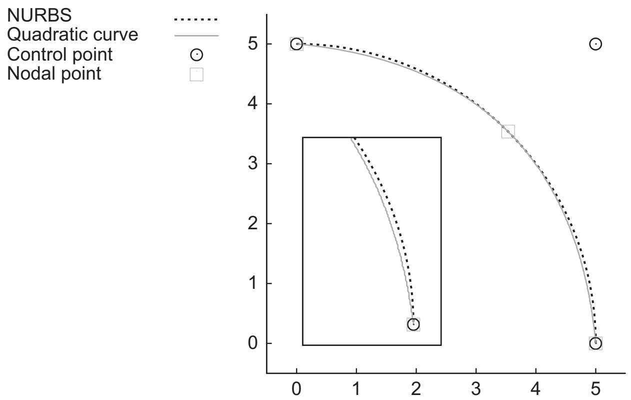

Fig. 3. Description of a quarter circle by one quadratic Serendipity element and one NURBS

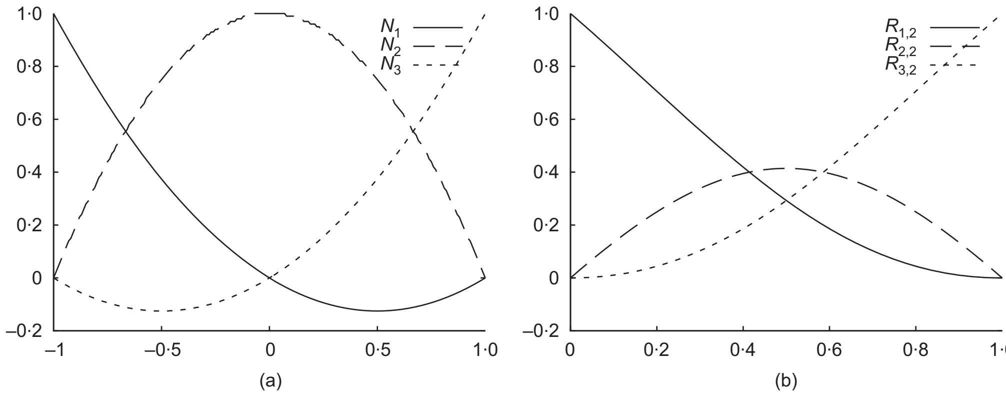

Fig. 4. Comparison of quadratic Serendipity (a) and NURBS (b) basis functions for quarter circle

Knot insertion

The technique of knot insertion involves inserting local coordinates into the knot vector, thereby increasing the number of basis functions. For the example of a quarter circle this gives $\Xi_d = \{0, 0, 0, 0{\cdot}5, 1, 1, 1\}$. The order is kept the same as that for describing the geometry (i.e. $p_d = p$). However, as a consequence, the number of control points for the displacements increases to $I_d = I + 1$. The new weights associated with the control points are computed by a linear combination of the original values (see Piegl & Tiller (1997)). Figure 5(a) shows the resulting basis functions. Comparison with Fig. 4 reveals that the solution space is enriched by one additional basis function. Since the number of control points is increased by one, the number of unknowns is increased by two. Consequently, one more collocation point is required. This method is similar to *h*-refinement in the conventional BEM but does not involve the refinement of a mesh.

Order elevation

The technique of order elevation involves increasing the order of the basis functions. This mechanism elevates the order of the NURBS by replicating existing knots. To generate cubic basis functions out of quadratic basis functions (i.e. $p_d = p + 1$), the existing knots are replicated once, which leads to a corresponding knot vector of $\Xi_d = \{0, 0, 0, 0, 1, 1, 1, 1\}$. The basis functions are shown in Fig. 5(b). It can be seen that the order has been increased to three (cubic) and this goes hand-in-hand with an increase in control points by one and the number of unknowns by two. This is equivalent to a *p*-refinement in conventional BEM.

k-refinement

This technique is simply the combination of order elevation and knot insertion. Knots are inserted after elevating the order.

EXTENSION TO NON-LINEAR AND INHOMOGENEOUS PROBLEMS

Extension of the BEM to problems involving non-linear material behaviour and inhomogeneous soil conditions require evaluation of the volume integral in equation (1). Efficient methods have been developed for the evaluation of these integrals and the BEM has been applied successfully to practical problems in tunneling (Beer *et al.*, 2008; Ribeiro *et al.*, 2008; Riederer *et al.*, 2008, 2009). Implementation of these methods in the supergeometric concept presented here does not involve significant innovations, so is not discussed further.

EXAMPLE

The purpose of the example chosen here is to demonstrate both what the authors consider the main advantage of the isogeometric method for application in geotechnical work (namely accurate description of the boundary shape and excavation tractions with a few parameters) and the proposed supergeometric solution refinement process. Two adjacent tunnels are analysed. The ground is subjected to a vertical compressive virgin stress σ_v of magnitude 1·0 MPa and a horizontal virgin stress σ_h of magnitude 0·25 MPa. The elastic material properties are $\nu = 0{\cdot}25$ and $E = 10\,000$ MPa. This model can be described exactly by 12 NURBS curves of order two (see Fig. 6). The reference

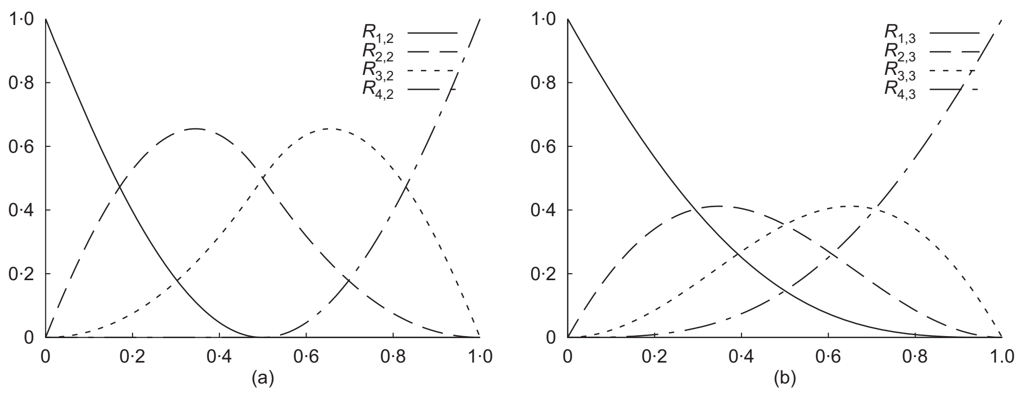

Fig. 5. Basis functions generated by knot insertion (a) and order elevation (b) for the example of a quarter circle

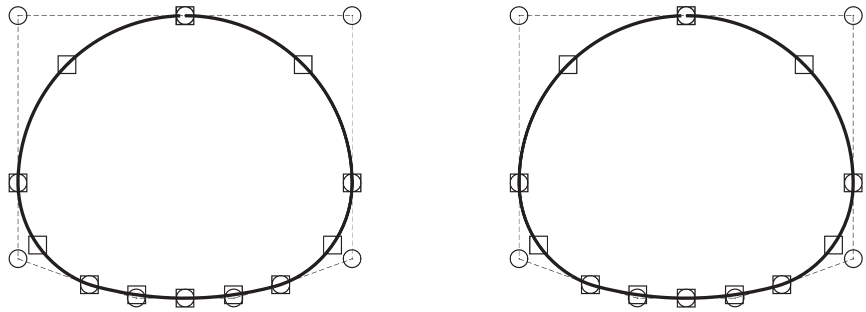

Fig. 6. Discretisation of geometry of two adjacent tunnels by 12 NURBS showing the control points (circles) and the computed collocation points (squares)

solution is calculated using a conventional isoparametric BEM analysis with an extremely fine mesh (240 quadratic elements); k-refinement was used to enrich the description of the variation of unknowns.

Figure 7 shows the convergence of the solution as a plot of the L^2 norm of the difference in the value of displacement as compared to the converged solution along the boundary versus the degrees of freedom (dof). It can be seen that the results of the proposed supergeometric method do not differ from the isogeometric method and start with a much lower error when compared to a conventional isoparametric BEM analysis. Convergence rates are equal to or better than those for isoparametric analysis.

SUMMARY AND CONCLUSIONS

The purpose of this article was to show the benefits that can be gained by using a proposed supergeometric method for the numerical simulation of underground construction. A simple 2D example in tunnelling demonstrated that, compared with isoparametric analysis, a significant gain in accuracy is possible with few parameters. Moreover, the results of the proposed supergeometric method do not differ from the isogeometric method. In other words, refinement of the geometry description is not required to improve the results. Maintaining the initial – already exact – representation of the geometry is beneficial with regard to both the refinement process and the computational effort (computation of the Jacobian and the outward normal), with the expectation of considerable computational savings for large 3D examples. 3D non-linear applications will be reported in the future.

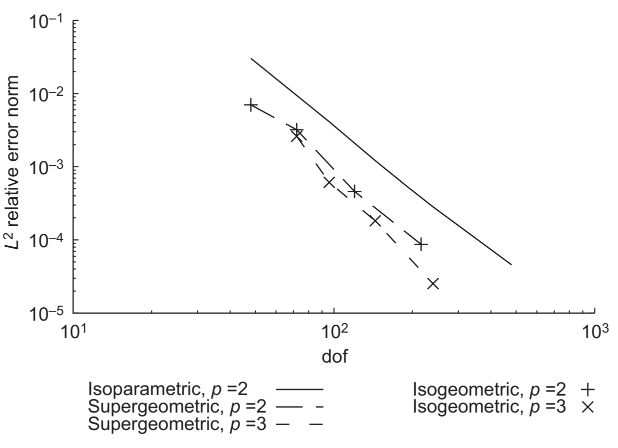

Fig. 7. Plot of error versus degrees of freedom (dof): comparison of isoparametric and isogeometric/supergeometric method (using k-refinement)

Acknowledgement

The work is supported by the Austrian science fund (FWF project P24974-N30).

REFERENCES

Beer, G., Smith, I. & Duenser, C. (2008). *The boundary element method with programming*. Vienna: Springer.

Belibassakis, K. A., Gerostathis, T. P., Politis, C. G., Kaklis, P. D., Ginnis, A. I. & Mourkogiannis, D. N. (2009). A novel BEM-ISOGEOMETRIC method with application to the wavemaking resistance problem of bodies at constant speed. *Proc. 13th Congress of Int. Maritime Assoc. of Mediterranean*.

Gioda, G. & Swoboda, G. (1999). Developments and applications of the numerical analysis of tunnels in continuous media. *Int. J. Num. Anal. Methods Geomech.* **23**, No. 14, 1393–1405.

Hughes, T. J. R., Cottrell, J. A. & Bazilevs, Y. (2005). Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engng* **194**, No. 39–41, 4135–4195.

Li, K. & Qian, X. (2011). Isogeometric analysis and shape optimization via boundary integral. *Comput. Aided Des.* **43**, No. 11, 1427–1437.

Piegl, L. & Tiller, W. (1997). *The NURBS book,* 2nd edn. New York: Springer.

Ribeiro, T., Beer, G. & Duenser, C. (2008). Efficient elastoplastic analysis with the boundary element method. *Comput. Mech.* **41**, No. 5, 715–732.

Riederer, K., Duenser, C. & Beer, G. (2008). Modeling of inhomogeneities and reinforcements in elasto-plastic problems with the BEM. *Proc. BeTeq 2008*, pp. 179–184.

Riederer, K., Duenser, C. & Beer, G. (2009). Simulation of linear inclusions with the BEM. *Engng Anal. Bound. Elements* **33**, No. 7, 959–965.

Scott, M. A., Simpson R. N., Evans J. A., Lipton, S., Bordas, S. P. A., Hughes, T. J. R. & Sederberg, T. W. (2013). Isogeometric boundary element analysis using unstructured T-splines. *Comput. Methods Appl. Mech. Engng* **254**, February, 197–221.

Simpson, R. N., Bordas, S. P. A., Trevelyan, J. & Rabzuck, T. Y. (2012). A two-dimensional isogeometric boundary element method for elastostatic analysis. *Comput. Methods Appl. Mech. Engng* **209–212**, February, 87–100.