Note on Parity Factors of Regular Graphs

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Abstract

In this paper, we obtain a sufficient condition for the existence of parity factors in a regular graph in terms of edge-connectivity. Moreover, we also show that our condition is sharp.

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1. Preliminaries

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph $G$ is called the \textit{order} of $G$ and is denoted by $n$. The number of edges of $G$ is called the \textit{size} of $G$ and is denoted by $e$. For a vertex $v$ of graph $G$, the number of edges of $G$ incident to $v$ is called the \textit{degree} of $v$ in $G$ and is denoted by $d_G(v)$. For two subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of $G$ joining $S$ to $T$.

Let $H$ be a function associating a subset of $\mathbb{Z}$ to each vertex of $G$. A spanning subgraph $F$ of graph $G$ is called an $H$-factor of $G$ if

\begin{equation}
 d_F(x) \in H(x) \quad \text{for every vertex } x \in V(G).
\end{equation}

For a spanning subgraph $F$ of $G$ and for a vertex $v$ of $G$, define

\[ \delta(H; F, v) = \min\{|d_F(v) - i | i \in H_v\}, \]

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and let $\delta(H; F) = \sum_{x \in V(G)} \delta(H; F, x)$. Thus a spanning subgraph $F$ is an $H$-factor if and only if $\delta(H; F) = 0$. Let

$$\delta_H(G) = \min \{ \delta(H; F) \mid F \text{ are spanning subgraphs of } G \}.$$ 

A spanning subgraph $F$ is called $H$-optimal if $\delta(H; F) = \delta_H(G)$. The $H$-factor problem is to determine the value $\delta_H(G)$. An integer $h$ is called a gap of $H(v)$ if $h \notin H(v)$ but $H(v)$ contains an element less than $h$ and an element greater than $h$. Lovász [11] gave a structural description on the $H$-factor problem in the case where $H(v)$ has no two consecutive gaps for all $v \in V(G)$ and showed that the problem is NP-complete without this restriction. Moreover, he also conjectured that the decision problem of determining whether a graph has an $H$-factor is polynomial in the case where $H(v)$ has no two consecutive gaps for all $v \in V(G)$. Cornuéjols [5] proved the conjecture.

Let therefore $g, f : V \rightarrow \mathbb{Z}^+$ such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod{2}$ for every $v \in V$. Then a spanning subgraph $F$ of $G$ is called a $(g, f)$-parity-factor, if $g(v) \leq d_F(v) \leq f(v)$ and $d_F(v) \equiv f(v) \pmod{2}$ for all $v \in V$. Clearly, a $(g, f)$-parity-factor is a special kind of $H$-factor and it has been shown that the decision problem of determining whether a graph has a $(g, f)$-parity factor is polynomial.

Let $a, b$ be two integers such that $1 \leq a \leq b$ and $a \equiv b \pmod{2}$. If $g(v) = a$ and $f(v) = b$ for all $v \in V(G)$, then a $(g, f)$-parity-factor is called an $(a, b)$-parity factor. Let $n \geq 1$ be odd. If $a = 1$ and $b = n$, then an $(a, b)$-parity factor is called a $(1, n)$-odd factor. There is also a special case of the $(g, f)$-factor problem which is called the even factor problem, i.e., the problem with $g(v) = 2, f(v) \geq |V(G)|$ and $f(v) \equiv g(v) \pmod{2}$ for all $v \in V(G)$.

Feischner gave a sufficient condition for a graph to have an even factor in terms of edge connectivity.

**Theorem 1.1** (Feischner [8]; Lovász [12]). *If $G$ is a bridgeless graph with $\delta(G) \geq 3$, then $G$ has an even factor.*

For a general graph $G$ and an integer $k$, a spanning subgraph $F$ such that

$$d_F(x) = k \quad \text{for all } x \in V(G)$$

is called a $k$-factor. In fact, a $k$-factor is also a $(k, k)$-parity factor.

The first investigation of the $(1, n)$-odd factor problem is due to Amahashi [2], who gave a Tutte type characterization for graphs having a global odd factor.

**Theorem 1.2** (Amahashi). *Let $n$ be an odd integer. A graph $G$ has a $(1, n)$-odd factor if and only if

$$o(G - S) \leq n |S| \quad \text{for all subsets } S \subset V(G).$$

(2)*

For general odd value functions $h$, Cui and Kano [6] established a Tutte type of theorem.

**Theorem 1.3** (Cui and Kano, [6]). *Let $h : V(G) \rightarrow N$ be odd value function. A graph $G$ has a $(1, h)$-odd factor if and only if

$$o(G - S) \leq h(S) \quad \text{for all subsets } S \subset V(G).$$

(3)
Now there are many results on consecutive factors (i.e. \((g, f)\)-factor). But the research progress on non-consecutive factors is slow. In non-consecutive factor problems, \((g, f)\)-parity factors have many similar properties with \(k\)-factors. So we believe that many results on \(k\)-factors can be extended to \((g, f)\)-factor. In this paper, we will extend a result on \(k\)-factors of regular graphs to the \((g, f)\)-parity-factors.

Now let us recall one of the classical results due to Petersen.

**Theorem 1.4** (Petersen [13]). Let \(r\) and \(k\) be integers such that \(1 \leq k \leq r\). Every \(2r\)-regular graph has a \(2k\)-factor.

Considering the edge-connectivity, Gallai [7] proved the following result.

**Theorem 1.5** (Gallai [7]). Let \(r\) and \(k\) be integers such that \(1 \leq k < r\), and \(G\) an \(m\)-edge-connected \(r\)-regular graph, where \(m \geq 1\). If one of the following conditions holds, then \(G\) has a \(k\)-factor.

(i) \(r\) is even, \(k\) is odd, \(|G|\) is even, and \(\frac{r}{m} \leq k \leq r(1 - \frac{1}{m})\);

(ii) \(r\) is odd, \(k\) is even and \(2 \leq k \leq r(1 - \frac{1}{m})\);

(iii) \(r\) and \(k\) are both odd and \(\frac{r}{m} \leq k\).


**Theorem 1.6** (Bollobás, Saito and Wormald). Let \(r\) and \(k\) be integers such that \(1 \leq k < r\), and \(G\) be an \(m\)-edge-connected \(r\)-regular graph, where \(m \geq 1\) is a positive integer. Let \(m^* \in \{m, m+1\}\) such that \(m^* \equiv 1 \pmod{2}\). If one of the following conditions holds, then \(G\) has a \(k\)-factor.

(i) \(r\) is odd, \(k\) is even and \(2 \leq k \leq r(1 - \frac{1}{m^*})\);

(ii) \(r\) and \(k\) are both odd and \(\frac{r}{m^*} \leq k\).

In this paper, we extend Theorems 1.5 and 1.6 to \((a, b)\)-factors. The main tool in our proofs is the following theorem of Lovász (see[11]).

**Theorem 1.7** (Lovász [11]). \(G\) has a \((g, f)\)-parity factor if and only if for all disjoint subsets \(S\) and \(T\) of \(V(G)\),

\[
\delta(S, T) = f(S) + \sum_{x \in T} d_G(x) - g(T) - e_G(S, T) - \tau \geq 0,
\]

where \(\tau\) denotes the number of components \(C\), called \(f\)-odd components of \(G - (S \cup T)\) such that \(e_G(V(C), T) + f(V(C)) \equiv 1 \pmod{2}\). Moreover, \(\delta(S, T) \equiv f(V(G)) \pmod{2}\).
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2. Main Theorem

**Theorem 2.1.** Let \( a, b \) and \( r \) be integers such that \( 1 \leq a \leq b < r \) and \( a \equiv b \pmod{2} \). Let \( G \) be an \( m \)-edge-connected \( r \)-regular graph with \( n \) vertices. Let \( m^* \in \{m, m+1\} \) such that \( m^* \equiv 1 \pmod{2} \). If one of the following conditions holds, then \( G \) has an \((a, b)\)-parity factor.

(i) \( r \) is even, \( a, b \) are odd, \(|G|\) is even, \( \frac{r}{m} \leq b \) and \( a \leq r(1 - \frac{1}{m}) \);

(ii) \( r \) is odd, \( a, b \) are even and \( a \leq r(1 - \frac{1}{m}) \);

(iii) \( r, a, b \) are odd and \( \frac{r}{m^*} \leq b \).

By Theorem 1.6, (ii) and (iii) are true. Now we prove (i). Let \( \theta_1 = \frac{a}{r} \) and \( \theta_2 = \frac{b}{r} \). Then \( 0 < \theta_1 \leq \theta_2 < 1 \). Suppose that \( G \) contains no \((a, b)\)-parity factors. By Theorem 1.7, there exist two disjoint subsets \( S \) and \( T \) of \( V(G) \) such that \( S \cup T \neq \emptyset \), and

\[-2 \geq \delta(S, T) = b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) - \tau, \tag{4}\]

where \( \tau \) is the number of \( a \)-odd (i.e. \( b \)-odd) components \( C \) of \( G - (S \cup T) \). Let \( C_1, \cdots, C_\tau \) denote \( a \)-odd components of \( G - S - T \) and \( D = C_1 \cup \cdots \cup C_\tau \).

Note that

\[-2 \geq \delta(S, T) = b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) - \tau
= b|S| + (r - a)|T| - e_G(S, T) - \tau
= \theta_2 r|S| + (1 - \theta_1) r|T| - e_G(S, T) - \tau
= \theta_2 \sum_{x \in S} d_G(x) + (1 - \theta_1) \sum_{x \in T} d_G(x) - e_G(S, T) - \tau
\geq \theta_2(e_G(S, T) + \sum_{i=1}^{\tau} e_G(S, C_i)) + (1 - \theta_1)(e_G(S, T) + \sum_{i=1}^{\tau} e_G(T, C_i)) - e_G(S, T) - \tau
= \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) - 1) + (\theta_2 - \theta_1)e_G(S, T)
\geq \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) - 1).\]

Since \( G \) is connected and \( 0 < \theta_1 \leq \theta_2 < 1 \), so \( \theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) > 0 \) for each \( C_i \).
Hence we will obtain a contradiction by showing that for every \( C = C_i, 1 \leq i \leq \tau \), we have

\[\theta_2 e_G(S, C) + (1 - \theta_1)e_G(T, C) \geq 1. \tag{5}\]
These inequalities imply

$$-2 \geq \delta_G(S, T) \geq \sum_{i=1}^{r-2} (\theta_2 e_G(S, C_i) + (1 - \theta_1) e_G(T, C_i) - 1)$$

$$> \sum_{i=1}^{r-2} (\theta_2 e_G(S, C_i) + (1 - \theta_1) e_G(T, C_i) - 1) - 2 \geq -2,$$

which is impossible.

Now, we will prove the 5 is true. Since \( C \) is an \( a \)-odd component of \( G - (S \cup T) \), we have

$$a|C| + e_G(T, C) \equiv 1 \pmod{2}. \quad (6)$$

Moreover, since

$$r|C| = \sum_{x \in V(C)} d_G(x) = e_G(S \cup T, C) + 2|E(C)|,$$

we have

$$r|C| = e_G(S \cup T, C) \pmod{2}. \quad (7)$$

It is obvious that the two inequalities \( e_G(S, C) \geq 1 \) and \( e_G(T, C) \geq 1 \) imply

$$\theta_2 e_G(S, C) + (1 - \theta_1) e_G(T, C) \geq \theta_2 + 1 - \theta_1 = 1.$$

Hence we may assume \( e_G(S, C) = 0 \) or \( e_G(T, C) = 0. \)

We consider the condition (i). If \( e_G(S, C) = 0 \), then \( e_G(T, C) \geq m \). Since \( a \leq r(1 - \frac{1}{m}) \), then \( \theta_1 \leq 1 - \frac{1}{m} \) and so \( 1 \leq (1 - \theta_1)m \). By substituting \( e_G(T, C) \geq m \) and \( e_G(S, C) = 0 \) into (5), we have

$$(1 - \theta_1) e_G(T, C) \geq (1 - \theta_1)m \geq 1.$$

If \( e_G(T, C) = 0 \), then \( e_G(S, C) \geq m \). Since \( \frac{r}{m} \leq b \), hence \( \theta_2 m \geq 1 \), and so we obtain

$$\theta_2 e_G(S, C) \geq \theta_2 m \geq 1.$$

Consequently, condition (i) guarantees (5) holds and thus (i) is true. The proof is completed. \( \Box \)

**Remark:** The edge connectivity conditions in Theorem 2.1 are sharp.

We will give the construction for condition (i) of Theorem 2.1. For (ii) and (iii), the constructions are similar. Let \( r \geq 2 \) be an even integer, \( a, b \geq 1 \) two odd integers and \( 2 \leq m \leq r - 2 \) an even integer such that \( b < r/m \) or \( r(1 - \frac{1}{m}) < a \). Since \( G \) has an \( (a, b) \)-parity factor if and only if \( G \) has an \( (r - b, r - a) \)-parity factor, so we can assume \( b < r/m \). Let \( J(r, m) \) be the complete graph \( K_{r+1} \) from which a matching of size \( m/2 \) is deleted. Take \( r \) disjoint copies of \( J(r, m) \). Add \( m \) new vertices and connect each of these vertices to a vertex of degree \( r - 1 \) of \( J(r, m) \). This gives an \( m \)-edge-connected \( r \)-regular graph denoted by \( G \). Let \( S \) denote the set of \( m \) new vertices and \( T = \emptyset \). Let \( \tau \) denote the number of components \( C \), which are called \( a \)-odd components of \( G - (S \cup T) \) and \( e_G(V(C), T) + a|C| \equiv 1 \pmod{2} \). Then we have \( \tau = r \), and

$$\delta(S, T) = b|S| + \sum_{x \in T} d_{G-S}(x) - a|T| - \tau(S, T) = bm - r < 0.$$

So by Theorem 1.7, \( G \) contains no \((a, b)\)-parity factors.
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