A soil-water characteristic curve with hydraulic hysteresis is derived using fractals by treating pores as either bodies or throats. As suction is increased along the main drying curve, drying of a body is controlled by the largest throat connected to it. As suction is reduced along the main wetting curve, the absorbed water collects in the smallest curve, drying of a body is controlled by the largest throat in order of size. The curve is fitted to data for a silt loam. The fit is reasonable for main drying and wetting curves, but less good for scanning curves.

**KEYWORDS:** fractals; partial saturation; suction

INTRODUCTION

A soil-water characteristic curve (SWCC) relates suction to water content or degree of saturation, and is important in understanding water storage in the ground and soil strength variations in infrastructure (Khalili & Zargarbashi, 2010).

SWCCs depend on pore size distributions (Gitirana & Fredlund, 2004; Marinho, 2005), and many expressions have been derived by soil scientists using fractals, motivated by self-similar fractal geometry of pore sizes measured experimentally (Tyler & Wheatcroft, 1989; Perfect & Kay, 1995; Bird et al., 2000; Hunt & Gee, 2002, Perfect, 2005; Wang et al., 2005; Cihan et al., 2007; Yu et al., 2009; Russell, 2010). A fractal pore size distribution exists when the number of pores larger than a given size is proportional to that size in a power law, with the exponent representing the fractal dimension. Fractal-based SWCCs are appealing, as the defining parameters are linked to microstructural properties. However, most SWCCs used within the geotechnical engineering community are phenomenological in origin (Wheeler et al., 2003; Gitirana & Fredlund, 2004; Khalili et al., 2008; Pedroso & Williams, 2010). Although they may model hysteresis (Li, 2005), or incorporate grain size descriptors (Kamiya et al., 2003), they rely on fitting parameters with no direct link to microstructural properties.

Here a new fractal description of pore size distributions is presented for non-swelling soils, and is used to define separate wetting, drying and scanning curves. Deriving a SWCC using fractals is not new but, for the first time, the hysteretic loop observed during a drying–wetting–drying cycle is captured. The defining parameters are linked to microstructural properties. The SWCC is compared with experimental data.

**A FRACTAL DESCRIPTION OF A SOIL CONTAINING PORE BODIES AND PORE THROATS**

In this study, pores are classified as either bodies or throats (similar to Conner et al., 1986; Tsakiroglou & Ioannidis, 2008). A body has a number of smaller throats connected to it. A body cannot change to be a throat, or vice versa. Bodies and throats of different sizes exist, and obey fractal distributions. Each body or throat size is denoted by an order. Order \( k \) represents the largest size, order \( k - 1 \) represents the second largest size and so on, down to order 0 representing the smallest size.

In a soil of volume \( V \), throats and bodies of order \( k \) have an overall volume \( \mu_k V \), where \( \mu \) is a material parameter. \( \mu V \) is divided into parts belonging to throats and bodies, \( \mu q V \) and \( \mu (1 - q) V \) respectively, where \( q \) is a material parameter. \( \mu \) and \( q \) are assumed constant for all orders of size.

The total numbers of bodies (or throats) of order \( k \) (with size \( d_k \)) are found by dividing the total volume of bodies (or throats) by that of a single body (or throat), which is assumed equal to \( \Lambda d_k^3 \). \( \Lambda \) is a dimensionless geometric shape factor, assumed constant for all bodies and throats, and is unimportant in the derivations that follow. The total numbers of bodies or throats of order \( k \) become \( \mu (1 - q) V / (\Lambda d_k^3) \) and \( \mu q V / (\Lambda d_k^3) \) respectively. \( n \) represents the ratio between individual body or throat volumes of successive orders (a material constant) so that bodies and throats of order \( k - 1 \) have size \( d_k / n^{1/3} \). The total volumes of bodies and throats of order \( k - 1 \) are \( \mu p (1 - q) V \) and \( \mu q V \) respectively, where \( p \) represents the ratio between total body (or throat) volumes of successive orders (a material constant). The total numbers of bodies and throats of order
Subscripts b and t indicate an association with bodies and throats respectively. For \( np > 1 \) (a mathematical requirement of fractal geometries) these simplify to

\[
N_b(L > d_b) = \frac{\mu(1-q)V}{\Lambda d_b^k} \left[ 1 + np + (np)^2 + \ldots + (np)^{k-i} \right]
\]

\[
= \frac{\mu(1-q)V(np)^{(k-i)}}{\Lambda d_b^k} \sum_{j=0}^{k-i} (np)^j
\]

(2)

\[
N_t(L > d_t) = \frac{\mu q V(np)^{(k-i)}}{\Lambda d_t^k} \sum_{j=0}^{k-i} (np)^j \ln \left( \frac{np}{np-1} \right)
\]

(3)

Subscripts b and t indicate an association with bodies and throats respectively. For \( np > 1 \) (a mathematical requirement of fractal geometries) these simplify to

\[
N_b(L > d_b) = \frac{\mu(1-q)V}{\Lambda d_b^k} \left( \frac{np}{np-1} \right)
\]

(4)

\[
N_t(L > d_t) = \frac{\mu q V(np)^{(k-i)}}{\Lambda d_t^k} \left( \frac{np}{np-1} \right)
\]

(5)

From equations (4) and (5), when a large number of orders exist, the ratio of the number of bodies or throats of order \( i - 1 \) or higher to the number of bodies or throats of order \( i \) or higher is \( np \). Recall also that the ratio of the body or throat size of order \( i - 1 \) to body or throat size of order \( i \) is \( n^{-1/3} \). For distributions of body or throat sizes obeying the fractal distributions

\[
N_b(L > d_b) \propto d_b^{-D_b}
\]

and

\[
N_t(L > d_t) \propto d_t^{-D_t}
\]

where \( D_b \) and \( D_t \) denote the fractal dimensions of the bodies and throats respectively, the fractal dimensions are equal and given by

\[
D_b = D_t = D = 3 \left( 1 + \frac{\ln p}{\ln n} \right)
\]

(7)

The material parameters are summarised in Table 2 with their meanings and ranges of permissible values.

FROM FRACTAL PORE BODIES AND THROATS TO A SOIL-WATER CHARACTERISTIC CURVE

The SWCC comprises a main wetting curve, a main drying curve and scanning curves. Derivations are made with reference to Figs 1 and 2. Fig. 1 presents curves in the double logarithmic suction against degree of saturation \((ln S - ln S_i)\) plane. Seven different stages during a wetting–drying–wetting cycle are labelled I to VII. Fig. 2 shows a simple idealised connected sequence of bodies and throats of different sizes at the seven stages, starting from an arbitrary position on the main wetting curve (Stage I). The sizes and
volumes of bodies and throats are labelled in Fig. 2. A grey body or throat indicates wet, and white indicates dry.

The assumed linear connection of smaller and larger pores in Fig. 2 is highly idealised, and not an exact representation of pore connectivity in soils, where most bodies are connected to more than two throats, permitting a multidimensional emptying/filling pattern. The idealisation is useful to study pore scale properties influencing macroscopic measurements of soil-water retention.

During wetting, absorbed water collects in the smallest bodies and throats first, and then gradually fills larger bodies and throats in order of size. This is not influenced by bodies being connected to smaller throats, as all parts of the body and throat spaces are assumed accessible (Conner et al., 1986).

At Stage I, where the \((s, S_r)\) state is on the main wetting curve and \(s = s_x\), all bodies and throats larger than or equal to \(d_x\) are dry, as \(S_r\) ensures that bodies and throats of size \(d_x\) do not fill with water. Therefore all bodies and throats of size \(d_x / n^{1/3}\) or smaller have filled when \(s = s_x\). There is an inverse proportionality between \(s_x\) and the smallest drained pore size \(d_x\). Notice that pores are filled even if they are not connected, as absorbed water collects in the smallest bodies and throats first, and then fills larger bodies and throats (Conner et al., 1986). This seems at odds with Fig. 2. However, as already mentioned, real soils have more pore connections than that shown in this simplified representation, permitting successively larger pores to be filled in order of size.

Stage II is reached by further wetting and reducing \(s\) to \(s_x / n^{1/3}\); the value associated with bodies or throats one order larger. While moving from Stage I to II, the state stays on the main wetting curve.

The volumes of wetted bodies and throats can be expressed in incremental form as

\[
\delta V_b = -\Lambda C D d_b^{-3} \delta d_b \quad (8a)
\]

and

\[
\delta V_t = -\Lambda C D d_t^{-3} \delta d_t \quad (8b)
\]

Before integrating these terms to find \(S_t\) (as in Russell, 2010), simplifications are introduced, noting that, during wetting, bodies and throats of a certain size can be treated as the one pore unit, and supposing that the smallest body or throat size is so small that it can be assumed to have null size. It follows that the values of \(S_t\) at Stage I, when \(s = s_x\), and at Stage II, when \(s = s_x / n^{1/3}\), are

\[
S_{t(I)} = \left( \frac{d_x / n^{1/3}}{d_{\text{max}}} \right)^{3-D} \quad (9a)
\]

and

\[
S_{t(II)} = \left( \frac{d_x / n^{1/3}}{d_{\text{max}}} \right)^{3-D} \quad (9b)
\]

in which \(d_{\text{max}}\) is the largest pore size. In the ln \(s\)–ln \(S_r\) plane the slope of the main wetting curve is

\[
\alpha = \frac{\ln (S_{t(II)}) - \ln (S_{t(I)})}{\ln (s_{\text{III}}) - \ln (s_{\text{I}})} = -(3 - D) \quad (10)
\]

A more general definition for the main wetting curve is

\[
S_t = \begin{cases} 
1 & \text{for } s < s_{\text{ex}} \\
\left( \frac{s}{s_{\text{ex}}} \right)^{D-3} & \text{for } s > s_{\text{ex}} 
\end{cases} \quad (11)
\]

in which \(s_{\text{ex}}\) is the suction associated with air expulsion in the largest pores,

\[
s_{\text{ex}} = -\frac{4T \cos \theta}{d_{\text{max}}} \quad (12)
\]

where \(T\) is the surface tension of water and \(\theta\) is the contact angle between water and soil.

Stage III on the main drying curve is reached by increasing \(s\) to \(s_x\). Because a body is connected to a number of throats, drying of a body is controlled by the largest throat connected to it. It is enough that the largest throat drains for the body to drain. Therefore the largest throat size (rather than body size) must be directly related to the state at which the body drains. At Stage III, bodies of size \(d_x n^{1/3}\) have drained, as have the largest throats of size \(d_x\) connected to them. \(S_t\) is then

\[
S_{t(III)} = (1 - q) \left( \frac{d_x}{d_{\text{max}}} \right)^{3-D} + q \left( \frac{d_x / n^{1/3}}{d_{\text{max}}} \right)^{3-D} \quad (13)
\]

Stage IV is reached following further drying, to when \(s = s_x n^{1/3}\). Stage V is reached following even more drying, to when \(s = s_x n^{2/3}\). While moving from Stage III to Stage V, the state stays on the main drying curve. The values of \(S_t\) at Stages IV and V are

\[
S_{t(IV)} = (1 - q) \left( \frac{d_x / n^{1/3}}{d_{\text{max}}} \right)^{3-D} + q \left( \frac{d_x / n^{2/3}}{d_{\text{max}}} \right)^{3-D} \quad (14a)
\]

and

\[
S_{t(V)} = (1 - q) \left( \frac{d_x / n^{2/3}}{d_{\text{max}}} \right)^{3-D} + q \left( \frac{d_x / n}{d_{\text{max}}} \right)^{3-D} \quad (14b)
\]

The slope of the main drying curve is also \(-(3 - D)\), and a general definition is

\[
S_t = \begin{cases} 
1 & \text{for } s < s_{\text{ae}} \\
\left( \frac{s}{s_{\text{ae}}} \right)^{D-3} & \text{for } s > s_{\text{ae}} 
\end{cases} \quad (15)
\]

in which \(s_{\text{ae}}\) is suction associated with air entry. To establish
the relationship between $s_{ae}$ and $s_{ex}$, notice that $n^{1/3}$ represents the ratio between $s$ on the main wetting curve when $S_r = 1$ and $s$ on the main drying curve after the largest pore throats have drained (Fig. 1). It follows that

$$s_{ae} = s_{ex} n^{1/3} \left( 1 - \frac{\beta}{\pi} \right)$$

Equations (11) and (15) happen to be of the same form as the Brooks & Corey (1964) SWCC when the residual $S_r$ is zero.

Moving from Stage II to Stage III, or more generally as wetting is changed to drying and the state moves from the main wetting curve to the main drying curve (or vice versa), the state moves along a scanning curve. When straight in the $\ln s - \ln S_r$ plane, all scanning curves have the slope

$$\beta = \ln \left( \frac{S_{l(III)}}{S_{r(III)}} \right) - \ln \left( S_{l(III)} \right) - \ln \left( s_{ex} \right) = \frac{\ln \left( 1 - q + q \left( \frac{1}{n^{1/3}} \right)^{3-D} \right)}{n^{1/3}}$$

(17)

The state when $S_r = 1$ and $s = s_{ae}$ is inaccessible, as the largest throats drain as soon as $s$ is increased above $s_{ex}$. The inaccessible region is bound by the dashed line in Fig. 1. $s_{ae}$ is dependent on $d_{max}$ as well as on the other material parameters $n$, $q$, and $D$.

In studies by Perfect (2005) and Cihan et al. (2007), different assumptions regarding geometry and interconnectivity of pores and drainage sequences were made. Although they attributed hysteresis to incomplete drying of pores of a certain size as suction increases, as is also done here, their derivation and validation were limited to the main drying curve. Their expression for scale-invariant draining becomes the same as equation (15) when their parameter $P$ control-
ling the volumetric fraction of pores of a certain size that
drain at a certain suction, is equal to \( q \) and \( \phi \), noting that
\( s_{ae} \) and their \( h_{\text{min}} \) have the same role. However, influences of
\( n_{1/3} \) and their scale parameter \( b \) on the relative magnitudes of
\( s_{ae} \) and \( s_{ex} \) may differ slightly, as the slight lowering of \( S_{r} \)
due to drainage of the largest throats prior to the largest
bodies is allowed in this study, but not in theirs.

The SWCC is now fitted to data for a silt loam from Topp
(1971) in Fig. 3. The main drying and wetting curves have
been fitted, by trial and error, using the material parameters
listed in Table 2. The constant slope for scanning curves
does not permit a good fit. However, constant slopes are
appealing and convenient, especially for computational ana-
lyses involving coupled hydro-mechanical behaviour. The

\[
\begin{align*}
\text{Degree of saturation, } S_r & \quad \text{Suction, } s: \text{kPa} \\
1.00 & \quad 0.95 \\
0.90 & \quad 0.85 \\
0.80 & \quad 0.75 \\
0.70 & \quad 0.65 \\
0.60 & \quad 0.55 \\
0.50 & \quad 0.45 \\
0.40 & \quad 0.35 \\
0.30 & \quad 0.25 \\
0.20 & \quad 0.15 \\
0.10 & \quad 0.05 \\
0.00 & \quad 0.00
\end{align*}
\]

\[
\begin{align*}
\text{Degree of saturation, } S_{\text{cr}} & \quad \text{Suction, } s: \text{kPa} \\
1.00 & \quad 0.95 \\
0.90 & \quad 0.85 \\
0.80 & \quad 0.75 \\
0.70 & \quad 0.65 \\
0.60 & \quad 0.55 \\
0.50 & \quad 0.45 \\
0.40 & \quad 0.35 \\
0.30 & \quad 0.25 \\
0.20 & \quad 0.15 \\
0.10 & \quad 0.05 \\
0.00 & \quad 0.00
\end{align*}
\]

The intention is not to achieve perfection, but to present an
adequate and simple explanation of soil-water characteristics
based on fractals.

CONCLUSION

The pore space may be modelled by treating pores as
either bodies or throats. Fractal distributions for bodies and
throats may be defined, and fractal dimensions become equal
when the ratio of body and throat numbers at all size scales
is constant.

The fractal definitions permit derivation of main wetting
and drying curves. As \( s \) is increased along the main drying
curve, drying of a body is controlled by the largest of the

\[
\begin{align*}
\text{Degree of saturation, } S_r & \quad \text{Suction, } s: \text{kPa} \\
1.00 & \quad 0.95 \\
0.90 & \quad 0.85 \\
0.80 & \quad 0.75 \\
0.70 & \quad 0.65 \\
0.60 & \quad 0.55 \\
0.50 & \quad 0.45 \\
0.40 & \quad 0.35 \\
0.30 & \quad 0.25 \\
0.20 & \quad 0.15 \\
0.10 & \quad 0.05 \\
0.00 & \quad 0.00
\end{align*}
\]

\[
\begin{align*}
\text{Degree of saturation, } S_{\text{cr}} & \quad \text{Suction, } s: \text{kPa} \\
1.00 & \quad 0.95 \\
0.90 & \quad 0.85 \\
0.80 & \quad 0.75 \\
0.70 & \quad 0.65 \\
0.60 & \quad 0.55 \\
0.50 & \quad 0.45 \\
0.40 & \quad 0.35 \\
0.30 & \quad 0.25 \\
0.20 & \quad 0.15 \\
0.10 & \quad 0.05 \\
0.00 & \quad 0.00
\end{align*}
\]

Fig. 3. The SWCC fitted to the data of Topp (1971) for a silt loam: (a) scanning data are for drying paths, (b) scanning data
are for wetting paths
throats connected to it. As $s$ is reduced along the main wetting curve, water collects in the smallest bodies and throats first, and then fills larger bodies and throats. The curves are straight lines in the $\ln s$–$\ln S_b$ plane, and have slopes that are functions of the fractal dimension. Scanning curves arise, and have slopes dependent on the volumetric fraction of throats and fractal properties.

For real soils there would be natural rounding of the sharp corners, and repeated drying–wetting reversals may show a slight non-linearity. In any case, the analysis shows how the general features of the SWCC may be linked to fractal pore scale characteristics.

**NOTATION**

- $C$: constant of proportionality
- $D_n$, $D_i$: fractural dimensions of bodies and throats respectively
- $d_{bi}$: sizes of bodies and throats of order $i$
- $d_i$: size of bodies of order $k$
- $d_{max}$: largest pore size
- $d_s$: the largest drained pore size when suction is $s_s$
- $k$: order of largest size
- $L$: size of body or throat
- $N_b$, $N_i$: total numbers of bodies and throats
- $n$: ratio between individual body or throat volumes of successive orders
- $p$: ratio between total body (or throat) volumes of successive orders
- $q$: material parameter
- $S$: degree of saturation
- $s$: suction
- $s_{ae}$: suction associated with air entry
- $s_{as}$: suction associated with air expulsion
- $s_r$: an arbitrary suction
- $T$: surface tension of water
- $V$: overall soil volume
- $V_{wb}$, $V_{wh}$: volumes of wetted bodies and throats
- $\alpha$: slope of main wetting curve
- $\theta$: contact angle between water and soil
- $\Lambda$: dimensionless geometric shape factor
- $\mu$: material parameter
- $\phi$: porosity of soil

**REFERENCES**


