Application of Smooth Tests of Goodness of Fit to Generalized Linear Models

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The thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying subject to the provisions of the Copyright Act 1968.

Paul Rippon
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## Contents

Abstract 1

1 Introduction 3
   1.1 Statistical Models 3
   1.2 Guide to Examiners 10

2 Likelihood 15
   2.1 The Likelihood Function 15
   2.2 Likelihood Ratio, Wald and Score Tests 19
   2.3 Smooth Tests of Goodness of Fit 29

3 Smooth Testing for Poisson Regression 37
   3.1 Poisson Regression 38
   3.2 A Smooth Test of the Distributional Assumption 39
   3.3 Example: Bladder Cancer 51

4 Smooth Testing for GLMs 55
   4.1 Generalized Linear Models 55
   4.2 Derivation of the Smooth Test 58
   4.3 Applying the Smooth Test 71
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Smooth Testing: Count Response</td>
<td>81</td>
</tr>
<tr>
<td>5.1</td>
<td>Size Study</td>
<td>83</td>
</tr>
<tr>
<td>5.2</td>
<td>Using Bootstrap p-values</td>
<td>92</td>
</tr>
<tr>
<td>5.3</td>
<td>A Smooth Test for the Poisson Distribution</td>
<td>97</td>
</tr>
<tr>
<td>5.4</td>
<td>Goodness of Fit for Poisson Regression Models</td>
<td>100</td>
</tr>
<tr>
<td>5.5</td>
<td>Power Study for Poisson Regression</td>
<td>104</td>
</tr>
<tr>
<td>5.6</td>
<td>Poisson Regression Examples</td>
<td>118</td>
</tr>
<tr>
<td>5.7</td>
<td>Negative Binomial Regression Examples</td>
<td>129</td>
</tr>
<tr>
<td>5.8</td>
<td>Chapter Summary</td>
<td>135</td>
</tr>
<tr>
<td>6</td>
<td>Smooth Testing: Binary Response</td>
<td>137</td>
</tr>
<tr>
<td>6.1</td>
<td>Binomial Regression</td>
<td>137</td>
</tr>
<tr>
<td>6.2</td>
<td>Smooth Testing for Binomial Regression Models</td>
<td>142</td>
</tr>
<tr>
<td>6.3</td>
<td>A Smooth Test for the Binomial Distribution</td>
<td>143</td>
</tr>
<tr>
<td>6.4</td>
<td>Binomial Regression Applications</td>
<td>145</td>
</tr>
<tr>
<td>6.5</td>
<td>Power Study for Logistic Regression</td>
<td>159</td>
</tr>
<tr>
<td>6.6</td>
<td>Chapter Summary</td>
<td>163</td>
</tr>
<tr>
<td>7</td>
<td>Smooth Testing: Continuous Response</td>
<td>165</td>
</tr>
<tr>
<td>7.1</td>
<td>Unknown Dispersion Parameter</td>
<td>165</td>
</tr>
<tr>
<td>7.2</td>
<td>Adapting the Smooth Test</td>
<td>166</td>
</tr>
<tr>
<td>7.3</td>
<td>Smooth Testing for Normal Response Models</td>
<td>169</td>
</tr>
<tr>
<td>7.4</td>
<td>A Smooth Test for the Normal Distribution</td>
<td>174</td>
</tr>
<tr>
<td>7.5</td>
<td>Normal Response Examples</td>
<td>175</td>
</tr>
<tr>
<td>7.6</td>
<td>Smooth Testing for Gamma Response Models</td>
<td>188</td>
</tr>
</tbody>
</table>
CONTENTS

7.7 A Smooth Test for the Gamma Distribution . . . . . . . . . . . . . . 196
7.8 Gamma Response Examples . . . . . . . . . . . . . . . . . . . . . 200
7.9 Chapter Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . 205

8 Conclusion and Further Research 207
8.1 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 207
8.2 Further Research . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 208

A Fitting and Assessing GLMs 217
A.1 Exponential Family of Distributions . . . . . . . . . . . . . . . . . . 217
A.2 Fitting a Generalized Linear Model . . . . . . . . . . . . . . . . . . 219
A.3 Assessing GLMs . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 223
A.4 Poisson Regression Example . . . . . . . . . . . . . . . . . . . . . . . 228

B Simulation of Probability Distributions 233
B.1 Bootstrap p-values . . . . . . . . . . . . . . . . . . . . . . . . . . . . 234
B.2 Estimating Probabilities . . . . . . . . . . . . . . . . . . . . . . . . . 245
B.3 Estimating Quantiles . . . . . . . . . . . . . . . . . . . . . . . . . . . 248
B.4 Estimating Power . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 254

C Moments and Cumulants 259
C.1 Moments about the Origin . . . . . . . . . . . . . . . . . . . . . . . . 259
C.2 Moments about the Mean . . . . . . . . . . . . . . . . . . . . . . . . . 260
C.3 Factorial Moments . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 263
C.4 Cumulants . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 266
D Orthogonal Polynomials

D.1 Orthogonal polynomials .............................................. 272
D.2 Computing Sequences of Orthogonal Polynomials .............. 275
D.3 Examples ................................................................. 295

E Developed Software

E.1 The SmoothGLM R package ........................................... 310
E.2 The Rippon R package ................................................ 316

F Useful Results

F.1 Matrix Algebra .......................................................... 317
F.2 Proof of Weightings Condition ...................................... 325
F.3 The Gamma and Related Functions ................................. 326

G Generalized Score Test: Behrens-Fisher

G.1 Introduction ............................................................. 329
G.2 Normal Populations with Equal Variances ....................... 330
G.3 Normal Populations with Different Variances .................... 335
G.4 A Generalized Score Test ............................................. 341
G.5 Power Study ............................................................. 348
G.6 Size Study ............................................................... 354
G.7 Conclusion ............................................................... 360

H Quasi-Likelihood

H.1 Estimating Functions .................................................. 361

Bibliography

370
Abstract

Statistical models are an essential part of data analysis across many diverse fields. They are used to test research hypotheses, aid decision making, estimate effect sizes and/or improve understanding of the underlying processes generating the data of interest. However it is essential to critically assess any fitted model, confirming that the model really is compatible with the data, before meaningful conclusions are possible.

Generalized linear models (GLMs) provide a flexible modelling framework encompassing many commonly used models including the normal linear model, logistic regression model and Poisson regression model. This thesis explores how the smooth testing concept – originally proposed by Neyman (1937) and further developed by Rayner et al. (2009) among others – can be used to test the distributional assumption in a GLM. However sensible interpretation of this test, or any other test used to assess the fit of a GLM, must recognize that:

- the stochastic, deterministic and link components that make up a GLM should all be considered when assessing model validity,
- the validity of any one of these three components cannot be sensibly considered in isolation as it is confounded by the validity of the other two.

It is therefore important to consider how the smooth test developed in this thesis might be usefully incorporated into an overall model development strategy for GLMs, either replacing or supplementing existing diagnostic tools. Simulation studies demonstrate that the power of the smooth test is competitive with other existing tests. However, it also offers the possibility of improved diagnostic ability.
through the breakdown of the overall smooth test statistic into a sum of squares of interpretable components. The SmoothGLM package has been developed which implements the smooth test in a form that can be easily applied to models fitted using the standard glm() function within the R statistical computing environment.