COMPLEX SERIES FOR $1/\pi$

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Abstract. Many series for $1/\pi$ were discovered since the appearance of S. Ramanujan’s famous paper “Modular equations and approximation to $\pi$” published in 1914. Almost all these series involve only real numbers. Recently, in an attempt to prove a series for $1/\pi$ discovered by Z.-W. Sun, the authors found that a series for $1/\pi$ involving complex numbers is needed. In this article, we illustrate a method that would allow us to prove series of this type.

1. Introduction

G. Bauer [2] is likely to be the first mathematician to have discovered a series for $1/\pi$ in the form

\[ \sum_{k=0}^{\infty} a_k(A + Bk)X^k = \frac{C}{\pi}, \]

where \(\{a_k\}\) is a sequence of rational numbers, and \(A, B, C\) and \(X\) are real algebraic numbers. Bauer’s series is

\[ \frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{k!} (4k + 1), \]

where

\((a)_0 = 1 \quad \text{and} \quad (a)_k = a(a + 1) \cdots (a + k - 1) \quad \text{for} \quad k \geq 1.\]

It was, however, the paper of S. Ramanujan [10] that popularize the study of series of the type (1.1).

Many new series of the form (1.1) are found after Ramanujan’s work (see [4], [5], [8], and [11]), with the most recent discovery being those found empirically by Z.-W. Sun [12] and proved by these authors in [7]. All such series share one common property that the coefficients are all real.

In [9], J. Guillera and W. Zudilin discovered the first series for $1/\pi$ with complex coefficients, namely,

\[ \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{k!} \left(\frac{49 - 13\sqrt{-7}}{64} + \frac{105 - 21\sqrt{-7}}{32} k\right) \left(\frac{47 + 45\sqrt{-7}}{128}\right)^k = \frac{\sqrt{7}}{\pi}. \]

This series was shown to be equivalent to another series involving only real numbers and the proof of the latter series follows from application of the Wilf–Zeilberger method.
Let
\[ pF_{p-1}\left(\begin{array}{c}
a_1, a_2, \ldots, a_p \\ b_2, \ldots, b_p
dot\end{array}\right| z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_2)_k \cdots (b_p)_k} \frac{z^k}{k!}, \quad |z| < 1, \]
and let
\[ (1.3) \quad P_k(x) = 2F_1\left(-k, k+1 \bigg| \frac{1-x}{2}\right). \]
Recently, Z.-W. Sun [12] discovered many new series for $1/\pi$ associated with $T_k(b, c)$, where
\[ T_k(b, c) = \left(b^2 - 4c\right)^{k/2} P_k\left(\frac{b}{(b^2 - 4c)^{1/2}}\right). \]
Using (1.3), H.H. Chan, J. Wan and W. Zudilin [7] converted Sun's series to series involving $P_k(x)$, one of which is
\[ (1.4) \quad \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k(\frac{3}{4})_k}{k!^2} P_k\left(\frac{-7i}{33\sqrt{15}}\right)(13 + 80k)\left(\frac{11\sqrt{-15}}{147}\right)^k = \frac{7\sqrt{43(3 + 2\sqrt{5})}}{8\pi}. \]
To prove (1.4), we need two series analogous to (1.2), namely,
\[ (1.5) \quad \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k(\frac{1}{4})_k(\frac{3}{4})_k}{k!^3} (52 \pm 12\sqrt{-3} + (320 \pm 55\sqrt{-3})k) \times \left(\frac{2(5 \pm \sqrt{-3})}{7\sqrt{3}}\right)^4k = \frac{98\sqrt{3}}{\pi}. \]
It suffices to prove any one of the above series since one is the conjugate of the other.

The proof of (1.5) were sketched briefly in [7]. In this note, we will discuss a method to establish identities such as (1.5). Our proof is different from that given in [7] and is applicable to a more general collection of series similar to (1.5).

2. Functions and forms associated with $\Gamma_0(2)$ and a transformation formula

Our main aim is to prove (1.5) and these series arise from the study of Ramanujan’s quartic theory of elliptic functions [3]. We recall some of the facts from [3].

For $|q| < 1$, define
\[ f(-q) = \prod_{j=1}^{\infty} (1 - q^j). \]
When $q = e^{2\pi i \tau}$ with $\text{Im} \ \tau > 0$, we find that
\[ q^{1/24}f(-q) = \eta(\tau), \]
where $\eta(\tau)$ is the Dedekind $\eta$-function. It is well known that $\eta(\tau)$ [1, Theorem 3.1] satisfies the transformation formula

$$
(2.1) \quad \eta \left( -\frac{1}{\tau} \right) = \sqrt{-i\tau} \eta(\tau).
$$

Let

$$
Z(q) = \frac{f^8(-q) + 32qf^8(-q^4)}{f^4(-q^2)}
$$

and

$$
(2.2) \quad X(q) = 4x(q)(1 - x(q))
$$

where

$$
(2.3) \quad \frac{1}{x(q)} = 1 + \frac{f^{24}(-q)}{64qf^{24}(-q^2)}.
$$

In [3], we know that

$$
Z(q) = 3F_2 \left( \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \gvert \begin{array}{c} \begin{array}{c} \frac{1}{2} \tau \\ \sqrt{2} \end{array} \end{array} X(q) \right).
$$

To extract the number $\pi$ from these functions, we need the transformation formula for $A(q)$ and this follows immediately from (2.1). More precisely, we have

$$
\frac{1}{\tau} G \left( -\frac{1}{\tau} \right) = \sqrt{\frac{2}{\pi i}} + \tau G(\tau).
$$

Differentiating the above with respect to $\tau$, we deduce that

$$
\left. \frac{1}{\tau} \cdot \frac{q}{Z} \frac{dZ}{dq} \right|_{q=e^{2\pi i/(\sqrt{2})}} = \frac{\sqrt{2}}{\pi i} + \tau \left. \frac{q}{Z} \frac{dZ}{dq} \right|_{q=e^{2\pi i/(\sqrt{2})}}.
$$

To simplify notations, let

$$
G(\tau) = \left. \frac{q}{Z} \frac{dZ}{dq} \right|_{q=e^{2\pi i/(\sqrt{2})}}.
$$

Then the transformation can be rewritten as

$$
(2.4) \quad \left. \frac{1}{\tau} G \left( -\frac{1}{\tau} \right) = \frac{\sqrt{2}}{\pi i} + \tau G(\tau). \right.
$$

In the next section, we will express $G(\tau)$ and $G(-1/\tau)$ in terms of hypergeometric function and its derivative.

3. Some intermediate identities

Set

$$
\tau_1 = \frac{\sqrt{-15} - 1}{2\sqrt{2}}, \quad \tau_2 = \frac{\sqrt{-5/3} - 1}{2\sqrt{2}}, \quad \text{and} \quad \tau_3 = \frac{\sqrt{-15} + 1}{2\sqrt{2}}.
$$

From (2.4), we deduce that

$$
G \left( -\frac{1}{\tau_1} \right) = \tau_1 \frac{\sqrt{2}}{\pi i} - \left( \frac{7}{4} + \frac{\sqrt{-15}}{4} \right) G(\tau_1)
$$
and
\[ G(\tau_2) = \left( \frac{1}{\tau_2} \right)^2 G(\tau_1) + \frac{\sqrt{2}}{\tau_2 \pi i}, \]
where we have used \(-1/\tau_2 = \tau_1 + \sqrt{2}\), which implies that
\[ G\left( -\frac{1}{\tau_2} \right) = G(\tau_1). \]

Hence, we find that
\[ G\left( -\frac{1}{\tau_1} \right) = \frac{\sqrt{-15} - 1}{2\pi i} - \left( \frac{7}{4} + \frac{\sqrt{-15}}{4} \right) G(\tau_1) \]
and
\[ G(\tau_2) = \left( -\frac{3}{4} + \frac{3\sqrt{-15}}{4} \right) G(\tau_1) + \left( -\frac{3}{2} - \frac{1}{2\sqrt{-15}} \right) \frac{1}{\pi i}. \]

Now, let
\[ (3.3) \quad M_N(q) = \frac{Z(q)}{Z(q^N)}. \]
Then we find that
\[ \frac{q}{M_N(q)} \frac{dM_N(q)}{dq} = \tilde{Z}(q) - N\tilde{Z}(q^N), \]
where
\[ \tilde{Z}(q) = \frac{q}{Z(q)} \frac{dZ(q)}{dq}. \]

Letting \( q = e^{2\pi i \tau/\sqrt{2}} \), we deduce that
\[ G(\tau) - NG(N\tau) = \tilde{M}_N(\tau), \]
where
\[ (3.4) \quad \tilde{M}_N(\tau) = \frac{q}{M_N(q)} \frac{dM_N(q)}{dq}. \]

When \( N = 2 \), we have
\[ (3.5) \quad G\left( -\frac{1}{\tau_1} \right) - 2G(\tau_3) = \tilde{M}_2 \left( -\frac{1}{\tau_1} \right) \]
and when \( N = 3 \),
\[ (3.6) \quad G(\tau_2) - 3G(\tau_3) = \tilde{M}_3(\tau_2). \]

Using (3.1) and (3.2), we would have two identities relating \( G(\tau_1) \) and \( G(\tau_3) \) (see (4.5) and (4.6) below).
A modular equation of degree $N$ in the theory of signature 4 is a relation between $x(q)$ and $x(q^N)$, where $x(q)$ is given by (2.3). Modular equation of degree $N$ is not unique. In order to establish the two series for $1/\pi$ in (1.5), we will need the following modular equations:

**Theorem 4.1.** Let $\alpha = x(q)$ and $\gamma = x(q^2)$. Then
\begin{equation}
64\gamma - 80\gamma\alpha + 18\gamma\alpha^2 - 81\gamma^2\alpha + 144\gamma^2 - 64\gamma^2 - \alpha^2 = 0.
\end{equation}

**Theorem 4.2.** Let $\alpha = x(q)$ and $\beta = x(q^3)$. Then
\begin{equation}
\alpha^4 + \beta^4 + 141056\beta^3\alpha^3 + 19206\beta^2\alpha^2 - 4096\alpha\beta + 36864\beta^4\alpha^4
- 3972(\beta^3\alpha + \alpha^3\beta) + 36480(\alpha^4\beta^2 + \beta^4\alpha^2) - 73728(\beta^4\alpha^3 + \alpha^4\beta^3)
+ 384(\alpha^4\beta + \beta^4\alpha) + 7680(\alpha^2\beta + \beta^2\alpha) - 63360(\alpha^3\beta^2 + \beta^3\alpha^2) = 0.
\end{equation}

Let $F(\tau) = x(q)$, with $q = e^{2\pi i \tau/\sqrt{2}}$. Since
\[3\tau_2 = \tau_3 - \sqrt{2},\]
we find that $F(\tau_2)$ and $F(\tau_3)$ satisfies (4.2). In a similar way, we conclude that $F(-1/\tau_1)$ and $F(\tau_3)$ satisfies (4.1). Now, using (2.1), we find that
\[F\left(-\frac{1}{\tau_1}\right) = 1 - F(\tau_1).
\]
Hence, we deduce that
\begin{equation}
F(\tau_2) = 1 - F\left(-\frac{1}{\tau_2}\right) = 1 - F(\tau_1),
\end{equation}
where we have used
\[-\frac{1}{\tau_2} = \tau_1 + \sqrt{2}.
\]
Hence, the two relations we obtained reduced to two equations involving $F(\tau_1)$ and $F(\tau_3)$. Solving these equations, we conclude that
\[F(\tau_1) = \frac{1}{2} - \frac{32}{147}\sqrt{5} - \frac{11}{294}\sqrt{-15}.
\]
By taking conjugation, we find that
\[F(\tau_3) = \frac{1}{2} - \frac{32}{147}\sqrt{5} + \frac{11}{294}\sqrt{-15}.
\]
By (4.3), we deduce that
\[F(\tau_2) = \frac{1}{2} + \frac{32}{147}\sqrt{5} + \frac{11}{294}\sqrt{-15}.
\]
We next describe how we obtain an expression for $M_N$ defined by (3.3). It is known [3] that if $x = x(q)$, then
\[q\frac{dx}{dq} = Z(x(1-x) = \frac{Z}{4}X,
\]
where $X$ is defined by (2.2). If $N$ is a positive integer greater than 1, then
\[
q^N \frac{d x(q^N)}{d(q^N)} = \frac{Z(q^N)}{4} X(q^N).
\]
This yields
\[
(4.4) \quad M_N = \frac{1}{N} \frac{d x(q)}{d(q^N)} \frac{X(q^N)}{X(q)}.
\]
But if we are given a modular equation of degree $N$, then the right-hand side can be expressed explicitly in terms of $X(q)$ and $X(q^N)$ and hence we have an expression of $M_N$ in terms of $X(q)$ and $X(q^N)$. From (4.4), we can then derive an explicit expression of $d M_N / d X(q)$ in terms of $X(q)$ and $X(q^N)$ and this in turn yields the expression for $\tilde{M}_N$ defined by (3.4). We will carry out these computations and determine the right-hand side of (3.5).

Differentiating (4.1) with respect to $\alpha$, we conclude that
\[
\frac{d^2 \gamma}{d \alpha} = \frac{80 \gamma - 36 \alpha + 162 \gamma^2 - 144 \gamma^2 + 2 \alpha}{64 - 80 \alpha + 18 \alpha^2 - 162 \alpha^2 + 288 \gamma \alpha - 128 \gamma^2}.
\]
Hence,
\[
M_2 = \frac{1}{2} \frac{64 - 80 \alpha + 18 \alpha^2 - 162 \alpha^2 + 288 \gamma \alpha - 128 \gamma}{64 - 80 \alpha + 18 \alpha^2 - 162 \alpha^2 + 288 \gamma \alpha - 128 \gamma} \frac{\gamma (1 - \gamma)}{\alpha (1 - \alpha)}.
\]

Differentiating $M_2$ with respect to $\alpha$, and letting $\alpha = F(-1/\tau_1)$ and $\gamma = F(\tau_2)$, we conclude that
\[
(4.5) \quad G \left( -\frac{1}{\tau_1} \right) - 2G(\tau_3) = \left( \frac{11}{49} + \frac{\sqrt{5}}{7} + \frac{\sqrt{-15}}{21} - \frac{\sqrt{-3}}{147} \right) Z(\tau_1).
\]

In a similar way, we use (4.2) and the relation between $Z(\tau_1) = Z(-1/\tau_2)$ and $Z(\tau_2)$ to deduce that
\[
(4.6) \quad G(\tau_2) - 3G(\tau_3) = \left( \frac{4}{49} \sqrt{5} - \frac{20}{49} \sqrt{-3} + \frac{27}{49} \sqrt{-15} + \frac{30}{49} \right) Z(\tau_1).
\]

Next, using (3.1) and (3.2) in (4.5) and (4.6) to remove the term $G(\tau_3)$, we find that
\[
- \left( \frac{15}{4} + \frac{9}{4} \sqrt{-15} \right) G(\tau_1) - \left( \frac{27}{49} - \frac{13}{49} \sqrt{5} - \frac{3}{49} \sqrt{-15} - \frac{39}{49} \sqrt{-3} \right) Z(\tau_1) = -\frac{9i + \sqrt{15}}{2\pi}.
\]

Finally, observing that
\[
Z(\tau_1) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}) k! (\frac{1}{4} k! (\frac{3}{4} k!) k)}{k!^{15}} (4F(\tau_1)(1 - F(\tau_1)))^k
\]
\footnote{This would be too complicated to present here.}
and

\[ G(\tau_1) = (1 - 2F(\tau_1)) \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)_k \frac{(\frac{3}{4})_k}{k!^3} k \left( 4F(\tau_1)(1 - F(\tau_1)) \right)^k, \]

we obtain the desired series for \( 1/\pi \).

Remarks.

1. The degrees of modular equations to be used to prove complex series for \( 1/\pi \) are not as obvious as in the real series. In the real series for \( 1/\pi \), if \( \tau = \sqrt{-pq}/2 \) where \( p \) and \( q \) are primes, then it is clear that we need modular equations of degree \( p \) and \( q \). In the complex case, we observe that the norms of \( -1/\tau_1 \) and \( \tau_2 \) are 2 and 3, respectively. These norms determine the degrees of modular equations we used. We stress that the method presented here can also be applied to series for \( 1/\pi \) with complex coefficients that belong to the theory of elliptic function in other alternative bases.

2. In most of the proofs of Ramanujan-type series for \( 1/\pi \), the most complicated expression arises from differentiating \( M_N \). In the quartic theory, this complication can be avoided by writing the derivative of \( M_N \) in terms of the expression

\[ f_N(q) = \frac{N L(q^N) - L(q)}{(N - 1) \sqrt{Z(q)Z(q^N)}} \]

where

\[ L(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}. \]

This identity, after some simplifications, is \[3, Eq. (4.36)]

\[ \frac{N L(q^N) - L(q)}{Z(q)} = (1 - 3x(q^N)) \frac{N}{M_N} - (1 - 3x(q)) \frac{N}{M_N} - 6 \frac{\tilde{M}_N}{Z(q)}. \]

The expression \( f_N \) can then be expressed in terms of \( x(q) \) and \( x(q^N) \) and we can then derive the value of \( \tilde{M}_N \) at corresponding values of \( q \) without differentiating \( M_N \). In this article, we would require the formulas

\[ f_2^2(q) = 1 + 3\sqrt{x(q^2)} \]

and

\[ f_3^2(q) = 1 + 3\sqrt{x(q)x(q^3)}. \]

The two values that we needed are then

\[ f_2(e^{2\pi i(-1/\tau_1)}) = \frac{\sqrt{105} + 2\sqrt{21} + 2\sqrt{-35} - \sqrt{-7}}{14} \]
and

\[ f_3(e^{2\pi i\tau}) = \frac{\sqrt{105} - 3\sqrt{-7}}{14}. \]

These values would then lead to the values on the right-hand sides of (4.5) and (4.6).

3. A different approach to derive series for \(1/\pi\) with complex coefficients and argument is based on algebraic transformations of hypergeometric and related series of modular origin; the required details of the method can be found in [6]. For example, starting from real Ramanujan’s series

\[
\sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)^k}{k!^3} (5 - \sqrt{5} + 20k) \left(\frac{\sqrt{5} - 1}{2}\right)^6 = \frac{2\sqrt{5}}{\pi} \sqrt{2 + \sqrt{5}},
\]

and applying subsequently the transformation

\[
3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \mid 1, 1 \bigg| \frac{27p^4(1-p^2)(2+p)^4(1+2p)}{(2+2p-p^2)^6}\right) = \frac{(2+2p-p^2)^2}{4(1+2p)} 3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid 1, 1 \bigg| \frac{4p^3(1-p^2)(1+p)^2(2+p)}{(1+2p)^2}\right)
\]

at

\[ p = \frac{(1 + \sqrt{5})\sqrt{5} - 2 - \sqrt{22 - 10\sqrt{5}} - 1}{2} \]

(cf. [7]) and then the generating function

\[
\sum_{k=0}^{\infty} \delta_k u^k = \frac{1}{1 - 4u} 3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \mid 1, 1 \bigg| \frac{108u^2}{(1 - 4u)^3}\right)
\]

of the Domb numbers [6] at \(u = (3 - 2i - \sqrt{5 - 10i})/32\), we obtain the following two complex series:

\[
\sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)^k(k_1^3)}{k!^3}(3(401 - i - (109 - 69i)\sqrt{1 + 2i}) + 5830k) \times \left(\frac{27(2530 + 1451i - 65(30 - i)\sqrt{1 + 2i})}{495 - 4888i}\right) = \frac{3321 - 381i + 81(33 - 17i)\sqrt{1 + 2i}}{4\pi}
\]

and

\[
\sum_{k=0}^{\infty} (69 + 13i - (23 - 7i)\sqrt{1 + 2i} + 170k) \delta_k \left(\frac{3 - 2i - (1 - 2i)\sqrt{1 + 2i}}{32}\right)^k = \frac{66 + 42i + 12(1 - 4i)\sqrt{1 + 2i}}{\pi}.
\]
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