HEINE’S BASIC TRANSFORM AND
A PERMUTATION GROUP FOR $q$-HARMONIC SERIES

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Abstract. Let $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$. The new bound $\mu(h_p(1)) \leq 2.46497868\ldots$ for the irrationality measure of the $q$-harmonic series $h_p(1) = \sum_{n=1}^{\infty} q^n/(1 - q^n)$ is proved. The essential ingredient of the proof is a permutation group originated by classical transform of the Heine series.

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1. Introduction. More than 150 years ago, E. Heine considered [He1] the series

$$\phi_1\left(\frac{q^a, q^b}{q^c} \bigg| q, z\right) = 1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} z^2 + \ldots,$$

where $|q| < 1$, $|z| < 1$, $c \neq 0, -1, -2, \ldots$, and proved [He2] several results for it. In particular, he obtained the transformation formula

$$\phi_1\left(\frac{q^a, q^b}{q^c} \bigg| q, z\right) = \frac{(q^a; q)_\infty}{(q^c; q)_\infty} \frac{(q^b z; q)_\infty}{(z; q)_\infty} \cdot \phi_1\left(\frac{q^{c-a}, z}{q^b z} \bigg| q, q^a\right),$$ (1)

where $(x; q)_\infty := \prod_{n=1}^{\infty} (1 - x q^{n-1})$. Today generalized Heine series become an actively investigating part of modern mathematics and a lot of papers and monographs (see, e.g., [Ex], [Fi], [GR]) are devoted to their study. The aim of this note is to make a use of Heine’s transform (1) in deducing a sharp irrationality measure for the $q$-harmonic series

$$h_p(1) := \sum_{n=1}^{\infty} \frac{1}{p^n - 1} = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} = \frac{q}{1 - q} \cdot \phi_1\left(\frac{q, q}{q^2} \bigg| q, q\right), \quad p = \frac{1}{q} \in \mathbb{Z} \setminus \{0, \pm 1\}.$$ (2)
As it is easily seen, the series $h_p(1)$ as function of $p$ is irrational. The irrationality of the number $h_2(1)$ (i.e., $q = 1/2$ in (2)) was first proved by P. Erdős, who also stated the problem of extending his result to an arbitrary integer $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. This problem (and even in more general settings) was solved by J.-P. Bézivin [Bé] and in a quantitative form by P. Borwein [Bo]. A sharp irrationality measure

$$
\mu(h_p(1)) \leq \frac{2\pi^2}{\pi^2 - 2} = 2.50828476\ldots
$$

was obtained by P. Bundschuh and K. Väänänen [BV] with a help of explicit Padé approximations to the $q$-logarithm function and a certain $q$-arithmetic observation. (Here $\mu = \mu(\alpha)$ denotes the irrationality exponent of an irrational number $\alpha$ that is the least possible exponent such that for any $\varepsilon > 0$ the inequality $|\alpha - a/b| \leq b^{-(\mu + \varepsilon)}$ has only finitely many solutions in integers $a, b$.) The works [BV], [MV], and [Ass] contain estimates for irrationality exponents of the $q$-logarithm values. Finally, introducing a $q$-arithmetic approach in [Zu1] resulted in a slight improvement of the estimate (3) and of the estimate in [Ass] for a $q$-analogue of $\log 2$ (the result for $\mu(h_p(1))$ in [Zu1] is wrong due to a computational error; see the remark at the end of Section 3 below).

The ‘ordinary’ arithmetic approach occurs as a part of the group-structure approach proposed by G. Rhin and C. Viola in [RV1], [RV2] for obtaining quantitative results for the values $\zeta(2)$ and $\zeta(3)$ of Riemann’s zeta function. Recently, the author [Zu2] extended the method of [RV1] to a suitable $q$-analogue of $\zeta(2)$. The permutation group in [RV1], [Zu2] is rather rich to get nice estimates for irrationality exponents in both ordinary and $q$-(basic) cases. A simpler group (of order 12) for the $q$-harmonic series that appears below also leads to a quantitative result.

**Theorem 1.** The irrationality exponent of $h_p(1)$ satisfies the estimate

$$
\mu(h_p(1)) \leq 2.46497868\ldots
$$

However, the group has no ordinary analogue, hence it has not appeared before in an arithmetic study. As pointed out by the referee, it is worth saying that the estimate in Theorem 1 is uniform in $p$.

2. $q$-Basis. Throughout the paper $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$. As usual (see, e.g., [GR]), define the shifted $q$-factorial

$$(a; q)_0 = 1, \quad (a; q)_n := \prod_{\nu=1}^{n} (1 - aq^{\nu-1}) \quad \text{for } n = 1, 2, \ldots,$$

and Jackson’s $q$-gamma function

$$
\Gamma_q(t) := \frac{(q; q)_{\infty}}{(q^t; q)_{\infty}}(1 - q)^{1-t}.
$$
Then $q$-extensions of factorial and binomial coefficients read as follows:

$$[n]_q! := \Gamma_q(n + 1) = \frac{(q; q)_n}{(1 - q)^n} = p^{-n(n-1)/2}[n]_p!,$$

$$\begin{bmatrix} n \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k \cdot (q; q)_{n-k}} = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!} = p^{-k(n-k)}\begin{bmatrix} n \end{bmatrix}_p,$$

where $k = 0, 1, \ldots, n$ and $n = 0, 1, 2, \ldots$.

Since $p^n - 1 = \prod_{l \mid n} \Phi_l(p)$, where

$$\Phi_l(p) := \prod_{k=1}^l (p - e^{2\pi ik/l}) \in \mathbb{Z}[p], \quad l = 1, 2, 3, \ldots,$$

are (irreducible over $\mathbb{Z}$) cyclotomic polynomials, we deduce the following claims:

- the polynomials (4) and only they are irreducible divisors of the polynomial

$$[n]_p! = \prod_{\nu=1}^n \frac{p^\nu - 1}{p - 1} \in \mathbb{Z}[p];$$

moreover,

$$\operatorname{ord}_{\Phi_l(p)}[n]_p! = 0, \quad \operatorname{ord}_{\Phi_l(p)}[n]_p! = \left\lfloor \frac{n}{l} \right\rfloor, \quad l = 2, 3, 4, \ldots,$$

where $\left\lfloor \cdot \right\rfloor$ denotes the integral part of a number;

- the polynomial $D_n(p) := \prod_{l=1}^n \Phi_l(p)$ is the least common multiple of the polynomials $p^k - 1$, $k = 1, \ldots, n$.

In the above notation, the $q$-arithmetic approach is characterized by the following assertions.

**Lemma 1** ([BV], Section 2; [Ass], Lemma 2). There holds the limit relation

$$\lim_{n \to \infty} \frac{\log |D_n(p)|}{n^2 \log |p|} = \frac{3}{\pi^2}.$$  

**Lemma 2** ([Zu1], Lemma 1). For each demi-interval $[u,v) \subset (0,1) \text{ with } u,v \in \mathbb{Q}$, there holds the limit relation

$$\lim_{n \to \infty} \frac{1}{n^2 \log |p|} \sum_{l \mid \{n/l\} \in [u,v)} \log |\Phi_l(p)| = \frac{3}{\pi^2} (\psi'(u) - \psi'(v)) = \frac{3}{\pi^2} \int_u^v d(-\psi'(x)),$$

where $\{a\} = a - [a]$ and $\psi(x)$ is the logarithmic derivative of Euler’s gamma function.
3. Linear forms involving $q$-harmonic series. Let $a_0, a_1, a_2,$ and $b$ be positive integers satisfying the condition
\[ a_1 + a_2 \leq b. \] (6)

Consider the rational function
\[ R(T) = \frac{(qT; q)_{a_1-1}}{(q; q)_{a_1-1}} \cdot \frac{(q; q)_{b-a_2-1}}{(q^{a_2} T; q)_{b-a_2-1}} \cdot T^{a_0}, \]
so that
\[ R(q^l) = \frac{\Gamma_q(b-a_2)}{(1-q)\Gamma_q(a_1)} \cdot \frac{\Gamma_q(t+a_1)\Gamma_q(t+a_2)}{\Gamma_q(t+1)\Gamma_q(t+b)} \cdot q^{a_0 l}. \] (7)

Denote by $a_1^* \leq a_2^*$ the ordered version of the set $a_1, a_2,$ i.e., $\{a_1^*, a_2^*\} = \{a_1, a_2\}$. Condition (6) implies that $R(T)T^{-a_0} = O(T^{-1})$ as $T \to \infty,$ hence we can write down the partial-fraction decomposition
\[ R(T) = T^{a_0} \sum_{k=a_2^*}^{b-1} \frac{A_k}{1-q^k T}, \]
where
\[ A_k = \left. \left( R(T)T^{-a_0}(1-q^k T) \right) \right|_{T=q^{-k}} \]
\[ = (-1)^{a_1+a_2+k+1} q^{(a_1-2k)(a_1-1)/2+(b-a_2)(b-a_2+1)/2} \times \left[ \begin{array}{c} k-1 \\ a_1-1 \end{array} \right] \left[ \begin{array}{c} b-a_2-1 \\ b-k-1 \end{array} \right]_q \]
\[ = (-1)^{a_1+a_2+k+1} p^{a_1(a_1-1)/2-(b-a_2)(b-a_2+1)/2+(b-k)(b-k-1)/2} \times \left[ \begin{array}{c} k-1 \\ a_1-1 \end{array} \right] \left[ \begin{array}{c} b-a_2-1 \\ b-k-1 \end{array} \right]_p. \] (8)

Furthermore, consider the series
\[ F(a; b) = F(a_0, a_1, a_2; b) := \sum_{l=0}^{\infty} R(q^l) = \sum_{t=1-a_1^*}^{\infty} R(q^l) \]
\[ = \sum_{k=a_2^*}^{b-1} A_k \sum_{t=1-a_1^*}^{\infty} \frac{q^{a_0 l}}{1-q^{k+l}} = \sum_{k=a_2^*}^{b-1} A_k q^{-a_0 k} \sum_{l=k-a_1^*+1}^{\infty} \frac{q^{a_0 l}}{1-q^l} \]
\[ = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \left( \sum_{l=1}^{k-a_1^*} \frac{q^l}{1-q^l} - \sum_{l=1}^{k-a_1^*} \frac{q^l}{1-q^l} \sum_{l=k-a_1^*+1}^{\infty} \frac{q^l-q^{a_0 l}}{1-q^l} \right) \]
\[ = A(p) h_p(1) - B_1(p) - B_2(p), \] (9)

where
\[ A(p) = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k}, \quad B_1(p) = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \sum_{l=1}^{k-a_1^*} \frac{1}{p^l-1}, \] (10)
and

\[ B_2(p) = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \sum_{l=k-a_1^*+1}^{\infty} \sum_{j=1}^{a_0-1} q^{jl} = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \sum_{j=1}^{a_0-1} \frac{(q^j)^{k-a_1^*+1}}{1-q^j} \]

\[ = \sum_{k=a_2^*}^{b-1} \sum_{j=1}^{a_0-1} A_k p^{a_0 k-j(k-a_1^*)} \frac{1}{p^j - 1}. \quad (11) \]

Now, mention that \( a_0 k \geq a_0 a_2^* \) and \( a_0 k - j(k - a_1^*) > a_0 k - a_0 (k - a_1^*) = a_0 a_1^* \) for \( a_2^* \leq k < b, 1 \leq j < a_0 \). Hence, taking \( M_0 = a_0 a_1^* + a_1(a_1-1)/2 - (b-a_2)(b-a_2-1)/2 \), from (8) we deduce the inclusions

\[ p^{-M_0} A_k p^{a_0 k} \in \mathbb{Z}[p], \quad p^{-M_0} A_k p^{a_0 k-j(k-a_1^*)} \in \mathbb{Z}[p]. \]

Applying the results of Section 2 to formulae (10) and (11) we obtain

\[ p^{-M_0} A(p) \in \mathbb{Z}[p], \]
\[ p^{-M_0} D_{b-a_1^*-1}(p) \cdot B_1(p) \in \mathbb{Z}[p], \quad p^{-M_0} D_{a_0-1}(p) \cdot B_2(p) \in \mathbb{Z}[p]. \]

Therefore representation (9) yields the following assertion.

**Lemma 3.** With some suitable integer

\[ M = M(\mathbf{a}; b) \geq a_0 a_1^* + \frac{a_1(a_1-1)}{2} - \frac{(b-a_2)(b-a_2-1)}{2} \quad (12) \]

there holds the inclusion

\[ p^{-M} D_{\max\{a_0-1, b-a_1^*-1\}}(p) \cdot F(\mathbf{a}; b) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p]. \]

Thanks to (7), the quantity \( F(\mathbf{a}; b) \) can be identified with the Heine series:

\[ F(a_0, a_1, a_2; b) = \frac{\Gamma_q(a_2) \Gamma_q(b-a_2)}{(1-q) \Gamma_q(b)} \cdot \phi_1 \left( \frac{q^{a_1}, q^{a_2}}{q, q^{-a_0}} \right). \quad (13) \]

The lower estimate in (12) for \( M(\mathbf{a}; b) \) is rather rough and we require the following sharp form of it.

**Lemma 4.** Suppose that

\[ a_1 \leq a_2, \quad a_1 + a_2 \leq b \leq a_0 + a_2. \quad (14) \]

Then we have

\[ p^{-M} D_{\max\{a_0-1, b-a_1^*-1\}}(p) \cdot F(\mathbf{a}; b) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p], \quad (15) \]
where
\[ M = M(a; b) := \frac{a_1(a_1 - 1)}{2} + a_0a_1 + (b - a_2)(a_2 - a_1). \] (16)

In addition, for any \( p \in \mathbb{Z} \setminus \{0, \pm 1\} \), the estimates
\[ |F(a; b)| = |p|^{O(b)}, \quad |A(p)| \leq |p|^{(a_0 + a_1 + a_2)b - (a_1^2 + a_2^2 + b^2)/2 + O(b)} \] (17)
hold with some absolute constant in \( O(b) \).

Proof. The first condition in (14) allows us to write \( a_1^* = a_1, \ a_2^* = a_2 \) in (10), (11), and to apply Lemma 3 from \([Zu1]\) to the quantity \( B_2(p) \) after interchanging summation in (11):
\[ B_2(p) = p^{a_1(a_1 - 1)/2 + a_0a_1 + (b - a_2)(a_2 - a_1)} \sum_{j=1}^{a_0-1} \frac{1}{p^j - 1} \sum_{l=0}^{a_2-1} (-1)^l p^{(a_1 - l)(a_1 - l - 1)/2} \times \left[ \frac{b - a_2 + l - 1}{a_1 - 1} \right] \left[ \frac{a_2 - 1}{l} \right] \left( p^{a_0 - j - 1}; p^{-1} \right)_{b - a_2 + l}. \]

From this formula we deduce the inclusion
\[ p^{-M_2} D_{a_0-1}(p) \cdot B_2(p) \in \mathbb{Z}[p], \quad \text{where} \quad M_2 = \frac{a_1(a_1 - 1)}{2} + a_0a_1 + (b - a_2)(a_2 - a_1), \]
while formulae (8), (10) and the inequality
\[ \frac{(b - k)(b - k - 1)}{2} + a_0k \geq \frac{(b - a_2)(b - a_2 - 1)}{2} + a_0a_2 \quad \text{for} \quad k \geq a_2 \geq b - a_0 \]
yield the inclusions
\[ p^{-M_1}(p) \cdot A(p) \in \mathbb{Z}[p], \quad p^{-M_1} D_{b - a_2 - 1}(p) \cdot B_1(p) \in \mathbb{Z}[p], \]
where \( M_1 = \frac{a_1(a_1 - 1)}{2} + a_0a_2 \).

Thus using the fact \( \min\{M_1, M_2\} = M_2 \) under the hypothesis (14), we arrive at the desired inclusion (15).

To prove the second part of the lemma, we would like to adopt the construction and results of \([Zu1]\). There we consider the family of series
\[ I(n; m) = I(n_0, n_1, n_2; m) \]
\[ = (q; q)^{n_2}_{n_0} q^{-n_0(n_1 - n_0)} \sum_{s = n_1 + 1}^{\infty} \frac{(1 - q^{n_1 - n_0 + 1 - s}) \cdots (1 - q^{n_1 - s})}{(q^{-s - 1})(q^{-s} - q) \cdots (q^{-s} - q^{n_2})} q^{(m - 1)s} \]
\[ = (-1)^{n_0} q^{n_0(n_0 + 1)/2 + (n_1 + 1)(n_2 - n_0 + m)} \]
\[ \times \frac{(q; q)_{n_1}(q; q)_2}{(q; q)_{n_1 + n_2 + 1}} \cdot \varphi_1 \left( q^{n_0 + 1}, q^{n_1 + 1} \left| q, q^{n_2 - n_0 + m} \right. \right), \] (18)
where \( n_0, n_1, n_2, \) and \( m \) are positive integers satisfying the conditions
\[
n_1 \geq n_0, \quad n_2 \geq n_0, \quad m > n_0. \tag{19}
\]
Comparing representations (13) and (18) we conclude that
\[
F(a; b) = (-1)^{n_0} p^{n_0(n_0+1)/2 + (n_1+1)(n_2-n_0+m)} I(n; m),
\]
where
\[
n_0 = a_1 - 1, \quad n_1 = a_2 - 1, \quad n_2 = b - a_2 - 1, \quad m = a_0 + a_1 + a_2 - b,
\]
and conditions (19) become conditions (14). Therefore the estimates (17) are consequences of the corresponding results for the quantity \( I(n; m) \) (see [Zu1], Lemmas 6 and 7), and the lemma follows.

**Remark.** In [Zu1] we made a mistake in computing the exponent of \( p \) when applied the identity of [Zu1], Lemma 3. The correct application of the identity leads to the inclusions
\[
p^{(n_1-n_0)(m-n_0+1)} D_{\text{max}} \{n_1+n_2-n_0,m\} \cdot I \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p],
\]
and to the corresponding changes of Propositions 1, 2 in [Zu1]. Fortunately, these changes do not influence on the result of Theorem 2 in [Zu1] (concerning the irrationality exponent of \( \ln p(2) \)), and Theorem 1 in the present work considerably improves the wrong result of Theorem 1 in [Zu1].

**4. Permutation group for \( q \)-harmonic series.** Heine’s transform (1) yields the stability of the quantity
\[
\frac{F(a_0, a_1, a_2; b)}{\Gamma_q(a_0) \Gamma_q(a_2) \Gamma_q(b-a_2)} = \frac{1}{(1-q) \Gamma_q(a_0) \Gamma_q(b)} \cdot \phi_1 \left( \frac{q^{a_1} q^{a_2}}{q} \left| \begin{array}{c} q, q^{a_0} \end{array} \right. \right) \tag{20}
\]
under the action of
\[
\tau: (a_0, a_1, a_2; b) \mapsto (a_1, b - a_1, a_0 + a_2).
\]
In addition, the quantity (20) is obviously stable under the action of the permutation
\[
\sigma: (a_0, a_1, a_2; b) \mapsto (a_0, a_2, a_1; b)
\]
interchanging the parameters \( a_1 \) and \( a_2 \). Let \( \mathfrak{G} \) denote the group generated by \( \tau, \sigma \); the group \( \mathfrak{G} = \langle \tau, \sigma : \tau^6 = \sigma^2 = (\tau \sigma)^2 = \text{id} \rangle \) has order 12 (see [Fi], Section 20).

To interpret \( \mathfrak{G} \) as a permutation group, we now introduce the tuple \( c \) of the six additional parameters
\[
c_{00} = a_0 + a_1 + a_2 - b - 1, \quad c_{01} = a_0 - 1, \quad c_{11} = a_1 - 1, \quad c_{21} = a_2 - 1,
\]
\[
c_{12} = b - a_1 - 1, \quad c_{22} = b - a_2 - 1,
\]
and take \( H(c) := F(a; b) \). Then
\[
\tau = (c_{22} c_{21} c_{01} c_{12} c_{00}), \quad \sigma = (c_{11} c_{21})(c_{12} c_{22})
\]
are permutations of the tuple \( c \) of orders 6 and 2, respectively, and the \( \mathfrak{G} \)-stability of the quantity (20) can be summarized by the following assertion.
Lemma 5. The quantity

\[ \frac{H(c)}{\Pi_q(c)}, \]

where \( \Pi_q(c) := [c_{01}]_q! [c_{21}]_q! [c_{22}]_q! \),

is stable under the action of \( G \).

By definition of the \( q \)-factorial coefficient, \( \Pi_q(c) = p^{-N(c)} \Pi_p(c) \), where

\[ N(c) := \frac{c_{01}(c_{01} + 1) + c_{21}(c_{21} + 1) + c_{22}(c_{22} + 1)}{2}. \]

We also require some additional characteristics: \( M(c) := M(a; b) \) is the ‘suitable’ integer of Lemma 3 if \( a_1 + a_2 \leq b \) or is determined by formula (16) if stronger conditions (14) hold;

\[ m(c) := \max\{c_{00}, c_{01}, c_{11}, c_{21}, c_{12}, c_{22}\}, \]

\[ s_+(c) := c_{01} + c_{21} + c_{22} = a_0 + b - 3, \]

\[ s_-(c) := c_{00} + c_{11} + c_{12} = a_0 + a_1 + a_2 - 3, \]

\[ s(c) := s_+(c) - s_-(c) = b - a_1 - a_2. \]

Then \( m(c) \) is \( G \)-stable, while the quantities \( s_\pm(c) \) and \( s(c) \) obey the following rules.

Lemma 6. The following relations hold:

\[ s_+(\tau c) = s_-(c), \quad s_-(\tau c) = s_+(c), \quad s(\tau c) = -s(c), \]

\[ s_+(\sigma c) = s_+(c), \quad s_-(\sigma c) = s_-(c), \quad s(\sigma c) = s(c), \]

where \( gc \) denotes the image of \( c \) under the action of a permutation \( g \in G \).

Proof by direct computation.

As it is easily seen, condition (6) is equivalent to condition \( s(c) \geq 0 \); hence we are able to write the conclusions of Lemmas 3 and 4 in the form

\[ p^{-M(c)} D_{m(c)}(p) \cdot H(c) \in \mathbb{Z}[p]h_p(1) + \mathbb{Z}[p], \quad (21) \]

if (and only if) \( s(c) \geq 0 \). This fact and Lemma 6 mean that we might use the group \( \mathfrak{G}_+ = \langle \tau^2, \sigma \rangle \subset G \) of order 6 instead of the total group \( G \) if \( s(c) > 0 \). The case \( s(c) = 0 \) is out of our further interest since it implies the relations \( c_{00} = c_{01}, \)
\( c_{11} = c_{22}, c_{21} = c_{12}, \) and as a consequence

\[ \Pi_p(gc) = \Pi_p(c) \quad \text{for all} \quad g \in G. \]
If \( s(c) > 0 \), we obtain at most three different values of \( \Pi_p(\sigma^4 g c) \), \( g \in \mathfrak{G}_+ \), namely,

\[
\Pi_p = \Pi_p(c) = \Pi_p(\sigma^4 c) = [c_{01}]_p! [c_{21}]_p! [c_{22}]_p!,
\]

\[
\Pi'_p = \Pi_p(\tau^2 c) = \Pi_p(\sigma c) = [c_{01}]_p! [c_{11}]_p! [c_{12}]_p!,
\]

\[
\Pi''_p = \Pi_p(\tau^4 c) = \Pi_p(\sigma^2 c) = [c_{00}]_p! [c_{12}]_p! [c_{22}]_p!.
\]

For each \( l = 2, 3, \ldots, m(c) \), take

\[
\nu_l := \max_{g \in \mathfrak{G}_+} \operatorname{ord}_{\Phi_l(p)}(\Pi_p(\sigma^4 g c)) / \Pi_p(\sigma^4 g c),
\]

\[
= \max \left\{ 0, \left\lfloor \frac{c_{21}}{l} \right\rfloor + \left\lfloor \frac{c_{22}}{l} \right\rfloor - \left\lfloor \frac{c_{11}}{l} \right\rfloor - \left\lfloor \frac{c_{12}}{l} \right\rfloor, \right. \\
\left. \left\lfloor \frac{c_{01}}{l} \right\rfloor + \left\lfloor \frac{c_{21}}{l} \right\rfloor - \left\lfloor \frac{c_{00}}{l} \right\rfloor - \left\lfloor \frac{c_{12}}{l} \right\rfloor \right\} ,
\]

and set

\[
\Omega(p) := \prod_{l=2}^{m(c)} \Phi_l^{\nu_l}(p) \in \mathbb{Z}[p].
\]

**Lemma 7.** There holds the inclusion

\[
p^{-M(c)} D_{m(c)}(p) \Omega^{-1}(p) \cdot H(c) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p],
\]

provided that \( s(c) > 0 \).

**Proof** follows lines of the proof of Proposition 2 in [Zu2]. The inclusion (21) and Lemma 5 yield

\[
p^{-M(gc) - N(gc) + N(c)} D_{m(gc)}(p) \cdot \Pi_p(gc) / \Pi_p(c) \cdot H(c) = p^{-M(gc)} D_{m(gc)}(p) \cdot H(gc)
\]

\[
\in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p]
\]

for all \( g \in \mathfrak{G}_+ \). Since cyclotomic polynomials enters the \( p \)-factorial \([n]_p!\) in accordance with formula (5) and these polynomials are coprime with the polynomial \( p \in \mathbb{Z}[p] \), we arrive at the desired claim (23).

**5. q-Conclusion.** We now take a tuple of the new positive integers (directions) \( \alpha_0, \alpha_1, \alpha_2, \) and \( \beta \) satisfying the conditions

\[
\alpha_1 \leq \alpha_2, \quad \alpha_1 + \alpha_2 < \beta \leq \alpha_0 + \alpha_2,
\]

and to each integer \( n = 1, 2, \ldots \) assign the old parameters \( a \) and \( b \) in accordance with the following rule:

\[
a_j = \alpha_j n + 1, \quad j = 0, 1, 2, \quad b = \beta n + 2.
\]
Then setting

\[ c_{00} = \alpha_0 + \alpha_1 + \alpha_2 - \beta, \quad c_{j1} = \alpha_j, \quad j = 0, 1, 2, \quad c_{j2} = \beta - \alpha_j, \quad j = 1, 2, \]

and \( m = m(c) \) introduce the quantities

\[ H_n := H(c \cdot n) = F(a; b) = A_n h_p(1) - B_n, \quad n = 1, 2, \ldots. \]

Finally, mention that

\[ p^{-M(a;b)} D_{mn}(p) \Omega^{-1}(p) \cdot H_n \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p] \subset \mathbb{Z} h_p(1) + \mathbb{Z}, \quad n = 1, 2, \ldots, (24) \]

by Lemma 7, and that

\[ \lim_{n \to \infty} \frac{\log |H_n|}{n^2 \log |p|} = 0, \quad \limsup_{n \to \infty} \frac{\log |A_n|}{n^2 \log |p|} \leq (\alpha_0 + \alpha_1 + \alpha_2)\beta - \frac{\alpha_1^2 + \alpha_2^2 + \beta^2}{2} =: C_1 \]

by Lemma 4. In addition, \( \nu_l = \omega(n/l) \) in (22), where

\[ \omega(x) := \max \{ 0, |c_{21} x| + |c_{22} x| - |c_{11} x| - |c_{12} x|, |c_{01} x| + |c_{21} x| - |c_{00} x| - |c_{12} x| \} \]

is a 1-periodic integral-valued function; therefore,

\[ \lim_{n \to \infty} \frac{\log |p^{-M(a;b)} D_{mn}(p) \Omega(p)|}{n^2 \log |p|} = \frac{1}{2} \alpha_1^2 + \alpha_0 \alpha_1 + (\beta - \alpha_2)(\alpha_2 - \alpha_1)
\]

\[ - \frac{3}{\pi^2} \left( m^2 - \int_0^1 \omega(x) d(-\psi'(x)) \right) =: C_0 \]

by Lemmas 1, 2, and 4. By standard arguments, relations (24)–(26) yield the estimate \( \mu(h_p(1)) \leq C_1/C_0 \), provided that \( C_0 > 0 \).

Take

\[ \alpha_0 = 14, \quad \alpha_1 = 12, \quad \alpha_2 = 14, \quad \beta = 27 \]

so that

\[ c_{00} = 13, \quad c_{01} = 14, \quad c_{11} = 12, \quad c_{21} = 14, \quad c_{12} = 15, \quad c_{22} = 13. \]

Then

\[ C_1 = 545.5, \quad C_0 = 221.30008816 \ldots, \]

hence \( \mu(h_p(1)) \leq C_1/C_0 = 2.46497868 \ldots \). Here \( \omega(x) = 1 \) for \( x \in [0, 1) \) belonging the set

\[ \left[ \frac{11}{14}, \frac{12}{14} \right) \cup \left[ \frac{1}{3}, \frac{1}{5} \right) \cup \left[ \frac{4}{11}, \frac{5}{11} \right) \cup \left[ \frac{7}{11}, \frac{8}{11} \right) \cup \left[ \frac{10}{11}, \frac{11}{11} \right) \cup \left[ \frac{13}{11}, \frac{14}{11} \right) \cup \left[ \frac{17}{11}, \frac{18}{11} \right) \cup \left[ \frac{4}{7}, \frac{5}{7} \right) \cup \left[ \frac{9}{14}, \frac{2}{3} \right) \cup \left[ \frac{5}{7}, \frac{6}{7} \right) \cup \left[ \frac{11}{14}, \frac{12}{14} \right) \cup \left[ \frac{13}{14}, \frac{14}{14} \right). \]
The proof of Theorem 1 is complete.

Remark. The series (13) involving linear forms in 1 and $q$-harmonic series can be represented as some $q$-integral (see, e.g., [Ex], Section 2.5.1). This $q$-integral representation is very similar to that used in [RV1] and [RV2] for describing the permutation groups for $\zeta(2)$ and $\zeta(3)$. Inspite of this similarity, there exists no general pattern to change the variable of $q$-integration (see [Ask] and [Ex], Section 2.2.4). Therefore the hypergeometric construction proposed in this paper as well as in our previous works [Zu1], [Zu2] looks a very natural way to extend the group-structure approach to solving new number-theoretic problems.

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References