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SCALE FUNCTIONS AND TREE ENDS

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Abstract

A class of totally disconnected groups consisting of partial direct products on an index set is examined. For such a group, the scale function is found, and for automorphisms arising from permutations of the index set, the tidy subgroups are characterised. When applied to the case where the index set is a finitely-generated free group and the permutation is translation by an element $x$ of the group, the scale depends on the cyclically reduced form of $x$ and the tidy subgroup on the element which conjugates $x$ to its cyclically reduced form.

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0. Introduction and notation

It was shown by van Dantzig in 1931 that each totally disconnected locally compact group has a base of neighbourhoods of the identity consisting of compact open subgroups, [13]. He also gave an example of a totally disconnected locally compact group which fails to have a normal compact open subgroup. This example is the semidirect product $G \rtimes_{\alpha} \mathbb{Z}$, where

$$G = \left\{ g \in \prod_{z} \mathbb{Z}/(2\mathbb{Z}) : \exists N \text{ such that } g(k) = 0 \text{ for } k < N \right\}$$

and the automorphism $\alpha$ is the translation defined by $\alpha(g)(k) = g(k + 1)$. The subgroups $G_N = \{ g : g(k) = 0 \text{ for } k < N \}$ form a base of neighbourhoods of the identity for a topology on $G$, in which each $G_N$ is compact and open, and $G \rtimes_{\alpha} \mathbb{Z}$ is equipped with the product topology.
Results in the paper [15] imply that van Dantzig’s example is in fact typical of the way in which a totally disconnected locally compact group may fail to have a normal compact open subgroup. It was shown that, if \( G \) is any totally disconnected locally compact group and \( x \) is an element of \( G \), then there is a compact open subgroup \( U \) of \( G \) such that:

T1. \( U = U_+ U_- \), where \( U_\pm = \bigcap_{n \geq 0} x^{\pm n} U x^{\mp n} \); and
T2. \( U_+ = \bigcup_{n \geq 0} x^n U_+ x^{-n} \) and \( U_- = \bigcup_{n \geq 0} x^{-n} U_- x^n \) are closed subgroups of \( G \).

A subgroup satisfying T1 and T2 is said to be tidy for \( x \). Note that if \( x U x^{-1} = U \), then \( U \) is tidy for \( x \) and \( U_+ \), \( U_- \), \( U_++ \) and \( U_- \) equal \( U \). Conversely, if \( U = U_+ = U_- \), then \( x \) normalises \( U \).

It was further shown in [15] that the index \( s(x) = [x U_+ x^{-1} : U_+] \) is independent of the choice of subgroup tidy for \( x \) and defines a continuous function \( s : G \to \mathbb{Z}^+ \) such that

S1. \( s(x) = 1 = s(x^{-1}) \) if and only if there is a compact open subgroup \( U \) of \( G \) with \( x U x^{-1} = U \).

Regarding \( \alpha \) as an element of \( G \rtimes_\alpha \mathbb{Z} \) in the van Dantzig example, \( U_+ = G_0 \) is tidy for \( \alpha \), \( U_+ = G \), \( U_- = \{e\} \), \( s(\alpha) = 2 \) and \( s(\alpha^{-1}) = 1 \).

The function \( s \) is called the scale function of \( G \). In the case when \( x \) is not periodic, \( \langle x, U_+ \rangle \) is closed and is isomorphic to \( U_+ \rtimes_\alpha \mathbb{Z} \), where \( \alpha \) is the automorphism of \( U_+ \) defined by \( \alpha(u) = x u x^{-1} \), \( (u \in U_+) \). Thus any element \( x \) of \( G \) which fails to normalise any compact open subgroup of \( G \) belongs to a closed subgroup \( U_+ \rtimes_\alpha \mathbb{Z} \) (or \( U_- \rtimes_\alpha \mathbb{Z} \)) which has the same form as van Dantzig’s example.

These results do not completely answer the question of when totally disconnected locally compact groups have normal compact open subgroups. It can happen that each element of a group \( G \) normalises some compact open subgroup, but that \( G \) has no normal compact open subgroup. The scale function of a group in which each element normalises some compact open subgroup is identically 1 and so, following [10], we shall call such a group uniscalalar. Examples of uniscalalar groups having no normal compact open subgroup are given in [16] and [6, Section 6]. (Note that the main theorem in [16] was proved earlier in [7].) However in all known examples \( G \) is not compactly generated and it is an important question in the structure theory of totally disconnected locally compact groups to decide whether there are compactly generated uniscalalar groups which have no normal compact open subgroups. A partial answer is given in [6] where it is shown that each compactly generated, uniscalalar, rank 1 \( p \)-adic Lie group does have a compact open normal subgroup. This question is a special case of the problem of how the local, or element by element, structure described by the tidy subgroups of \( G \) may be assembled to give a global description of \( G \).

The present paper generalises van Dantzig’s construction with the aim of using the groups found to help to answer some of these global structure questions.
1. The extension of van Dantzig's construction and preliminary remarks

For the extension of van Dantzig's construction we consider groups of the form $G \rtimes A$, where $A$ is a discrete group and $G$ is a restricted product, indexed by a set $X$, of copies of a finite group $K$. The restricted product is defined as follows. We suppose $X$ to be partitioned into subsets $P$ and $S$ and define

$$G = \prod_{S,P} K = \prod_{X} K = \sum_{S} K \times \prod_{P} K.$$ 

The topology on $G$ is defined to be the product of the discrete topology on $\sum_{S} K$ and the product topology on $\prod_{P} K$. For $Y \subseteq X$, define $G_{\{Y\}} = \{ f \in G : f = e \text{ off } Y \}$; then $G_{\{Y\}}$ is compact if and only if $Y \setminus P$ is finite and $G_{\{Y\}}$ is open if and only if $P \setminus Y$ is finite. As $Y$ ranges over all sets of finite difference with $P$, the sets $G_{\{Y\}}$ form a base for the topology at $e \in G$. In a situation where the roles of the sets $P$ and $S$ are reversed, we will use the symbol \[\prod\] in place of \[\prod\].

The action of $A$ by automorphisms of $G$ is induced by an action of $A$ on $X$. For each bijection $\alpha : X \to X$ and $f \in G$, define $\alpha(f)$ by $\alpha(f)(x) = f(\alpha^{-1}(x))$. It follows from the description of the topology of $G$ above that $\alpha(f) \in G$ and $\alpha^{-1}(f) \in G$ if and only if the symmetric difference $P \Delta \alpha(P)$ is finite and in this case the map $f \mapsto \alpha(f)$ is a continuous automorphism of $G$. Hence, given an action of $A$ on $X$ such that $P \Delta \alpha(P)$ is finite for every $\alpha$ in $A$, there is an induced action of $A$ by automorphisms of $G$. The semidirect product $G \rtimes A$ is defined to be the set $G \times A$ equipped with the product topology and the multiplication $(g_1, \alpha_1)(g_2, \alpha_2) = (g_1\alpha_1(g_2), \alpha_1\alpha_2)$. It is a totally disconnected locally compact group. The identity element in $G$ will be denoted by $e$ and that in $A$ by $i$. The identity in $G \rtimes A$ then is $(e, i)$.

This construction is our extension of van Dantzig's example. The original example may be retrieved by taking $X = \mathbb{Z} = A$, $P = \mathbb{N}^+$ and $\alpha(n) : k \mapsto k - n$, $(n \in A, k \in X)$ in our construction. Note that: if $S$ is finite, then $\prod_{X} K$ is isomorphic to the compact group $\prod_{X} K$ and; if $P$ is finite, then $\prod_{X} K$ is isomorphic to the discrete group $\sum_{X} K$. In these cases we do not get any new types of totally disconnected groups and so we will usually consider cases in which both $P$ and $S$ are infinite.

The aim now is to analyse the examples $G \rtimes A$:

- to identify tidy subgroups for elements $x$ in these groups;
- to describe the scale function for these groups; and
- investigate how the tidy subgroups depend on $x$.

**Proposition 1.1.** Let $Y$ be a subset of $X$ such that $Y \Delta P$ is finite and let $(g, \alpha)$ belong to $G \rtimes A$. Then $G_{\{Y\}}$ is tidy for $(g, \alpha)$ if and only if it is tidy for $(e, \alpha)$. There may be subgroups tidy for $(e, \alpha)$ which are not tidy for $(g, \alpha)$.

**Proof.** It is immediate from the definitions that for any $g \in G$, $(g, \alpha)G_{\{Y\}}(g, \alpha)^{-1}$
Taking \( x = (g, \alpha) \) and \( U = G_{[\gamma]} \), it follows that \( U_+ \), \( U_- \), \( U_{++} \) and \( U_{--} \) are independent of \( g \). Hence \( G_{[\gamma]} \) satisfies T1 and T2 with \( x = (g, \alpha) \) if and only if T1 and T2 are satisfied with \( x = (e, \alpha) \).

Every subgroup is tidy for the identity \((e, i)\). We give an example of a compact open subgroup which is not tidy for \((g, i)\). For the example, let \( X \) be a single point, \( A \) be trivial and \( K = S_3 \) be the group of permutations of \([1, 2, 3]\). Let \( H = \{e, (12)\} \) and \( g = (123) \). Then \( gHg^{-1} \cap H = \{e\} \) and it follows that \( H \times \{i\} \) is not tidy for \((g, i)\). Although this example is discrete, even finite, it can be used as the basis of nondiscrete examples.

If the finite group \( K \) happens to be abelian, then any subgroup \( U \subset G \times \{i\} \) is tidy for \((g, \alpha)\) if and only if it is tidy for \((e, \alpha)\). We shall see that \((g, \alpha)\) always has tidy subgroups of the form \( G_{[\gamma]} \), from which follows the

**Corollary 1.2.** For each \((g, \alpha)\) in \( G \times A \) we have \( s((g, \alpha)) = s((e, \alpha)) \).

Proposition 1.1 shows that in order to identify some tidy subgroups for arbitrary elements of \( G \times A \) it suffices to identify tidy subgroups of the form \( G_{[\gamma]} \) for the elements \((e, \alpha)\). We may work inside \( G \) for this and consequently may simplify notation as follows: the compact open subgroup \( U \subset G \) will be said to be tidy for \( \alpha \) if \( U \times \{i\} \) is tidy for \((e, \alpha)\). Observe that the criteria for \( U \) to be tidy for \( x \) are stated in terms of the inner automorphism \( g \mapsto xgx^{-1} \) \((g \in G)\) and so \( U \) will be tidy for \( \alpha \) if and only if \( U \) satisfies T1 and T2 with \( U_{\pm} = \bigcap_{n \geq 0} \alpha^{\pm n}(U) \), \( U_{++} = \bigcup_{n \geq 0} \alpha^n(U_+) \) and \( U_{--} = \bigcup_{n \geq 0} \alpha^{-n}(U_-) \). Similarly, the scale of the automorphism \( \alpha \) will be the scale of \((e, \alpha)\), which is \( s(\alpha) = [\alpha(U_+) : U_+] \).

The identification of tidy subgroups for individual elements will be seen to reduce to van Dantzig’s example and so we begin with a complete description of this case. It is necessary only to identify the subgroups tidy for the single automorphism \( \alpha \) induced by the translation \( k \mapsto k - 1 \) of \( \mathbb{Z} \).

**Proposition 1.3.** Suppose \( U \) is a compact open subgroup of \( \prod_z K \). Then \( U \) is a tidy subgroup for \( \alpha \) if and only if

\[
U = \sum_{j < n} \{e\} \times U_0 \times \prod_{j \geq n} K,
\]

where \( m, n \in \mathbb{Z} \), \( n \geq m \) and \( U_0 \subseteq K^{n-m} \) is such that \( \{e\} \times U_0 \subseteq U_0 \times K \).

**Proof.** Since \( U \) is compact and open, there exist \( m, n \) such that \( \prod_{j \geq n} K \subseteq U \subseteq \prod_{j \geq m} K \). Put \( U_0 = U \cap \prod_{j = 0}^{n-1} K \subseteq K^{n-m} \). Note that

\[
U_\pm = \bigcap_{k \geq 0} \alpha^{-k}(U) \subseteq \bigcap_{k \geq 0} \alpha^{-k} \left( \prod_{j \geq m} K \right) = \bigcap_{k \geq 0} \bigcap_{j \geq m+k} K = \{e\}.
\]
If $U$ is tidy, then $U = U_+ \cup U_- = U_+ = \bigcap_{k \geq 0} \alpha^k(U)$, and so $U \subseteq \alpha(U)$. Consequently $\{e\} \times U_0 \subseteq U_0 \times K$. Conversely, if $\{e\} \times U_0 \subseteq U_0 \times K$ then $\alpha(U) \supseteq U$, so $U_+ = U$. Moreover, $\alpha^k(U_+) \supseteq \prod_{j \geq n-k} K$ so $\bigcup_{k \geq 0} \alpha^k(U_+) = \prod K$, which is closed, as is $\bigcup_{k \geq 0} \alpha^{-k}(U_-) = \{e\}$. Hence $U$ is tidy.

There are several ways in which such a subgroup $U_0 \subseteq K^{n-m}$ may arise.

(a) If $m = n$ then $U = \prod_{j \geq n} K = G_{[(n, \infty)]}$.
(b) If $\{K_i\}_z$ is an increasing sequence of subgroups of $K$ varying from $\{e\}$ to $K$, then $\prod K_i$ is a tidy subgroup for $\alpha$.
(c) If $\varphi : K \to K$ is a group homomorphism such that $\varphi^{n-m}$ is the trivial homomorphism, then

$$U_0 = \{\varphi^{n-m-1}(k), \varphi^{n-m-2}(k), \ldots, \varphi(k), k : k \in K\}$$

has the desired property.

The group $G \rtimes Z$ thus has subgroups, as in (a), which are tidy for every element of the group. That is not the case in general. In the next section we characterise tidy subgroups of individual elements in $G \rtimes A$ and then investigate global properties of particular examples in later sections.

2. Scale functions of automorphisms of $\prod_x K$

The purpose of the present section is to determine $s(\alpha)$ where $\alpha$ is an automorphism of $\prod_x K$ as considered in the introduction. Beyond this, we will see a characterisation of the compact open subgroups that are tidy for $\alpha$. There are few surprises here—this case reduces to a finite product of fundamental cases, including the groups of the type considered in Section 1.

Before proceeding, we have a lemma, whose proof follows directly from the definitions of ‘tidy’ and ‘scale function’.

**Lemma 2.1.** Suppose for $i = 1, 2$ that $G_i$ is a totally disconnected group and $\alpha_i \in \text{Aut}(G_i)$ has a tidy subgroup $U_i$. Put $G = G_1 \times G_2$, $U = U_1 \times U_2$, a compact open subgroup of $G$ and $\alpha = \alpha_1 \otimes \alpha_2 \in \text{Aut}(G)$. Then $U$ is a tidy subgroup for $\alpha$ and $s(\alpha) = s(\alpha_1)s(\alpha_2)$.

The action of $\{\alpha^n : n \in \mathbb{Z}\}$ on $X$ defines orbits $\mathcal{O}_i = \{\alpha^n(z_i)\}_{n \in \mathbb{Z}}$ for $i$ in some index set $I$. Define $\mathcal{O}_i^+ = \{\alpha^n(z_i) : n \geq 0\}$ and $\mathcal{O}_i^- = \mathcal{O}_i \setminus \mathcal{O}_i^+$. Each element of the finite set $P \Delta \alpha(P)$ can be written as either $\alpha^k(z_i) \in P$ with $\alpha^{k-1}(z_i) \notin P$ or vice-versa. Consequently, there are only finitely many places where orbits cross from $P$ into $S$ or from $S$ into $P$. For a single infinite orbit $\mathcal{O}_i$, this means that either $\alpha^k(z_i) \in P$ for all
sufficiently large $k$ or $\alpha^k(z_i) \in S$ for all sufficiently large $k$. Equivalently, precisely one of $\mathcal{O}_i^+ \setminus P$ or $\mathcal{O}_i^- \setminus S$ is finite. A similar dichotomy holds for $\mathcal{O}_i^-$. We partition $I$ into 6 parts: $I = I_P \cup I_S \cup I_{P,P} \cup I_{S,P} \cup I_{P,S} \cup I_{S,S}$ where:

(i) $I_P$ and $I_S$ are those $i$ for which $\mathcal{O}_i$ is finite, with $I_P$ being those for which $\mathcal{O}_i \subseteq P$ and $I_S$ being those for which $\mathcal{O}_i \cap S \neq \emptyset$, and

(ii) each $I_{Q,R}$ consists of those $i$ for which $\mathcal{O}_i$ is infinite and $\mathcal{O}_i^- \setminus Q$ and $\mathcal{O}_i^+ \setminus R$ are finite.

Again using the finiteness of $P \triangle \alpha(P)$, we see that there are only finitely many $i \in I_S$ with $\mathcal{O}_i \cap P$ nonempty, and likewise for $i \in I_{S,S}$ having $\mathcal{O}_i \cap P$ nonempty and $i \in I_{P,P}$ having $\mathcal{O}_i \cap S$ nonempty. Also the cardinalities $n_+ = |I_{P,S}|$ and $n_- = |I_{S,P}|$ are finite. Consequently,

$$P' = \bigcup_{i \in I_P} \mathcal{O}_i \cup \bigcup_{i \in I_{S,P}} \mathcal{O}_i \cup \bigcup_{i \in I_{S,P}} \mathcal{O}_i^+ \cup \bigcup_{i \in I_{S,P}} \mathcal{O}_i^-$$

has finite difference with $P$, making $G_{(|P|)}$ a compact open subgroup of $G$. We will use the following decomposition of $G$

$$G_{(|P|)} = G_P \times G_S \times G_{P,P} \times G_{S,P} \times G_{P,S} \times G_{S,S}$$

where each $G_{*,*}$ is the obvious factor. This type of subscripting will also be used to denote sets in a partition of $X$ consisting of the union of the corresponding orbits. This gives us, for instance, $X_{P,P} = \bigcup_{i \in I_{P,P}} \mathcal{O}_i$ and $G_{P,P} = G_{|X_{P,P}|}$.

**Theorem 2.2.** With notation as above, $G_{(|P|)}$ is a tidy subgroup for $\alpha$, $s(\alpha) = |K|^{n_+}$ and $s(\alpha^{-1}) = |K|^{n_-}$.

**Proof.** Clearly, $G_{(|P|)} = G_P \times \{e\} \times G_{P,P} \times \prod_{i \in I_{S,P}} G_{|\mathcal{O}_i^+|} \times \prod_{i \in I_{S,S}} G_{|\mathcal{O}_i^-|} \times \{e\}$, corresponding to the decomposition of $G$ above. Since the first three and the last factors of $G_{(|P|)}$ are $\alpha$-invariant, they are tidy for the corresponding restriction $\alpha_{*,*}$ with $s(\alpha_{*,*}) = s(\alpha_{*,*}^{-1}) = 1$. Next note that for each $i \in I_{P,S}$, the action of $\alpha$ on the invariant subgroup $G_{|\mathcal{O}_i|}$ is a shift as in Section 1. Consequently $G_{|\mathcal{O}_i|}$ is tidy for $\alpha_i$, the restriction of $\alpha$ to $G_{|\mathcal{O}_i|}$, with $s(\alpha_i) = |K|$ and $s(\alpha_i^{-1}) = 1$. The situation for $i \in I_{S,P}$ is similar, but with $s(\alpha_i) = 1$ and $s(\alpha_i^{-1}) = |K|$. The result now follows from Lemma 2.1. \(\square\)

To simplify what follows, we will assume for the rest of this section that $P = P'$. This does not cause any loss of generality, $P$ and $P'$ have finite difference, and so define
the same totally disconnected locally compact group $G$. Moreover, the partitioning of the orbits and resulting decomposition of $G$ are identical.

The subgroups $U$ of $G$ that are tidy for $\alpha$ can be characterised using a similar strategy to the above, and again this reduces to a product of subgroups of the type considered in Section 1. However, the decomposition of $G$ on which the structure of $U$ is based need not be as fine as that above. To obtain an appropriate decomposition, we reindex $G_{s,p}$ and $G_{p,s}$ to each be a single partial direct product. For instance

$$G_{s,p} = \prod_{i \in \mathbb{I}_{s,p}} K \cong \prod_{\mathbb{I}} K = \prod_{\mathbb{I}} K_n,$$

on which $\alpha$ acts by translation. The tilde $\sim$ will be used to denote the isomorphisms $G_{s,p} \to \prod_{\mathbb{I}} K_n$ and $G_{p,s} \to \prod_{\mathbb{I}} K_n$, as appropriate.

Much of the analysis of a tidy subgroup $U$ relies on establishing relationships between the orbits $\mathcal{O}_i$ and the support set of $U$. In the following, $Q$ is this support set, and $R$ is the largest subset of $X$ such that $G[R] \subseteq U$. Then since $G[R] \subseteq U \subseteq G[Q]$, with $U$ being compact and open in the product topology, the sets $P$, $Q$ and $R$ differ in only finitely many points.

**Lemma 2.3.** With $U$ a tidy subgroup with support set $Q$ as above,

(i) finite orbits are either totally contained in $Q$ or disjoint from $Q$,

(ii) $Q \cap X_{S,S} = \emptyset$,

(iii) $\alpha^k(Q \cap X_{S,P}) \subseteq Q$ for all $k \geq 0$, and

(iv) $\alpha^{-k}(Q \cap X_{P,S}) \subseteq Q$ for all $k \geq 0$.

**Proof.** The support sets of $U_+$ and $U_-$ are subsets of $\bigcap_{n=0}^{\infty} \alpha^n(Q)$ and $\bigcap_{n=0}^{\infty} \alpha^{-n}(Q)$ respectively. Since $U = U_+ U_-$, we have that

$$Q = \bigcap_{n=0}^{\infty} \alpha^n(Q) \cup \bigcap_{n=0}^{\infty} \alpha^{-n}(Q).$$

Each of the conclusions follows immediately. □

An immediate consequence of this is that any tidy subgroup $U$ is contained within $G_p \times G_s \times G_{p,p} \times G_{s,p} \times G_{p,s}$. With $G_{p,p}$ we can actually do better, and obtain $G_{p,p}$ as a factor of $U$, and consequently of $U_+$ and $U_-$.

**Lemma 2.4.** $X_{p,p}$ is a subset of $R$.

**Proof.** By definition of the topology on $G$, $Z = X_{p,p} \setminus R$ is finite. Put $Y = X_{p,p} \setminus (\bigcup_{k \geq 0} \alpha^k(Z))$. Then $G[Y] < U_+$ and so $\bigcup_{k \geq 0} G[\alpha^k(Y)] < U_{++}$. Since $\alpha$ is just a shift on each orbit $\mathcal{O}_i \subset X_{p,p}$, $\bigcup_{k \geq 0} \alpha^k(Y) = X_{p,p}$. Hence $\bigcup_{k \geq 0} G[\alpha^k(Y)]$ is dense in $G[X_{p,p}]$. Since $U_{++}$ is closed, we have $G[X_{p,p}] < U_{++}$. 

However, $G_{\{x_{r}, p\}}$ is compact and so there is $m$ such that $\alpha^{-m}(G_{\{x_{r}, p\}}) < U_{+} < U$. Since $G_{\{x_{r}, p\}}$ is invariant under $\alpha$, it follows that $G_{\{x_{r}, p\}} < U$.

To complete the classification of tidy subgroups, we need to consider the finite orbits and those orbits passing from $P$ to $S$ or vice-versa.

**Theorem 2.5.** A subgroup $U$ of $G$ is tidy for $\alpha$ if and only if $U = U_{0} \times G_{P, P} \times U_{S, P} \times U_{P, S} \times \{e\}$ where $U_{0}$ is a compact open subgroup of $G_{P} \times G_{S}$ invariant under $\alpha$ and $\alpha^{-1}$, $U_{S, P}$ is supported on $X_{S, P}$ and satisfies $\alpha(U_{S, P}) < U_{S, P}$ and $U_{P, S}$ is supported on $X_{P, S}$ and satisfies $\alpha(U_{P, S}) > U_{P, S}$.

**Proof.** By Lemma 2.1, the stated conditions are sufficient for $U$ to be tidy for $\alpha$. Conversely, suppose $U$ is tidy for $\alpha$. Put

$U_{0} = \{g \in G_{P} \times G_{S} : \exists h \in U \text{ such that the restriction of } h \text{ to } X_{P} \times X_{S} \text{ equals } g\}$,

$U_{S, P} = \{g \in G_{S, P} : \exists h \in U \text{ such that the restriction of } h \text{ to } X_{S, P} \text{ equals } g\}$

and

$U_{P, S} = \{g \in G_{P, S} : \exists h \in U \text{ such that the restriction of } h \text{ to } X_{P, S} \text{ equals } g\}$.

Since Lemma 2.4 shows that $G_{P, P} < U$, to prove the Theorem it will suffice to show that each of $U_{S, P}$ and $U_{P, S}$ is a subgroup of $U$ and that $U_{0}$ is a subgroup of $U_{+} \cap U_{-}$.

To show this for $U_{0}$, let $g$ be in $U_{0}$ and choose $h \in U$ whose restriction to $X_{P} \times X_{S}$ equals $g$. Factor $h$ as $h^{+}h^{-}$ where $h^{+} \in U_{+}$ and $h^{-} \in U_{-}$. Since $G_{P, P} < U$, it may be supposed that $h$ equals $e$ on $X_{P, P}$ and, by Lemma 2.3, that $h^{+}$ and $h^{-}$ equal $e$ on $X_{P, P} \cup X_{S, P}$ and $X_{P, P} \cup X_{P, S}$ respectively. Let $g^{+} \in U_{0}$ agree with $h^{+}$ on $X_{S} \cup X_{P}$. Then $h^{+} = g^{+}f^{+}$, where $f^{+} \in G_{P, P}$. Now $\alpha^{-n}(f^{+})$ converges to the identity as $n \to \infty$ and, since $g^{+}$ is supported on finite $\alpha$-orbits, $\alpha^{-n}(g^{+}) \to g^{+}$ as $n \to \infty$. It follows that $g^{+} = \lim_{n \to \infty} \alpha^{-n}(h^{+})$, which belongs to $U_{+} \cap U_{-}$. Similarly, $g_{-} = \lim_{n \to \infty} \alpha^{n}(h_{-})$ belongs to $U_{+} \cap U_{-}$. Therefore $g = g^{+}g^{-}$ belongs to $U_{+} \cap U_{-}$.

Next let $g$ be in $U_{S, P}$ and choose $h \in U$ whose restriction to $X_{S, P}$ equals $g$. Factor $h$ as $h^{+}h^{-}$ where $h^{+} \in U_{+}$ and $h^{-} \in U_{-}$. Then it may be supposed that $h$, $h^{+}$ and $h^{-}$ equal $e$ on $X_{P} \cup X_{S} \cup X_{P, P}$. Now $h^{+}$ also equals $e$ on $X_{S, P}$ and $h_{-}$ does on $X_{P, S}$. Hence we have $g = h_{-}$ and thus belongs to $U_{-} < U$. Similarly, $U_{P, S} < U_{+} < U$.

The structure of the subgroups $U_{S, P}$ and $U_{P, S}$ may be described more explicitly with the aid of Proposition 1.3 since $\tilde{U}_{S, P} \subseteq \prod_{K^{n}} G^{n}$ and $\tilde{U}_{P, S} \subseteq \prod_{K^{n}} G^{n}$ are tidy under translation.

3. Groups acting on graphs

Now suppose that we have a finitely-generated group $G = \langle a_{1}, \ldots, a_{n} \rangle$, and a right $G$-set $X$. Let $\Gamma$ be the Cayley graph of the action of $G$ on $X$, so that vertices
x, y ∈ X are adjacent when there is some k with xa_k = y or xa_k^{-1} = y. In this section we consider partial direct sums as previously, where the index set is a graph and the mapping α is given as right-translation by an element a ∈ G. Note that this is not an automorphism of the graph G. We find that the possibilities for the set P in the construction of the partial direct sum can be related to the structure of the graph Γ.

In a graph Γ with vertex set X, we define:

(i) a path to be a 1-1 mapping ξ : N → X such that consecutive terms are adjacent vertices in Γ,

(ii) two vertices x and y to be in the same component of Γ if there is a sequence of vertices x_0, x_1, ..., x_n with x_0 = x and x_n = y and such that x_{k-1} is adjacent to x_k for k = 1, ..., n,

(iii) the components of Γ to be the equivalence classes under the equivalence relation on vertices of being in the same component,

(iv) two paths ξ and ζ to be disconnected by a set Y ⊆ X when there exist arbitrarily large i, j such that ξ_i and ζ_j lie in different components of Γ \ Y,

(v) two paths to be equivalent, ξ ~ ζ, when there is no finite set disconnecting them, and

(vi) an end of Γ to be an equivalence class of paths in Γ.

The set of all ends is denoted Ω and the set X = X ∪ Ω can be endowed with a natural topology so that X is dense. If X is a connected graph, which occurs if G acts transitively, then X is compact. In this topology, points x ∈ X are isolated and for a finite set Y ⊆ X, the closure of a component Z of Γ \ Y includes precisely those ends whose paths eventually lie in Z. Then a base of neighbourhoods of ω ∈ Ω can be taken to be \( \{ Z_Y : Y \text{ finite} \} \), where Z_Y is the component of Γ \ Y in which the paths of ω eventually lie. It can be shown that X is metrisable—see for example [17, equation (2.1)] or [3, Proposition IV.6.7]. For a Cayley graph, it follows immediately from the definition that each vertex x is only a finite distance from its right translate xa. Hence, if \( \{ x_i \} \) is a sequence converging to ω ∈ Ω, then the sequence \( \{ xa \} \) (which need not be a path) will also converge to ω.

**Proposition 3.1.** If P ⊆ X then P \ Pa is finite for all a ∈ G if and only if P is an open set in X.

**Proof.** Suppose P \ Pa is finite for all a ∈ G, then so is P \ Pa = (P \ Pa^{-1})a. Put F = \( \bigcup a \) (Pa_k ∪ Pa_k^{-1}) \ P, a finite set with P ⊆ X \ F. Now, any y ∈ X connected by an edge to some x ∈ P is either in P or in F. It follows that each component of X \ F is either wholly contained in P or does not meet P and so P is composed of components. Hence P is open.

Now suppose P \ Pa is infinite for some a ∈ G. Let ω ∈ Ω be a limit point of P \ Pa, so that ω ∈ P \ X. If \( \{ x_i \} \) is a sequence converging on ω, then
\(\{x_i a^{-1}\}_i^\infty \subseteq P a^{-1} \setminus P\) is a sequence also converging on \(\omega\). However, \(\{x_i a^{-1}\}_i^\infty \subseteq X \setminus P\), and so \(\omega\) is not an interior point of \(\overline{P}\). Hence \(\overline{P}\) is not open. \(\square\)

In the situation where \(\Gamma = G\) and \(\alpha = \rho_x\), translation by an element \(x \in G\), the structure of the space of ends of \(G\) can yield information on the scale function.

**Proposition 3.2.** Suppose \(H\) is a subgroup of \(G\) such that \(H\) has a single end. Then \(s(\rho_x) = 1\) for all \(x \in H\).

**Proof.** Let \(y \in G\) be such that \(\mathcal{O}_i = \{yx^n\}_0^\infty\) is infinite. We show that \(\mathcal{O}_i\) is an orbit with \(i \in \mathcal{I}_{P,P}\) or \(i \in \mathcal{I}_{S,S}\), by the classification scheme in Section 2.

Since \(H\) has only one end, so does the coset \(yH\), say \(\omega \in \Omega\). Supposing \(\omega \in \overline{P}\), a closed and open set in \(G \cup \Omega\), we have that both sequences \(\{yx^n\}_0^\infty\) and \(\{yx^{-n}\}_0^\infty\) lie in \(P\) after a finite number of terms. Consequently \(i \in \mathcal{I}_{P,P}\). A similar argument gives \(i \in \mathcal{I}_{S,S}\) in the case when \(\omega \notin \overline{P}\). \(\square\)

Now a finitely generated infinite group \(G\) has either one, two or infinitely many ends, see [3, Theorem IV.6.10]. If \(G\) has two ends, then it has an infinite cyclic subgroup of finite index, see [3, Theorem IV.6.12], and the scale function and tidy subgroups may be computed using the techniques of Section 1. If \(G\) has infinitely many ends, then it is essentially a non-trivial free product, see [3, Theorem IV.6.10]. The ends of groups were studied in [4] and crucial steps towards the results on the number of ends taken in [8, 12, 1].

### 4. Scale functions on free groups and free products

We now consider several examples where \(A = X\) is a group, with the action being right multiplication, that is, the action of \(x\) in \(A\) is given by \(y \mapsto yx^{-1}\). With this action we have that \((e, x)G_{[y]}(e, x)^{-1} = G_{[yx^{-1}]}\), \((x \in A, Y \subset X)\). Typically, \(P\) will consist of a finite number of branches of the Cayley graph of \(A\).

In all these examples, the group \(A\) is a free group or a free product of groups. In the case of free groups, elements of the group will be written as words in the generators and their inverses, and we will assume that the words are reduced—that is, of minimal length. For two words \(u\) and \(x\), with \(u\) non-empty, we will say the count of \(u\) in \(x\) is the number of times \(u\) appears as a subword of \(x\). As we are interested in the orbits \(\{yx^n\}\), we will also need to know the asymptotic behaviours of the count of a word \(u\) in the reduced word of \(yx^n\). For this we define a cyclic reduction of an element \(x\), this being a word \(w\) of shortest length in the conjugacy class of \(x\). Note that the cyclic reduction of a word is defined only up to cycling of the letters, for example \(abc\) and \(bca = a^{-1}abca\) are both cyclic reductions of \(abc\). Then the cyclic count of a word
u in x is the number of cyclic occurrences of u in the cyclic reduction w of x—this includes normal instances of u as a subword of w and those instances of u that wrap around from the end to the beginning of w. This means that the cyclic count is the number c such that for all sufficiently large n, the difference between the count of u in x^n and the count of u in x^{n+1} is c. If S ⊆ H, then the total count of S in x is the sum of the counts of all elements u ∈ S in the word x and the total cyclic count is defined similarly.

For instance, if a, b, c are generators of \( \mathbb{F}_3 \), then

(i) a cyclic reduction of \( w = abcbabca^2bac^{-1}b^{-1}a^{-1} \) is \( babca^2ba \) and the cyclic count of ab in w is 3;
(ii) the cyclic count of ababab in ab is 1; and
(iii) the total cyclic count of \( \{ab\}^\infty \) in ab is infinite.

In the cases dealt with below, the total cyclic count is finite. The computation of \( s(x) \) in terms of a cyclic count begins by assuming that x is cyclically reduced. This does not affect the value of the scale function, as it is invariant under conjugation.

In the case of the free product \( B \ast C \), a non-empty reduced word consists of a word with symbols alternating between \( B \setminus \{e\} \) and \( C \setminus \{e\} \) and the length of such a word will be the number of symbols. The count of a word u of length one is not well-defined because B and C need not be free but for u with length at least two the number of occurrences of u in x can be counted. A cyclically-reduced word will either have length one or have even length because any word \( b_1c_2\cdots b_nc_{n+1} \) of length \( 2n+1, n > 1 \), may be cyclically reduced to the word \( b_1c_2\cdots b_nc_{n+1}c_1 \) of length \( 2n \).

The first example to be discussed is the mixed case where A is the free product \( B \ast C \), with B a free group and C a discrete group. In this case the letters in reduced words will be generators of \( B \) or their inverses and elements of \( C \). ‘Length’, ‘count’ and so on are defined accordingly.

**Example 4.1.** Take \( B = \{b^n\} \cong \mathbb{Z} \) and C any discrete group. Let \( A = X = \mathbb{Z} \) and let \( P = \{y \in X : \text{ the reduced word for } y \text{ is of the form } bw\} \). Let x be in A and write \( x = zwz^{-1} \) where w is cyclically reduced. Then:

(i) \( \log_{|K|} s(x) \) equals the cyclic count of b in x; and
(ii) a tidy subgroup for x is \( G_{|Pz^{-1}|} \).

**Proof.** (i) Suppose to begin with that x is cyclically reduced. For \( x = b \), the only orbit passing from P to S is \( \{b^n\} \), and so \( s(b) = |K| \). Consequently for \( x = b^n \), \( s(b^n) = |K|^n \) if \( n > 0 \) and \( s(b^n) = 1 \) if \( n \leq 0 \).

For other cyclically reduced words we can suppose that \( x = b^{q_1}c_1b^{q_2}c_2\cdots b^{q_k}c_k \). The cyclic count of b in x is \( \sum_{q_i > 0} q_i \). Then \( y \in P \) satisfies \( xy^{-1} \notin P \) if and only if \( y = b^ic_1b^{q_i+1}c_{j+1}\cdots b^{q_k}c_k \) where \( 1 \leq j \leq k \) and \( 1 \leq i \leq q_j \). The value of \( \log_{|K|} s(x) \) is at most the number of such \( y \), which is \( \sum_{q_i > 0} q_i \). Since x is cyclically reduced we
have for such \( y \) that \( yx^{-n} \in S \) when \( n \) is positive and \( yx^{-n} \in P \) when \( n \) is negative. Hence \( \log_{g(x)} s(x) \) is exactly \( \sum_{q_i > 0} q_i \) as required. The result now follows for general \( x \) because the scale function is invariant under conjugation.

(ii) It was shown in the first part that, if \( x \) is cyclically reduced, then all \( x \)-orbits enter or leave \( P \) exactly once or not at all. Hence \( G_{[P]} \) is tidy for \( x \) when \( x \) is cyclically reduced. If \( x = zwz^{-1} \), where \( w \) is cyclically reduced, then \( (e, z)G_{[P]}(e, z)^{-1} = G_{[Pz^{-1}]} \) is tidy for \( x \). \( \square \)

It has been seen in other examples that tidy subgroups are a type of normal form for elements of totally disconnected locally compact groups. In [15] it was seen that, in automorphism groups of trees, identifying tidy subgroups for an automorphism \( x \) corresponds to identifying the unique path in the tree such that \( x \) is a translation along the path. In [5] it was seen that, in \( p \)-adic Lie groups, identifying tidy subgroups for \( x \) corresponds to finding the Jordan canonical form for the adjoint representation of \( x \) on the Lie algebra. In the present case we see that identifying tidy subgroups corresponds to finding the cyclically reduced form for a word \( x \).

Another characterisation of the scale function and tidy subgroups is given in [14]. It is shown that for any totally disconnected locally compact group \( G \)

\[
s(x) = \min \{ [x U x^{-1} : U \cap x U x^{-1}] : U \text{ is a compact open subgroup of } G \} \quad (x \in G)
\]

and the minimum is attained at precisely those compact open subgroups which are tidy for \( x \). In view of this characterisation, and of the fact that the scale function is invariant under conjugation, it is not surprising that identifying tidy subgroups in this example involves minimising the cyclic word length.

The proof of (i) shows that all cyclically reduced elements of \( A \) have \( G_{[P]} \) as a tidy subgroup. However, as we shall see, there is no subgroup of \( G \) which is tidy for every element of \( A \). For a pair of elements \( x_1, x_2 \) in a totally disconnected locally compact group define

\[
d(x_1, x_2) = \min \{ [U_1 : U \cap U_2] [U_2 : U \cap U_2] : U_i \text{ tidy for } x_i, \ i = 1, 2. \}
\]

Then \( d(x_1, x_2) \) is a measure of how far \( x_1 \) and \( x_2 \) are from having a common tidy subgroup.

The computation of this value involves the cyclic reduction of pairs of elements of \( B \ast C \). Among the conjugates of a pair \( (x_1, x_2) \) are pairs \( (w_1, zw_2z^{-1}) \) where \( w_1 \) and \( w_2 \) are cyclically reduced and \( z \) has minimum length. Define a cyclic reduction of \( (x_1, x_2) \) to be such a pair. The element \( z \) will be said to be a comparator of \( x_1 \) and \( x_2 \).

Some properties of the comparator and notation will be required in the following discussion.

Suppose that the comparator \( z = b^{q_1} c_1 b^{q_2} c_2 \cdots b^{q_k} c_k \) is a reduced word, where we may have \( q_1 = 0 \) or \( c_k = 1 \) but \( q_j \neq 0 \) and \( c_j \neq 1 \) otherwise. There are further
restrictions on $z$ depending on $w_1$ and $w_2$. The possibilities and restrictions for cyclically reduced $w_2$ are:

1. $w_2 = b^*$, in which case $c_k \neq i$ (otherwise $b^{q_i}$ can be commuted past $b^*$ and the length of $z$ reduced);
2. $w_2 \in C$, in which case $c_k = i$ (otherwise $c_k$ and $c_k^{-1}$ can be absorbed into $w_2$ and the length of $z$ reduced); and
3. $w_2 = b'c \cdots c'b^*$, where $r$ and $s$ are not both 0 and have the same sign, in which case either: (a) $c_k \neq i$, or (b) $c_k = i$ and $s = 0$ if $q_k$ has the same sign as $r$ and $r = 0$ if $q_k$ has the opposite sign as $s$.

In all cases except (3a) when either $r$ or $s$ is zero, $zw_2z^{-1}$ is in reduced form for all non-zero $n$. In case (3a) with $r = 0$, $c_k$ and $d_k$ can be combined to reduce the length of $zw_2z^{-1}$ by 1. Note however that $c_k = i$ because then $w_2$ could be replaced with a cyclically equivalent word and the length of $z$ reduced. Thus in this case there is a contraction in $zw_2z^{-1}$ but no cancellation. No further reduction of $zw_2z^{-1}$ can be made and similarly if $s = 0$. Thus in all cases there is no cancellation possible in $zw_2z^{-1}$. It may be seen in the same way that there is no cancellation in $zw_2z^{-1}$ for any $n \neq 0$.

(iii) Let $x_1, x_2 \in A$. Then $\log_{|K_1|} d(x_1, x_2)$ equals the total count of $\{b, b^{-1}\}$ in a comparator of $x_1$ and $x_2$.

**Proof.** The function $d$ is conjugation invariant and so we have that $d(x_1, x_2) = d(w_1, zw_2z^{-1})$ where $(w_1, zw_2z^{-1})$ is a cyclic reduction of $(x_1, x_2)$. We compute this latter value.

Since $w_1$ and $w_2$ are cyclically reduced, $G_{|P|}$ is tidy for $w_1$ and $G_{|P_z^{-1}|}$ is tidy for $zw_2z^{-1}$. Denote these groups as $U_1$ and $U_2$ respectively. Then

$$\log_{|K_1|}[U_1 : U_1 \cap U_2] = #(P \setminus Pz^{-1}) \quad \text{and} \quad \log_{|K_1|}[U_2 : U_1 \cap U_2] = #(Pz^{-1} \setminus P).$$

An element $y \in P$ satisfies $yz^{-1} \notin P$ if and only if $y = b^i c_j b^{q_{j+1}} c_{j+1} \cdots b^{q_k} c_k$ where $1 \leq j \leq k$ and $1 \leq i \leq q_j$. Hence $#(Pz^{-1} \setminus P)$ equals $\sum_{q_j > 0} q_j$, which is the count of $b$ in $z$. Now $#(P \setminus Pz^{-1}) = #(Pz \setminus P)$ which, by the same argument, equals the count of $b$ in $z^{-1}$. This is just the count of $b^{-1}$ in $z$ and so $\log_{|K_1|} d(w_1, zw_2z^{-1})$ is at most the total count of $\{b, b^{-1}\}$ in $z$.

Now let $U_1$ be any subgroup tidy for $w_1$ and $U_2$ be any subgroup tidy for $zw_2z^{-1}$. Suppose at first that both $w_1$ and $w_2$ have infinite order. Let $y \in P$ be such that $yz^{-1} \notin P$. Then $y = b^i c_j b^{q_{j+1}} c_{j+1} \cdots b^{q_k} c_k$ as above, so that $y$ is a right subword of $z$. Since there is no cancellation in $zw_2z^{-1}$ for any non-zero $n$, it follows that there is no cancellation in $yw_2^n$ for any $n$ and hence that $yw_2^n \in P$ for every $n$. It follows that $yz^{-1}(zw_2z^{-1})^n \in Pz^{-1}$ for every $n$. Since this orbit is infinite, we have $G_{|yz^{-1}|} \subset U_2$. Similarly, $yz^{-1} = b^{-q_j} c_{j-1} b^{-q_{j-1}} \cdots c_1^{-1} b^{-q_1}$ is a right subword of $z^{-1}$. Since there
is no cancellation in \( z^{-1}w_i^n z \) for \( n \neq 0 \), \( yz^{-1}w_i^n \not\in P \) for every \( n \). Since this orbit is infinite, we have \( G_{(yz^{-1})} \cap U_1 = \{e\} \). We have shown then that \( G_{(Pz^{-1}\setminus P)} \cap U_1 = \{e\} \)
and \( G_{(Pz^{-1}\setminus P)} \subset U_2 \) and it follows that \( \log_{|K|}[U_2 : U_1 \cap U_2] \geq \#(Pz^{-1} \setminus P) \). It may be shown that in a similar way that \( \log_{|K|}[U_1 : U_1 \cap U_2] \geq \#(P \setminus Pz^{-1}) \). Therefore \( \log_{|K|} d(w_1, zw_2z^{-1}) \) is at least the total count of \( \{b, b^{-1}\} \) in \( z \) when \( w_1 \) and \( w_2 \) have infinite order.

The remaining case is when either \( w_1 \) or \( w_2 \) has finite order. Suppose, without loss of generality, that \( w_1 \) has finite order. Then, by [9, Corollary 4.1.4], \( w_1 \) is conjugate to an element of \( C \) and so, since \( w_1 \) is cyclically reduced, it follows that in fact \( w_1 \) belongs to \( C \). The above discussion of the comparator shows that \( q_i \neq 0 \) in this case. The \( w_1 \)-orbits are finite and \( U_1 \) is an infinite product of finite groups invariant under \( w_1 \).

As a first subcase, consider when \( w_2 \) has infinite order. Then, as seen above, \( G_{(Pz^{-1}\setminus P)} \subset U_2 \) and \( G_{(Pz^{-1}\setminus P)} \cap U_2 = \{e\} \). It cannot be shown that \( G_{(Pz^{-1}\setminus P)} \cap U_1 \) must be the trivial subgroup but we can show that this intersection may be \textit{assumed} to be trivial without increasing \( [U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] \). To this end, let \( yz^{-1} \) be in \( Pz^{-1} \setminus P \) and note that the equation we are trying to prove holds if \( w_1 \) is the identity, so that we may suppose that \( (w_1) \) has at least two elements. Then, for \( w \in (w_1) \setminus \{t\} \), the element \( yz^{-1}wzw_2^n \) has no cancellations because \( q_i \neq 0 \), \( w \in C \) and \( zw_2 \) has no cancellations. Hence \( yz^{-1}wzw_2^n \) is not in \( P \), because \( yz^{-1} \) isn’t, and it follows that \( yz^{-1}w(zw_2z^{-1})^n \) is not in \( Pz^{-1} \) for any \( n \). Since this orbit is infinite, it follows that \( G_{(yz^{-1}w)} \cap U_2 = \{e\} \). Denote \( U_1 \cap G_{(yz^{-1}w_1)} \) and \( U_2 \cap G_{(yz^{-1}w_1)} \) by \( U'_1 \) and \( U'_2 \) respectively. Then we have seen that \( U_1 \) is the product over the orbit of copies of some subgroup, \( L \) say, of \( K \) and \( U'_2 = G_{(yz^{-1}w_1)} \). Hence \( [U'_1 : U'_1 \cap U'_2] \geq |L| \) and \( [U'_2 : U'_1 \cap U'_2] = |K|/|L| \). It follows that if \( U_1 \) is now replaced by the group which is trivial on \( yz^{-1}(w_1) \), and agrees with \( U_1 \) elsewhere, then the new group is also tidy for \( w_1 \) and \( [U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] \) is not increased. Doing this for each \( yz^{-1} \in Pz^{-1} \setminus P \) we arrive at a group \( U_1 \) which is tidy for \( w_1 \) and satisfies \( G_{(Pz^{-1}\setminus P)} \cap U_1 = \{e\} \) without increasing \( [U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] \). Similarly, it may be shown that \( U_1 \) may be assumed to satisfy \( G_{(Pz^{-1}\setminus P)} \subset U_1 \) without increasing \( [U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] \). For this, show that for each \( y \in P \setminus Pz^{-1} \) and \( w \in (w_1) \setminus \{t\} \) we have \( yw(zw_2z^{-1})^n \in Pz^{-1} \) for every \( n \), from which it follows that \( U''_1 = U_2 \cap G_{(yw_1)} = G_{(yw_1)(\{y\})} \). Hence \( [U''_1 : U''_1 \cap U''_2] = |L| \) and \( [U''_2 : U''_1 \cap U''_2] \geq |K|/|L| \), where \( U''_1 = U_1 \cap G_{(yw_1)} \) is the product of copies of \( L \). Then \( U_1 \) may be replaced by a group whose intersection with \( G_{(yw_1)} \) is the product of copies of \( K \) without increasing \( [U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] \). Therefore, \( \log_{|K|} d(w_1, zw_2z^{-1}) \) is at least the total count of \( \{b, b^{-1}\} \) when \( w_2 \) has infinite order.

The second subcase is when \( w_1 \) and \( w_2 \) both have finite order. It may be \textit{assumed} that \( G_{(Pz^{-1}\setminus P)} \cap U_1 = \{e\} \), \( G_{(Pz^{-1}\setminus P)} \subset U_1 \), \( G_{(Pz^{-1}\setminus P)} \subset U_2 \) and \( G_{(Pz^{-1}\setminus P)} \cap U_2 = \{e\} \) and this suffices to show that \( \log_{|K|} d(w_1, zw_2z^{-1}) \) is at least the total count of \( \{b, b^{-1}\} \).
To see, for example, that we may assume that \( G_{P \setminus Pz^{-1}} \subset U_1 \), let \( y \in P \setminus Pz^{-1} \), let \( L_1 = U_1 \cap G_{[y]} \) and \( L_2 = U_2 \cap G_{[y]} \). Then for each \( u \in \langle w_1 \rangle \setminus \{i\} \) we have \( yu \in P \) because \( w \in C \), so that there is no cancellation. Furthermore, \( q_i \neq 0 \) because \( w_1 \in C \) and so there is no cancellation in \( yu \in z \). It follows that \( yu \in Pz^{-1} \) as well. Similar arguments show that

\[
S_y = \{yu_1v_1 \cdots u_j : u_j \in \langle w_1 \rangle \setminus \{i\}, v_j \in \langle zw_2z^{-1} \rangle \setminus \{i\}\}
\]

is contained in \( P \cap Pz^{-1} \). By construction, \( S_y \) is partitioned into \( \langle zw_2z^{-1} \rangle \)-cosets and \( \{y\} \cup S_y \) into \( w_1 \)-cosets. It may be shown that, if \( U_1' = U_1 \cap G_{[y] \cup S_y} \) and \( U_2' = U_2 \cap G_{[y] \cup S_y} \), then \([U_1' : U_1' \cap U_2'] [U_2' : U_1' \cap U_2'] \geq |K'|/|L_2|\). Replacing \( U_1 \) by the group which agrees with \( G_{[y] \cup S_y} \) on \( \{y\} \cup S_y \) and with \( U_1 \) elsewhere, and \( U_2 \) by the group which agrees with \( G_{S_y} \) on \( S_y \) and with \( U_2 \) elsewhere, the new groups are still tidy for \( w_1 \) and \( zw_2z^{-1} \) respectively and we now have \([U_1' : U_1' \cap U_2'] [U_2' : U_1' \cap U_2'] = |K'|/|L_2|\). Repeating for each \( y \in P \setminus Pz^{-1} \), we have \( G_{P \setminus Pz^{-1}} \subset U_1 \) and \([U_1 : U_1 \cap U_2] [U_2 : U_1 \cap U_2] \) has not been increased. Similar arguments show that the other assumptions may also be made and so \( \log_{|K|} d(w_1, zw_2z^{-1}) \) is at least the total count of \( \{b, b^{-1}\} \) when \( w_1 \) and \( w_2 \) both have finite order.

It follows from the above discussion that if \( x \) has infinite order and is in \( C \), then \( x \) normalises a unique compact open subgroup of \( G \).

It may seem that this example is rather special but in fact the same discussion applies whenever we take \( P = \{y \in X : \text{the reduced word for } y \text{ has the form } sw\} \) where \( s \) is a reduced word ending in \( b \), because, if \( z \) is any reduced word, then \#(\( P \triangle Pz^{-1} \)) equals the total count of \( \{b, b^{-1}\} \) in \( z \). We now give some examples where \( P \) has several branches.

**Example 4.2.** Let \( A = X = \mathbb{F}_k = \langle a_1, \ldots, a_k \rangle \) and let \( P \) be those \( w \in \mathbb{F}_k \) whose representations as reduced words begin with one of \( a_1, a_2, \ldots, a_k \).

(i) \( \log_{|K|} s(x) = (\text{total cyclic count of } \{a_i : 1 \leq i \leq k\} \text{ in } x) - (\text{total cyclic count of } \{a_j^{-1} a_i : 1 \leq i \neq j \leq k\} \text{ in } x) \).

(ii) Let \( x = zwz^{-1} \) where \( w \) is the cyclic reduction for \( x \). If \( w \in P \cap Pz^{-1} \), then \( G_{((P \cup \{i\}) \setminus Pz^{-1})} \) is tidy for \( x \) and \( G_{Pz^{-1}} \) is not. If \( w \in P \cup Pz^{-1} \), then \( G_{Pz^{-1}} \) and \( G_{((P \cup \{i\}) \setminus Pz^{-1})} \) are both tidy for \( x \). If \( w \notin P \cup Pz^{-1} \), then \( G_{Pz^{-1}} \) is tidy for \( x \) and \( G_{((P \cup \{i\}) \setminus Pz^{-1})} \) is not.

(iii) Let \( x_1 \) and \( x_2 \) be in \( A \), and let \( (w_1, zw_2z^{-1}) \) be the cyclic reduction of the pair \((x_1, x_2)\). Then

\[
\log_{|K|} d(x_1, x_2) = (\text{total count of } \{a_i, a_i^{-1} : 1 \leq i \leq k\} \text{ in } z) - (\text{total count of } \{a_j^{-1} a_i, a_i a_j^{-1} : 1 \leq i \neq j \leq k\} \text{ in } z) + \varepsilon(w_1, w_2, z),
\]

where \(-2 \leq \varepsilon(w_1, w_2, z) \leq 2\).
**Proof.** (i) We may suppose that \( x \) is cyclically reduced and write \( x = a_{j_1}^{m_1} a_{j_2}^{m_2} \cdots a_{j_n}^{m_n} \), where \( j_i \neq j_{i+1}, i = 1, \ldots, n - 1 \) and, if \( n > 1 \), \( j_n \neq j_1 \). Define

\[
P_i = \{ y \in A : \text{the reduced word for } y \text{ has the form } a_i w \},
\]

so that \( P = \bigcup_{i=1}^k P_i \).

Let \( y \in P_i \) for some \( i \) and suppose that \( y x^{-1} \notin P_i \). As in the previous example, the number of such \( y \) equals the count of \( a_i \) in \( x \), which is also the total cyclic count. As before, it follows from the fact that \( x \) is cyclically reduced that \( y x^{-p} \notin P_i \) for \( p \geq 0 \) and \( y x^{-p} \in P_i \) for \( p < 0 \). Hence the orbit \( \{ y x^{-p} \} \) will contribute to the value of \( s(x) \) unless it enters \( P_j \) for some \( j \neq i \). Since \( x \) is cyclically reduced, the words \( (y x^{-1}) x^{-p}, p > 0 \), are reduced. Hence there are two ways in which the orbit may enter \( P_j \): a) \( y x^{-1} = i \) and \( x^{-1} \in P_j \) or b) \( y x^{-1} \) is in \( P_j \). Now the first possibility occurs if and only if \( x \in P_i \) and \( x^{-1} \in P_i \), which means that \( j_1 = i \) and \( m_1 > 0 \) and \( j_n = j \) and \( m_n < 0 \). The second possibility occurs when \( y \) has the form \( a_{j_r}^{m_r} a_{j_{r+1}}^{m_{r+1}} \cdots a_{j_n}^{m_n} \), where \( r > 1, j_r = i \) and \( m_r > 0 \) and \( j_{r-1} = j \) and \( m_{r-1} < 0 \). These possibilities coincide with the cyclic occurrences of \( a_j^{-1} a_i \) in \( x \), so that the number of times the orbit leaves \( P_i \) and enters \( P_j \) is the cyclic count of \( x \). Hence \( \#(P x^{-1} \setminus P) \) is equal to the difference between the total cyclic count of \( \{ a_i : 1 \leq i \leq k \} \) in \( x \) and the total cyclic count of \( \{ a_j^{-1} a_i : 1 \leq i \neq j \leq k \} \) in \( x \). Since, as we have seen, each orbit which leaves \( P \cup \{ i \} \) does not return this number equals \( \log_{|k|} s(x) \).

(ii) Let \( w \) be cyclically reduced. Then the argument in the previous paragraph shows that a \( w \)-orbit which leaves \( P \) does not return unless \( w \in P \cap P^{-1} \), in which case one orbit leaves \( P \), passes through \( i \) and then returns to \( P \). Hence if \( w \notin P \cap P^{-1} \), then \( G_{\{P\}} \) is tidy for \( w \) and, if \( w \in P \cap P^{-1} \), then \( G_{\{P\}} \) is not tidy for \( w \) but \( G_{\{P \cup \{i\}\}} \) is.

If \( w \in P \), then \( \iota \in P w^{-1} \) and, if \( w \in P^{-1} \), then \( \iota \in P w \). Hence in both cases \( G_{\{P \cup \{i\}\}} \) is tidy for \( w \). However \( G_{\{P \cup \{i\}\}} \) is not tidy for \( w \) if \( w \notin P \cup P^{-1} \) because in that case \( w^n \notin P \) for every \( n \).

Therefore, for cyclically reduced \( w \): if \( w \in P \cap P^{-1} \), then \( G_{\{P \cup \{i\}\}} \) is tidy for \( w \) but \( G_{\{P\}} \) is not; if \( w \in P \cup P^{-1} \), then \( G_{\{P \cup \{i\}\}} \) and \( G_{\{P\}} \) are both tidy for \( w \); and if \( w \notin P \cup P^{-1} \), then \( G_{\{P\}} \) is tidy for \( w \) but \( G_{\{P \cup \{i\}\}} \) is not. Conjugating \( w \) by \( z \) yields the claim for \( x \).

(iii) Let \((w_1, z w_2 z^{-1})\) be the cyclic reduction of \((x_1, x_2)\). Then \( z^{-1} w_1 z \) is a reduced word. It follows, as in the previous example, that \( \{ y w_1^n \}_{n \in \mathbb{Z}} \subset P \) for every \( y \in P \setminus P z^{-1} \) and that \( \{ y z^{-1} w_1^n \}_{n \in \mathbb{Z}} \cap P = \emptyset \) for every \( y z^{-1} \in P z^{-1} \setminus \{ P \cup \{i\} \} \). Hence, for every subgroup \( U_1 \) tidy for \( w_1 \), \( G_{\{P \setminus P z^{-1}\}} \subset U_1 \) and \( G_{\{P z^{-1} \setminus (P \cup \{i\})\}} \cap U_1 = \{ e \} \). Similarly, since \( z w_2 z^{-1} \) is a reduced word, for every set \( U_2 \) tidy for \( z w_2 z^{-1} \), we have \( G_{\{P z^{-1} \setminus P\}} \subset U_2 \) and \( G_{\{P (P z^{-1} \setminus U_2)\}} \cap U_2 = \{ e \} \). These observations may be used as in the previous example to show that \( d(x_1, x_2) \) is attained when \( U_1 \) and \( U_2 \) are among the subgroups tidy for \( x_1 \) and \( x_2 \) given in (ii).
If \( U_1 = G_{[P]} \) and \( U_2 = G_{[Pz^{-1}]} \), then
\[
\log_{|K|}[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] = \#(P \triangle Pz^{-1})
\]
and this number equals
\[
(\text{total count of } \{a_i, a_i^{-1} : 1 \leq i \leq k\} \text{ in } z) \\
- (\text{total count of } \{a_j^{-1}a_i, a_i^{-1}a_j^{-1} : 1 \leq i \neq j \leq k\} \text{ in } z).
\]
The \( \varepsilon(w_1, w_2, z) \) term arises because, depending on \( w_1, w_2 \) and \( z, d(x_1, x_2) \) may be attained when \( U_1 = G_{[P \cup \{i\}]} \) or \( U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \). There are numerous cases to be considered.

1. When \( w_1, w_2 \notin P \cup P^{-1} \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 0 \).
2. When \( w_1 \notin P \cup P^{-1} \) and \( w_2 \in P \triangle P^{-1} \) and:
   a) \( z^{-1} \in P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and \( \varepsilon(w_1, w_2, z) = -1 \);
   b) \( z^{-1} \notin P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 0 \).
3. When \( w_1 \notin P \cup P^{-1} \) and \( w_2 \in P \cap P^{-1} \) and:
   a) \( z^{-1} \in P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and \( \varepsilon(w_1, w_2, z) = -1 \);
   b) \( z^{-1} \notin P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 1 \).
4. When \( w_1, w_2 \in P \triangle P^{-1} \) and:
   a) \( z^{-1} \in P \) and \( z \in P \), then \( U_1 = G_{[P \cup \{i\}]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and \( \varepsilon(w_1, w_2, z) = -2 \);
   b) \( z^{-1} \in P \) and \( z \notin P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and \( \varepsilon(w_1, w_2, z) = -1 \);
   c) \( z^{-1} \notin P \) and \( z \in P \), then \( U_1 = G_{[P \cup \{i\}]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 0 \);
   d) \( z^{-1} \notin P \) and \( z \notin P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 0 \).
5. When \( w_1, w_2 \in P \triangle P^{-1} \) and \( w_2 \in P \cap P^{-1} \) and:
   a) \( z^{-1} \in P \) and \( z \in P \), then \( U_1 = G_{[P \cup \{i\}]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and \( \varepsilon(w_1, w_2, z) = -2 \);
   b) \( z^{-1} \in P \) and \( z \notin P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and \( \varepsilon(w_1, w_2, z) = -1 \);
   c) \( z^{-1} \notin P \) and \( z \in P \), then \( U_1 = G_{[P \cup \{i\}]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 0 \);
   d) \( z^{-1} \notin P \) and \( z \notin P \), then \( U_1 = G_{[P]}, U_2 = G_{[Pz^{-1}]} \) and \( \varepsilon(w_1, w_2, z) = 2 \).
6. When \( w_1, w_2 \in P \cap P^{-1}, \) then \( U_1 = G_{[P \cup \{i\}]}, U_2 = G_{[Pz^{-1} \cup \{z^{-1}\}]} \) and when:
   a) \( z^{-1} \in P \) and \( z \in P \), we have \( \varepsilon(w_1, w_2, z) = -2 \);
   b) \( z^{-1} \in P \) and \( z \notin P \), we have \( \varepsilon(w_1, w_2, z) = 0 \);
   c) \( z^{-1} \notin P \) and \( z \in P \), we have \( \varepsilon(w_1, w_2, z) = 0 \);
   d) \( z^{-1} \notin P \) and \( z \notin P \), we have \( \varepsilon(w_1, w_2, z) = 2 \).

The remaining cases may be obtained from these by interchanging the roles of \( w_1 \) and \( w_2 \). \( \square \)

A similar analysis applies whenever \( P \) has several branches, that is, when \( P_{s_1, \ldots, s_n} = \{y \in X : \text{the reduced word for } y \text{ has the form } s_j w \text{ for some } j\} \). Let \( J \) denote the family of last letters of the \( s_j \)'s counted according to multiplicity and let \( D = \{s_j^{-1}s_i :
1 \leq i \neq j \leq n). Then

$$\log_{|K|} s(x) = (\text{total cyclic count of } J \text{ in } x) - (\text{total cyclic count of } D \text{ in } x).$$

An extreme case is when \( \{s_j\} \) is just the set of generators and their inverses, so that \( P = \mathbb{F}_k \setminus \{t\} \). Then \( J = \{s_j\} \) and the cyclic count of \( J \) in \( x \) is just the length of the cyclic reduction of \( x \). \( D \) is the set of all length 2 words and the cyclic count of \( D \) in \( x \) is also just the length of the cyclic reduction of \( x \). Hence the scale function is identically 1, which could have been seen immediately in this case because \( \prod_x K \) is a normal compact open subgroup.

**Example 4.3.** For any discrete groups \( B \) and \( C \), let \( A = X = B \ast C \) and let \( P \) consist of those reduced words beginning with a symbol from \( B \). Then \( \log_{|K|} s(x) \) is the total cyclic count of words \( cb \) in \( x \), where \( b \in B \) and \( c \in C \).

We first check that this is a valid example, in that \( P \triangle P_x \) is finite for all \( x \). Suppose \( y \in P \) and \( y \notin P_x \). Then \( y \) begins with \( b \in B \) but \( yx^{-1} \) does not begin with a symbol from \( B \). Hence either \( y = x \) or \( y \) must be a proper right subword of \( x \), of which there are only finitely many. Hence \( P \setminus P_x \) is always finite. Also \( P_x \setminus P = (P \setminus P_x) x \), which is finite so \( P \triangle P_x \) is finite.

**Proof.** By cyclic reduction we can suppose that either \( x \in B, x \in C \), or \( x = b_1c_1 \cdots b_kc_k \) with alternating symbols from \( B \) and \( C \). In the first case \( P x^{-1} = P \cup \{t\} \) and \( P \cup \{t\} \) is invariant under \( x \), so that \((e, x) \) normalises \( G_{[P \cup \{t\}]} \) and \( s(x) = 1 \). In the second case \( P x^{-1} = P \), \((e, x) \) normalises \( G_{[P]} \) and \( s(x) = 1 \).

In the third case, suppose \( y \) is such that \( y \in P \) but \( yx^{-1} \notin P \). Then we must have \( y = b_jc_j \cdots b_kc_k \), where \( 1 \leq j \leq k \). Moreover, \( yx^n \in P \) and \( yx^{-n} \notin P \) for all \( n \geq 1 \). Therefore there are exactly \( k \) orbits \( \{yx^{-n}\} \) such that for all sufficiently large \( n \), \( yx^n \in P \) and \( yx^{-n} \notin P \). Hence \( s(x) = |K|^k \), as required. \( \square \)

## 5. Construction of uniscalar groups

The preceding examples have non-trivial scale function but it seems possible that the construction described in the first section could be used to construct a compactly generated uniscalar group without compact open normal subgroups and thus to answer the question discussed in the introduction. In order to say how this might be done, we first formulate some properties of group actions.

**Definition 5.1.** Let the group \( A \) act on the set \( X \) and let \( P \subset X \).

(i) \( P \) is almost invariant if \( P \triangle a.P \) is finite for every \( a \in A \).
(ii) \( P \) is \textit{locally nearly invariant} if for each \( a \in A \) there is \( Q_a \subset X \) such that \( P \triangle Q_a \) is finite and \( a.Q_a = Q_a \).

(iii) \( P \) is \textit{nearly invariant} if there is \( Q \subset X \) such that \( P \triangle Q \) is finite and \( a.Q = Q \) for every \( a \in A \).

It is clear that, if \( P \) is nearly invariant, then it is locally nearly invariant and that if \( P \) is locally nearly invariant it is almost invariant. The condition that \( P \) be almost invariant is necessary for the group \( G \rtimes A \) constructed in Section 1 to be a topological group. This group is uniscalar if and only if \( P \) is locally nearly invariant and has a compact open normal subgroup if, and only if, \( P \) is nearly invariant. Further, \( G \rtimes A \) is compactly generated provided that \( A \) is finitely generated and its action on \( X \) is transitive. Hence a positive answer to the following question would produce an example of a compactly generated uniscalar group which does not have a compact open normal subgroup.

\textbf{Question 5.2.} \textit{Is there a finitely generated group} \( A \) \textit{acting transitively on a set} \( X \) \textit{with a} \( P \subset X \) \textit{which is locally nearly invariant but not nearly invariant?}

It is clear that \( X \) must be countable and that, if \( P \) is to be not nearly invariant, that both \( P \) and \( X \setminus P \) must be infinite. In this case \( P \) is called a \textit{moiety}.

If \( P \subset X \) is almost invariant for an action of \( A \) on \( X \) and if for each \( a \in A \) and \( x \in X \) the orbit \( \{a^n.x : n \in \mathbb{Z} \} \) is finite, then \( P \) is locally nearly invariant. It would be particularly interesting to find an action of this type where \( P \) is not nearly invariant because then for each \( a \in A \) the group \( G \rtimes A \) would have a base of neighbourhoods of the identity consisting of compact open subgroups normalised by \( a \) but would have no compact open normal subgroup.

In answer to these questions, Meenaxi Bhattacharjee and Dugald Macpherson have constructed an example satisfying these conditions in [2]. It follows then that there is a compactly generated uniscalar totally disconnected locally compact group which does not have a compact open normal subgroup. On the other hand, it is shown by Anne Parreau in [11] that every compactly generated, uniscalar \( p \)-adic Lie group has a compact open normal subgroup.

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\textbf{References}


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