Robust Non-Rigid Surface Matching
and
Its Application to Scoliosis Modelling

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TABLE OF CONTENTS

TABLE OF CONTENTS..................................................................................iii
LIST OF TABLES ..........................................................................................vii
LIST OF FIGURES ......................................................................................ix
ABSTRACT .................................................................................................xiii
ACKNOWLEDGEMENTS..............................................................................xv
DEDICATION ...............................................................................................xvii

CHAPTER 1
INTRODUCTION .......................................................................................1

1.1 Overview ..........................................................1
1.2 Hypothesis and Proposed Research ....................2
1.3 Problem Statement ........................................3
1.4 Goal and Scope of Study ................................4
1.5 Methodology .......................................................5
1.5.1 Data Acquisition .........................................6
1.5.2 Surface Matching Algorithm .......................8
1.5.3 Scoliotic Shape Change and Noise Level Determination ......11
1.5.4 Determination of Unseen Noise – Outliers ..........12
1.5.5 Verification of Results ................................12

1.6 Benefits and Contributions ......................................14
1.7 Organisation of the Thesis ...................................15
1.8 Chapter Summary ...............................................15

CHAPTER 2
INTRODUCTION TO SCOLIOSIS AND SURFACE MATCHING ...............17

2.1 Introduction ...............................................................17
2.2 Scoliosis ......................................................................17
2.2.1 Scoliosis Monitoring and Assessment ..........20
2.2.2 Constraints in Scoliosis Modelling and Monitoring ........23
2.3 Surface Matching without Control points for
   Deformation Detection .......................................................28
CHAPTER 3
SELECTION OF THE OPTIMUM OBJECTIVE FUNCTION AND TRANSFORMATION PROCEDURE ................................................. 45
3.1 Introduction ........................................................................... 45
3.2 Transformation Function I –
Modified Iterative Closest Point (MICP) ............................... 47
  3.2.1 Integration of Invariant Feature, Principal Curvature,
to Modify the Classical ICP .................................................. 50
  3.2.2 The Basic Estimation Model ........................................... 51
3.3 Transformation Function II –
Least Z Coordinate Difference (LZD) .................................. 54
3.4 Transformation Function III – Combined ICP-LZD ............... 59
3.5 Transformation Function IV –
Least Normal Distance Different (LNDD) ............................ 64
3.6 Transformation Results ....................................................... 69
3.7 Chapter Summary .............................................................. 79

CHAPTER 4
NON-RIGID SURFACE MATCHING ..................................................... 81
4.1 Introduction ........................................................................... 81
4.2 Affine Transformation .......................................................... 82
  4.2.1 Generic Affine Transformation ..................................... 85
  4.2.2 Results and Discussions ............................................... 89
4.3 New Non-Rigid Surface Matching Algorithm with New Parameters ... 91
  4.3.1 Results and Discussions ............................................... 103
4.4 Chapter Summary .............................................................. 113
CHAPTER 5
ROBUST NON-RIGID SURFACE MATCHING ........................................115
5.1 Introduction ..........................................................................................115
5.2 The Proposed Robust Estimator –
Least Trimmed Squares (LTS) .................................................................117
5.3 Robust Non-Rigid Surface Matching
Using LTS Estimator ....................................................................................120
5.4 Existing Robust Estimator Applied in Least Squares Matching ..........124
5.5 Results and Discussion ...........................................................................126
  5.5.1 Trial on Simulated Human Back Surface ......................................126
  5.5.2 Robust Non-Rigid Surface Matching to
      Unmask Scoliotic Deformities ...............................................................131
5.6 Chapter Summary ....................................................................................133

CHAPTER 6
CONCLUSIONS AND FUTURE WORK ......................................................135
6.1 Overview ...............................................................................................135
6.2 Conclusions and Findings .....................................................................135
6.3 Future Work ...........................................................................................140

REFERENCES .............................................................................................143

APPENDIX A
DERIVATION OF THE OBSERVATION EQUATION OF MICP 155

APPENDIX B
DERIVATION OF THE OBSERVATION EQUATION OF LNDD 163

APPENDIX C
DERIVATION OF THE OBSERVATION EQUATION OF LNDD
WITH NON-RIGID TRANSFORMATION PARAMETERS 171

APPENDIX D
LIST OF PUBLICATION ............................................................................179
LIST OF TABLES

Table 3.1: Test Case I: rigid transformation results for different
transformation function using synthetic data ........................................71
Table 3.2: Test Case II: rigid transformation results for different
transformation function using actual scoliosis data ........................71
Table 4.1: Comparison of experimental results of Rigid LNDD
and Non-Rigid LNDD .................................................................89
Table 4.2: New transformation parameters and their transformation activities ......92
Table 4.3: Matching results from one of the test cases ..............................104
Table 4.4: Comparison of r.m.s. error between rigid and non-rigid
matching algorithms using four different sets of actual scoliosis data .....105
Table 4.5: Parameter results from surface matching of scoliosis data
set B using non-rigid matching algorithm .........................................107
Table 4.6: Comparison of r.m.s. error between non-rigid
and CRNr matching algorithm ......................................................111
Table 4.7: Comparison of non-rigid transformation parameters
between non-rigid and CRNr matching algorithm ...........................111
Table 5.1: Comparison between known parameters and their estimated
parameters computed by different robust LNDD algorithms.
the deformation percentage is 9% .............................................127
Table 5.2: Capability of matching the surfaces under different
deformation percentages by using different robust LNDD algorithm ......130
Table 5.3: Summary of r.m.s. error of different matching algorithm
in matching the same actual scoliosis data sets .............................132
LIST OF FIGURES

Figure 1.1: Research methodology flow chart ..............................................7
Figure 1.2: Surface topography of a trunk surface ....................................8
Figure 1.3: Example of alignment of two scan, previous scanned data (blue) and current scanned data (red), from different time epoch. The data points, highlighted in the black circle, are the example of unseen noise (outliers) ..........................................................10
Figure 2.1: Comparison of normal spine (right) with scoliotic spine (left) ..........19
Figure 2.2: Cobb angle measurement ..........................................................20
Figure 2.3: Example of external asymmetric view from a scoliotic patient ..........21
Figure 2.4: Illustration of the rigid transformation that rotates and translates two different reference systems into common frame. (a) Before the transformation. (b) After the transformation ............32
Figure 3.1: MICP Principle .................................................................48
Figure 3.2: Concept of MICP ...............................................................49
Figure 3.3: LZD Principle .................................................................55
Figure 3.4: Concept of LZD ...............................................................56
Figure 3.5: Combined ICP-LZD Principle .................................................60
Figure 3.6: Matching Concept of Combined ICP-LZD ...............................61
Figure 3.7: Assumptions have been made in the experiment in getting the vertical difference and surface slope ........................................64
Figure 3.8: Correspondence is established between points on S₂ with patches on S₁. Surface S₂ is rigidly moved to match with S₁ by minimising the normal distance D .................................66
Figure 3.9: (a) Synthetic data sets for simulation studies. Reference (red) and Model (black) surface before transformation. (b) Reference and Model surface after transformation using (i) Modified ICP, (ii) LZD, (iii) ICP-LZD and (iv) LNDD. Matching results show that all four transformation functions have relatively good matching results. It can be concluded that all four functions are validated and well-defined .......................72
Figure 3.10: (a) Two real scoliosis data sets for transformation functions comparison studies. Reference (S₁) and Model surface (S₂) before transformation. (b) S₁ and S₂ after transformation using (i) MICP, (ii) LZD, (iii) ICP-LZD, and (iv) LNDD. MICP has the worst result with the largest r.m.s. value, indicating huge divergences in numerous parts. LZD and LNDD have much better results, and LNDD has smallest r.m.s. value ……………………73

Figure 3.11: (a) Corresponding point search using point-point distance metric. (b) Corresponding point search using point-to-point distance metric with attached curvature information. The different shape marks indicate different curvature ……………………………………….75

Figure 3.12: (a) Matching achieved successfully in “clean data”. (b) In Test Case II, some points, in green box, need to be slid away from their corresponding point to obtain global convergence ………….75

Figure 3.13: (a) Common process of obtaining slope of two points. (b) Curvature (green line) is the actual geometric properties of two points on a surface. Use of slope, red line, may lead to inaccurate matching ……………………………………………………………77

Figure 3.14: (a) Original surface with possible outliers, highlighted in purple circle. (b) Extra outliers (highlighted in green circle) are generated from the process interpolating the original data into a regular grid format ….77

Figure 4.1: Concatenated scaling factor into transformation parameter assures best shape fit. Model surface (red) is “stretched” toward cover same surface area of Reference surface. (a) Surface matching result using rigid LNDD. (b) Surface matching result using non-rigid LNDD. Green boxes show the best shape fit after surface matching ………………………………………………………90

Figure 4.2: Illustration of different shearing parameters together with their transformation. Green arrows indicate the forces that shear the object ………………………………………………………93

Figure 4.3: Flowchart of non-rigid surface matching programme ………………..102
Figure 4.4: Contour topographic map of data set B, (a) before the matching, (b) matched by using rigid algorithm, (c) matched by using non-rigid algorithm. The circles highlight the comparison of matched results of deformed areas between rigid and non-rigid algorithm. Solid lines represent surface $S_2$ and dash lines are surface $S_1$.

Figure 4.5: Comparison results of Difference Map (filled contour) defined in Berg (2002) with the results from non-rigid matching (normal and dashed lines). A very close deformities distribution pattern can be recognised on both results indicating this new non-rigid matching algorithm is able to detect surface deformities caused by scoliosis. In the Difference Map, darker colours indicate more serious deformations and would be expected to coincide with the bigger separations between the unbroken and broken contours.

Figure 4.6: Contour topographic map of transformed data set B. Green lines represent transformation results using non-rigid transformation and red lines are transformation results using CRNr. The results indicate that both transformation algorithms give very close matching results.

Figure 5.1: Simulated human back surface for testing purpose. (a) Original Surface, (b) Deformed Surface.

Figure 5.2: Matching results of simulated human back surface using different matching algorithms (non-robust and robust). The results are illustrated in contour topographic map plotting. (a) using non-rigid LNDD algorithm, (b) using rigid M-LNDD algorithm, (c) using rigid GM-LNDD algorithm, and (d) using TrLNDD algorithm. It has to be mentioned that the changes in Z coordinate will affect the contour plotting. If the surfaces can not be matched correctly, it will generate “noisy-like” non-smooth data as in (a).

Figure 5.3: Illustration of the matching results using non-rigid and TrLNND algorithms. (a) non-rigid matching algorithm, (b) TrLNND. TrLNND shows its capacity in matching the deformed areas and generates smoother contours.
ABSTRACT

With the advancement of digital photogrammetry techniques, development of digital surface topographies becomes easier and straightforward, leading to more desirable applications in the medical field. The potential application of digital photogrammetry to the medical field is expected to be able to manipulate and analyse the surface data, which in many cases requires comparisons with previously derived data for the purpose of identifying surface change. A potential application in the medical field may result from the extensive use of surface topographies of back shapes to monitor scoliosis.

A new spatial data manipulation tool in the form of a non-rigid surface matching algorithm with new parameters has been investigated, aimed at replacing the classical least squares 3D surface matching approach which allows positional fit rather than shape fit. A computer program has been written to implement the matching algorithm. So far, the analysis of shape change to identify scoliotic progress has not been satisfactorily solved. As a contribution to this task, the capacity of the non-rigid matching algorithm to find the match and simultaneously model the scoliotic deformities has been assessed.

A complete review of the advantages and disadvantages of the classical surface matching algorithm has been analysed and presented in this work. Current constraints faced in the scoliosis modelling and monitoring has been discussed as the preparation for designing the new matching algorithm. The main conclusion of this review is that the surface matching is a feasible tool for solving the modelling and monitoring constraints faced by the researchers.

Various surface matching algorithms with different objective functions and transformation procedure have been investigated. This investigation serves the purpose to determine the optimal transformation objective function and transformation procedure that is best to be implemented in the proposed non-rigid surface matching algorithm. Four different surface matching algorithms based on a rigid transformation, involving six unknown parameters have been developed and compared. These four algorithms are Modified Iterative Closest Point (MICP), Least Z Difference (LZD),
Combined ICP-LZD, and Least Normal Distance Difference (LNDD). LNDD was chosen as the required transformation objective function and transformation procedure due to its simplicity, efficiency and the accuracy of the resulting calculation. This selection is supported by the studies carried out by Schenk (2000) where the outcome is matched with the result reported here.

The scoliosis data sets are automatically matched by a least squares non-rigid matching algorithm involving nine unknown parameters with three scales and six shears. The effectiveness of this algorithm comes from its ability to match the deformed surface to the un-deformed surface and detect the deformation magnitude. This non-rigid matching algorithm has been trialled using models with predictable topographic deformation. There is evidence that surface deformities can be modelled. In addition, to demonstrate the capability of this new non-rigid matching algorithm in scoliosis modelling, the author has performed experimental comparison with classical rigid matching algorithm using four different scoliosis data sets. The non-rigid matching algorithm returned r.m.s. values which were improved by about 10% for all data. Analysis indicates that this new non-rigid matching algorithm has proven to be a very successful tool and is an improvement on the classical approach. The new parameters are able to model and delineate the possible shape changes caused by scoliosis. Scaling factor can be interpreted as the dilation caused by natural growth (seen noise) in all three direction, while shearing parameters can be used to depict the deformation caused by scoliosis. The results show that this new non-rigid algorithm not only assures the best positional fit but also the best shape fit. This research is among the first using non-rigid transformation for surface matching without control points.

Finally, the matching accuracy and robustness is improved by integrating Least Trimmed Squares (LTS) into the non-rigid matching algorithm. This robust algorithm is called Trimmed Least Normal Distance Difference (TrLNDD). The TrLNDD is able to detect local deformation covering up to 50% of the surfaces being matched, the highest value reported in the literature. Improvement of about 79% of r.m.s. value can be detected indicating LTS estimator is highly desirable to be integrated into existing matching algorithm in detecting the surface deformation. True scoliotic change can be revealed by eliminating the unseen noise (outliers) by using the TrLNDD algorithm.
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DEDICATION

TO MY BELOVED WIFE

PUI YEE

AND SON

HAW JET
CHAPTER 1

INTRODUCTION

1.1 Overview

Everyone’s spine has a natural curve and no one’s spine is perfectly straight. Some curvature in the neck, upper trunk and lower trunk is considered as normal. Humans need these spinal curves to help the upper body maintain proper balance and alignment over the pelvis. However, when there are abnormal side-to-side (lateral) curves in the spinal column, this is referred to as scoliosis. Unlike poor posture, scoliosis cannot be corrected simply by learning to stand straight. In fact, it requires advice and treatments to avoid a worsening situation because it can cause deformities in spinal and trunk surface (in this thesis, this is referred to as the back surface of the patient).

Scoliosis is a fairly common condition that usually appears during adolescence – children aged 10 – 18 years. Although it affects both males and females, the condition is more prevalent in females (Pearsall, 1992). According to the Scoliosis Association of Australia (http://www.scoliosis.org.au/pages/default.cfm?page_id=4797), 90 percent affected people are female and approximately 4 percent of the general population is affected. Females are also more likely to have severe scoliosis than males. The National Scoliosis Foundation report (http://www.scoliosis.org/) shows that one in ten people will have scoliosis, and two to three people in a thousand may require treatment.

Scoliosis spinal deformities are described using Cobb curve angle standard (Goldberg, 2001; Korovessis, 1996; Sapkas, 2003). X-ray is the traditional method to obtain the spine curvature to determine the Cobb angle. However, excessive exposure to X-ray ionizing radiation can be hazardous to the children’s health. Furthermore, indices from Cobb angles do not describe trunk surface deformities which are the major concern for the patients and their parents. A reliable and non-invasive technique is therefore
required as a complementary to this traditional method to reduce the number of radiographs taken and to quantify the trunk surface deformities.

With the advancement of digital photogrammetric techniques, development of digital surface topographies becomes easier and straightforward, leading to more desirable applications in the medical field. The potential application of digital photogrammetry to the medical field is expected to be able to manipulate and analyse the surface data, which in many cases requires comparisons with previously derived data for the purpose of identifying surface change. A potential application in the medical field may result from the extensive use of surface topographies of back shapes to monitor scoliosis.

A new spatial data manipulation tool in the form of a non-rigid surface matching algorithm with new parameters was investigated, aimed at replacing the classical least squares 3D surface matching approach which allows positional fit rather than shape fit. A computer program will be written to implement the matching algorithm. So far, the analysis of shape change to identify scoliotic progress has not been satisfactorily solved. As a contribution to this task, the capacity of the non-rigid matching algorithm to find the robust match and simultaneously model the scoliotic deformities will be assessed.

1.2 Hypothesis and Proposed Research

In tracking the progression of scoliosis, the amount of change in the deformity provides the most important information necessary for making treatment decisions. Scanning of the trunk surface of scoliosis patients does not adequately quantify the changes in deformity over time (Berg, 2002; Jaremko, 2002; Liu, 2001; Pearsall, 1992). In order to provide better qualitative and quantitative measurement, natural history like normal growth and scoliotic shape changes should be included during the monitoring as well (Grivas, 2007). Although surface topography has been useful in quantifying trunk deformity, the more important issue is whether there has been true change in the condition. True change is referred to as the scoliotic deformities including the scoliotic shape change. True change, however, may be masked by normal growth and outliers. Although normal growth may contribute to the change of trunk surface, the goal of this research is to determine the true scoliotic change. For this reason, normal growth is referred as non-scoliotic change. Thus, the major concern here is how surface topography can be used to model and quantify accurately the trunk deformity without
any “seen and unseen noise level”. Noise usually means random errors in the data. In this work, however, the author uses the term noise in a broader sense of random errors denoting all examples with erroneous attribute values and non-scoliotic change. This definition includes not only erroneous data (outliers) but also non-scoliotic representatives (normal growth). Seen noise level in this context can be referred as natural history and unseen noise can be defined as apparent or hidden outliers. This noise level should then be used as the threshold to determine the true change.

The author is proposing to develop change parameters useful in quantifying trunk deformity due to scoliotic deformities. Pairs of surface topographies from different epochs will be matched together using least squares approach to get the closest fit. Change parameters will be designed and employed to determine the non-scoliotic and true scoliotic deformities. Identified noise level, seen and unseen noise, will be eliminated to detect the true change. The ability of the developing change parameters will be tested and results will be verified. This research will serve as a test to evaluate the sensitivity of surface topography and least squares approach in detecting the true change in scoliosis.

1.3 Problem Statement

Scoliosis is an abnormal lateral curvature of the spine, generally first noticed as a change in the shape of the back (Bunnell, 1984; Karachalios, 1999; Stokes, 1989). Longitudinal patient follow-up is essential because the scoliotic deformities change so rapidly while the patient is growing. The golden method in assessing scoliotic deformation is using the Cobb angle which will require X-ray scanning. Measured back surface topographies have been used to monitor the external manifestation of scoliosis to avoid excessive exposure to large doses of radiation which may increase the risk of breast cancer (Levy, 1996; Doody, 2000). For this reason, matching the back surface topographies in two successive clinical visits to monitor the scoliotic changes has been seen by the author as advantageous and has been investigated in the work reported here. A more detailed discussion regarding constraints faced in current scoliosis modelling and monitoring can be found in Chapter 2 Section 2.2.2.
Comparison of measured back surface topographies in scoliosis patients can be complicated by the existence of sources of apparent shape changes other than the scoliotic deformities. These may include natural changes in the patient due to growth or the true scoliotic deformities. The critical requirement for a new matching algorithm is to enable changes due to scoliosis to be modelled from observed surface topographies using the tool of automated surface-to-surface alignment. Recognising that two changed surfaces cannot be expected to be simple rigid replicas of each other, a non-rigid surface matching algorithm is highly favourable.

The classical surface matching algorithm, which rigidly matches the surfaces (Karras, 1993; Mitchell, 1998; Pilgrim, 1996; Rosenholm, 1988), is not able to perform the non-rigid surface matching. Furthermore, the use of surface gradients as the value expressing geometric characteristic may degrade the performance of this matching algorithm. If no distinct surface gradients exist on the surface, the matching algorithm may fail. This is because the algorithm solution depends on the strong matrix coefficients derived from the surface gradient. In addition, the best definition of geometric characteristic between two points on a continuous surface is the curvature but not the surface gradient. In Chapter 2 Section 2.4.3, the readers can get more ideas regarding the constraints encountered by current surface matching algorithm.

1.4 Goal and Scope of Study

The ultimate goal of this research is to develop and modify a new algorithm based on the least squares method that is suitable to match the surfaces into same reference frame for non-rigid deformation monitoring purposes. Scoliosis data will be used in this research to achieve the research goal. The main reason to use scoliosis data is because the curvature on the spine will cause non-rigid deformation on the patient’s back surface. A non-rigid matching algorithm is required to detect the scoliosis deformation. This match with the research goal reported here. Furthermore, The University of Newcastle and the scoliosis research group from Glenrose Rehabilitation Hospital, Canada, have been working collaboratively to develop matching algorithm to match the scoliosis data. Actual scoliosis data can be obtained from this research group.
The author would like to emphasize that this research concentrates on enhancing current rigid surface matching using simple least squares so it can be applied in the non-rigid surface matching environment. Complex non-rigid algorithms applied in medical image matching are not within the research scope (Habib, 2001; Lester, 1999; Pagani, 2007; Thompson, 2007; Yanovsky, 2009; Young, 2009; Zacharaki, 2009; Zitova, 2003).

Objectives below are set up in order to achieve the research goal:

- To develop a mathematical model to determine the true change of scoliosis.
- To develop change parameters that are useful in estimating scoliotic deformities.
- To assess non-rigid matching of two changed surfaces in modelling trunk deformity.
- To obtain the robustness of the matching algorithm using the least trimmed squares estimator.

1.5 Methodology

The goal of this research is to develop a surface alignment algorithm that will adequately describe the true change as it is hoped that this will allow better quantification of trunk deformity and correlate more closely with the patient’s perception of deformity. The author is proposing using the tool of non-rigid surface-to-surface alignment to quantify this deformity from trunk surface topographies.

A few tasks must be done before making an algorithm that can automatically detect changes. The appropriate parameters have to be selected and the noise has to be removed in order to have an accurate analysis and subsequently a robust algorithm. The following sections describe each of the tasks in detail and the methodology flow chart is described in Figure 1.1.
1.5.1 Data Acquisition

Because a laser scanner is a non-invasive device and can be operated by non-specialist personnel, it seemed an ideal addition to the assessment and observation of trunk surface deformity in a pediatric orthopedic department. Results from the laser scan are available within minutes allowing comparison with previous results. Thus, deformity progression can be immediately recognised. This research will begin with acquiring trunk surface topographies from scoliosis patients. Surface topographies can be obtained by measurements on patient’s trunk with a laser scanner and described as a set of X, Y and Z coordinates.

The scoliosis Research Group from Glenrose Rehabilitation Hospital, Canada has been doing active research on this field. It has a well setup research laboratory capturing three dimensional representations of the surface features of the trunk from scoliosis patient. All of the three dimensional back surface data will be acquired from this research group and converted into surface topographies as shown in Figure 1.2. A Minolta Vivid 700 portable non-contact scanner has been used to capture a three-dimensional scan of scoliosis patient’s back surface. This three-dimensional scan containing X, Y and Z coordinates are described in Cartesian coordinate format. This data set is stored in non-proprietary format (in ASCII) with “.dat” suffix. This will be the major data in support of this study.

A pair of generated surface topographies from different epochs will be matched together to monitor the changes. The matching process involves finding the relative positions of the pair of surface topographies at their location of closest match. This will be done in the second stage which is Surface Matching.
Figure 1.1: Research methodology flow chart.
One of the most difficult problems to solve in this study is to obtain the position of closest fit for pairs of surface topographies from different epochs. It requires a rigorous approach and an optimal quality in obtaining closeness-of-fit index for the surface matching. This is the purpose of this study to develop a mathematical surface to surface alignment algorithm that is based on a least squares solution for the closest surface fit. Many matching methods are available, such as Iterative Closest Point (ICP) developed by Besl and McKay (Besl, 1992), but the least squares approach is considered best suited for this research for couple of reasons. Firstly, the least squares approach does not interpret the surface signal or extract the surface features which is not required in this research. Secondly, since the data derived from laser scanner represent the object surface, the problem should be defined as surface matching problem. Furthermore, it minimises the vertical surface separations by a successive refinement of initial estimated parameters (Mitchell and Chadwick, 1998) and these separations can contribute to the deformation and noise parameter determination in the later stage.

Figure 1.2: Surface topography of a trunk surface.
However, this transformation objective function is replaced due to the drawbacks encountered in the classical approach. Instead of minimising the vertical separation as the transformation function, the new non-rigid matching algorithm minimises the normal distance, D, between the surfaces. Comparison between the transformation functions is discussed and presented in Chapter 3. In order to give the readers an idea about the classical transformation function, the author will continue to use this transformation objective function, which minimises vertical separation, to illustrate the methodology.

Figure 1.3 illustrates the key idea of this matching algorithm. A pair of surface topographies for the same patient from different epochs is matched together. The least squares approach is applied in the matching process. If rigid transformation is applied, the geometric relationship between the surfaces can be defined using a 7-parameters transformation, involving three rotations, three translations and a scale. However, these parameters can be extended or reduced depending on the solution demands. In this work the scale parameter is assumed to be close to one because the same scanner is used to scan the same physical surface. Thus, the least squares approach is actually establishing six transformation parameters: three rotations and three translations in X, Y and Z directions. The mathematical equation is as follows:

\[
\delta Z = a_1 \Delta \omega + a_2 \Delta \phi + a_3 \Delta \kappa + a_4 \Delta T_x + a_5 \Delta T_y + a_6 \Delta T_z
\]

(1.1)

where, \( \Delta \) signifies the refinement of initial estimated parameters; \( \omega, \phi, \) and \( \kappa \) are the conventional photogrammetric rotations about the X, Y and Z axes, respectively; \( T_x, T_y, \) and \( T_z \) represent the translations in X, Y and Z axes, respectively; and \( a_1 \) to \( a_6 \) are coefficients. The parameters transform the second surface to the first surface topography to generate the vertical separation between these surfaces.

Once the pair of surface topographies is matched together, the depth component (vertical separation) can be calculated to yield the residual surface difference, \( \delta Z \). Residual surface difference can be studied to quantify the trunk deformity in scoliosis. It is hoped that the result can help model the true change that affect the trunk deformity.
After automated matching, results from residual surface differences can serve as a difference index to detect the change. Assuming the automated matching is done thoroughly, the difference index is calculated to see how the two surfaces differ. However, the difference index does not pertain totally to scoliotic change. Changes such as growth and outliers could be taken into account. If these two changes are big enough in comparison to scoliotic change, the difference index will be inaccurate. It can be concluded that it is insufficient to apply rigid transformation to monitor correctly the true change due to the influence of non-scoliotic change and outliers. This research seeks a method in which growth variation and outliers are filtered out so that true scoliotic change can be determined by the difference index. In other words, a non-rigid transformation equation for a “perfect” match will be designed. Transformation parameters will be refined in the next stage to determine the noise level. The determined noise will be eliminated to reveal the true scoliotic change. These parameters will be included in equation 1.1 for non-rigid matching.

Figure 1.3: Example of alignment of two scan, previous scanned data (blue) and current scanned data (red), from different time epochs. The data points, highlighted in the black circle, are the examples of unseen noise (outliers).
1.5.3 Scoliotic Shape Change and Noise Level Determination

After the surface matching stage, the depth component in Z axis can be used to generate residual surface differences and this index can be compared to determine the change along this direction. However, this result is inevitably influenced by not only the treatment but also the natural history of the patients. The result can be blinded by noise caused by natural history including growth variation. Furthermore, any unseen noise shall not be excluded that may influence the result. Let an arbitrary patient’s data be denoted as ST. A pair of ST will be captured in different epochs to generate the residual surface difference. Let \( ST_1 \) be the first captured data in time \( t_1 \) and \( ST_2 \) will be captured at time \( t_2 = t_1 + \Delta t \). Theoretically, \( ST_1 \) and \( ST_2 \) are essentially the same because they are both the subset from ST, the same patient. However, in quantitative sense, \( ST_1 \) and \( ST_2 \) are different because they experienced changes including scoliotic shape change and natural growth, if any, over a period of time \( \Delta t \). If every point on \( ST_2 \) undergoes a rigid transformation into \( ST_1 \), we can represent it with the following equation:

\[
[ST_2] = [G][Ssc][ST_1]
\]

(1.2)

where, \([G]\) denotes the growth matrix that models growth; and \([Ssc]\) represents the scoliosis matrix that models scoliotic shape change.

The major task in this stage is to determine the unwanted noise level and eliminate it to generate “clean” true change that is caused by scoliotic deformation only. This is the concern from the clinician and patients. New refinements including deformation parameter and noise parameter will be generated based on the comparison result. Refined transformation parameters will be employed in another surface matching process and this process will continue until a satisfactory noise level elimination complete. It is hope that the final result can reveal the true change of scoliotic deformation.

If equation 1.2 is appropriately designed, tested and converted into parameters based on the hypotheses mentioned above, it can be used in least squares matching for non-rigid
surface transformation to detect the true change of scoliotic deformation. The new equation will be:

\[ \delta Z = a_1 \Delta \omega + a_2 \Delta \phi + a_3 \Delta \kappa + a_4 \Delta T_x + a_5 \Delta T_y + a_6 \Delta T_z + a_7 \Delta G + a_8 \Delta Ssc \]

(1.3)

Ideally, the second surface will “perfectly” match with the first surface and produce zero difference index if appropriate parameters have been found. The true change caused by scoliotic deformation can simply calculated by extracting the “Ssc” parameters with negligible error.

### 1.5.4 Determination of Unseen Noise - Outliers

While the surface difference detection is desirable in surface matching, outlier detection is also a demanding component of the surface matching procedure. As discussed above, the matching results may be blinded by outliers. For example, large residuals may be interpreted as being surface differences or changes, and may then still be included in the matching. On the contrary, the large differences may be excluded from the matching process. Thus it is crucial to include the outliers detection component into the surface matching procedure. A simple classical outliers detection technique, which the residuals that differ from their mean by the value greater than three times their standard deviation, will be examined first. This is follow by the M-estimator and least mean squares outliers elimination methods. Finally, these methods will be compared with Least Trimmed Squares, which is the outliers elimination method that will be applied in this research. Literature reviews regarding outliers detection techniques are discussed in Chapter 2. Detailed application and results can be referred in Chapter 5.

### 1.5.5 Verification of Results

In the final stage, assessment check will be performed to verify the final result. The verification steps are as follows:
i. **Goodness of Match - Root-mean-square error**

Root-mean-square (r.m.s.) calculation will be applied to test the accuracy of the initial estimated parameters for every matching. Least squares surface matching is based on the approximation estimator to get the closest fit. It should be noted that, if the causes of differences, either due to scoliotic deformities or noise, could be completely modeled, the r.m.s. difference index would tend towards a value of zero. Furthermore, the r.m.s. error can be used to test the validity of the parameters that are hypothesised.

ii. **Introduce transformations to the data to test the ability of the algorithm to recover the transformation parameters**

Certain amounts of transformations will be introduced into one of the surface topography data to test whether the algorithm can return input values of these transformations. This can be easily achieved by transforming the surface with known transformation parameters. These transformation parameters will be recorded for comparison with the result after the surface matching. This transformed surface will be matched with another “identical” surface. The result will be inspected to understand the ability of the program to recover the transformation parameters. If the program can successfully return this input value, it will not only verify the algorithm and program but also test the acceptable degree and validity of the concept.

iii. **Forward and reverse matching**

Forward matching is a process to match the Model Surface ($S_2$) with Reference Surface ($S_1$). In contrast, reverse matching is a reverse process to match $S_1$ with $S_2$. The primary reason for doing forward and reverse matching is to verify the validity and accuracy of refinement that model the deformation parameter and noise parameter.
1.6 Benefits and Contributions

Clinicians and scoliosis patients are primarily concerned with the cosmetic appearance, and many patients ask how their condition has changed. Indices from residual surface difference can provide the information about the trunk deformation because this reflects the extent of true change that caused by scoliotic deformation. This technique developed in this study, thus, provides an educational method for clinicians, patients and families to understand changes in the surface deformity due to natural history or effects of treatment.

Assessments of scoliosis are routinely done by means of clinical examination and full spinal X-ray. Multiple exposures to ionization radiation, however, can be hazardous to the child and are costly. With the increase confidence of quantifying scoliosis changes using surface topography, frequency of the growing children exposure to undesirable radiation can be drastically reduced.

It has long been debated whether X-ray examination should be replaced by surface topography as a tool for scoliosis deformation monitoring (Doody, 2000; Levy, 1996). However, it is unlikely that surface topography will supplant radiography because they are not measuring the same aspect of deformities. X-ray is necessary to make a diagnosis concerning the spinal deformity while surface topography is used to clarify the back surface deformity. However, surface topography is useful in patient monitoring as an alternative to radiography. This study will help to determine how to well use surface topography as a tool in scoliosis deformation monitoring. It can also determine the clinical value and position of surface topography in managing scoliosis. This study will also help to identify the useful surface topography parameters as there is still lack of consensus in the current literature. It can help to determine the sensitivity of surface topography in detecting the change that caused by scoliosis deformation.

For these purposes, a new robust non-rigid matching algorithm has been proposed, developed and implemented in this study, which provides both the better positional and shape fits. It is presented in Chapters 3, 4 and 5 respectively. The experimental results are very promising, demonstrating that this new robust non-rigid matching algorithm is able to improve the precision of the matching (defined by the residuals of the separation)
and the accuracy of the matching (which compares the r.m.s.). The author believes the shear parameters that utilised in the matching algorithm will serve as a powerful shape matching tool and can be used to aid in many existing techniques towards effective shape analysis. This new robust non-rigid matching algorithm should give new insights to many problems in surface matching. The successful robust non-rigid matching algorithm can supplement the classical rigid matching algorithm for surface deformation monitoring, whilst requiring no prior knowledge about the surfaces.

1.7 Organisation of the Thesis

The chapters are organised as follows:

The introduction to surface matching without control points and overview of current surface matching techniques and outliers detection method are presented in Chapter 2. A description, comparison and selection of transformation objective functions are discussed in Chapter 3. The proposed new non-rigid surface matching algorithm can be found in Chapter 4. The robustness of the new non-rigid surface matching algorithm is discussed and presented in Chapter 5. Finally the general conclusions of this work are presented in Chapter 6.

1.8 Chapter Summary

Although there has been active research in surface matching in different field, surface matching without control points in biomedical field is still uncommon and limited research has been reported. Research in biomedical field deserves special attention especially in scoliosis because it proved to be useful in detecting the scoliosis magnitude and progression. However, there is still a lack of research in defining the true scoliosis deformation change. True change can be distorted by existing noise in the data. The author proposes a method using non-rigid least squares matching to determine the existing noise and model the scoliotic shape change. By eliminating this noise, true scoliosis deformation change can be easily determined and its magnitude and progression can be judged.
CHAPTER 2

INTRODUCTION TO SCOLIOSIS AND SURFACE MATCHING

2.1 Introduction

This chapter will start with a discussion about scoliosis, followed by a brief introduction to surface matching without control points. Examples of current surface matching algorithms, namely Iterative Closest Point (ICP) and Least Z Difference (LZD), are discussed. A more detailed algorithm formulation is presented in Chapter 3. For the sake of readers understanding about outliers detection methods, previous outliers detection methods that incorporated into the LZD surface matching algorithm are presented in the last section of this chapter. It is worth mention that the discussion of scoliosis in this chapter is based on the author’s abstraction of information from the literatures. For further and detailed information about scoliosis, the readers are advised to seek medical advice from the medical doctors.

2.2 Scoliosis

Scoliosis is a lateral or sideways curve in spine with concordant vertebral rotation that is apparent when viewing the spine from behind the person who has it. It usually causes the spine to assume an “S” (double curve) or a long “C” (single curve) shape instead of being straight. When the spine curve happens in an S-shape, the curve may not be noticeable and the person can appear straight because the two curves counteract each other. This lateral curvature, either single or double curve, affects the rib cage and presents as a deformity of the trunk. Figure 2.1 shows the example of a normal human spine compared with a scoliotic spine.

There are many causes of scoliosis, but by far the most common (approximately 80 percent) has no known cause, called ‘idiopathic’ (Scoliosis Association of Australia;
National Scoliosis Foundation). Because of this reason, scoliosis is generally referred as Adolescent Idiopathic Scoliosis (AIS). Of the three types of scoliosis, namely infantile, juvenile and adolescent idiopathic scoliosis, AIS is the most common type. Because idiopathic scoliosis has no known cause, it is impossible to completely prevent it from occurring and clinical monitoring is the only intervention necessary for most patients.

Scoliosis deformities are an evolving process which must be detected and treated as early as possible. It involves mild to severe deformation of the spine that leads to noticeable trunk deformity (Scoliosis Association of Australia; National Scoliosis Foundation). Trunk deformity has the distinct characteristic that it creates asymmetries on the various regions in the back. These include scapula, shoulder, waist and hip asymmetries, where one side is higher than the other (cf. Figure 2.1). A mild degree of scoliosis is common among the scoliotic patients. Scoliosis generally does not require any specific treatment; however severe scoliosis does indeed need treatment like surgery to correct the spine's curvature. Once young children have been detected with spinal deformity, they are required for intensive follow up review for many years and often into their adult life because deformation of the spine generally increases with time (Goldberg, 2001).

Scoliosis deformity is described using the Cobb curve angle that quantifies the extent of lateral curvature within the affected region of the spine and it is measured in degrees. Measurements obtained from the Cobb angle, however, do not fully describe three-dimensional geometry of the spine and associated deviations (Pearsall, 1992). Furthermore these measurements do not describe the trunk surface deformities and the cosmetic appearance, which is the major concern of the scoliosis patients and their families. Figure 2.2 shows the example on how the Cobb angle measured from the spine.
In general, curves that are less than 25 degrees are not treated but the deformity progression will be observed. However, some curves get progressively worse, and curves over 30 degrees need to be carefully watched because they can worsen rapidly. If the risk of further progression is considered high, first bracing, then surgery, are considered. Once the spine has developed a significant curve, even surgery will not be able to make it perfectly straight again. Thus, identification of scoliotic condition needs physical intervention that requires precise quantification and monitoring of scoliosis progression. It is worth to mention that trunk surface deformities are caused by the lateral curve in the spine. The research reported in this thesis is focus on detection of the deformation that caused by the scoliosis on the trunk surface. Spine deformation detection is not within the scope of this research.

Figure 2.1: Comparison of normal spine (right) with scoliotic spine (left). (Source from [www.youngwomenshealth.org](http://www.youngwomenshealth.org))
2.2.1 Scoliosis Monitoring and Assessment

Scoliosis is not a disease but it is a descriptive term to describe the curvature that happens on humans’ spines. However, one should note that all spines have curves and healthy spines have front-to-back curves. It is often difficult for the untrained eyes to detect a person who has scoliosis. Monitoring the development of scoliosis is a challenge. The best way to look for a scoliosis is to look at the back from behind as the person bends forward. It is then easy to see the curve as one side of the rib cage will project more than the other. If there is any suggestion of scoliosis medical opinion, plain X-rays of the spine may be taken to detect the extent of a scoliosis. Figure 2.3 shows a scoliotic patient with an obvious rib prominence when she bends forward and the radiograph indicates the spinal curvature of a scoliotic patient.

The most common treatment for scoliosis is based on “observation” in which the patient is watched for any progression in the curvature. Progression in the curvature of the spine can be grouped from mild to severe deformation. Bracing may be indicated if mild
deformation is noted. It is not an attempt to straighten the spine but to hold the curve in place in the hope of stopping the progression. If the progression is found to worsen, an operation may be needed for straightening the spine. However, treatment for non-severe scoliosis patients depends solely on observation. Observation can be done either using X-ray or non-invasive surface scanning.

The use of radiographs (X-ray) to monitor the development of idiopathic scoliosis is the traditional method (Goldberg, 2001; Liu, 2001). A standard posterior-anterior two dimensional (2D) radiograph of the spine usually is taken and subsequently used to determine the Cobb angle. The magnitude of the curve determined from the Cobb angle is used to determine the appropriate treatment. The rule of thumb in clinical intervention is that curves of 0 to 20 degrees should be treated by observation only (Liu, 2001). Intermittent follow up evaluation is recommended and often continues into the patient’s adult life. In other words, patients are required to attend radiographs scanning which may be hazardous to their health during every medical visit for their scoliosis deformities progression monitoring.
Frequent X-ray screening may cause the patient exposure to high radiation. This is especially important for young women, because developing breast tissue has increased sensitivity to radiation, and repeated exposure in adolescence can increase the risk of breast cancer later on. Clinical visit is thus limited to once in every six months to reduce the cumulative X-ray exposure (Doody, 2000, Levy, 1996, Pazos, 2005). However, six months may be too long to monitor the progression of scoliosis, especially the external asymmetric, trunk surface deformities. A safe, reliable, non-invasive and truly three dimensional investigation would be highly useful in the assessment of the adolescents with scoliosis.

The idea that develops non-invasive techniques to complement or to replace existing scoliosis measurement procedure (X-ray scanning) has been pursued for decades (Aggarwala, 2002; Berg, 2002; Dawson, 1993; Denton, 1992; Jaremko, 2001; Jaremko, 2002; Liu, 2001; Moore, 2000; Pazos, 1996; Pearsall, 1992; Ponset, 2000; Theologis, 1997). The major advantage of non-invasive techniques compared to existing scoliosis measurement is that it could be used more frequently allowing the practitioner to better document the changes of scoliosis. This would enhance not only the patient’s treatment but the understanding of scoliosis progression. This can be done because non-invasive techniques can drastically reduce the number of radiographs taken during the patient’s treatment period.

Specific 3-D measurement techniques, based on optical or opto-electronic principles of image capture, can today provide a complete set of results to quantify scoliosis, without any risk for the patient. Laser scanning has been introduced in capturing the 3D trunk or 3D back surface (surface topography) for scoliosis treatment and monitoring. Since spinal curvature just affects the posterior of the body, only back surface topography, rather the 360 degrees trunk surface, will be used in this study.

Surface topography is an instant success with patients and parents, in that they prefer not to be exposed to radiation and can believe that their complaint about deformity is being taken into account as well as the less easily accessible radiographic parameters. Surface topography is fundamentally an attempt to quantify what the clinicians and the patients are seeing, the cosmetic appearance. Furthermore surface topography is a valid
measure of trunk surface deformity and can be used in place of radiography for monitoring patients, particularly in minor cases and in young children.

2.2.2 Constraints in Scoliosis Modelling and Monitoring

Review of previous research on scoliosis surface deformation monitoring has been conducted. Several problems have been identified and outlined as follows:

i. Fiducial markers are not ethical.

Surface deformation analysis involves quantitative assessment for difference detection from two identical/non-identical surfaces at different epochs. It usually requires fixed control points to enable the positional relationship between surfaces to be recovered. However, in cases especially in biomedicine, it is unethical and often not feasible to place permanently marked control points on the patients. In such cases, surface deformation analysis can be accomplished by employing surface matching technique (Berg, 2002; Karras, 1993; Mitchell, 1998; Mitchell, 2006). Surface matching can be referred to as registration without control points, a method used to find the transformation parameters to minimise the separation and incorporate difference detection between surfaces. Most of the surface matching applications are used in area of photogrammetry and remote sensing (Akca, 2003; Ebner, 1988; Gruen, 2005; Karras, 1993; Maas, 2000; Miller, 2008; Mills, 2003; Mitchell, 1999; Pilgrim, 1996a, 1996b; Postolov, 1999; Rosenholm, 1988; Zhang, 2006), while only few are made clinically relevant and even rare in scoliosis application (Berg, 2002; Habib, 2001; Mitchell, 1998; Mitchell, 2006).

ii. X-rays are hazardous to children’s health and six month period for next medical visit is considered too long in contrast with adolescent growing speed.

Scoliosis causes lateral curvature on spine. Depending on the magnitude of the spinal curvature, patients typically undergo routine diagnostic radiographic examinations of the spine throughout the growth spurt until the spine curvature
stabilises (Doody, 2000; Levy, 1996; Pazos, 2005). They are repeatedly exposed to ionizing radiation throughout the diagnosis, which is hazardous to their long term health.

It is generally acknowledged that high radiation exposure is associated with increased cancer risk in children. Studies from Doody (2000), Levy (1996) and Pazos (2005) suggest that the clinical visit for scoliosis patients should limited to every six months. The intention is to reduce the frequency of radiography taken and limit cumulative radiation exposure. However, a six months interval is believed to be too long to follow up the scoliosis progression closely especially for children who are experiencing a golden period for their body development where growth is rapid. A non-invasive method to help in monitoring the progression in between this period will be preferable. This will not only reduce the high dose radiation exposure and risk but to provide additional complementary information for scoliosis diagnosis and management.

The major impetus for introducing non-invasive surface scanning is to reduce the radiation exposure in growing children to detect and monitor the progression of scoliosis. In this method, the patient’s trunk is scanned with a laser scanner to generate surface topography data. This surface topography produces a three-dimension map that will be used to describe and quantify the trunk surface deformity. Quantification of trunk surface deformity may aid diagnosis and monitoring of scoliosis and thereby minimise the use of radiographs. A method of documentation and monitoring of deformity that is inexpensive and noninvasive, as well as repeatable and reliable, should be welcome by the clinicians, patients and their parents. Surprisingly, little on the quantitative measurement of back surface deformation has been published.

Results from Pratt’s study (2002) deserve special attention in surface topography scoliosis research. In this study, results show that patient and parent perceptions of deformity were significantly related to surface topography before the surgery, but not with radiologic parameters of deformity. Therefore, as the authors have suggested, one should consider including surface topography in treatment decisions and follow-up.
iii. **Cobb angle does not describe the external asymmetries.**

Clinicians and patients are interested in measuring and understanding the progression and deformation of scoliosis. Basically, scoliosis involves internal (spinal deformity) and external (trunk deformity) changes. Internal changes in spinal deformity, both the spine angles and twists can be seen from X-rays and defined using Cobb angle, however external changes are often the primary concern of adolescent patients and their families because of its attendant social and psychological issues (Pratt, 2002). Current methods to measure external changes are based on non-invasive surface scanning.

Anyone with a view of scoliosis in which all deformity is mechanical and secondary to the spinal curve will not find topography a useful addition, because it will seem superfluous to needs and irritating in its failure to yield a radiation-free Cobb angle. However, the child with scoliosis is dissatisfied with the trunk deformity and not the Cobb angle (Goldberg, 2001). Furthermore, treatment attempts to improve both spinal and trunk deformities, but internal indices from Cobb angle do not describe the external deformity.

iv. **Current problem for surface matching in scoliosis monitoring.**

Matching the trunk surfaces in two successive clinical visits was proposed to monitor the change in the trunk surface features (Aggarwala, 2002; American Academy of Orthopaedic Surgeons, accessed 2007; Berg, 2002). Currently there are two methods to align the surface topographies to monitor scoliotic changes: manual and semi-automated modes (Aggarwala, 2002; American Academy of Orthopaedic Surgeons, accessed 2007; Berg, 2002; National Scoliosis Foundation, accessed 2007). The manual alignment method involves superimposing and manipulating the two images to obtain the closest fit. This manual mode is prone to error because it depends solely on operator interpretation. It is susceptible to the operator’s ability to align the surfaces manually without inducing distortion. This method is largely based on subjective visual observation rather than objective calculations. The semi-automated
technique is dependent upon accurately locating common points on both images. Fiducial markers are placed on the patient’s back before the scanning. Human error can be occurred in this stage while placing the fiducial markers on the patient. It is also unethical and often not feasible to place permanently marked control points on the patients. Therefore, both the manual or semi-automated modes are susceptible to human error that may lead to a misinterpretation of the progression of scoliosis. Thus, there is a need for a high accuracy method to match the surfaces and accommodate the questions like the non-rigid scoliotic shape change.

Berg (2002) have proposed a newer technique which involves subtracting aligned surface maps of the trunk surface to obtain “difference maps” that can be used to assess changes in trunk surface features over time. The first surface is fixed, and the second surface is manually manipulated to match with the first as the best fit. Once the two surface maps of the trunk surface are aligned together, the depth component is subtracted to provide a difference map. This technique yields a continuous measure of the changes in trunk surface that are difficult to capture from physical measures of deformity. Although the difference map provides a visual instrument showing how the trunk deformity has progressed, several limitations have been identified,

a. the process of aligning surface maps is based on manual best-fit technique which solely depends on individual visual interpretation that is prone to error and bias;
b. subtracting surface maps to obtain difference maps is not robust and repeatable making the maps variable;
c. it is difficult to obtain quantitative measures of change from difference maps;
d. the matching technique does not adequately account for presence of shape change.

Berg (2002) claim that surface matching in scoliosis monitoring must be fitted manually because it is difficult to reliably and accurately fit with common landmarks, fit to an analytical model, or ensure steep gradients exist from visit to
visit. They also comment that back surface fitting is unable to conform to any simple surface matching algorithm due to the complication of shape changing of the whole trunk that is unrelated to the effects due to scoliosis or through natural change. However the authors did not comment on how to reliably and accurately eliminate these noises (growth and/or outliers) to determine the true change on the trunk caused by scoliosis. It is commonly believed that manual fixing is prone to error and bias because it is based on individual visual observation and interpretation without accurate measurement. It is more important that we can not verify our result and identify the error with manual fixing. Obviously, automatic matching is still the preferred way as it can reduce individual error and the results can be compared and verified. However, modification of matching algorithm is necessary to accommodate the scoliotic shape changes. Extra parameters are required to accurately model the natural history.

v. **Less confidence in quantifying the true changes.**

Given the inability to accurately understand the progress for each patient, tools are required to detect past progression and accurate parameters for monitoring the change of scoliotic condition. Patients and their anxious parents sitting in the clinic waiting room are more concerned about the cosmetic appearance changes rather than the spinal curvature. The search for a useful role for surface topography in scoliosis management is not new, perhaps one shall consider how confidently scoliosis progression can be detected by using surface topography.

Although there are investigations which have developed indices to quantify the trunk deformities based on surface topography to estimate Cobb angle of scoliotic spinal deformity (Goldberg, 2001, Jaremko, 2001, Jaremko 2002), limited research has been conducted on quantifying confidently the true deformities change on trunk surface. Accurate and precise quantification will help the clinicians and patients for better treatment decision, not only based on spinal correction but also aesthetic improvement.

vi. **Hampered by lack of universal parameters to quantify the changes**
Although a variety of surface topography parameters has been proposed by different authors including “difference maps” by Berg (2002), however, there is still no consensus as to which parameter can best describe scoliosis deformity problem. It is more challenging that there is still no universally accepted parameter to define the surface deformity yet. Thus there is a need of a better method to mathematically model such as the shape change descriptions and natural influences to monitor the progression of trunk deformity.

This study is designed to examine the extent to which change in surface topographic parameters reliably detects change in the trunk and thus could be used to enhance monitoring of deformity and as a prerequisite for follow up radiographs. Surface topography is not found to be a quick fix but requires considerable effort to bring it to a useful point, both in the initial gaining of expertise in the simple operation and in the interpretation of the results. Surface topography is fundamentally an attempt to quantify what the clinician and parents are seeing, cosmetic appearance. It is unlikely that surface topography will supplant radiography because they are not measuring the same aspect of deformities. However, surface topography is useful in patient monitoring as an alternative to radiography.

2.3 Surface Matching without Control Points for Deformation Detection

Surfaces play an important role in diverse applications. Generally, many photogrammetric applications work with surfaces that are acquired by different sensors and/or at different epochs. There is an increasing need to combine and match these surfaces together and compare them (Akca, 2003; Gruen, 2005; Miller, 2008; Mitchell, 1998; Mitchell, 2006; Zhang, 2006). However, this procedure is usually referred to as registration especially in computer vision. Registration of surface models without the use of control points has been investigated by interested researchers in both computer vision and photogrammetry (Besl, 1992; Chen, 1992; Campbell, 2001; Ebner, 1988; Gruen, 2005; Habib, 2001; Karras, 1993; Maas, 2000; Miller, 2008; Mitchell, 1998; Pilgrim, 1996a, b; Postolov, 1999; Okatani, 2002; Rosenholm, 1988; Szeliski, 1996; Zhang, 1994, Zhang, 2006). But for most photogrammetric purposes, it is the
quantification of differences between the surfaces, rather than the surface registration itself, which is of greater concern.

Due to differences in data acquisition methodologies, surfaces involved in photogrammetric applications generally have different geometric and radiometric features. Surface matching is a fundamental procedure for these applications because manipulation of these height data requires them to be aligned relative to the same reference frame. An accurate matching strategy is crucial especially for change detection analysis because this requires a rigid transformation that pre-matches the surfaces.

Detecting differences or deformations between surfaces without control points is desirable for many industrial applications such as orthophoto production, object recognition, city modeling and soil erosion monitoring. Traditional techniques for surface deformation detection involve extensive use of control points. However, in many circumstances, establishment of control points is cumbersome, unreliable and sometime impossible. Therefore, surface matching without use of control points is of great concern in many industries. The essential task of detecting surface difference without control points is to match the surfaces precisely with no derived information in place of original and high data redundancy. According to the abstraction level of information used for matching, surface matching could belong to one of the three categories below:

i) Same physical surface acquired at different epochs in similar position, i.e. in the case of monitoring surface deformation and soil erosion.

ii) Same physical surface acquired by independent systems, i.e. in case of object recognition and security analysis.

iii) Different surfaces acquired by independent systems, i.e. in case of quality control.

With the advancement of surface related sensors, the collection of surface information becomes more easy and straightforward then ever in history. The extension of this advancement is the increasing demand for rapid generation of 3D surface models and analyses. The 3D surface model generated is actually a group of 3D point clouds
comprised of X, Y and Z Cartesian coordinate representing the object surface. Any related problem thus should be defined as a surface matching problem instead of 3D point clouds matching.

Similar to the photogrammetric application, biomedical analyses also deals with surfaces such as in studies of diagnosis progression where changes between surfaces from different epochs are detected (Habib, 2001). Therefore, surface matching is an essential task for this research to match scoliosis data from different epochs.

There are various factors that make surface matching a difficult problem. Two of them are the major concern of this study. One of the factors is the existence of outliers – point with large residual or without correspondence either due to noise interference during the data capture or changes in sizes (growth variation) or surface deformation. Secondly, the geometric transformations may require incorporating non-rigid mappings in order to account for shape deformation. It is thought that changes on the back surface due to scoliosis deformation are more complicated than rigid transformation, but may be well modelled by relatively simple deformation model with non-rigid transformation. Thus, the major task in this research is to design the non-rigid matching parameters to detect the scoliotic changes and at the same time reject the noises that may affect the matching.

Historically, surface matching has been classified as being ‘rigid’ where surfaces are assumed to be of objects that simply need to be rotated and translated with respect to one another to achieve convergence or ‘non-rigid’ where convergence between shapes in two surfaces cannot be achieved without some localised shearing of the surfaces. The following section will discuss the rigid and non-rigid transformation used in surface matching. This will begin with the rigid transformation and its various matching techniques ranging in complexity and followed by a brief discussion about non-rigid transformation. A more detailed discussion and non-rigid surface matching algorithm implementation is discussed and presented in Chapter 4.

2.4 Rigid Transformation

A rigid transformation (conformal transformation) is a series of rotation and translation transformations. Generally, the scale is set to uniform scale and shear is excluded.
Surface shape is maintained because lengths between points and angles between vectors are preserved. However, position and orientation is altered according to the transformation parameters. Figure 2.4 illustrates the transformation process to transform two different reference systems representing two surfaces into common frame.

2.4.1 The Transformation Equation

Suppose that two surfaces, Reference Surface ($S_1$) and Model Surface ($S_2$) are to be matched. In matrix notation, the rigid transformation equation that mathematically transforms $S_2$ to $S_1$ can be written as:

$$(X', Y', Z')^T = s * R * (X, Y, Z)^T + T$$

(2.1)

where,
(X, Y, Z)$^T$ is the coordinate of a model surface point before transformation
(X', Y', Z')$^T$ is the coordinate of a model surface point after transformation
s is the scaling factor
T is the 3x1 linear translation matrix:

$$T = [T_x, T_y, T_z]^T$$

R is the 3x3 rotation matrix with rotation sequence $\omega$, $\varphi$, and $\kappa$ about the X, Y, and Z axis respectively.

$$R = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$
with:

\[ m_{11} = \cos(\phi)\cos(\kappa); \]
\[ m_{12} = \sin(\omega)\sin(\phi)\cos(\kappa) + \cos(\omega)\sin(\kappa); \]
\[ m_{13} = -\cos(\omega)\sin(\phi)\cos(\kappa) + \sin(\omega)\sin(\kappa); \]
\[ m_{21} = -\cos(\phi)\sin(\kappa); \]
\[ m_{22} = -\sin(\omega)\sin(\phi)\sin(\kappa) + \cos(\omega)\cos(\kappa); \]
\[ m_{23} = \cos(\omega)\sin(\phi)\sin(\kappa) + \sin(\omega)\cos(\kappa); \]
\[ m_{31} = \sin(\phi); \]
\[ m_{32} = -\sin(\omega)\cos(\phi); \]
\[ m_{33} = \cos(\omega)\cos(\phi); \]

Equation 2.1 can be written as:

\[ X' = s(m_{11}X + m_{12}Y + m_{13}Z) + T_x; \]
\[ Y' = s(m_{21}X + m_{22}Y + m_{23}Z) + T_y; \]
\[ Z' = s(m_{31}X + m_{32}Y + m_{33}Z) + T_z; \]

(2.2)

Figure 2.4: Illustration of the rigid transformation that rotates and translates two different reference systems into common frame. (a) Before the transformation. (b) After the transformation.
If no a priori knowledge of the positioning is known, a common approach for surface matching is to use techniques based on the intrinsic properties of datasets, or geometric matching. The common approach is to first find the corresponding geometry and then estimate the rigid transformation parameters by minimising the objective function. This is followed by back-substituting the rigid transformation parameters to match the surfaces. The general surface matching problem has bred a number of matching algorithms. The Least Z Difference (LZD) and Iterative Closest Point (ICP) are the typical algorithms for surface matching. The latter is the standard algorithm in Computer Vision and Pattern Recognition. LZD is preferred by the researchers in photogrammetry because the ICP’s efficiency is lower than LZD (Mitchell, 1999; Zhang, 2005). The next section details the existing surface matching algorithms, namely ICP and LZD, which minimise the square distances between pairs of corresponding geometric found in the datasets to be matched to obtain the rigid transformation parameters.

2.4.2 Iterative Closest Point (ICP)

The geometric point matching basically consists in minimising the square distances between pairs of corresponding points in the datasets to be matched. It can be defined as follows:

Given two datasets $S_1$ and $S_2$ and a set of points of $S_1$ which have been paired with a set of corresponding points on $S_2$, denoted respectively by $\{S_1i\}$ and $\{S_2i\}$. Find the rigid transformation, defined by the rotation $R$ and the translation $T$, which minimises the sum of squares of the objective function.

One of the most popular methods of geometric point matching is called Iterative Closest Point (ICP) algorithm developed by Besl and McKay (Besl, 1992), Chen and Medioni (Chen, 1992), and Zhang (Zhang, 1994). The ICP method is based on the concept of searching for pairs of nearest points in the two surfaces, and estimating the rigid transformation parameters which align the surfaces. The rigid transformation parameters are applied to the points on $S_2$. Then, one can refine the resulting transformation by applying this procedure iteratively. The mean-squares objective function must be solved
at each iteration n. The global transformation \((R, T)\) is incrementally updated as follows: \(R = RR_n\) and \(T = T + T_n\).

The methods described by these authors (Besl, 1992; Chen, 1992; Jost, 2002; Zhang, 1994) differ mainly in how they establish point correspondences. Besl (1992) and Zhang (1994) use the point-to-nearest point distance as the objective function. An alternative approach to this search schema was proposed by Chen (1992). Chen (1992) calculates the closest point for a point from one surface by using the distance between the surfaces in the direction normal to the first surface. This point-to-tangent plane distance idea was originally proposed by Potmesil (1983). Menq (1992) finds closest points between a point set and a set of parametric surface patches by solving non-linear equations. Finally, Champleboux (1992) converts a point set into an octree-spline and computes Euclidean distances.

Although ICP has been a popular basis for numerous versatile surface matching algorithms, the ICP algorithm is not in consideration for this research for a number of reasons:

i. ICP assumes that one point set is the subset of the other. The algorithm was originally intended for object recognition and, thus, matching a \(S_2\) dataset that is a subset of \(S_1\) is not the case in this research scenario. Furthermore, when this assumption is not valid, false matches are created.

ii. The ICP algorithm requires an exhaustive search for the nearest points which is computationally expensive. The computational effort increases with the number of points in the matching process. Zhang (2005) has compared both ICP and LZD in terms of the efficiency of the algorithm. The study reported that ICP took more than nine times longer than the LZD method to achieve convergence using the same dataset. This result is supported by Xu (2000).

iii. Emphasis is put on the estimation of six rigid transformation parameters without a uniform scale parameter. The inclusion of a scale parameter solely depends on the research scenario, however, scale parameters are useful especially in medical tasks because they can be used to model the variable like growth variation.

iv. ICP algorithm is sensitive to outliers and does not have the ability to detect the possible deformation existing on the surface.
However, the ICP algorithm has the advantage of being insensitive to viewpoint direction (Godin, 1995). As recommended by Mitchell (1998), however, in realistic industrial and medical task, advantage may be taken of the fact that surface models may be measured with a high degree of positional relocation. In other words, the transformation parameters to align the surfaces are close to zero, a scenario which certainly occurs in the research reported in this thesis. ICP’s relative complexity may be unnecessary for conventional 3D topographic surfaces. Accordingly, the ICP algorithm was not considered in this research.

2.4. 3 Least Vertical Separation (Z) Difference (LZD)

Gruen (1985) first addressed the problem of surface matching and its solution method as a straight extension of Least Squares Matching (LSM). The theory behind LSM of surfaces warrants development. LSM is a technique which enables a poorly controlled matching surface to be registered to a well-controlled reference surface. Due to its high level of flexibility and its powerful mathematical model, the LSM concept has been applied to many different types of measurement and feature extraction problems. Ebner (1988) were among the first to propose LSM using Digital Terrain Models (DTM) as control information. This was followed by Rosenholm (1988) where the functional model of Digital Elevation Model (DEM) matching has been formulated. In both experiments, better results were obtained using a DEM to orientate photogrammetric data than with conventional methods (Karras, 1993; Rosenholm, 1988).

The concept of LSM is quite similar to ICP. A reference surface is held fixed, while the model surface (matching surface) is fitted to the reference surface. The solution is founded on the standard 3D rigid coordinate transformation, which defines the three rotations ($\omega, \varphi, \kappa$), three translations ($T_X, T_Y, T_Z$) and scale factor ($s$) as discussed in section 2.4. In Ebner (1988) and Rosenholm (1988), the matching method basically estimates the 3D similarity transformation parameters between two DTMs / DEMs, minimising the sum of squares of differences along the Z-axes, a vertical differences between the points on the model surface and corresponding surface patches on the reference surface. This ignores surface separations in the X and Y directions and other possible conditions, such as minimising the angles between the surface normal. It is
assumed that the Z separation is minimised when the surfaces are registered in the same coordinate system. Good initial approximations are required for the unknown transformation parameters as the problem is non-linear.

The LZD algorithm can be stated as follows:

i. input: point-patch sets, $S_2 = \{s_{2i}\}$ with $N_{S2}$ points (data) and $S_1 = \{s_{1i}\}$ with $N_{S1}$ patches (triangulate and interpolate to obtain the Z that is vertically-closest to the point in $S_2$)

ii. output: A transformation $(R, T)$ that registers $S_1$ and $S_2$

iii. initialisation: iteration $n = 0$, $S_{20} = S_2$, $R_0 = (0; 0; 0)$ and $T_0 = (0; 0; 0)$

iv. iteration $n$:
   a. Compute the slope distance between the search point with its nearest point:
      Substitute the slope distance into the observation equations and linearise using Taylor's expansion to obtain the partial derivative coefficients.

   b. Compute the registration:
      Define the mean squared error of the couplings $\{s_{2i,0}; s_{1i}\}$ as a function of $R_n$ and $T_n$. Compute the rigid transformation $(R_n, T_n)$.

   c. Apply the registration:
      Apply the rigid transformation to obtain the set $S_{2,n+1} = \{S_{2i,n+1}\}$

   d. Iteration termination:
      Stop when the maximum number of iterations or a defined criterion is reached. Set $R = R_n$ and $T = T_n$

By using LZD, the product of the matching is the ability to detect changes, as the residuals from the least squares calculation are the surface separations. If a convergent
solution is achieved, these disparities may represent errors in the solution or actual differences between the two surfaces. Consequently, an outlier detection module is crucial to differentiate the matching error from the actual surface changes. (A brief discussion about outlier detection methods is presented in the following section and implementation is presented in Chapter 5.) If the two surfaces being matched are representations of the same surface captured at different epochs, the residuals potentially represent change, an ability which is highly desirable in this research scenario. In fact, employing LZD for surface matching has already been applied in medical tasks using similar mathematical concepts but with different practical experience.

According to the literature studies, the LZD algorithm is feasible and favourable for scoliosis deformation monitoring for the following reasons:

i. LZD is an automatic matching algorithm which can reduce the user bias in manual matching.

ii. Successful applications in medical tasks have been reported.

iii. Simple mathematical algorithm with excellent results. Avoids using complex algorithm like ICP.

iv. LZD can be readily reformed in order to detect surface differences. Product of the matching can be directly used to model the changes as the residuals are the surface separation.

v. All information (point – patch pairs) is used and the redundancy is rather high.

However, the author has identified several of problems faced by LZD which deserved more discussion:

i. Reliance on existence of distinctive surface texture

The crux of the LZD algorithm problem relies on the use of surface gradient (surface slope) for a solution to be achieved. If there is no distinctive surface texture (smooth surface), the algorithm may lead to incorrect convergence. As discussed in (Mills, 2003; Pilgrim 1996a, 1996b) the precision of the matching solution depends on the nature of the coefficients in the linearisation of the
surface matching observation equations. Of particular interest is the
determination that the solution relies heavily on the surface gradients in the X
and the Y axes. Consequently, the LZD approach may be impractical for smooth
surfaces. Mills (2003) has conducted a simple analysis to show the importance
of predominant surface texture in the LZD algorithm. The reported results
showed that two flat surfaces being matched will allow only three parameters, ω,
φ, and T_z, to be found. Mills (2003) concluded that the lack of gradients in
different directions give unrealistic translation errors as the surfaces are free to
slide across each other. This was proven by conducting a simple analysis in this
research. A 40x40 gridded flat surface has been used in this analysis. A replica
of this surface was induced with mathematical transformations. The result
matched with the declaration by Mills (2003), that is only ω, φ, and T_z were
recovered.

ii. Surface gradient is not the best geometric features as the fundamental matching
criterion.

The best description of two points on a surface is the curvature between them.
(More detail discussion can be found in Chapter 3 section 3.6.) However, one of
the advantages of LZD algorithm is its simplicity to use. The calculation of
curvature and incorporated as the fundamental matching criterion will result in
the LZD algorithm being more complicated which is unnecessary and
unfavourable. Accordingly, replacing curvature as the fundamental matching
criterion is bypassed here. Replacing surface gradient as the fundamental
matching criterion, however, is the major issue needing to be resolved in LZD.

iii. Unable to address non-rigid deformation problem.

The existing six rigid transformations are not able to model the shape changes of
the surface. They are positional parameters and not shape parameters.
Consequently, the classical rigid transformation algorithm is not ready to be
used in shape change modelling. This will require modification of the existing
algorithm to recover the shape change in the surface. This is crucial to integrate
non-rigid parameters to model the non-rigid deformations including shape changes.

iv. LZD suffers from the difficulty that the differences between the two surfaces are generally too large to be effective. That is, the users are typically trying to align two surfaces which are too different to be aligned. Large residuals might be considered as outliers and not included in the matching procedure.

In order to avoid reliance on surface texture, the proposed matching algorithm in this research estimates the 3D transformation parameters between two surfaces by minimising the sum of squares of the Euclidean distances normal to the surfaces. Schenk (1999) and Habib (2001) used the same proposed method, minimising the distances between normal of the two surfaces, with reference to absolute orientation of imagery and change detection. The reported results proved that this method has higher accuracy and efficiency. The author has compared the surface matching methods, namely, modified ICP (MICP), LZD, combined ICP-LZD and Least Normal Distance Difference (LNDD). The results were matched with the results reported by Schenk (1999) and Habib (2001). Detailed discussion and implementation is presented in Chapter 3.

Consequently, least squares minimisation of normal distance difference forms the basis of this research. While the initial LZD algorithm was designed for application to 2.5D datasets, the author has tested that this LNDD algorithm is suitable for application to 3D data which normally comprises X, Y, and Z Cartesian coordinates. The author has also integrated nine non-rigid transformation parameters to resolve the modelling problem encountered in classical LZD algorithm for modelling of non-rigid deformations. For robustness, Least Trimmed Squares was first incorporated into LSM. The proposed mathematical model is another generalisation of LZD. More details can be found in Chapters 4 and 5.

2.5 Non-Rigid Transformation

Non-rigid transformation is the antonym of rigid transformation. It will neither preserve the lengths between points nor angles between vectors. In other words, non-rigid
transformation will change the surface position, orientation and its shape. Commonly, the scaling factor is expanded to all three axes instead of single uniform scale. Shear is attached in the transformation to deform the surface shape to improve the matching function. Today rigid transformation is often extended to include affine transformation which includes scale factors and shears.

Much of the non-rigid transformation work is in medical image registration due to its research nature to change the appearance, by rotating, translating, stretching, shearing etc., of one image so it more closely resembles another so the pair can be directly compared, combined or analysed. Examples of non-rigid image registration are to monitor and quantify disease progression over time in the individual (Pagani, 2007; Thompson, 2007), to build statistical models of structural variation in a population (Young, 2009; Zacharaki, 2009) or to register brain images of the same subject acquired with different modalities (Shams, 2007; Wang, 2007; Yanovsky, 2009). For more specific aspects of image registration, the reader is referred to other reviews; there is good technical coverage in Hill (2001), Brown (1992), Lester (1999), Maintz (1998) and Zitova (2003). It is not within the scope of this research to fully derive peculiarities of the non-rigid image registration algorithms. However, it is to incorporate non-rigid transformation parameters into existing LSM theory to match surfaces experiencing non-rigid shape change.

Clearly most of the human body does not conform to a rigid approximation (Brown, 1992; Hill, 2001; Lester, 1999; Thompson, 2007; Young, 2009; Zitova, 2003) and much of the most interesting and challenging work in surface matching today involves the development of non-rigid surface matching techniques. Deformation caused by scoliosis is non-rigid and so rigid transformation is not sufficient to transform and match the surfaces together. This will require non-rigid transformation and affine transformation is classified as non-rigid transformation. This research is among the first using non-rigid transformation for surface matching without control points.

Based on the rigid transformation model with seven unknown parameters (three rotations, three translations and a scale factor), a general affine transformation model with nine parameters (three rotations, three translations and three scales) has been studied. A new non-rigid transformation with a similar number but different parameters
has been derived. These nine new parameters are six shear parameters concatenated with three scale parameters. More detailed implementation of non-rigid transformation model can be found in Chapter 4.

2.6 Outlier Detection Techniques and Methodologies

According to Xu (2000), two surfaces cannot be precisely matched until the difference between the surfaces has been identified and removed from the data sets used for matching. On the other hand, the difference cannot be detected until the two surfaces have been precisely matched. Therefore, the procedure of surface matching must be performed simultaneously with the procedure of difference detection. However, one should be aware that the difference between surfaces can be masked by outliers, large random errors generated in the data collection process. An outlier is defined as an observation that is numerically distant from the majority or “bulk” of the data, or which in some way deviates from the general pattern of the data (Maronna, 2006). That is, the detected surface differences from surface matching algorithm execution could be the actual difference, or the outliers, or the general noise. Moreover, the final result of surface matching can be adversely biased by outliers in the data.

According to Rousseeuw, (1987) and Maronna (2006), classical estimates such as the sample mean, the sample variance, sample covariance and correlations, or the least squares fit of a regression model, can be adversely influenced by outliers, even by a single one, and often fail to provide good fits to the data. A direct requirement of a good fit to the data is therefore the reliable detection of outliers. Robust parameter estimates are required to provide good fit to the data when the data contain outliers, as well as when the data are free of them. Short range laser scanner is used to capture 3D data of back surface to aid in the non-invasive monitoring of scoliosis. However, human subjects can be positioned differently each time and factors like scoliosis progression and natural growth might lead to anatomical changes. As a result, “raw” 3D data might be “blinded” by different kind of noise like outliers. It is desirable to integrate robust estimator into the surface matching algorithm to obtain the robustness in order to increase the final result confidence level.
There are two major approaches for dealing with outliers: (i) outlier detection and (ii) robust estimation (Pilgrim, 1996a, 1996b; Xu, 2000). In the outlier detection approach, one first tries to detect (and remove) outliers and then perform estimation with the “clean” observation. The robust approach employs estimators which are relatively insensitive to outliers to produce reliable estimation.

There are many statistics based on residuals, usually resulting from least squares adjustment, designed to measure the variability of observations. Two popular statistics are: (i) normalised residual developed by Baarda (1968) and (ii) studentised residual proposed by Pope (1976). Statistical testing procedure using either of these two statistics is known as data snooping. The major problem arising from data snooping is that it only applicable under the assumption that no more than one outlier is present. Although statistics for multiple outlier detection do exist, they are not practical because we do not know a priori how many outliers are present and where they are. However, this can be compromised by: first, iteratively performing single-case outlier detection; second, combining outlier detection techniques with robust estimator (Pilgrim, 1996b).

In robust estimation, outliers could be discovered as those having large residuals. Two important concepts related to a robust estimator are: (i) breakdown point and (ii) the relative statistical efficiency.

The breakdown point of an estimator is the smallest fraction of outlier contamination that can cause the estimates arbitrarily biased (Rousseeuw, 1987, 2006). For example, the breakdown point of least squares estimator is 0 since a single outlier can make the result invalid, while the median is a robust measure of central tendency which has breakdown point of 0.5 (50%), the highest possible value.

The relative statistical efficiency of an estimator is defined as the ratio between the lowest achievable variance for the estimated parameters and actual variance provided by the estimator (Huang, 1990; Meer, 1991). The highest possible efficiency of an estimator is 1.

To deal with possible deformation changes and the presence of possible outliers, several robust versions of LSM using different robust estimators can be found in the literature.
Karras (1993) are among the first to employ outlier detection to improve LSM precision. DEM matching technique was applied and extended with a data snooping technique to detect deformation of human body. The results showed that LSM with data snooping performs well when deformation is of large magnitude but small proportion. It gives inferior results for deformation of relatively small magnitude but large proportion, however. With the aim to improve the deformation detection in LSM, Pilgrim (1996a, b) has applied the M-estimator to obtain the robustness of the matching procedure. The classical LSM is converted into iterations of reweighted least squares matching. The weights are determined as and are updated during iterations according to the updated residuals and estimate of standard deviation of observations. The objective function of the LZD method can be written as:

$$\sum w_i dZ_i^2 = \min$$

(2.3)

where, $w_i$ denotes the weight with either 1 or 0, and $dZ$ is the difference in height value of each matched point.

When the weight is equal to 1 for all matched points, convergence is achieved for the two surfaces. It is reported that this robust matching procedure is capable of detecting difference covering up to 25% of surfaces being matched. The M-estimator has high statistical efficiencies, typically more than 0.9 (Kumar, 1994; Huang, 1990). However, it has breakdown points less than $1/(p+1)$, where $p$ is the number of parameters to be estimated (Meer et. al.1991). In Pilgrim’s research, $p$ is 6 and the breakdown point is consequently less than $1/7(\approx 0.143)$. In this research, $p$ is 9 and the breakdown point is $1/9(\approx 0.111)$. This low breakdown point is not desirable especially in this research where the surfaces are experiencing possible high deformation. As argued in Hampel (1986), many M-estimators are constructed to be efficient on the one hand and fairly robust on the other hand, with high breakdown point but low statistical efficiency or vice versa.

In another study conducted by Li (2001), the Least Median Squares (LMS) estimator was incorporated with the LZD algorithm and then proposed a new algorithm, called
LMS-LZD method. LMS is proposed and developed by Rousseeuw and Leroy (1987), which is given by:

\[
\text{Med } r_i^2 = \min
\]

(2.4)

where, Med means the median of the LMS estimator, and \( r_i^2 \) is the residual of observations.

In LMS, the estimates must yield the smallest value for the median of squared residuals computed for all observations. Accordingly, the LMS estimator has the highest possible breakdown point which is 0.5. This can be achieved because the LMS estimates need only to fit well half of the observations and ignore the other half. However, this has resulted LMS has relatively abnormally low relative statistical efficiency. This can be intuitively understood by noting that half of the observations have no actual influence on estimates.

2.7 Chapter Summary

The author has introduced the current methods used in surface matching without control points for deformation monitoring. The current constraints faced by the researchers in spatial data manipulation have also been discussed. Constraints in scoliosis modelling and monitoring have been presented. This introduction has highlighted the fact that surface matching is a feasible tool for matching the scoliosis data for modelling and monitoring purposes. The goal of this research is to modify and develop a new surface matching algorithm based on the least squares method that is suitable to match the scoliosis data into a same reference frame for deformation modelling and monitoring purposes. To achieve this, a non-rigid matching algorithm with new transformation objective function will be designed and developed. New transformation parameters will be employed to overcome the drawbacks faced by the classical matching algorithm. To obtain the robustness of the algorithm, least trimmed squares will be applied. This will be discussed in Chapters 3, 4 and 5.
CHAPTER 3

SELECTION OF THE OPTIMUM OBJECTIVE FUNCTION AND TRANSFORMATION PROCEDURE

3.1 Introduction

In this chapter, the author will discuss and present different rigid surface matching algorithms. The goal is to select the optimum objective function and transformation procedure that is best to be implemented in the proposed non-rigid surface matching algorithm. To achieve this, four different surface matching algorithms based on a rigid transformation, involving six unknown parameters with three rotations and three translations (scale is set to be equal to one as the surfaces have assumed to have captured from same object), have been developed and compared to test the optimum objective and transformation procedure for non-rigid transformation. These four algorithms are able to find the optimal transformation parameters without control points such that the differences between two sets of surface descriptions become minimal.

i) The first transformation function is a modified Iterative Closest Point (ICP) algorithm that minimise sum of the distance between temporarily paired points. Curvature features are used as the fundamental in the matching procedure to improve this matching algorithm. For briefness, this algorithm is called Modified Iterative Closest Point (MICP).

ii) The second transformation function is a matching algorithm that minimises the sum of Z coordinate differences between pairs of points, which lie on different surfaces but with close X and Y coordinates. For briefness, the author has given this method the name Least Z Difference (LZD). A brief discussion about ICP and LZD is presented in Chapter 2.
iii) The third approach is a combination of ICP and LZD algorithms. The idea to combine both algorithms can be attributed to the author’s awareness of the successful ICP and LZD algorithms in surface matching. The main practical difficulties of ICP are its complexity and sensitivity to surface deformation. The advantage of LZD lies in its ability to detect the surface deformation simultaneously and its simplicity. However, the use of triangulation and interpolation are one of the major issues associated with this matching algorithm. By integration of LZD into ICP, it is expected that it will simplify the function, reduce triangulation and interpolation processes, and the residual of separation is ready to be used to detect the surface change.

iv) The final approach that the author tested is to minimise the sum of the normal distances (Euclidean distance between the surfaces that are perpendicular to the first) between provisionally paired points and patches. The author calls this function as Least Normal Distance Difference (LNDD).

Such procedures, MICP, LZD, Combined ICP-LZD and LNDD, would determine the transformation procedure that minimise respective objective function according to least squares principle. These matching algorithms do not require control points to exist between surfaces in order to achieve matching. As long as the two surfaces are approximately aligned then successful matching can occur. This implies that the differences between the two surfaces are assumed to be local; hence, the remaining distances after establishing the transformation parameters are residuals. The method to decide the optimum objective function to use for non-rigid transformation is based on the r.m.s. error of residual of the final result for both synthetic and real data.

All the transformation models (rigid or non-rigid) were undertaken in the MATLAB programming environment. Oftentimes, it is required to organise coordinates into matrix form especially for large datasets to simplify the calculation. MATLAB contains mathematical and statistical functions to support all common engineering and scientific functions. Further to this, matrices are the fundamental data type in MATLAB making it an ideal programming environment for this research. Meanwhile all the graphical features that are required to visualise the final result are available in MATLAB.
3.2 Transformation Function I - Modified Iterative Closest Point (MICP)

Consider an object surface extracted from 3D point data as the Model Surface \( (S_2) \) and to be registered to a Reference Surface \( (S_1) \) extracted from another 3D point data set. For each point on \( S_2 \) the point on the \( S_1 \) which is closest to \( S_2 \) is defined as temporarily matching point. This particular point on \( S_1 \) can be classified as a correspondence point to the search point on \( S_2 \). Then, the rigid movement is applied to the model point so that the least squares distance criterion is satisfied for the defined correspondence point. For the newly set position for the model, a temporary match point is defined based on the shortest distance, and the above procedure is repeated until the stopping threshold is satisfied. This transformation algorithm has been widely used in surface registration especially in biomedical analysis and robotic engineering (Campbell, 2001; Hill, 2001; Jost, 2002; Lester, 1999; Mantz, 1998; Pagain, 2007; Shams, 2007; Thompson, 2007; Wang, 2007; Yanovsky, 2009; Young, 2009; Zacharaki, 2009; Zhang, 1994; Zitova, 2003).

In order to improve the accuracy of this algorithm, the author has modified the searching criterion for the correspondence point by adding geometric differential information as an extra constraint. Principal curvature, maximal and minimal curvature \((k_1 \text{ and } k_2)\), has been selected for this purpose because this geometric feature is invariant under rigid transformation. This approach is relatively simple, first k-nearest neighbour technique (KNN) is employed to search five nearest neighbour points for every model point on \( S_2 \)(the number of nearest neighbour points can increase according to the nature of application). This will follow by filtering out these five nearest neighbour points by comparing the principal curvature to get the corresponding point. The improvement of this method is that the point feature is directly deal with the point that has close relationship (particular in curvature). The least squares approach which minimizes the sum of the distances between points is applied to rigidly move \( S_2 \) to match with \( S_1 \). Figure 3.1 shows the flowchart of MICP matching programme and Figure 3.2 illustrates the concept of getting the corresponding point and minimising the separation between surfaces. Details of the matching algorithm are described in Section 3.2.1 and 3.2.2.
Start Programme

Read Reference Surface ($S_1$)

Read Model Surface ($S_2$)

Compute Principal Curvature ($k_1$ & $k_2$)

Compute Principal Curvature ($k_1$ & $k_2$)

Find Five Closest Points for Every Model Point Using K-Nearest Neighbour Technique (KNN)

Compare the principal curvature to get the corresponding point

Form L and A Matrices as in Equation 3.14

Solve Least Squares Solution as in Equation 3.15

Update Parameters

Transform $S_2$ with Updated Parameters

Convergence Achieved?

No

Start New Iteration with Transformed $S_2$

Yes

Display Results

Figure 3.1: MICP principle
Figure 3.2: Concept of MICP. (a) Search five closest points for each search point. (b) Filtering out these five nearest points by comparing the principal curvature to get the corresponding point. The different shape marks indicate different curvature. (c) Apply rigid movement until least squares distance criterion is satisfied.
3.2.1 Integration of Invariant Feature, Principal Curvature, to Modify the Classical ICP

Principal curvatures are defined as the extreme value (maximal and minimal) of normal curvature at a given point on a surface. The principal curvatures measure the maximum and minimum bending amount at different direction of a regular surface at each point. Principal curvatures are denoted as $k_1$ and $k_2$ and are invariant under rotation. This matches with the requirement for an invariant feature that can be attached to the correspondence point search that suggests better matches. Below is the formula to compute principal curvatures of a parametric surface $S$. For detailed presentation, refer to Manfredo’s (1976) book.

Any curve $C$ on a surface $S$ can be defined as $S : C(u,v)$. It can be represented in the parameterisation form:

$$u = u(t), \quad v = v(t)$$

(3.1)

The vectors $Cu$, $Cv$ forms a basis tangent space $T_s(C)$. To determine the element of arc of such a curve, this will lead us to the first and second fundamental form which is of basic importance in calculating the curvature and metric properties of a surface. The three coefficients of first fundamental form, $E$, $F$ and $G$, can be expressed in the basis $[Cu, Cv]$ as:

$$E = Cu \cdot Cu, \quad F = Cu \cdot Cv = Cv \cdot Cu, \quad G = Cv \cdot Cv$$

(3.2)

The second fundamental form has an intuitive interpretation in terms of the first fundamental. The three coefficients of second fundamental form are:

$$L = n \cdot E, \quad M = n \cdot F, \quad N = n \cdot G$$

(3.3)
where,

\[
n = \frac{Cu \times Cv}{\|Cu \times Cv\|}
\]

is the normal to the surface. In practice, we will first compute the Gaussian curvature (K) and Mean curvature (H):

\[
K = \frac{LN - M^2}{EG - F^2}; \quad H = \frac{GL + EN - 2FM}{2(EG - F^2)}
\]

(3.4)

because the Gaussian and mean curvature are related to \(k_1\) and \(k_2\) by

\[
K = k_1 \ast k_2; \quad H = \frac{(k_1 + k_2)}{2}
\]

(3.5)

then we compute the principal curvatures.

\[
k_1 = H + (H^2 - K)^{1/2}; \quad k_2 = H - (H^2 - K)^{1/2}
\]

(3.6)

### 3.2.2 The Basic Estimation Model

In Euclidean three-space, the distance between two points is defined as:

\[
d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}
\]

(3.7)

The problem statement is to estimate the final location and orientation of the model surface \(S_2(X, Y, Z)\), which satisfies the minimum condition of least squares matching with respect to reference surface \(S_1(X, Y, Z)\). The functional model is:
\[ S_1(X,Y,Z)^T = S_2(X,Y,Z)^T \]  
\[ (3.8) \]

\[ S_2(X,Y,Z)^T = T + mRS_{20}(X,Y,Z)^T \]  
\[ (3.9) \]

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
= \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix} + m
\begin{pmatrix}
 r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{pmatrix}
\begin{pmatrix}
X_0 \\
Y_0 \\
Z_0
\end{pmatrix}
\]
\[ (3.10) \]

where, \( S_1(X,Y,Z)^T \) is the coordinate of a point in \( S_1 \), \( S_2(X,Y,Z)^T \) is the coordinate of a point in \( S_2 \), \( r_j = R(\omega, \phi, \kappa) \) are the elements of the orthogonal rotation matrix (refer Chapter 2 Section 2.4.1 for details), \( (T_X, T_Y, T_Z)^T \) is the translation vector, \( m \) is the uniform scale factor, and subscript \( _0 \) indicates evaluation at the estimated match position.

In order to perform least squares estimation, equation 3.8 must be linearised by Taylor’s expansion, ignoring second and higher order terms.

\[
S_1(X,Y,Z) = S_{20}(X,Y,Z) + \left[ \left( \frac{\partial S_{20}(X,Y,Z)}{\partial X} \right) dX \right] + \left[ \left( \frac{\partial S_{20}(X,Y,Z)}{\partial Y} \right) dY \right] + \left[ \left( \frac{\partial S_{20}(X,Y,Z)}{\partial Z} \right) dZ \right]
\]
\[ (3.11) \]

where, subscript \( _0 \) indicates evaluation at the estimated match position, with:

\[
dX = (\partial X/\partial \pi_i) * dpi;
\]
\[
dY = (\partial Y/\partial \pi_i) * dpi;
\]
\[
dZ = (\partial Z/\partial \pi_i) * dpi;
\]

where, \( \pi_i \in \{ \omega, \phi, \kappa, T_X, T_Y, T_Z \} \) is the i-th transformation parameters in equation 3.11.

Differentiation of equation 3.11 gives:
\[ dX = a_{13}d\omega + a_{12}d\varphi + a_{11}d\kappa + T_X ; \]
\[ dY = a_{23}d\omega + a_{22}d\varphi + a_{21}d\kappa + T_Y ; \]
\[ dZ = a_{33}d\omega + a_{32}d\varphi + a_{31}d\kappa + T_Z ; \]

(3.12)

where, \( a_{ij} \) are the coefficient terms whose expansions are trivial (the solution is derived in Appendix A). Using the following notation:

\[ S_{2X} = \frac{\partial S_{20}(X,Y,Z)}{\partial X} = \frac{S_X - S_{20}X}{d} ; \]
\[ S_{2Y} = \frac{\partial S_{20}(X,Y,Z)}{\partial Y} = \frac{S_Y - S_{20}Y}{d} ; \]
\[ S_{2Z} = \frac{\partial S_{20}(X,Y,Z)}{\partial Z} = \frac{S_Z - S_{20}Z}{d} ; \]

(3.13)

where, \( d \) is the Euclidean distance between points and \( \{S_{2X}, S_{2Y}, S_{2Z}\} \) are numeric first derivatives of function \( S_2(X, Y, Z) \).

Substituting equation 3.13, equation 3.11 becomes:

\[ S_1(X,Y,Z) - S_{20}(X,Y,Z) = d \]
\[ = (S_{2X}.a_{11} + S_{2Y}.a_{21} + S_{2Z}.a_{31})d\omega \]
\[ + (S_{2X}.a_{12} + S_{2Y}.a_{22} + S_{2Z}.a_{32})d\varphi \]
\[ + (S_{2X}.a_{13} + S_{2Y}.a_{23} + S_{2Z}.a_{33})d\kappa + S_{2X}.T_X + S_{2Y}.T_Y + S_{2Z}.T_Z \]

(3.14)

Each temporarily paired point is setup with one observation equation of the form of equation 3.14 and can be arranged in matrix notation to give:

\[ V_{(i,j)} = A_{(i,6)} * \Delta_{(a,i)} + L_{(i,j)} \]

(3.15)
where,

\[ V_{(i, 1)} \] is the vector of residual

\[ A_{(i, 6)} \] is the coefficient matrix from equation 3.14

\[ \Delta_{(6, 1)} \] is the vector of correction to the initial estimated unknown parameters

\[ L_{(i, 1)} \] is the vector of observation which in this case is the Euclidean distance differences,

\[ d \]

\[ i \] is the number of points on the surface

### 3.3 Transformation Function II - Least Z Coordinate Difference (LZD)

The LZD transformation function is a little different compared with the other two transformation functions. Although matching of the model surface uses three dimensional coordinate transformation principles, the set of observation equations of this technique is confined to the surface separation in the Z coordinate direction at each observation point on one surface. Separation in the X and Y coordinate directions is ignored. The LZD programme structure is shown in Figure 3.3. Refer Chapter 2 Section 2.4.3 for the programme structure discussion. The concept in getting the corresponding point and minimising the separation between surfaces are illustrated in Figure 3.4.

A least squares minimisation procedure, making use of surface slopes as the value expressing the geometry characteristic, was used in LZD. The entire matching process relies on surface slope value to calculate estimates of the unknown transformation parameters. A very simple method was employed to calculate the slope values necessary in the matching algorithm. For each data point \( X_n \) a slope value is calculated using point \( X_{n-1} \) and \( X_{n+1} \). The gradient between these two points is calculated and assigned as the slope values in the X direction by \( \frac{dX}{dZ} \). The same process is applied in the Y direction by \( \frac{dY}{dZ} \). The surface slope was chosen as the value expressing the geometry characteristic to the matching procedure for a couple of reasons. First, the use of surface slopes can make almost direct use of the 3D coordinate data, making it appealing to be used. Further, there are variety of techniques can be employed for slope calculation, giving it high flexibility of use. Also, vector characteristics of slope can be used to enhance the determination of a successful match, which can be used to remove relief mismatch between surfaces (Rosenholm, 1988; Karras, 1993; Pilgrim 1996a, 1996b;
and Mitchell, 1999). Although slope in association with the observed data has a number of benefits as claimed by the authors, this matching technique encounters a major drawback which is not discussed. Further discussion on the drawback of using slope as the value expressing surface feature can be found in Section 3.6.

Figure 3.3: LZD principle
Figure 3.4: Concept of LZD. (a) $S_1$ is first triangulated or gridded. This is then interpolating $S_1$ to obtain the Z value that is vertically-closest to the search point. Filled green bubbles indicate the interpolated Z value. (b) Compute the surface slope for every search point in $S_2$, dark green lines. (c) Apply rigid movement until least squares distance criterion is satisfied.
Taking the third row of rigid transformation equation ignoring scale factor (assumed to be one), equation 2.2 gives:

\[
S_2Z = (m_{31}S_{20}X + m_{32}S_{20}Y + m_{33}S_{20}Z) + T_Z;
\]

(3.16)

where, \(m_{ij}\) are elements in Rotation matrix.

Linearising equation 3.16 using Taylor’s expansion ignoring second and higher order terms, results in:

\[
S_2Z = S_{20}Z + dT_Z + S_{20}X *[m_{21} * d\omega - \cos(\omega) * m_{11} * d\varphi + m_{32} * d\kappa] \\
+ S_{20}Y *[m_{22} * d\omega - \cos(\omega) * m_{12} * d\varphi + m_{31} * d\kappa] \\
+ S_{20}Z *[m_{23} * d\omega - \cos(\omega) * m_{13} * d\varphi]
\]

(3.17)

where, subscript \(0\) indicates evaluation at the initial approximation match position, \(d\omega, d\varphi, d\kappa\) and \(dT_Z\) are the corrections to the initial approximations. The units of \(d\omega, d\varphi, d\kappa\) are radians.

A general relationship exists for three dimensional surface in a cartesian coordinate system.

\[
S_2Z = f(S_2X, S_2Y)
\]

(3.18)

Linearise equation 3.18 by using Taylor’s expansion and the following equation result:

\[
S_2Z = f(S_2X, S_2Y)_0 + \left(\frac{\partial f}{\partial S_2X}\right)_x \ast X_x + \left(\frac{\partial f}{\partial S_2Y}\right)_y \ast Y_y
\]

(3.19)

where,
\[ X_s = S_2X - (S_2X)_0; \]
\[ Y_s = S_2Y - (S_2Y)_0; \]

\( \partial f / \partial S_2X \) and \( \partial f / \partial S_2Y \) are the surface slope in the X and Y directions respectively.

Equating equation 3.17 and equation 3.19 and substituting similar equations for \( S_2X \) and \( S_2Y \), the following equations result:

\[
\delta Z + \Delta V = \left[ \begin{array}{c}
S_{20}X \cdot \left(m_{21} + \partial f / \partial S_2Y \cdot m_{31}\right) + S_{20}Y \cdot \left(m_{22} + \partial f / \partial S_2Y \cdot m_{32}\right) \\
+ S_{20}Z \cdot \left(m_{23} + \partial f / \partial S_2Y \cdot m_{33}\right) \end{array} \right] d\omega \\
+ \left[ \begin{array}{c}
S_{20}X \cdot (-\cos(\omega) \cdot m_{11} + \partial f / \partial S_2X \cdot \cos(\kappa) \cdot m_{13} - \partial f / \partial S_2Y \cdot \sin(\omega) \cdot m_{13}) \\
+ S_{20}Y \cdot (-\cos(\omega) \cdot m_{12} + \partial f / \partial S_2X \cdot \sin(\kappa) \cdot m_{13} - \partial f / \partial S_2Y \cdot \sin(\omega) \cdot m_{12}) \\
+ S_{20}Z \cdot (-\cos(\omega) \cdot m_{13} + \partial f / \partial S_2X \cdot \cos(\varphi) - \partial f / \partial S_2Y \cdot \sin(\omega) \cdot m_{13}) \end{array} \right] d\varphi \\
+ \left[ \begin{array}{c}
S_{20}X \cdot (m_{32} + \partial f / \partial S_2X \cdot m_{32} - \partial f / \partial S_2Y \cdot m_{22}) \\
+ S_{20}Y \cdot (-m_{31} + \partial f / \partial S_2X \cdot m_{31} - \partial f / \partial S_2Y \cdot m_{21}) \end{array} \right] d\kappa \\
- dT_X \cdot \partial f / \partial S_2X - dT_Y \cdot \partial f / \partial S_2Y + dT_Z \]

(3.20)

where,
\( \omega \) is the rotation about X axis
\( \varphi \) is the rotation about Y axis
\( \kappa \) is the rotation about Z axis
\( T_X \) is the translation in X axis
\( T_Y \) is the translation in Y axis
\( T_Z \) is the translation in Z axis
\( \Delta V \) is the residuals
\( d\omega, d\varphi, d\kappa, dT_X, dT_Y, \) and \( dT_Z \) are corrections to the initial approximation
\( \delta Z \) is the vertical difference between surfaces

For every matched point, one observation equation of the form of equation 3.20 is setup and arrange in matrix form as equation 3.15 for least squares solution. The tool to
construct L and A matrices is the 3-dimensional rigid transformation. The transformation relates the element points in Model Surface \( S_2 \) with the element points in \( S_1 \). The least squares approach obtains the solution for vector \( \Delta \) by minimising the vertical difference \( dZ \) in element vector L. For detailed derivation about LZD algorithm, the readers can refer to dissertation published by Pilgrim, 1991.

### 3.4 Transformation Function III - Combined ICP-LZD

Combined ICP-LZD is a method that estimates the geometric relationship transformation parameters between two surfaces by finding the correspondence nearest point, minimising vertical difference between surfaces using the least squares approach. This method involves mathematical orientation of a surface with continuous refinement of the transformation parameters until the closest coincidence with another surface is found.

Figure 3.5 shows the principle of Combined ICP-LZD and Figure 3.6 illustrates the matching concept of this algorithm. Let the Reference Surface be \( S_1 \) and Model Surface be \( S_2 \). Both surfaces are described by \( m \) and \( n \) discrete points respectively that are randomly distributed. Suppose that both surfaces are, in fact, describing the same object surface, however, they are captured in different epochs causing possible dissimilarity in term of data density, distribution and accuracy. Both surfaces are not 100% aligned and transformation parameters need to be established for surface matching to detect surface deformation. The problem is cast as an adjustment problem where \( S_2 \) is transformed to \( S_1 \) in a way minimising the vertical difference between the two surfaces. The solution is iterated approaching to vertical difference minimum or the surfaces convergence. In the first iteration, initial approximation parameters must be provided. The three rotation and three translation parameters are set as \([0,0,0,0,0,0]^T\). The convergence behaviour basically depends on the quality of the initial approximation parameters.
Start Programme

Read Reference Surface (S₁)

Read Model Surface (S₂)

Compute the Slope Distance in X and Y directions for every Search Point

Compute the Nearest Point for Every Search Point in S₂ from S₁

Compute the Vertical Difference between S₁ and S₂

Form L and A Matrices as in Equation 3.20

Solve Least Squares Solution as in Equation 3.15

Update Parameters

Transform S₂ with Updated Parameters

Convergence Achieved?

No

Yes

Display Results

Start New Iteration with Transformed S₂

Figure 3.5: Combined ICP LZD principle
Figure 3.6: Matching concept of Combined ICP-LZD. (a) Search closest point on $S_1$ for each search point. (b) Compute the surface slope for every search point in $S_2$, dark green lines. (c) Apply rigid movement until least squares distance criterion is satisfied.
The nearest point matching proceeds as follows:

- Compute the coefficients of six transformation parameter coefficients. For every point in $S_2$, find the nearest point to obtain the surface slope in $X$ and $Y$ direction. These values are substituted into the observation equation to get the coefficient.
- Compute the nearest point for every search point in $S_2$ from $S_1$ (Figure 3.6). Calculate the vertical difference between $S_1$ and $S_2$.
- Calculate the transformation parameters using the least squares approach by minimising vertical difference. Return the transformation parameters which match the surfaces.
- Transform $S_2$ according to the transformation parameters found.
- Iterate this process until satisfactory convergence has been reached.

Because of high density point clouds that can be generated by laser scanner, this algorithm assumes that by finding the nearest point for each point matched is sufficient to establish the transformation parameters. In matrix notation, the 3-dimensional rigid transformation equation can be written as equation 3.9. For every point matched in the matching process, an observation equation is set up as in equation 3.20. This observation equation is repeated for $i$ pair points matched which being given in the general form necessary for least square adjustment as in equation 3.15.

The author assumed that the density of the laser scanned data is high enough to compensate for the random error generated by the process of finding the vertical difference, $dZ$. The vertical difference represents the vertical separation between $S_2$ and corresponding point in $S_1$. The corresponding point in $S_1$ normally needs to be found by triangulating and interpolating available surround points. This will reduce the accuracy and precision of the matching. Due to this reason, during the searching for corresponding nearest point in $S_1$, the nearest point found is the point that is close to the search point in $S_2$ (assumed position) but not the actual interpolated position of search point in $S_1$ (red point in Figure 3.7). The Vertical difference is calculated based on the coordinate of the assumed point position. Random error is generated here but the author believes that this error is negligible due to the high data density with comparatively small spacing. Because this algorithm is not using the actual interpolated position in $S_1$,
the author proposed to obtain the surface slope using nearest point algorithm in $S_2$. The author assumed that both surfaces are having very close surface slope similarity in X and Y direction. The program will search for the nearest point in $S_2$, calculate the surface slope difference between $dX/dZ$ and $dY/dZ$ in X and Y direction respectively. Thus, the derivations and the experiments performed here are based on several important assumptions as follows:

- Although both surfaces are not 100% aligned, they are relatively close and parallel.
- A point in $S_1$ can be the nearest point for more than one point in $S_2$.
- This algorithm is using the nearest point (assumed position) but not the interpolated actual position of the search point in $S_1$ (refer Figure 3.7). This process is inevitably generates the random error. However, the author considers this random error is negligible in getting the vertical difference. The reason behind this is that the surfaces are not 100% aligned, the nearest point in $S_1$ is possibly very close to the actual point.
- Gradients in both surfaces have high similarity with minor deformation. Thus, either of the surfaces can be used to generate the surface slope in both X and Y direction. Because this algorithm is not using the actual interpolated position in $S_1$, the surface slope is generated from the search point in $S_2$. 


Figure 3.7: Assumptions have been made in the experiment in getting the vertical difference and surface slope. The program needs to match points in $S_2$ to points in $S_1$. The blue point in $S_1$ is the nearest point to the search point in $S_2$. However, this is the assumed position of this search point in $S_1$. If absolute matching without any transformation is performed, the position that is perpendicular to the search point is the actual position in $S_1$ (red point position). Random error is generated because of the difference between actual and assumed position. Since both surfaces are not 100% aligned, we consider this random error is negligible. Surface slope in direction X and Y is calculated from $S_2$ instead of $S_1$. 
3.5 Transformation Function IV - Least Normal Distance Difference (LNDD)

The mechanism for this transformation function is simple. The correspondence between surfaces is established by associating a patch on Reference Surface $S_1$ with a point on Model Surface $S_2$. There are many ways to represent a patch $P$ on $S_1$. One way to represent $P$ is specify 3 non-collinear points $p_0 = (X_0, Y_0, Z_0), p_1 = (X_1, Y_1, Z_1), p_2 = (X_2, Y_2, Z_2)$ as the vertices of a triangle, the most primitive of a planar objects. As a result, the reference surface $R$ will be triangulated using Delaunay Triangulation to generate patches. Another reason to use Delaunay triangulation is to avoid the loss of accuracy from interpolating the raw data.

Correspondence between surface $S_1$ and surface $S_2$ is established by minimising the separation between the point and the surface patch. The separation is measured as a set of normal distances $D$ (the normal distance to a plane is a vector perpendicular to that surface) between the point and the surface patch (Figure 3.8). If the point belongs to a patch, the normal distance should be zero (or close to 0 because of deformation or other phenomenon happening on the surface) after performing a 3D rigid transformation that moves the point from $q$ to $q'$. This condition is known as the Coplanarity condition (Habib, 2001) and is mathematically described in equation 3.21, which states that the separation enclosed by a point and a corresponding patch is zero; in other words, the normal distance $D$ between them should be zero.

\[
\begin{vmatrix}
X_{p0} & Y_{p0} & Z_{p0} \\
X_{p1} & Y_{p1} & Z_{p1} \\
X_{p2} & Y_{p2} & Z_{p2}
\end{vmatrix}
= \begin{vmatrix}
X_{q'} - X_{p0} & Y_{q'} - Y_{p0} & Z_{q'} - Z_{p0} \\
X_{p1} - X_{p0} & Y_{p1} - Y_{p0} & Z_{p1} - Z_{p0} \\
X_{p2} - X_{p0} & Y_{p2} - Y_{p0} & Z_{p2} - Z_{p0}
\end{vmatrix} = 0
\]  

(3.21)
The patch in Figure 3.8 is defined by three points \( p_0 = (X_0, Y_0, Z_0), p_1 = (X_1, Y_1, Z_1), \) and 
\( p_2 = (X_2, Y_2, Z_2), \) \( \{p_0, p_1, p_2\} \in S_1. \) If \( p_a, p_b \) and \( p_c \) are the position vector of \( p_0, p_1 \) and \( p_2 \) respectively, then:

\[
\begin{align*}
v_1 &= p_b - p_a = (X_{p1} - X_{p0}, Y_{p1} - Y_{p0}, Z_{p1} - Z_{p0}) \\
v_2 &= p_c - p_a = (X_{p2} - X_{p0}, Y_{p2} - Y_{p0}, Z_{p2} - Z_{p0})
\end{align*}
\]

(3.22)

A normal vector to a plane, \( N, \) is given by cross product of the two vectors \( v_1 \) and \( v_2. \)

\[
N = v_1 \times v_2 = (a, b, c)
\]

(3.23)

where,

Figure 3.8: Correspondence is established between points on \( S_2 \) with patches on \( S_1. \) Surface \( S_2 \) is rigidly moved to match with \( S_1 \) by minimising the normal distance \( D. \)
\[ a = (Y_{p2} - Y_{p0})(Z_{p1} - Z_{p0}) - (Y_{p1} - Y_{p0})(Z_{p2} - Z_{p0}) \]
\[ b = (X_{p1} - X_{p0})(Z_{p2} - Z_{p0}) - (X_{p2} - X_{p0})(Z_{p1} - Z_{p0}) \]
\[ c = (X_{p2} - X_{p0})(Y_{p1} - Y_{p0}) - (X_{p1} - X_{p0})(Y_{p2} - Y_{p0}) \]

Next, calculate an expression for a unit normal vector, i.e., a normal vector of length one. It is simply \( N \) divided by its own length. The author will use notation \( n \) as the unit normal vector:

\[ n = \frac{N}{\|N\|} = \frac{(a, b, c)}{(a^2 + b^2 + c^2)^{\frac{1}{2}}} \quad (3.24) \]

The distance of a point to a plane is defined as:

\[ D = n \cdot (q - p) \quad (3.25) \]

where,

- \( n \) is the unit normal vector
- \( q \) is an arbitrary 3D point
- \( p \) is a point on the plane

Since the length of \( n \) is one, this distance is simply the absolute value of \( n \cdot (q - p) \):

\[ D = |n \cdot (q - p)|
= |n \cdot (X_q - X_p, Y_q - Y_p, Z_q - Z_p)|
= |a(X_q - X_p) + b(Y_q - Y_p) + c(Z_q - Z_p)| \sqrt{(a^2 + b^2 + c^2)^{\frac{1}{2}}}
= |a(X_q) + b(Y_q) + c(Z_q) - d| \sqrt{(a^2 + b^2 + c^2)^{\frac{1}{2}}}
\]

\[ (3.26) \]

where,

\[ d = a(-X_p) + b(-Y_p) + c(-Z_p) \]
Substituting equation 2.2 into equation 3.26 ignoring scale factor, the following equation results:

\[
D = \frac{a[(m_{11}X + m_{12}Y + m_{13}Z) + T_x] + b[(m_{21}X + m_{22}Y + m_{23}Z) + T_y] + c[(m_{31}X + m_{32}Y + m_{33}Z) + T_z] - d}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}
\]

\[(3.27)\]

Rearrange the functional model and following the step in equations 3.10, 3.11 and 3.12, differentiate equation 3.27 to obtain the partial derivative of six unknown parameters ω, φ, κ, T_x, T_y and T_z. However, the numerator of D is an absolute value and is not definable at point equal to zero. As a result, the signum function is used to differentiate D. In mathematics, the signum function is a mathematical function that extracts the sign of a real number. The signum function of a real number A is defined as:

\[\text{sign}(A) = \begin{cases} 
-1 & \text{when } A < 0, \\
0 & \text{when } A = 0, \\
1 & \text{when } A > 0,
\end{cases}\]

Let the variable η (eta) be the expression inside the absolute value sign in equation 3.27. Differentiation of equation 3.27 becomes:

\[
dX = \text{sign}(\eta)[a_{11}d\omega + a_{12}d\phi + a_{13}d\kappa + T_x]; \\
dY = \text{sign}(\eta)[a_{21}d\omega + a_{22}d\phi + a_{23}d\kappa + T_y]; \\
dZ = \text{sign}(\eta)[a_{31}d\omega + a_{32}d\phi + a_{33}d\kappa + T_z];
\]

\[(3.28)\]

where, \(a_{ij}\) are the coefficient terms whose expansions are trivial (the solution is derived in Appendix B). Using the following notation:
\[
S_X = \frac{\partial S_{20}(X, Y, Z)}{\partial X} = \frac{a}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}; \\
S_Y = \frac{\partial S_{20}(X, Y, Z)}{\partial Y} = \frac{b}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}; \\
S_Z = \frac{\partial S_{20}(X, Y, Z)}{\partial Z} = \frac{c}{(a^2 + b^2 + c^2)^{\frac{1}{2}}};
\]

(3.29)

where, \{S_X, S_Y, S_Z\} are numeric first derivatives of function \(S_2 (X, Y, Z)\).

Finally the derivative of \(D\) with respect to all the 6 transformation parameters is shown below:

\[
\begin{align*}
\frac{\partial D}{\partial \omega} &= \text{sign}(\eta)[a \cdot a_{11} + b \cdot a_{21} + c \cdot a_{31}] / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \\
\frac{\partial D}{\partial \varphi} &= \text{sign}(\eta)[a \cdot a_{12} + b \cdot a_{22} + c \cdot a_{32}] / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \\
\frac{\partial D}{\partial \kappa} &= \text{sign}(\eta)[a \cdot a_{13} + b \cdot a_{23} + c \cdot a_{33}] / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \\
\frac{\partial D}{\partial T_x} &= \text{sign}(\eta) \cdot a / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \\
\frac{\partial D}{\partial T_y} &= \text{sign}(\eta) \cdot b / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \\
\frac{\partial D}{\partial T_z} &= \text{sign}(\eta) \cdot c / (a^2 + b^2 + c^2)^{\frac{1}{2}};
\end{align*}
\]

(3.30)

For every temporarily matched point-patch, one observation equation of the form of equation 3.30 is setup and arranged in the matrix form of equation 3.15 for least squares solution.

### 3.6 Transformation Results

Two different test cases were performed to determine the optimum objective function that could be used in the non-rigid transformation. The assessment is comparatively simple: the transformation function with the smallest r.m.s. of residual \(V\) and which
graphically converges for both synthetic and real data will be chosen as the optimum objective function. These two different test cases are as below:

**Test Case I – Synthetic Data:** Two identical surfaces with one transformed by known transformation parameters and one retains in its original position and orientation. The transformed surface is transformed rigidly until it converges with the original surface. This test served the purpose of validating transformation functions by comparing the recovered transformation parameters with the known transformation parameters and the r.m.s. of residual $V$. The author tagged this transformation as “clean” as there is no outlier or deformation on the data. These surfaces are 50 x 50 grid DEM comprised of 2500 points.

**Test Case II – Real Scoliosis Data:** Two surfaces from the same object captured at different epochs. The transformation parameters are unknown. A similar procedure to the previous test case is performed where the transformation function is iterated until convergence is achieved. Compare the r.m.s. of residual $V$ to determine the surface deformation. Because the data were captured at different epochs, unexpected outliers and deformations may exist on the data.

The test cases reported in Chapter 3 is to determine the optimum objective function that could be used in the non-rigid transformation. This does not involve to validating or verifying the transformation results. Thus the number of points is not reported here. In Chapter 4, the number of points for every data set was reported. Table 3.1 contains the results obtained from the Test Case I and Table 3.2 shows the results acquired from Test Case II. Graphical matching results for Test Case I can be visualised in Figure 3.9. Test Case II is using real scoliosis data which will be the main data source for this research. The final matching results for Test Case II are graphically illustrated in Figure 3.10.

The experiment conducted with synthetic data revealed that the four transformation functions are able to recover all the six transformation parameters. The initial values of the parameters were offset by 0.1, 0.2 and 0.3 degrees for the angles along the X, Y and Z axes respectively and 3, 4, and 5 for the translation along the X, Y, and Z axes respectively. Because “clean” data was used, the geometric features for all points are retained and no outliers exist. All parameters were determined correctly, refer to Table
3.1. Although it is not 100% perfect the results are considered good in term of convergence as the r.m.s. error is close to zero. It can be concluded that all these four transformation functions are successfully well defined.

Table 3.1: Test Case I: rigid transformation results for different transformation function using synthetic data.

<table>
<thead>
<tr>
<th>Known Transformation Parameters</th>
<th>Transformation Function</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MICP</td>
</tr>
<tr>
<td>Rotation X (0.100 degrees)</td>
<td>-0.156</td>
</tr>
<tr>
<td>Rotation Y (0.200 degrees)</td>
<td>-0.160</td>
</tr>
<tr>
<td>Rotation Z (0.300 degrees)</td>
<td>-0.322</td>
</tr>
<tr>
<td>Translation X (3.000 mm)</td>
<td>-2.988</td>
</tr>
<tr>
<td>Translation Y (4.000 mm)</td>
<td>-4.052</td>
</tr>
<tr>
<td>Translation Z (5.000 mm)</td>
<td>-5.020</td>
</tr>
<tr>
<td>r.m.s. for Residual V (mm)</td>
<td>0.024</td>
</tr>
<tr>
<td>Number of Iteration to Converge</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3.2: Test Case II: rigid transformation results for different transformation functions using actual scoliosis data.

<table>
<thead>
<tr>
<th>Unknown Transformation Parameters</th>
<th>Transformation Function</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MICP</td>
</tr>
<tr>
<td>Rotation X (radians)</td>
<td>-0.050</td>
</tr>
<tr>
<td>Rotation Y (radians)</td>
<td>-0.038</td>
</tr>
<tr>
<td>Rotation Z (radians)</td>
<td>0.024</td>
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<td>Translation X (mm)</td>
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<td>Translation Z (mm)</td>
<td>-4.080</td>
</tr>
<tr>
<td>r.m.s. for Residual V (mm)</td>
<td>9.645</td>
</tr>
<tr>
<td>Number of Iteration to Converge</td>
<td>5</td>
</tr>
</tbody>
</table>
Figure 3.9: (a) Synthetic data sets for simulation studies. Reference (red) and Model (black) surface before transformation. (b) Reference and Model surface after transformation using (i) Modified ICP, (ii) LZD, (iii) ICP-LZD and (iv) LNDD. Matching results show that all four transformation functions have relatively good matching results. It can be concluded that all four functions are validated and well-defined.
Figure 3.10: (a) Two real scoliosis data sets for transformation functions comparison studies. Reference ($S_1$) and Model surface ($S_2$) before transformation. (b) $S_1$ and $S_2$ after transformation using (i) MICP, (ii) LZD, (iii) ICP-LZD, and (iv) LNDD. MICP has the worst result with the largest r.m.s. value, indicating huge divergences in numerous parts. LZD and LNDD have much better results, and LNDD has smallest r.m.s. value.
The transformation results from Table 3.2 indicate that the MICP experiences some difficulty to converge towards a good result. From the graphical visualisation (Figure 3.10(b)), the MICP obviously performs poorly in that there remain huge divergences in numerous parts of the two surfaces. In fact, many of the ICP algorithms applied in biomedical and robotic analysis have been modified to suit to the nature of application for better convergence results. Furthermore, ICP algorithm was originally intended for object recognition (Besl, 1992). For this reason, ICP assumes that one point set is a subset of the other. However, the data used in the test cases (both Test Case I and II) are from the same surfaces with almost 100% same surface coverage. It will not be surprising that MICP will create false matches.

Figure 3.11 shows how the point-to-point corresponding search utilising curvature information as additional information works compared with classical ICP. Although extra dissimilarity measures may expect higher accuracy and better convergence, deformation on the surface may upset this expectation. Deformation on the data surface might change the geometric features of points. As a consequence thereof, moving a search point to the nearest possible match point or its vicinity points can occur in more than one direction. Some of these points need to be sacrificed to slide away from the closest point to acquire global convergence. This is illustrated in Figure 3.12(b). This problem can be solved by utilising robust matching that excludes outliers and deformed points from the convergence iteration. The research interest at this stage is to determine the optimum objective function that could be applied in non-rigid transformation and the author will not yet consider the robustness of this transformation function. Accordingly, applying MICP will not be considered in non-rigid transformation.
Figure 3.11: (a) Corresponding point search using point-point distance metric. (b) Corresponding point search using point-to-point distance metric with attached curvature information. The different shape marks indicate different curvature.

Figure 3.12: (a) Matching achieved successfully in “clean data”. (b) In Test Case II, some points, in green box, need to be rejected from their corresponding point to obtain global convergence.
Results from LZD revealed that it is considerably a good transformation function in that it is able to recover the transformation parameters with low r.m.s. error even there are outliers and deformations existed in the data. However there is a major drawback encountered from relying on slope features as the fundamental matching criterion. Slope is defined as the ratio of the altitude change to the horizontal distance between two points on a line. Slope is normally used to describe the steepness, incline or grade of a straight line. This may be inappropriate to define geometric characteristic of two points on a surface in a straight line, particularly in nature. Therefore the result may be just an approximation to the real surface and this will raise the issue of the matching accuracy. Figure 3.13(a) shows how slope value of a line is calculated. Figure 3.13(b) below explains the common process of obtaining slope information on a surface. However, this research is dealing with scoliosis data which consists of intricately curved surfaces. Clearly, the best description of two points on a surface is the curvature between them, the green line in Figure 3.13(b), and not the slope of the line that connects the points. This suggests that curvature may be a more meaningful descriptor of a surface than slope.

Moreover, the classical LZD approach is to interpolate both data sets into a regular grid format followed by determining the Z difference at the grid posts. However there are problems with this simple approach. First, the interpolation to generate regular grid data will affect the comparison of Z difference because this process might introduce unexpected and unacceptable errors. More critical is the restriction to compare differences only along the Z axis. In extreme examples of a vertical surface, LZD will fail to generate good results.
Figure 3.13: (a) Common process of obtaining slope of two points. (b) Curvature (green line) is the actual geometric properties of two points on a surface. Use of slope, red line, may lead to inaccurate matching.

Figure 3.14: (a) Original surface with possible outliers, highlighted in purple circle. (b) Extra outliers (highlighted in green circle) are generated from the process interpolating the original data into a regular grid format.
The results reported above demonstrate that combined ICP-LZD was successful to achieve the goal to match surfaces together. It produces better matching results compared to MICP and has very similar results compared with LZD (cf. Table 3.2). However, for the nearest point algorithm matching there are certain conditions which must be met for successful and accurate surface matching. Essentially, whether a surface matching solution converges to the correct solution depends primarily on the density of the data, and the method of calculating the surface slope.

The density of the data must be reasonably high in order to recover the transformation parameters. However, the density threshold remains unknown and this requires further analysis. In low density data matching, minimising surface normal is superior because it can recover all the transformation parameters compared with nearest point algorithm. The author suspects that in low density data matching, the nearest point algorithm is unable to get the “correct” nearest point in $S_1$ thus affecting the retrieval of the correct vertical difference. The “correct” nearest point obtained, could be a point that is a distance away from the search point due to the low data density. Once the vertical difference is insignificant, it is hard to recover transformation parameters because the matching algorithm gets the matching solution by minimising the vertical difference. The matching program will assume that the matching has already converged and stopped because of an insignificant vertical difference.

As discussed previously, the combined ICP-LZD results strongly highlight that an accurate solution depends on a strong matrix of coefficients of the unknown transformation parameters. The crucial elements in the coefficients can be seen in the Figure 3.13 to be the surface slope. Thus, surface slope has to be evaluated carefully to generate strong matrix coefficients for better results. Combined ICP-LZD is using the nearest point algorithm to get the surface slope in the program. The points are very close together due to high data density and it will generate relatively small difference of Z ($dZ$) yielding a large surface slope in X and Y direction ($dX/dZ$ and $dY/dZ$). This will generate a weak matrix coefficient and prevent the program to get the optimal result. Thus, the difference of Z is set to a threshold of one in the program to avoid weak matrix coefficients. Any value that is smaller than one will be rejected and the program will look for next nearest point until the difference exceeds the threshold. This will make sure that the program will generate a strong matrix of coefficient for a better and
accurate solution. Inevitably, computation time for looking for the best nearest point to define the surface slope will increase.

The classical matching algorithm example, ICP and LZD, interpolates the original surface onto a regular grid (gridding) because most subsequent processes assumes regularly spaced data. For example, determining geometrical features like curvature in ICP and determining Z differences in LZD require regular grid data. The topography of the true surface is assumed by the interpolation, but is unknown and always approximated. Because the scanned range data might include noise, the information computed (from interpolation) from them will be even more corrupted by that noise (Figure 3.14). Thus, it is a good choice to avoid using any secondary features and information derived from raw range data; instead, data points are directly used as matching units. The use of Delaunay triangulation algorithm in LNDD is to connect the points to find a convex hull, not for interpolation. To avoid the problems faced in the previous three functions, an improved solution is to compute the difference between the correspondences along surface normal at the original point location. This is matched with the transformation function, LNDD. The final results illustrate that LNDD has the lowest r.m.s. error and gives the clue LNDD is more suitable as the transformation function. In conclusion, minimising the distance assures the best positional fit while minimising differences in surface normal assures the best shape fit (Schenk, 2000). Oftentimes the selection of transformation function is based on the nature of application. This research uses scoliosis data which might experience unexpected and unknown deformation. Assurance of best shape fit is more important and meaningful than best positional fit. As a consequence, LDNN has been chosen as the transformation function for non-rigid transformation.

3.7 Chapter Summary

Comparison of four different transformation functions, namely MICP, LZD, ICP-LZD and LNDD, has been presented. Analysis using both synthetic and real scoliosis data indicates that LNDD has the smallest r.m.s. value. Moreover, LNDD is able to overcome the drawbacks that constrain the other transformation functions from obtaining the convergence. For this reason, LNDD is chosen as the objective function and transformation procedure to design and develop a new non-rigid surface matching
algorithm. To the knowledge of the author, similar algorithms which minimise the normal distance from points to patches of a reference surface have never been tested on real medical data sets. Moreover, the use of non-rigid parameters to improve matching is innovative as no evidence of the use of such techniques with matching algorithms was found in the literature.
CHAPTER 4

NON-RIGID SURFACE MATCHING

4.1 Introduction

Shape changes due to scoliotic deformities are non-rigid changes that have occurred on the back surface of the patient. Accordingly, a non-rigid mathematical model is required to model this. Rigid surface matching uses six parameters (three translations and three rotations) to describe the change from one surface to another. However, these parameters are all “movement” parameters that allow the two surfaces to be shifted along and rotated about the three axes (Mitchell, 2006). As explained by Schenk (2000), this algorithm positionally fits the surfaces but the shape fitting remains unsolved. Differences between the two surfaces caused by any change in shape will remain when the position of closest fit is found. This is because the classical approach assumes that the shapes remain the same and only their average position change is sought (Mitchell, 2006).

In this chapter, the author will discuss the new derived non-rigid transformation algorithm with different transformation parameters. These new parameters include six shear parameters concatenated with three scale parameters. Linearised observation equations were derived to estimate the 3D transformation parameters for a given pair of surfaces. A least squares algorithm that minimises normal distance as the objective function (as discussed in Chapter 3) between the surfaces was performed. In brief, the mathematical model estimates the correspondent point-patch between surfaces from a non-rigid transformation, minimising the sum of squares of the normal distance separations between the point-patch correspondences. Geometric deformations are simultaneously modelled via surface shaping parameters (the six shear parameters). According to the final results obtained, this new non-rigid algorithm has improved the accuracy of matching, refined the method of surface matching suitable for deformed surface and derived new parameters for deformation monitoring.
4.2 Affine Transformation

An Affine transformation is an extension of linear transformations and is classified as a non-rigid transformation. Matrix multiplication is usually used to represent the transformation. In an affine transformation, straight lines will be preserved while lengths and angles will be changed. Once an object is affine transformed, there are three geometric properties that remain invariant from transformation. These are ratio of area, ratio of lengths of directional line and parallel lines.

The most common affine transformation is a concatenated transformation of scale, rotation and translation. Scale shrinks or grows the object size; rotation rotates the object around the axis vector; while translation translates the object along the direction vector. Although common affine transformation can improve the matching function compared with linear transformation, it is not the best transformation when applied on surfaces subjected to deformation. Therefore, extra parameters need to be derived and concatenated to model the surface deformation. Extra parameters can be concatenated into a single matrix by multiplying them together, in the order they are to occur. Concatenation of a transformation is a combination of series of transformation. For instance:

\[ v' = (A1) * (v); \]
\[ v'' = (A2) * (v'); \]
\[ w = (A3) * (v''); \]

(4.1)

Substitute to get

\[ w = (A3) * (A2) * (v'); \]
\[ w = (A3) * (A2) * (A1) * (v); \]

(4.2)
One can combine this series transformation into single matrix that performs all of them, for example a Rotation Matrix, \( R \):

\[
R = (A3) \cdot (A2) \cdot (A1);
\]

\[
w = (R) \cdot (v);
\]

(4.3)

Although one can concatenate the scale, rotation and translation in any order, matrix products are non-commutative and different matrix orders applied on the same point will not yield the same result. For example, rotate \((1 0 0)\) by 90° around Z, scale by \((S_X, S_Y, S_Z)\) to get \((0 S_Y 0)\); meanwhile scale \((1 0 0)\) by \((S_X, S_Y, S_Z)\), rotate \((1 0 0)\) by 90° around Z, will get \((0 S_X 0)\). The generally desired transformations order in affine transformation is to be performed scale \((S)\), then rotation \((R)\) followed by translation \((T)\).

\[
A = (T) \cdot (R) \cdot (S);
\]

(4.4)

This single matrix can be used to obtain the transformation parameters on the surface. The extra parameters the author proposed to include are shear parameters. Experimental results show that there are significant improvements on the matching function.

Below is the matrix equivalent (depicted in homogenous matrix form to combine all the transformation parameters into single matrix) of the various transformations in 3D:

Generic affine matrix:

\[
Q = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34} \\
    0 & 0 & 0 & 1
\end{bmatrix} \cdot p
\]

(4.5)
Translation:

\[
Q = \begin{pmatrix}
1 & 0 & 0 & T_x \\
0 & 1 & 0 & T_y \\
0 & 0 & 1 & T_z \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot p
\]

(4.6)

Rotation:

\[
Q = \begin{pmatrix}
m_{11} & m_{12} & m_{13} & 0 \\
m_{21} & m_{22} & m_{23} & 0 \\
m_{31} & m_{32} & m_{33} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot p
\]

(4.7)

Scale:

\[
Q = \begin{pmatrix}
S_x & 0 & 0 & 0 \\
0 & S_y & 0 & 0 \\
0 & 0 & S_z & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot p
\]

(4.8)

Shear:

\[
Q = \begin{pmatrix}
1 & S_{h_{xy}} & S_{h_{xz}} & 0 \\
S_{h_{yx}} & 1 & S_{h_{yz}} & 0 \\
S_{h_{zx}} & S_{h_{zy}} & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot p
\]

(4.9)

For the detailed rotation matrix, see Chapter 2 section 2.4.1.
4.2.1 Generic Affine Transformation

In order to prove that the affine transformation can improve the matching algorithm, the author has applied a generic affine transformation on real scoliosis data. A scaling factor in all three axes is concatenated with rigid transformation parameters. The objective function was LNDD as decided in the previous chapter. Concatenating the scaling factor in desired affine transformation order gives the following equation:

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix} =
\begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix}
\begin{bmatrix}
S_X & 0 & 0 \\
0 & S_Y & 0 \\
0 & 0 & S_Z
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
+ 
\begin{bmatrix}
T_X \\
T_Y \\
T_Z
\end{bmatrix}
\]

(4.10)

where, \(S_X\), \(S_Y\), and \(S_Z\) are the scaling factor on the X, Y, and Z axes respectively.

Equation 4.10 can be written as:

\[
X' = (m_{11} \cdot S_X \cdot X + m_{12} \cdot S_Y \cdot Y + m_{13} \cdot S_Z \cdot Z) + T_X;
Y' = (m_{21} \cdot S_X \cdot X + m_{22} \cdot S_Y \cdot Y + m_{23} \cdot S_Z \cdot Z) + T_Y;
Z' = (m_{31} \cdot S_X \cdot X + m_{32} \cdot S_Y \cdot Y + m_{33} \cdot S_Z \cdot Z) + T_Z;
\]

(4.11)

If \(D\) is the Euclidean distance of a point, \((X_q, Y_q, Z_q)\), normal to a plane which passes through \((X_p, Y_p\) and \(Z_p)\) and for which a normal is given by vector \(n\),

\[
n = (a, b, c)
\]

then \(D\) is given by:

\[
D = \left| a(X_q) + b(Y_q) + c(Z_q) - d \right| \quad \frac{1}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}
\]

(4.12)
where,

\[ d = a(-X_p) + b(-Y_p) + c(-Z_p) \]

Substituting equation 4.11 into equation 4.12 gives,

\[ D = \sqrt{\left[ \frac{a(m_{11}S_xS_y + m_{12}S_y + m_{13}S_z + T_x) + b(m_{21}S_xS_y + m_{22}S_y + m_{23}S_z + T_y) + c(m_{31}S_xS_y + m_{32}S_y + m_{33}S_z + T_z) - d}{(a^2 + b^2 + c^2)^2} \right]^2} \]

(4.13)

Linearising the normal distance \( D \) by deriving with respect to all the nine transformation parameters by using Taylor’s expansion, keeping the first order and ignoring second and higher expansion terms, the following equation results:

\[ D^* \approx D_0 + (\frac{\partial D}{\partial S_x})\Delta S_x + (\frac{\partial D}{\partial S_y})\Delta S_y + (\frac{\partial D}{\partial S_z})\Delta S_z + (\frac{\partial D}{\partial \omega})\Delta \omega + (\frac{\partial D}{\partial \varphi})\Delta \varphi + (\frac{\partial D}{\partial \kappa})\Delta \kappa + (\frac{\partial D}{\partial T_x})\Delta T_x + (\frac{\partial D}{\partial T_y})\Delta T_y + (\frac{\partial D}{\partial T_z})\Delta T_z \]

(4.14)

where, \( D^* \) is the approximation of \( D \), \( D_0 \) is the distance evaluated at the initial value of the parameters of the transformation, \( \Delta S_x, \Delta S_y, \Delta S_z \) are the corrections to initial values of the parameters, \( \frac{\partial D}{\partial S_x}, \frac{\partial D}{\partial \omega}, \frac{\partial D}{\partial \kappa}, \frac{\partial D}{\partial T_x}, \) etc., are the partial derivatives with respect to the indicated unknowns evaluated at the initial approximations.

Let \( V \) be the difference between \( D^* \) and \( D \):

\[ D = D^* - V \]

(4.15)
Equation 4.14 can thus be written as:

\[
D = D_0 + \left( \frac{\partial D}{\partial S_x} \right) \Delta S_x + \left( \frac{\partial D}{\partial S_y} \right) \Delta S_y + \left( \frac{\partial D}{\partial S_z} \right) \Delta S_z + \left( \frac{\partial D}{\partial \omega} \right) \Delta \omega + \left( \frac{\partial D}{\partial \varphi} \right) \Delta \varphi + \left( \frac{\partial D}{\partial \kappa} \right) \Delta \kappa + \left( \frac{\partial D}{\partial T_x} \right) \Delta T_x + \left( \frac{\partial D}{\partial T_y} \right) \Delta T_y + \left( \frac{\partial D}{\partial T_z} \right) \Delta T_z - V
\]

(4.16)

With the model requirement that D = 0 (nil separation between surfaces or perfect match) then by equation 4.15, \( V = D^* \), so equation 4.16 becomes:

\[
V = D_0 + \left( \frac{\partial D}{\partial S_x} \right) \Delta S_x + \left( \frac{\partial D}{\partial S_y} \right) \Delta S_y + \left( \frac{\partial D}{\partial S_z} \right) \Delta S_z + \left( \frac{\partial D}{\partial \omega} \right) \Delta \omega + \left( \frac{\partial D}{\partial \varphi} \right) \Delta \varphi + \left( \frac{\partial D}{\partial \kappa} \right) \Delta \kappa + \left( \frac{\partial D}{\partial T_x} \right) \Delta T_x + \left( \frac{\partial D}{\partial T_y} \right) \Delta T_y + \left( \frac{\partial D}{\partial T_z} \right) \Delta T_z
\]

(4.17)

Introducing the partial derivatives with respect to the dependents \( \Delta S_x, \Delta S_y, \ldots, \Delta T_y \) and \( \Delta T_z \), it yields 9i equations with 9i unknown variables. i is the number of pair point-patch matched. The details of the partial derivatives equations are as follows:
\[ \frac{\partial D}{\partial S_x} = \text{sign}(\eta) \ast a \ast X / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial S_y} = \text{sign}(\eta) \ast b \ast Y / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial S_z} = \text{sign}(\eta) \ast c \ast Z / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial \omega} = \text{sign}(\eta) \ast [b \ast S_z \ast Z - c \ast S_y \ast Y] / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial \varphi} = \text{sign}(\eta) \ast [c \ast S_x \ast X - a \ast S_z \ast Z] / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial \kappa} = \text{sign}(\eta) \ast [a \ast S_y \ast Y - b \ast S_x \ast X] / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial T_x} = \text{sign}(\eta) \ast a / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial T_y} = \text{sign}(\eta) \ast b / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial T_z} = \text{sign}(\eta) \ast c / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]

(4.18)

An observation equation of the form of equation 4.18 is written for each correspondent point-patch. Arranging the observation equations in matrix notation for least squares solution.

\[ V_{(i,j)} = A_{(i,9)} \ast X_{(9,j)} + L_{(i,j)} \]

(4.19)

where:

\( V_{(i,1)} \) is the vector of residuals
\( A_{(i,9)} \) is the coefficient matrix from equation 4.19
\( X_{(9,1)} \) is the vector of corrections to the initial estimates of the unknown parameters
\( L_{(i,1)} \) is the vector of observations which in this case are the Euclidean distance differences, (\( D^* - D^0 \))
i is the number of pair point-patch matched
4.2.2 Results and Discussions

A comparison study with respect to rigid and non-rigid LNDD was carried out using the same actual scoliosis data. Table 4.1 shows the results comparing Rigid LNDD and Non-rigid LNDD. Results obtained from non-rigid transformation shown unexpected higher r.m.s. for residuals V, 0.045 unit of length higher than rigid transformation. Moreover, non-rigid LNDD requires more iteration to achieve the convergence. Although the difference between the r.m.s. error is not significant, the purpose to run generic affine transformation is to prove that a lower error could be expected in non-rigid transformations.

To understand the occurrence of higher error, a detailed study has been carried out. From the graphical visual (Figure 4.1), obviously extra correspondences were established in affine transformation. The model surface is “stretched” to cover the same surface area of the Reference surface especially in the edge area. Outliers are prone to exist in surface edges and this is easily to understand why non-rigid transformation gives higher error. Because of extra correspondences, additional iteration is expected. However this finding is critical to this research in that dilation on the surface could contribute to scoliosis deformation monitoring.

Table 4.1: Comparison of experimental results of Rigid LNDD and Non-Rigid LNDD.

<table>
<thead>
<tr>
<th>Unknown Transformation Parameters</th>
<th>Transformation Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rigid LNDD</td>
</tr>
<tr>
<td>Rotation X (radians)</td>
<td>-0.072</td>
</tr>
<tr>
<td>Rotation Y (radians)</td>
<td>-0.028</td>
</tr>
<tr>
<td>Rotation Z (radians)</td>
<td>0.013</td>
</tr>
<tr>
<td>Translation X (mm)</td>
<td>-50.600</td>
</tr>
<tr>
<td>Translation Y (mm)</td>
<td>88.391</td>
</tr>
<tr>
<td>Translation Z (mm)</td>
<td>-4.500</td>
</tr>
<tr>
<td>r.m.s. for Residual V (unit of length)</td>
<td>1.899</td>
</tr>
<tr>
<td>Number of Iteration to Converge</td>
<td>9</td>
</tr>
</tbody>
</table>
Scoliosis normally occurs in adolescence and children at these ages have faster body development where growth is rapid. Scoliosis data that is captured from the patients normally have more than six month interval. True scoliosis change monitoring should exclude outliers and any other non-scoliosis deformation factor that could influence the final result. Parameters are required to explain the true scoliosis change but at the same time extra parameters are needed to depict the non-scoliosis deformation like natural growth as outlined in the research objectives.

Figure 4.1: Concatenated scaling factor into transformation parameter assures best shape fit. Model surface (red) is “stretched” to cover same surface area of Reference surface. (a) Surface matching result using rigid LNDD. (b) Surface matching result using non-rigid LNDD. Green boxes show the best shape fit after surface matching.

However, the goal is to define parameters that are able to depict scoliosis deformation. Manipulation and calculation of this information is not in within the research scope. In this case, scaling could contribute to monitoring the natural growth of the children and
excluding of this factor (natural growth) may help in determining the true scoliosis change. Scaling factor should be kept in transformation parameters while reconsideration to keep rotation and translation is required. This is because positional fit would not carry any meaning for the application to monitor scoliosis deformation. In the next section, the author will discuss the new transformation algorithm that replace rotation and translation parameters with six shear parameters concatenating with scaling factor. This new transformation algorithm has shown promising results.

4.3 New Non-Rigid Surface Matching Algorithm with New Parameters

The crucial element of this study is the non-rigid surface matching algorithm, which is used to obtain not only the position of closest fit for pairs of measured back surface topographies but also simultaneously model the non-rigid shape changes. To achieve this, new shape transformation parameters were introduced and incorporated into the non-rigid surface matching algorithm. The existing six “movement” parameters were replaced. These new parameters are three scale and six shear parameters. Scale change can be used to represent patient growth, while shear can model the form of a lateral movement of the patient’s shoulders relative to the waist which is a typical scoliotic change.

Shearing an object slants or skews the object with a specified angle that is relative to a specified axis. Its effect leaves fixed all points on one axis and other points are shifted parallel to the axis by a distance proportional to their perpendicular distance from the axis.

Object shears relative to a reference point which varies depending on the shearing methods. Table 4.2 below summarises the transformation activities of these nine different parameters. Figure 4.2 depicts the various transformations for different shear parameters. Scoliosis is a three dimensional spinal deformity that results in the lateral curvature of the back along with trunk rotation about the vertebrae. If one establishes a coordinate system in the standing body, such that the X axis is across the body from right to left, the Y axis is upwards, and the Z axis is from front to back, then one observes that, for example, a lateral lean of the shoulders relative to the waist can be represented as a displacement in the X direction, with a magnitude which is a function
of the Y coordinate, while Z coordinates are unaffected, and is a shear in the coronal (X – Y) plane, $S_{XY}$. It is of interest as it is a typical scoliotic change (Pazos, 2005). (See Figure 4.2 for a definition of the three planes known medically as the coronal (X-Y), sagittal (Y-Z) and transverse (X-Z) planes.) A rotation of the trunk in the transverse (X-Z) plane ($S_{ZX}$) could also be of interest, and a forward lean is given by the shear, $S_{ZY}$, in the sagittal plane. It is believed that by applying shear parameters, the shearing motion can be detected and modelled.

Table 4.2: New transformation parameters and their transformation activities

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Transformation Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_X$</td>
<td>A scale of the X coordinate</td>
</tr>
<tr>
<td>$S_Y$</td>
<td>A scale of the Y coordinate</td>
</tr>
<tr>
<td>$S_Z$</td>
<td>A scale of the Z coordinate</td>
</tr>
<tr>
<td>$S_{XY}$</td>
<td>A shear of the X coordinate while moving along the Y axis</td>
</tr>
<tr>
<td>$S_{XZ}$</td>
<td>A shear of the X coordinate while moving along the Z axis</td>
</tr>
<tr>
<td>$S_{YX}$</td>
<td>A shear of the Y coordinate while moving along the X axis</td>
</tr>
<tr>
<td>$S_{YZ}$</td>
<td>A shear of the Y coordinate while moving along the Z axis</td>
</tr>
<tr>
<td>$S_{ZX}$</td>
<td>A shear of the Z coordinate while moving along the X axis</td>
</tr>
<tr>
<td>$S_{ZY}$</td>
<td>A shear of the Z coordinate while moving along the Y axis</td>
</tr>
</tbody>
</table>
Figure 4.2: Illustration of different shearing parameters together with their transformation. Green arrows indicate the forces that shear the object.
By applying the six shear parameters into the transformation algorithm, one is able to solve the six movement transformation parameters, rotation and translation, which are applied in classical rigid transformation. Rotation can be represented by the combination of different shear matrices. For example, a rotation around the Z axis can be decomposed into three shear matrices. The equations below show the three-shear decomposition of rotation around X, Y and Z axis.

A rotation around the X axis by an angle $\omega$ can be decomposed into the following three-shear matrix. Rotation around the X axis by an angle $\omega$ is given as:

$$
R_x(\omega) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\omega) & -\sin(\omega) & 0 \\
0 & \sin(\omega) & \cos(\omega) & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

(4.20)

This rotation matrix can also be represented by a string of three shear operations:

$$
R_x(\omega) = S_{YZ}(-\tan(\omega/2)) * S_{ZY}(\sin(\omega)) * S_{YZ}(-\tan(\omega/2))
$$

(4.21)

The following is the same equation written out with column vector matrices:
A rotation around the Y axis by an angle $\varphi$ can be represented by the following three shear operations:

$$R_y(\varphi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

\begin{align*}
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\tan(\varphi/2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\varphi) & 0 & \sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{align*}

(4.22)

And

$$R_y(\varphi) = S_{xz}(\tan(\varphi/2)) * S_{xz}(-\sin(\varphi)) * S_{xz}(\tan(\varphi/2))$$

(4.23)
\[ R_\gamma (\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 & 0 & \tan(\varphi / 2) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

(4.25)

A rotation around the Z axis by an angle \( \kappa \) can be represented by the following column vector matrix:

\[ R_Z (\kappa) = \begin{pmatrix} \cos(\kappa) & \sin(\kappa) & 0 & 0 \\ \sin(\kappa) & \cos(\kappa) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

(4.26)

And

\[ R_Z (\kappa) = S_{XY} (-\tan(\kappa / 2)) * S_{YY} (\sin(\kappa)) * S_{XY} (-\tan(\kappa / 2)) \]

(4.27)
\[
R_x(\kappa) = \begin{pmatrix}
\cos(\kappa) & \sin(\kappa) & 0 & 0 \\
\sin(\kappa) & \cos(\kappa) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 - \tan(\kappa / 2) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin(\kappa) & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(4.28)

The surface normal to a flat surface is a vector perpendicular to that surface. For a non-flat surface, the surface normal at a point P is a vector perpendicular to the tangent plane to that surface at P. A vector does not have position, only magnitude and direction. On the contrary, a translation only affects the position and leaves the vector unchanged. Since the objective function of this new algorithm is to minimise the normal distance, vector that perpendicular to the tangent plane, translation parameters are replaceable. To compensate, the centroid value of the matching surfaces can be used to provide a predetermined alignment.

The six “movement” parameters that are applied in rigid surface matching were replaced completely by the scaling and shearing parameters. However, the rotation parameters can be solved from the shear parameters if required. As well, to counteract the loss of translation parameters, the data from the two sets of surfaces were reduced to their respective centroid values, a move which did not alter the size of the shear or scale parameters. This also reduces the numerical size of the terms, thereby improving the condition of the least squares solution. The new non-rigid transformation equation not only assures the best positional fit but also the best shape fit. This new equation is delineated as follows:
where, \((X, Y, Z)^T\) is the coordinate of a point on \(S_2\) before transformation and \((X', Y', Z')^T\) is the coordinate of the same point on \(S_2\) after transformation. \(S_X, S_Y\) and \(S_Z\) are the scale parameters along the X, Y, and Z axes and \(Sh_{XY}, Sh_{XZ}, Sh_{YX}, Sh_{YZ}, Sh_{ZX}\) and \(Sh_{ZY}\) are shear parameters shown in Figure 4.2. The shears are relative displacement, so, like scale change, they are dimensionless. Equation 4.29 can be written as:

\[
\begin{align*}
X' &= S_X * X + Sh_{xy} * S_Y * Y + Sh_{xz} * S_Z * Z; \\
Y' &= Sh_{yx} * S_X * X + S_Y * Y + Sh_{yz} * S_Z * Z; \\
Z' &= Sh_{zx} * S_X * X + Sh_{zy} * S_Y * Y + S_Z * Z;
\end{align*}
\]  

\[\text{(4.30)}\]
Let $V$ be the difference between approximated normal distance $D^*$ and the exact normal distance $D$:

$$D = D^* - V$$

(4.32)

Equation 4.31 can thus be written as:

$$D = D_0 + \left( \frac{\partial D}{\partial S_X} \right) \Delta S_X + \left( \frac{\partial D}{\partial S_Y} \right) \Delta S_Y + \left( \frac{\partial D}{\partial S_Z} \right) \Delta S_Z$$

$$+ \left( \frac{\partial D}{\partial S_{xy}} \right) \Delta S_{xy} + \left( \frac{\partial D}{\partial S_{xz}} \right) \Delta S_{xz} + \left( \frac{\partial D}{\partial S_{yz}} \right) \Delta S_{yz}$$

$$+ \left( \frac{\partial D}{\partial S_{xxy}} \right) \Delta S_{xxy} + \left( \frac{\partial D}{\partial S_{xzr}} \right) \Delta S_{xzr} + \left( \frac{\partial D}{\partial S_{yzy}} \right) \Delta S_{yzy}$$

$$+ \left( \frac{\partial D}{\partial S_{xyr}} \right) \Delta S_{xyr} + \left( \frac{\partial D}{\partial S_{xzr}} \right) \Delta S_{xzr} + \left( \frac{\partial D}{\partial S_{yzy}} \right) \Delta S_{yzy} - V$$

(4.33)

With the model requirement that $D = 0$ (nil separation between surfaces or perfect match) but allowing for measurement noise $V$, rearranging equation 4.33, then the following equation results:

$$V = D_0 + \left( \frac{\partial D}{\partial S_X} \right) \Delta S_X + \left( \frac{\partial D}{\partial S_Y} \right) \Delta S_Y + \left( \frac{\partial D}{\partial S_Z} \right) \Delta S_Z$$

$$+ \left( \frac{\partial D}{\partial S_{xy}} \right) \Delta S_{xy} + \left( \frac{\partial D}{\partial S_{xz}} \right) \Delta S_{xz} + \left( \frac{\partial D}{\partial S_{yz}} \right) \Delta S_{yz}$$

$$+ \left( \frac{\partial D}{\partial S_{xxy}} \right) \Delta S_{xxy} + \left( \frac{\partial D}{\partial S_{xzr}} \right) \Delta S_{xzr} + \left( \frac{\partial D}{\partial S_{yzy}} \right) \Delta S_{yzy}$$

$$+ \left( \frac{\partial D}{\partial S_{xyr}} \right) \Delta S_{xyr} + \left( \frac{\partial D}{\partial S_{xzr}} \right) \Delta S_{xzr} + \left( \frac{\partial D}{\partial S_{yzy}} \right) \Delta S_{yzy}$$

(4.34)

Introducing the partial derivatives with respect to the dependents of $\Delta S_X, \Delta S_Y, ..., \Delta S_{xzr}$ and $\Delta S_{yzy}$, the following new equation results:
\[ \frac{\partial D}{\partial S_x} = \text{sign}(\eta) \ast (a \ast X + b \ast Sh_{xy} \ast X + c \ast Sh_{xz} \ast X) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial S_y} = \text{sign}(\eta) \ast (a \ast Sh_{xy} \ast Y + b \ast Y + c \ast Sh_{yz} \ast Y) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial S_z} = \text{sign}(\eta) \ast (a \ast Sh_{xz} \ast Z + b \ast Sh_{yz} \ast Z + c \ast Z) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial Sh_{xy}} = \text{sign}(\eta) \ast (a \ast S_y \ast Y) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial Sh_{xz}} = \text{sign}(\eta) \ast (a \ast S_z \ast Z) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial Sh_{yx}} = \text{sign}(\eta) \ast (b \ast S_x \ast X) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial Sh_{yz}} = \text{sign}(\eta) \ast (b \ast S_z \ast Z) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial Sh_{zx}} = \text{sign}(\eta) \ast (c \ast S_x \ast X) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]
\[ \frac{\partial D}{\partial Sh_{zy}} = \text{sign}(\eta) \ast (c \ast S_y \ast Y) / (a^2 + b^2 + c^2)^{\frac{1}{2}}; \]

(4.35)

An observation equation of the form of equation 4.35 is written for each element on the S₂, i.e. for every correspondent point-patch. Arranging the observation equations in matrix notation for a least squares solution as equation 4.19.

Figure 4.3 shows the flowchart of non-rigid surface matching programme. The programme starts with reading the data sets and assigns the first data set as Reference Surface (S₁) and second data set as Model Surface (S₂). Centroid value for both surfaces is first calculated and then this value will be used as predetermined alignment. S₁ is triangulated and stores the triangles in matrix of indices. Following this, the points of S₂ are processed one by one in a loop function. For each point, a triangle is sought. If a triangle is found, the a, b, c and d parameters of the plane of the triangle are calculated. These are then used to evaluate firstly D₀ for an initial value of the transformation parameters and secondly the derivatives of the linearisation. The results are stored in L vector of observables and in the A matrix respectively. A solution is found by the least squares solution as in equation 4.19. The solution vector X includes correction to the initial parameters used to calculate D₀ and its derivatives. The programme is stopped when the solution convergence is optimised, that is when testing shows that the separation of the two surfaces is minimal.
For each match, the program provides various statistics such as number of iterations, standard deviations of the transformation parameters, residuals (surface distance separations) and the r.m.s. error. For an indication of the closeness of any two surfaces, the surface separations derived from the alignment and their root mean square (r.m.s.) value will be examined. Successful modelling of all causes of change would reduce the r.m.s. and all surface separations to the noise level of the data.

A number of experiments have been executed. Firstly, the matching algorithm was applied to a number of synthetic datasets with known topographic changes. Secondly, the algorithm was applied to some real scoliosis datasets in order to evaluate the matching strategy and its ability to determine the scoliotic changes, and, in order to confirm that this matching algorithm is feasible and favourable for scoliosis deformation monitoring, the results were compared with the manual matching results defined by Berg, et al. (2002). Thirdly, the possible benefits of executing a rigid match followed by the non-rigid matching algorithm were assessed, given that the latter includes the influences of the rotation parameters.
Figure 4.3: Flowchart of non-rigid surface matching programme.
4.3.1 Results and Discussions

This new non-rigid transformation algorithm has been trialled using models with known topographic changes. These trials were intended to demonstrate the ability to match a pair of surfaces and to determine whether the shearing source can be modelled. Two identical 50 x 50 grid DEM surfaces (S₁ and S₂) were created. Certain amounts of scale and shear changes which might realistically occur were artificially introduced to S₂, which was then transformed by the non-rigid matching algorithm to match with S₁. Tests were undertaken with different values of known parameters. These tests served the purpose of validating the matching algorithm by comparing the recovered transformation parameters with the known transformation parameters and by assessing the r.m.s. error of surface separation, which would be zero (for artificial surfaces) if the surface differences are modelled perfectly. The program returned the input values (known parameters) of the introduced parameters of scale and shear, so that it could be accepted that the concept was valid and that the algorithm and program had been verified.

Meanwhile the algorithm has also been tested using reverse-matching (S₁ is matched with S₂) and negligible differences from the forward matching were detected. There is therefore evidence that shape changes due to scoliotic deformities can be modelled. For briefness, Table 4.3 shows one of the matching results of the many test cases performed. The algorithm is able to recover the known parameters in this example with an average standard of deviation less than 0.005 and average r.m.s. error less than 0.002 units of length. There is therefore evidence that shape changes due to scoliotic deformities can be recovered well, although not perfectly. A limitation to the accuracy is apparent with rigid matching also, and it is assumed that the problem of registering surfaces is the ill-posed, like that of matching images, as noted by Heipke (1997, page 2). From the results obtained, it can be concluded that this matching algorithm is validated and well-defined.
Table 4.3: Matching results from one of the test cases.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Known Parameters</th>
<th>Matching Results</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&lt;sub&gt;X&lt;/sub&gt; (%)</td>
<td>20</td>
<td>-19.8</td>
<td>0.29</td>
</tr>
<tr>
<td>S&lt;sub&gt;Y&lt;/sub&gt; (%)</td>
<td>10</td>
<td>-8.7</td>
<td>0.13</td>
</tr>
<tr>
<td>S&lt;sub&gt;Z&lt;/sub&gt; (%)</td>
<td>0</td>
<td>0</td>
<td>0.02</td>
</tr>
<tr>
<td>Sh&lt;sub&gt;XY&lt;/sub&gt;</td>
<td>0.010</td>
<td>-0.011</td>
<td>0.003</td>
</tr>
<tr>
<td>Sh&lt;sub&gt;XZ&lt;/sub&gt;</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Sh&lt;sub&gt;YX&lt;/sub&gt;</td>
<td>0.020</td>
<td>-0.020</td>
<td>0.004</td>
</tr>
<tr>
<td>Sh&lt;sub&gt;YZ&lt;/sub&gt;</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Sh&lt;sub&gt;ZX&lt;/sub&gt;</td>
<td>0.010</td>
<td>-0.013</td>
<td>0.002</td>
</tr>
<tr>
<td>Sh&lt;sub&gt;ZY&lt;/sub&gt;</td>
<td>-0.010</td>
<td>0.011</td>
<td>0.002</td>
</tr>
<tr>
<td>r.m.s. error</td>
<td>0.00168 units of length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iterations</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In addition, to demonstrate the capability of this new non-rigid matching algorithm in scoliosis modelling, the author has performed an experimental comparison with classical rigid matching algorithm using four different actual scoliosis data sources. The scans creating the surface topographies were acquired independently during two successive clinical visits of the scoliosis patient. Table 4.4 presents the matching r.m.s. errors, achieved by applying the rigid and the non-rigid matching algorithm. The table also shows the r.m.s. error improvement obtained by applying non-rigid transformation. Every data set has a different data density (number of points) as stated in the table. The author does not have any data with > 2000 points on hand, this is the reason why there is no test for data > 2000 points. It is assumed that systematic errors due to imperfect modelling will contaminate the r.m.s. value, causing it to be larger than the scanner measurement noise. Due to the reason that different scoliosis patients might have experienced different deformations, the r.m.s. error will deteriorate in data with severe deformations. While both methods perform fairly well in that they are capable of matching the surfaces satisfactorily, it is clear that there is a significant improvement in r.m.s. using the non-rigid surface matching algorithm.
The numerical results for the transformation parameters are presented in Table 4.5. However, these results were found to be too simplistic for immediate indication of change. Consequently, the quantitative output was studied in conjunction with graphical views in the form of contour topographic maps from that match to provide an overall picture of the matched results. Data set B was selected for this purpose and is presented in Figure 4.4. This figure shows two surface topographies from data set B, one being the reference $S_1$ (dash lines), the other being the model $S_2$ (solid lines). They have been registered using both the rigid and the non-rigid matching algorithms. The scans creating the surface topographies were acquired independently during two successive clinical visits of the scoliosis patient.

Table 4.4: Comparison of r.m.s. error between rigid and non-rigid matching algorithms using four different sets of actual scoliosis data.

<table>
<thead>
<tr>
<th>Scoliosis Data (Number of points)</th>
<th>Transformation Algorithm</th>
<th>r.m.s. improvement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rigid Transformation</td>
<td>Non-rigid Transformation</td>
</tr>
<tr>
<td>A1 - A2 (&lt; 5000)</td>
<td>2.544</td>
<td>2.160</td>
</tr>
<tr>
<td>B1 - B2 (&gt; 10000)</td>
<td>1.670</td>
<td>1.492</td>
</tr>
<tr>
<td>C1 - C2 (&gt;10000 &lt; 15000)</td>
<td>1.400</td>
<td>1.257</td>
</tr>
<tr>
<td>D1 - D3 (&gt; 15000)</td>
<td>2.964</td>
<td>2.386</td>
</tr>
</tbody>
</table>
Figure 4.4: Contour topographic map of data set B, (a) before the matching, (b) matched by using rigid algorithm, (c) matched by using non-rigid algorithm. The circles highlight the comparison of matched results of deformed areas between rigid and non-rigid algorithm. Solid lines represent surface $S_2$ and dash lines are surface $S_1$. 
Table 4.5: Parameter results from surface matching of scoliosis data set B using non-rigid matching algorithm

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Results</th>
<th>Changes Suggestion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_X$ (%)</td>
<td>-4.09</td>
<td>Narrower</td>
</tr>
<tr>
<td>$S_Y$ (%)</td>
<td>0.56</td>
<td>Taller</td>
</tr>
<tr>
<td>$S_Z$ (%)</td>
<td>-0.07</td>
<td>Thinner</td>
</tr>
<tr>
<td>$S_{XY}$</td>
<td>-0.011</td>
<td>Sideways Lean</td>
</tr>
<tr>
<td>$S_{XZ}$</td>
<td>0.002</td>
<td>Trunk Rotation</td>
</tr>
<tr>
<td>$S_{YX}$</td>
<td>0.032</td>
<td>Raising of Shoulder</td>
</tr>
<tr>
<td>$S_{YZ}$</td>
<td>0.014</td>
<td>Forward Bending</td>
</tr>
<tr>
<td>$S_{ZX}$</td>
<td>0.020</td>
<td>Trunk Rotation</td>
</tr>
<tr>
<td>$S_{ZY}$</td>
<td>-0.070</td>
<td>Forward Lean</td>
</tr>
</tbody>
</table>

From the comparison results, one can easily notice that there is a significant improvement using non-rigid surface matching algorithm. Obviously, both methods perform fairly well in that they are capable of matching the surfaces satisfactorily. However, the rigid algorithm cannot compensate for the lack of matching the deformed areas, whilst the non-rigid approach significantly leads to more physical plausible matching results on the deformed areas (This can be noticed in Figure 4.4). The non-rigid algorithm clearly indicates better alignment results for deformation from scoliosis. The r.m.s. values estimated from the non-rigid matching algorithm are improved by around 10% for all the data. The maximum improvement up to about 20% is obtained for data with more than 15000 data points (Table 4.4).

The classical approach obviously leaves large divergences between numerous parts of the two data sets (highlighted in circles in Figure 4.4) because it fails to match the parts that experienced deformation. In contrast, non-rigid matching shows its capability in dealing with high and locally distributed deformation which the rigid matching algorithm does not consider. Furthermore, the displacement after the two surfaces were matched using the non-rigid algorithm shows a rather better result in non-deformed and less deformed areas compared with the rigid matching result.
Note also that it is intended that the new parameters can be directly used to identify the scoliosis progression. As observed from Table 4.5, largest shear (highlighted in bold in the table) is -0.07 for Sh$_{ZY}$, so that the patient might have a forward lean, followed by a shear of 0.03 in the form of raising of one shoulder relative to the other, and 0.02 for Sh$_{ZX}$, a mild trunk rotation. These changes could not be identified by using the rigid transformation.

Special attention is required for the result that the patient may be experiencing a negative growth, being 4% narrower in the X direction. This negative growth may be caused by scoliotic deformation or other medical reasons. Careful analysis needs to be carried out to identify and understand the cause. However, it is not within the scope of this research to determine any medical diagnosis progression. In conclusion, the transformation parameters could help in explaining the deformation magnitude and characterising the deformation progression caused by scoliosis. The real strength of this new algorithm lies in its potential for delineating the deformation magnitude and direction which has not been achievable before.

The result obtained in this research has been compared with the manual matching method proposed by Berg et al. (2002). This comparison result is presented in surface topographic map in Figure 4.5. The comparison result confirms that this new non-rigid surface matching algorithm is feasible and favourable for scoliosis deformation monitoring.

It appears at this stage that the non-rigid transformation is more suitable than rigid transformation in matching surfaces that have experienced deformation. With the derivation of a new affine transformation algorithm, the author has achieved the major objectives of developing a mathematical model that is able to model the non-scoliosis change and deformation caused by the scoliosis change. In this newly derived transformation model, a scaling factor can be used to explain the dilation caused by natural growth and weight variation in X, Y, and Z directions respectively. Meanwhile, shearing parameters can be used to depict the deformation caused by scoliosis. However, the major drawback is that the rigid transformation parameters (the rotation and translation parameters that positionally fit the surfaces) can be affected by the positional parameters which are not recovered. The shear parameters alone are not able to
determine the deformation of the surface. To improve the algorithm, the positional parameters are sought first, in a rigid match, while the non-rigid deformations are sought in a subsequent non-rigid match, in what is called here as Combined Rigid and Non-rigid (CRNr) matching algorithm. In this combined matching algorithm, the rigid transformation is executed to fit the position followed by non-rigid transformation to fit the shape of the surfaces defining the deformation. By doing this, the extracted shear parameters may more exactly define the surface deformation caused by scoliosis. The results of this combined algorithm are shown in Table 4.6. A superimposed contour topographic map for both transformation algorithms, non-rigid and CRNr, was generated for comparison purpose. Table 4.7 summarises the results of the transformation parameters for these different transformation algorithms.
Figure 4.5: Comparison results of Difference Map (filled contour) defined in Berg (2002) with the results from non-rigid matching (normal and dashed lines). A very close deformities distribution pattern can be recognised on both results indicating this new non-rigid matching algorithm is able to detect surface deformities caused by scoliosis. In the Difference Map, darker colours indicate more serious deformations and would be expected to coincide with the bigger separations between the unbroken and broken contours.
Table 4.6: Comparison of r.m.s. error between non-rigid and CRNr matching algorithm.

<table>
<thead>
<tr>
<th>Scoliosis Data (Density)</th>
<th>Transformation Algorithm</th>
<th>r.m.s. improvement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Non-rigid Transformation</td>
<td>Combination of Rigid and Non-rigid Transformation</td>
</tr>
<tr>
<td>AS1 - AS2 (&lt; 5000)</td>
<td>2.160</td>
<td>2.124</td>
</tr>
<tr>
<td>BB1 - BB2 (&gt; 10000)</td>
<td>1.492</td>
<td>1.423</td>
</tr>
<tr>
<td>AF1 - AF2 (&gt;10000 &lt; 15000)</td>
<td>1.257</td>
<td>1.257</td>
</tr>
<tr>
<td>AF1 - AF3 (&gt; 15000)</td>
<td>2.386</td>
<td>2.351</td>
</tr>
</tbody>
</table>

Table 4.7: Comparison of non-rigid transformation parameters between non-rigid and CRNr matching algorithm

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Matching Algorithm</th>
<th>Non-Rigid</th>
<th>CRNr</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX (%)</td>
<td>-4.09</td>
<td>-3.06</td>
<td></td>
</tr>
<tr>
<td>SY (%)</td>
<td>0.56</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>SZ (%)</td>
<td>-0.07</td>
<td>-0.04</td>
<td></td>
</tr>
<tr>
<td>SHXY</td>
<td>-0.011</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>ShXZ</td>
<td>0.002</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>SHYX</td>
<td>0.032</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>SHYZ</td>
<td>0.014</td>
<td>0.014</td>
<td></td>
</tr>
<tr>
<td>SHZX</td>
<td>0.020</td>
<td>0.013</td>
<td></td>
</tr>
<tr>
<td>SHZY</td>
<td>-0.070</td>
<td>-0.000</td>
<td></td>
</tr>
</tbody>
</table>
In this experiment both transformation algorithm show a similar distribution pattern of surface separation, while the combination of rigid and non-rigid transformation algorithm gives only a slightly better r.m.s. improvement of up to 5% (Table 4.6). The difference between the r.m.s. errors is not significant. The contour topographic map in Figure 4.6 also suggests that there is no significant matching difference between these algorithms. Nevertheless, if we adjust our focus to the non-rigid transformation parameters (Table 4.7) we are able to get some clues about the “true” deformation progression as CRNr has revealed the “true” shearing deformation without the contamination of the rotation parameters found with the non-rigid matching algorithm. As before, the largest shearing occurs in the Y and Z coordinates along Z axis, being the
raising of one shoulder and a trunk rotation respectively, (highlighted in bold in Table 4.7), while good agreement between the two different transformations for the magnitude of shearing parameter $\text{Sh}_{YZ}$ (0.014, highlighted in red) is evident. These suggest that the patient might have mild lateral and forward bending together with mild trunk rotation. The $\text{Sh}_{ZY}$ shear parameter has been reduced extensively, as if the forward lean has now been replaced by a rotation. In conclusion, CRNr may be better than the non-rigid transformation in explaining the “true” deformation magnitude and characterising the “true” deformation progression caused by the scoliosis.

The important finding through this experiment is that the non-rigid transformation can be combined or isolated depending on the application requirement. Segregated rigid transformation parameters are able to explain the position and direction of the surfaces while non-rigid transformation parameters can be used to clarify the scoliotic shape change. For the scoliosis research, delineating the deformation parameter that is able to explain the scoliosis progress is more important than matching the surfaces together. Whereas a match by rigid parameters meant that bulk differences were removed and the deformation could be studied by looking at the residual differences, deformation magnitudes can be extracted directly from the shearing parameters. However, determination of transformation parameter values to provide specific scoliosis deformation information is outside the author’s specialty and selection of the most appropriate parameters can be expected to require input from medical professional in future. Very significant results for the new non-rigid transformation algorithm are its ability to match surfaces with new parameters and its ability to monitor the surface deformation.

4.4 Chapter Summary

A non-rigid matching algorithm has been developed which can match two surfaces and its parameters can be used to model the surface deformation simultaneously. This new algorithm uses nine transformation parameters including six shear parameters which have not been used before to match the surfaces. This matching algorithm provides both better positional and shape fits. Some fundamental experiments utilising synthetic and actual scoliosis data show the effectiveness of this non-rigid surface matching algorithm. The experimental results are very promising, demonstrating that this new non-rigid
matching algorithm is able to improve the precision of the matching (defined by the residuals of the separation) and the accuracy of the matching (which compares the r.m.s.). The analyses have also demonstrated that this new algorithm is able to avoid the drawbacks faced in classical surface matching algorithms.
CHAPTER 5

ROBUST NON-RIGID SURFACE MATCHING

5.1 Introduction

Least Squares Matching (LSM) is appealing to researchers in the fields of photogrammetry and remote sensing because of its simplicity, computational efficiency and also its suitability to cases in surface matching with small bias pose (e.g., Miller, 2008; Mitchell, 1998; Pilgrim, 1996a, 1996b; Zhang, 2006). An assumption of the normal distribution fits well with LSM that minimises the sum of squares of residuals (or surface separations in this research) between deformed and original surfaces. However it is well-known that a Least Squares (LS) estimator might be affected by the presence of a single outlier. Accordingly, the most critical issue for LSM is probably that of robustness, as the original algorithm assumes outlier-free data. These outliers cause deviations from the normal distribution and can lead to a poor or erroneous estimation of the matching algorithm parameters. Therefore they require special care (Kumar, 1994, Rousseeuw 1987).

Theoretically, the LSM algorithm is able to match the surfaces only if the surfaces are very similar except for outliers. As explained by Li (2001), in the case that the surfaces suffer from local deformation, parts of one surface differ from corresponding parts of the other surface, and the matching residuals can be grouped into two categories: residuals from identical parts, denoted as $R_I$, and residuals from non-identical parts, denoted as $R_D$. In general, elements in $R_I$ can be assumed to obey the normal distribution and elements in $R_D$ would usually differ significantly from zero. When $R_I$ is mixed with $R_D$, elements in $R_D$ thus could be considered as outliers. Consequently, local deformation could be detected as outliers in the observations of the matching process.

The classical surface matching algorithm, Least Z Difference (LZD), solves the matching problem in an LSM sense (Rosenholm, 1988; Karras, 1993; Pilgrim 1996a,
To deal with presence of possible outliers and perhaps possible deformation changes, several robust versions of LZD using different robust estimators can be found in the literature. Karras (1993) adopted data snooping, and proposed a new method for matching surfaces with some existing outliers. Pilgrim (1991, 1996a, b) improved LSM by using a Maximum Likelihood, M-estimator, and incorporated it into the least Z difference technique, called M-LZD. This method can detect deformation areas of up to 25%. Li (2001) integrated the Least Median Squares (LMS) -estimator with the random sample scheme and then proposed a new robust LZD algorithm, called LMS-LZD. The algorithm can detect deformation areas of no more than 50%. Zhang (2006) took the magnitude and the relationship of observations into account by introducing a differential model, and proposed another robust version of LZD, called Differential Model LZD (DM-LZD), which could detect slightly over 50% deformation areas. It has the highest ability of deformation detection so far, but it is rather complex. The author also found that the DM-LZD requires further validation. Choice of these robust estimators, however, usually lacks solid statistical justification and what technique is most appropriate remains an open problem (Ye, 2000). From conclusions on modern regression methods (Rousseeuw, 1987; Rousseeuw, 1999; Rousseeuw, 2006; Wilcox, 1997), the author found that the Least Trimmed Squares (LTS) technique is more appropriate for Least Squares 3D Surface Matching. What is more important is that this robust estimator fits favourably with the application reported here.

This chapter addresses the problem of Euclidean alignment of two roughly pre-aligned 3D surfaces in the presence of outliers and, possibly, shape changes (surface deformation). A simple, high-performance and robust method for multi-temporal laser scanned data matching is proposed. This new method integrates the LTS estimator (Rousseeuw, 1984) with non-rigid Least Normal Distance Difference (LNDD) algorithm. The author called this robust algorithm the Trimmed LNDD (TrLNDD). This new algorithm is based on the consistent use of the LTS approach in all phases of the operation. While running the LNDD algorithm, given two 3D point sets, R and M, the task is to find the Euclidean transformation that brings M into the best possible alignment with R. LNDD as proven in Chapter 3 has higher pull-in-range and better convergence in term of r.m.s. error and matching residual compared with LZD. The
concentration here is on the issue of robustness while at the same time preserving the matching algorithm’s structure and convergence.

To provide a comparison, the author has compared this robust algorithm, TrLNDD, with two other robust estimators. They are:

i) M-estimator: Huber’s estimator, and
ii) Generalised M-estimator (GM-estimator): Integration of the data snooping technique (Baarda’s statistic) and Tukey’s biweight estimator.

The final results are discussed in Section 5.5.

5.2 The Proposed Robust Estimator – Least Trimmed Squares (LTS)

The Least Trimmed Squares (LTS) estimator is proposed by Rousseeuw (1984) to make the least squares estimator (LS) more robust. It was introduced to repair the low efficiency of Least Median Squares (LMedS). LTS involves sorting the square errors (residuals) in an increasing order and minimising the sum of a certain number of smaller values. By excluding the largest squared residuals from the LS criterion function, LTS produces a fit insensitive to the outliers. LTS is defined as:

\[
\min \sum_{i=1}^{h} (r_i^2)_{ch} 
\]

(5.1)

where, \((r_i^2)_{1n} \leq (r_i^2)_{2n} \leq ... \leq (r_i^2)_{hn}\) are the ordered squared residuals.

1 \leq h \leq n is the point that can be tuned to discard the outliers depending on the contamination of the data. This is equivalent to finding the \(h\)-subset with smallest least squares objective function. The LTS is then least squares fit to these \(h\) points (for more detail, see Rousseeuw, 1984). When \(h = n\), all residuals are taken into account.
The breakdown point of LTS is \( h = (n + p + 1)/2 \), where \( p \) is the number of parameters. For a large data set, its breakdown point is \( \varepsilon^* = 50\% \), when \( h \approx n/2 \), equals that of the LMedS. This means that the contaminated data (both outliers and deformation) between the surfaces has to be below 50\%. It should be emphasized that the goal of the LTS is not just to “give up” a portion of data. The majority good match is used to detect and discard the outliers while selecting as many inliers as possible for another, more accurate match.

LTS has higher breakdown points than other robust estimator like M-estimator and GM-estimator. LSM has a breakdown point of 0\% while LTS has a breakdown point of 50\% as reported above. The higher the breakdown point of an estimator, the more robust it is. The breakdown point cannot exceed 50\% because if more than half of the observations are contaminated, it is not possible to distinguish between the underlying distribution and the contaminating distribution (Huber, 1981; Maronna, 2006). Therefore, the maximum breakdown point value is set to 0.5 or 50\%. Based on the nature of the scoliosis deformation condition where patients can experience a high ratio of scoliotic deformities, LTS is favoured for incorporation into the matching algorithm to detect possible outliers and shape changes. However, there is estimator which achieves such a breakdown point and it is called LMedS.

LMedS estimator was introduced by Rousseeuw (1984) to make the least squares estimator more robust, that is, to make the linear regression insensitive to outliers. LMedS minimises the median, that is, the value in the middle of the sorted sequence. The LMedS is defined as:

\[
\min_{i=1}^{n} \text{med} r_i^2
\]

(5.2)

where, \( \text{med} \) is the median of the residual, \( r_i^2 \) is the squared residual, and \( n \) is the number of residuals.
The breakdown point of the LMedS is $c^* = 50\%$. In other words, up to a half of the input data may be contaminated: until the inliers form a majority, the median is selected from the inliers.

The LMedS estimator is resistant to outliers and has a high breakdown point like LTS, but it has extremely low statistical efficiency, which means it tends to perform poorly when there are no outliers. This is because LMedS needs only to fit well half of the observations and ignore the other half. In other words, half of the observations have no actual influence on the estimations. Based on the nature of the scoliosis deformation condition, 0% of scoliotic deformities are almost impossible to occur, a high breakdown point is comparatively more important than having a high statistical efficiency.

However, there are many differences between LTS and LMedS (Giloni, 2002; Rousseeuw, 2006). It should be emphasised that all the discussion in this paper about the LMedS and LTS is in the context of Least Squares 3D Surface Matching. The author prefers the LTS because:

i) the result of the LMedS estimator is statistical, while that of LTS is computational.

ii) LTS’s objective function is smoother, making it less sensitive to local effects than LMedS;

iii) LTS has higher statistical efficiency and better precision. Therefore, LTS has better convergence rate compared to LMedS which makes it more suitable for an iterative estimator (Chetverikov, 2005). As provided by Rousseeuw (1987), LTS was introduced to repair the low efficiency of LMedS.

As discussed by Chetverikov (2005), the median is efficient in discarding strongly deviating outliers. However, the scoliotic deformities are not necessary strong outliers. Further the spectrum of deformation is often in a continuous pattern. The efficiency of the median as the core estimation criterion in such a situation is questionable. The trimmed squares seem to better fit the nature of the application reported here.

The final argument in favour of LTS is the optimal transformation can be computed using LTS for each iteration in exactly the same way as is done in LNDD. The basic
observation is that LTS fits the original scheme of LNDD without any significant modification. At each step of iteration, the optimal transformation can be computed for trimmed squares in exactly the same way as it is done in LNDD for all squares. However, LMedS does not facilitate this computation, which is usually associated with a random sampling scheme (Xu, 2000).

Having almost all merits of LMedS and better statistical efficiency, LTS is usually preferred to LMedS (Rousseeuw, 1987; Rousseeuw, 1999; Rousseeuw, 2006; Wilcox, 1997). Therefore, LTS is preferred to LMedS in studying robust non-rigid surface matching.

5.3 Robust Non-Rigid Surface Matching Using LTS Estimator

Given two short range laser scanned surfaces, the reference surface $R = \{P_i| i = 1,2,\ldots,m_1\}$ and the model surface $M = \{Q_j| j = 1,2,\ldots,m_2\}$. Here $m_1$ does not necessarily equal $m_2$, that is to say, the two surfaces need not be of the same size. Additionally, the points on the two surfaces with the same subscripts do not represent the same real physical points (real correspondence). Similar to most 3D surface matching algorithms, the proposed algorithm also assumes that the two surfaces to be matched have been roughly aligned or that the transformation between them is close to zero. This can be easily achieved by calculating the centroid values of the surfaces and using these values as the predetermined alignment function.

For matching $M$ with $R$, the modified LNDD with nine different transformation parameters is preferred to the classical six parameters LZD. The Euclidean (Normal Distance) difference, $d_{ND}$, between the point-patch of each pair is calculated using LNDD algorithm. For a more detailed formulation, see Section 4.3 of Chapter 4. For the LTS method, the sum of squared residuals is replaced by a trimmed sum of squared residuals. That is, the squared residuals are ordered from low to high and the sum is computed from the lowest residual to a specific threshold $h$. It should be noted that all points located outside the overlapped region will be dropped by setting their weight ($w$) to zero. Therefore, the optimal transformation can be derived by minimising the sum of least $h$ squared $d_{ND}$ according to the LTS estimator, while the classical LSM / LZD algorithm finds the transformation by minimising the sum of all squared difference, $d_Z$.
(the difference in Z-direction). The required least $h$ squared $dND$ can be found using the equation 5.1 described in Section 5.2.

The absolute value of $dND$ located in the deformed region is usually larger than that in the unchanged region to some extent, and will be discarded while selecting the least $hdND$. Additionally, the new method retains the macro framework of the LNDD algorithm. Therefore, the new algorithm, called TrLNDD in this research, is robust and easy-to-use. One of the advantages of using LTS is that the breakdown point of the estimator can be varied from 0% to 50% (0.0 - 0.5) by including more or less squared residuals in the sum. This can be done by adjusting $h$ in the LTS formulation. Different subset selection schemes have different converging rates. The flexibility of breakdown point is highly required especially in the scoliosis deformation monitoring because different patients might experience different percentages of scoliotic deformities.

Rousseeuw (1999) have introduced a fast algorithm for LTS computations called FAST-LTS, one of the most efficient algorithms available. The implementation here is based on FAST-LTS. From this solution, a preliminary estimate of the concentration residual error scale, $S_{LTS}^0$ can be estimated using the following equation 5.3:

$$S_{LTS}^0 = d_{h,n} \sqrt{\frac{1}{h} \sum_{i=1}^{h} \left( (r_i)^2 \right)_{i,n}}$$

(5.3)

where, the term $\sum_{i=1}^{h} \left( (r_i)^2 \right)_{i,n}$ is meant to indicate a summation of the ordered squared residuals from 1 to $h$. The constant $d_{h,n}$ is computed using equation 5.4:

$$d_{h,n} = \frac{1}{\sqrt{1 - \frac{2n}{hc_{h,n}} \phi \left( \frac{1}{c_{h,n}} \right)}}$$

(5.4)
where, the function $\phi$ is the standard normal density function in equation 5.5:

$$
\phi(x) = \sqrt{\frac{1}{2\pi}} e^{-\frac{x^2}{2}}
$$

(5.5)

In equation 5.4, the constant $c_{h,n}$ is computed using equation 5.6:

$$
c_{h,n} = \frac{1}{\Phi^{-1}\left(\frac{h+n}{2n}\right)}
$$

(5.6)

where, $\Phi(x)$ is the standard normal distribution function. The constant, $d_{h,n}$ and $c_{h,n}$ are chosen to make the scale estimator consistent with the Gaussian model. A final scale estimate is computed using equation 5.7:

$$
S_{LTS} = \sqrt{\frac{\sum_i w_i r_i^2}{\sum_i w_i - p}}
$$

(5.7)

with $p$ equal to the number of parameters and $r_i$ equal to the $i$th residual and:

$$
w_i = \begin{cases} 
0 & \text{if } \left|\frac{r_i}{S_{LTS}^0}\right| > 2.5 \\
1 & \text{otherwise}
\end{cases}
$$

(5.8)

Correspondences with residuals beyond ±2.5 robust scaled residuals are suspicious outliers.

The robust functions can be incorporated in the existing algorithm through the technique of iteratively reweighted least squares (IRLS). The conventional least squares solution for a system of weighted observation is given by:
\[ X = -\left( A^T W^{-1} A \right)^{-1} A^T W^{-1} L \]

(5.9)

where, \( X \) is the vector of least squares estimator, \( A \) is the coefficient matrix, \( W \) is the weight matrix of the observations, and \( L \) is the Euclidean distance differences.

It should be noted that one should perform robust matching after several matching iterations. This is because at the beginning of the matching process, the Euclidean distance between the temporarily paired point-patches is caused almost totally by orientation difference but not by outliers or local deformations. The algorithm of TrLNDD can be summarised as follows:

\textit{TrLNDD Algorithm}

Step 1: Find point-patch pair located in the reference and model surface and compute the Euclidean difference of each pair.

Step 2: Estimate the transformation parameters using non-robust least squares to obtain initial residuals.

Step 3: Residuals for all of the paired point-patch are computed and used to calculate the trimmed sum of squared residuals as in equation 5.1.

Step 4: Weight the observations using FAST-LTS algorithm as discussed in equation 5.8.

Step 5: Compute the optimal transformation associated with the LTS estimator as in equation 5.9. After the computation, outliers are detected and rejected, and the final least squares solution is obtained for inliers only.

Step 6: If any terminal condition is satisfied, terminate the iteration. Otherwise, apply the derived transformation to the data, and then return to Step 3.
5.4 Existing Robust Estimator Applied in Least Squares Matching

Two different robust estimators were trialled to obtain the robustness of the LNDD matching algorithm. This trial serves as the comparison to obtain the best robust estimator that can be incorporated into the non-rigid matching algorithm.

The first estimator is from the M-estimator group developed by Huber (1981) and other statisticians (Hampel, 1986). M-estimators normally have high statistical efficiency, typically more than 0.9. Local deformation of small size is likely to be detected with high statistical efficiency. However, M-estimators have relatively lower breakdown point compared with other robust estimators. Rigorously, M-estimators have breakdown point of as low as 0 in the presence of high leverage outliers (Rousseeuw, 1987). The breakdown point of M-estimators is less than $1/p + 1$, where $p$ is the number of parameters to be estimated (Meer, 1995). In this study, $p$ is 9 and thus the breakdown point is less than $1/9 + 1$ (0.10 or 10%). Such a low level of breakdown point is not desirable especially in scoliosis deformation monitoring because the data might experience high leverage changes due to scoliotic deformities.

In computation, M-estimators are usually converted to iterations of reweighted least squares. The weight of the residuals is updated during iterations according to the updated residuals and estimate of standard deviation of observations. In this trial, Tukey’s biweight estimator has been employed in the matching algorithm to obtain the robustness of the algorithm. The weight function of Tukey’s biweight is defined as:

$$w(u) = \begin{cases} \left(1 - \frac{u^2}{c^2}\right)^2 & |u| \leq c \\ 0 & |u| > c \end{cases}$$

(5.10)

where, $w$ is the weight ranging from 0 to 1, $c$ is the tuning constant and is set to $4.685 \sigma$, $u$ is the standardised least squares residual at every iteration divided by its standard deviation, $\sigma$, $u_i = \frac{r_i}{\sigma}$. 
The weighting function is incorporated through IRLS as in equation 5.9. The weights are not held fixed, but alter from iteration to iteration in response to the changing residuals. The resultant robust matching algorithm is called M-LNDD.

The second robust estimator is an estimator from GM-estimators. It is a combination of M-estimator and data snooping. As discussed by Li (2000), an important consideration in data snooping is that one should look not just for large residuals, but for residuals that are large in comparison to their own standard deviations. The incorporation of data snooping into an M-estimator can be achieved by replacing $u_i$ in equation 5.10 by $u_i = r_i/\sigma_i$, which is Baarda’s statistic. That is, the $i$ residual is divided by its own standard deviation. This is intended to bound the influence of leverage points which is the weakness in M-estimators. As provided by Rousseeuw (1987), a GM-estimator tolerates more outliers compared with M-estimators. Meanwhile, a simple modification was done to achieve higher breakdown point in this GM-estimator. The $\sigma_i$ is computed as:

$$\sigma_i = 1.4826 \times \text{med}|r_i|$$

(5.11)

which is fairly robust with a breakdown point of 0.5. The constant is set to 1.4826 to achieve consistency at normal error distribution (Rousseeuw, 1987). This robust matching algorithm is called GM-LNDD.

M-estimators (Pilgrim, 1991, 1996a, 1996b) and GM-estimators (Li, 2001) have been previously used in Least Squares 3D surface matching. This is the main reason why the M-estimator and GM-estimator were only selected in the comparison test. One can compare the results in terms of compatibility between incorporation of robust estimators in rigid and non-rigid surface matching algorithm.

As reported in the previous section, although LMedS achieves a higher breakdown point compared with other robust estimator (except LTS), LTS was designed and proposed to repair the weakness of LMedS. The author also found that LTS can fit better in the
LNDD algorithm without any major modification. Consequently, LMedS is bypassed in the comparison test with LTS.

5.5 Results and Discussion

This robust non-rigid surface matching algorithm has been trialled using both simulated and actual scoliosis data. The purpose to run the trial on simulated data is to validate the robust algorithm and verify the matching results. A trial on actual scoliosis data is to assess the robust algorithm’s capacity in detecting the scoliotic deformities. The results are discussed in the following sub-sections:

5.5.1 Trial on Simulated Human Back Surface

The surface examined here is a simulated human back surface with 50 X 50 grid points as shown in Figure 5.1(a). Outliers and deformation quantities are added to the Z coordinates to generate another simulated human back surface with deformations relative to the original surface (Figure 5.1 b). This deformed surface is then scaled and sheared by known parameters. The robust matching algorithms are required to compute the transformation parameters correctly.

Table 5.1 shows the known transformation parameters values and their estimates computed by different robust LNDD algorithms in a set of tests. The deformation percentage is set to 9% (0.09) and the local deformation lies at the middle of the surface. Since the M-estimator has a maximum of 10% breakdown point, the chosen deformation percentage is the best breakdown point value to avoid any matching failure. It can be easily noticed that all robust matching algorithms out-perform the non-robust LNDD algorithm. The estimates from the non-robust LNDD algorithm are significantly affected by the deformation. The deformed surface is tilted and elevated to match well with the original surface. It is worth to mention that the negative value in the result indicates the patient might has hump back due to scoliosis deformation and resulted him / her looks shorter.

A clear graphical visualisation can be observed in Figure 5.2(a). Because the results are illustrated in contour topographic map plotting, tilted and elevated points will generate
noisy maps like in Figure 5.2(a). All the three robust LNDD algorithms yield reasonable results in spite of the presence of local deformations (Figure 5.2 b, c and d). TrLNDD has the clearest and smoothest contour plot compared with the results obtained using M-LNDD and GM-LNDD. However, possibly due to low breakdown point and leverage point sensitivity, results obtained from the M-LNDD algorithm has a lower accuracy compared with the other two robust LNDD algorithms. Among these matching algorithms, TrLNDD has the lowest r.m.s. error.

Table 5.1: Comparison between known parameters and their estimated parameters computed by different robust LNDD algorithms. The deformation percentage is 9%.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Known Parameters</th>
<th>Non-robust LNDD</th>
<th>M-LNDD</th>
<th>GM-LNDD</th>
<th>TrLNDD</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX</td>
<td>1.010</td>
<td>-1.028</td>
<td>1.019</td>
<td>1.007</td>
<td>1.009</td>
</tr>
<tr>
<td>SY</td>
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<td>0.892</td>
<td>1.005</td>
<td>1.010</td>
<td>-1.001</td>
</tr>
<tr>
<td>SZ</td>
<td>1.001</td>
<td>1.013</td>
<td>1.022</td>
<td>1.005</td>
<td>1.002</td>
</tr>
<tr>
<td>Sh_{XY}</td>
<td>0.010</td>
<td>0.032</td>
<td>0.030</td>
<td>0.015</td>
<td>0.009</td>
</tr>
<tr>
<td>Sh_{XZ}</td>
<td>0.000</td>
<td>0.007</td>
<td>0.010</td>
<td>0.006</td>
<td>0.003</td>
</tr>
<tr>
<td>Sh_{YX}</td>
<td>0.000</td>
<td>-0.172</td>
<td>0.002</td>
<td>0.002</td>
<td>0.003</td>
</tr>
<tr>
<td>Sh_{YZ}</td>
<td>0.100</td>
<td>-0.020</td>
<td>0.081</td>
<td>0.110</td>
<td>0.097</td>
</tr>
<tr>
<td>Sh_{ZX}</td>
<td>0.150</td>
<td>0.101</td>
<td>0.110</td>
<td>0.132</td>
<td>0.152</td>
</tr>
<tr>
<td>Sh_{ZY}</td>
<td>0.000</td>
<td>0.009</td>
<td>0.005</td>
<td>-0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>r.m.s. error</td>
<td>-</td>
<td>2.9871</td>
<td>1.7227</td>
<td>1.2815</td>
<td>0.8969</td>
</tr>
</tbody>
</table>

Deformation percentage: 9%
Figure 5.1: Simulated human back surface for testing purpose. (a) Original Surface. (b) Deformed Surface.
Figure 5.2: Matching results of simulated human back surface using different matching algorithms (non-robust and robust). The results are illustrated in contour topographic map plotting. (a) using non-rigid LNDD algorithm, (b) using rigid M-LNDD algorithm, (c) using rigid GM-LNDD algorithm, and (d) using TrLNDD algorithm. It has to be mentioned that the changes in Z coordinate will affect the contour plotting. If the surfaces can not be matched correctly, it will result in noise in the topographic maps as in (a).
Table 5.2 depicts the capability of different robust LNDD algorithms to match the surfaces under different deformation percentage. The YES/NO condition indicates that the robust algorithm could match the surfaces correctly depending on the size, shape and position of the deformations.

Table 5.2: Capability of matching the surfaces under different deformation percentages by using different robust LNDD algorithm.

<table>
<thead>
<tr>
<th>Deformation Percentage</th>
<th>M-LNDD</th>
<th>GM-LNDD</th>
<th>TrLNDD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 5%</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>6 - 9%</td>
<td>YES</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>10 – 15%</td>
<td>YES/NO</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>16 – 20%</td>
<td>NO</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>21 – 25%</td>
<td>NO</td>
<td>YES/NO</td>
<td>YES</td>
</tr>
<tr>
<td>26 – 35%</td>
<td>NO</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>36 – 45%</td>
<td>NO</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>46 – 50%</td>
<td>NO</td>
<td>NO</td>
<td>YES/NO</td>
</tr>
<tr>
<td>&gt; 50%</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
</tr>
</tbody>
</table>

As the percentage of local deformations changed, the three robust LNDD algorithms behaved quite differently. Obviously, the deformation percentage plays a dominant role in determining the successful matching of these three robust algorithms. M-LNDD failed when the deformation percentage was over 11%. As stated above, this is due to the fact that M-LNDD has a low breakdown point of 10% in this application. In the test cases when the deformation percentage was less than 20%, GM-LNDD always succeeded in matching the surfaces satisfactorily. The better performance of GM-LNDD compared with M-LNDD is of course a result of the integration the Baarda’s statistic into the algorithm. The TrLNDD algorithm performs reasonably well even if the deformation percentage is as large as 47%.

Very similar results were obtained in the studies by Pilgrim (1991, 1996a, 1996b), Li (2000), and Zhang (2006). This indicates the robust estimator can perform well under
both rigid and non-rigid matching algorithm. In conclusion, the LTS estimator can be incorporated into the newly developed non-rigid matching algorithm to obtain a Robust Least Squares 3D Surface Matching algorithm. The next step is to test this robust matching algorithm’s capability in scoliosis deformation monitoring.

5.5.2 Robust Non-Rigid Surface Matching to Unmask Scoliotic Deformities

Actual scoliosis data sets were used to examine the capacity of TrLNDD algorithm in determining the scoliosis deformation. The matching results are illustrated in Figure 5.3 using contour topographic maps. Table 5.3 summarises the achieved r.m.s. error of different matching algorithm, namely Rigid, Rigid LTS, Non-Rigid (LNDD) and Non-Rigid LTS (TrLNDD). Preliminary experimental results show that LTS scheme permits correct matching even when the outliers exceed 50%. This is due to the adjustability of $h$ in the LTS formulation. Since in scoliosis deformation detection, outliers and surface changes happened frequently, there is a need to relax LSM’s error model and use more robust estimators. One can observe from the graphical visualisation that the TrLNDD is able to match the deformed areas in a smoother contour compared to non-robust LNDD algorithm.

Further to this from Table 5.3, there is a significant improvement in r.m.s. error for both rigid and non-rigid algorithms that have LTS incorporated. Improvements of about 79% can be detected indicating robust estimator is highly desirable to be incorporated into existing matching algorithm in detecting the surface deformation. LTS is clearly one of the estimators to achieve robust surface matching.
Table 5.3: Summary of r.m.s. error of different matching algorithms in matching the same actual scoliosis data sets.

<table>
<thead>
<tr>
<th>Matching Algorithm</th>
<th>Rigid LNDD</th>
<th>Rigid LTS</th>
<th>Non-Rigid LNDD</th>
<th>Non-Rigid LTS (TrLNDD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>r.m.s. error</td>
<td>1.6702</td>
<td>0.3600</td>
<td>1.4916</td>
<td>0.3290</td>
</tr>
</tbody>
</table>

Figure 5.3: Illustration of the matching results using non-rigid and TrLNDD algorithms. (a) non-rigid matching algorithm, (b) TrLNDD. TrLNDD shows its capacity in matching the deformed areas and generates smoother contours.
5.6 Chapter Summary

The problem of detecting local deformation between two surfaces has been addressed. This problem is resolved by means of robust surface matching in this research. Several robust estimators have been employed and tested including M-estimators, GM-estimators and LTS. The major difference between these estimators is the breakdown point. The TrLNDD is able to detect local deformation covering up to 50% of the surfaces being matched. This adaptive robust scheme has significantly higher accuracy than non-robust algorithms. Advantages are illustrated on both synthetic and real data.

The author believes the TrLNDD developed here is suitable for scoliosis deformation monitoring to detect scoliotic deformities. The TrLNDD algorithm is able to eliminate the unseen noise, outliers, effect on the matching results. Thus, it can be concluded that the goal of the research has been achieved where the true scoliotic change can be revealed by eliminating the noise in the data using TrLNDD algorithm.

Since Least Squares 3D Surface Matching is a fundamental step of many photogrammetric and remote sensing problems, and TrLNDD algorithm can be fitted into the Least Squares 3D Surface Matching framework, conclusions of this TrLNDD algorithm may extend to other fields than those discussed in this thesis. Preliminary experimental results on both synthetic and actual scoliosis data verified the effectiveness of LTS in Least Squares 3D Surface Matching. As this TrLNDD algorithm exhibits excellent surface matching results, the author expects to see it applied in more detailed scoliosis deformation monitoring.
CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Overview

The main goal of this research was to develop a robust non-rigid surface matching algorithm that has the ability to solve the non-rigid matching problem and simultaneously detect deformations. This has been achieved by developing a Trimmed Least Normal Distance Difference (TrLNDD) algorithm, which was implemented and successfully applied to the scoliosis deformation detection problem. This matching algorithm iteratively minimises the Euclidean distance that is normal to the surface between a pair of corresponding 3D surfaces. In order to design a more accurate and applicable algorithm for surfaces experiencing non-rigid deformation, new transformation parameters have been designed and incorporated into the matching algorithm. Based on the statistical analysis of the results obtained from the experiments, it was concluded that the new algorithm outperforms the classical rigid matching algorithm. To further improve this non-rigid matching algorithm, Least Trimmed Squares has been applied to increase the robustness of the algorithm.

6.2 Conclusions and Findings

The development described here is of a non-rigid form of least squares 3D surface matching, which has been driven by its possible medical benefit for scoliosis patients, but the concept may be applicable in other deformation studies too.

As preparation for designing the matching algorithm, the author has analysed the merits and demerits of the classical surface matching algorithm and current constraints faced in the scoliosis modelling and monitoring. This has been discussed and presented in Chapter 2. This discussion has highlighted the fact that surface matching is a feasible
and favourable tool for matching the scoliosis data for modelling and monitoring purposes.

It was first necessary to choose a suitable objective function and efficient transformation procedure which would be able to be applied in the non-rigid surface matching algorithm. Various possible objective functions with different transformation procedure which could be used were examined, including MICP, LZD, Combined ICP-LZD and LNDD. Minimising the surface normal (LNDD) was chosen as the required objective function and transformation procedure due to its simplicity (compared to MICP and LZD), efficiency (defined by the residuals of the separation) and the accuracy (which compares the r.m.s. error) of the resulting calculation. This is supported by Schenk (2000) where a similar study was trialled on small artificial data sets in a comparison study of matching function. The outcome is matched with the result reported here. This objective function, minimising the surface normal, is regarded as being more rigorous than the classical approach, as explained by Schenk (2000). Meanwhile, it was decided that the transformation model would be undertaken in the MATLAB programming environment due to the fundamental data type supported in MATLAB.

A new matching algorithm involving a non-rigid transformation using nine new parameters, namely three scales and six shears, has been designed, developed and presented. The effectiveness of this algorithm comes from its ability to match the deformed surface to the un-deformed surface and detect the deformation magnitude. The validity of the procedure has been verified with artificial surface data sets. Its effectiveness has been demonstrated by showing results from matching deformed scoliosis data. An apparent improvement in r.m.s. over the rigid transformation is achieved, and this is taken as evidence that a non-rigid transformation provides a superior model of the back’s shape change. An improvement by around 10% of r.m.s. values for all data has been reported. The maximum improvement up to about 20% is obtained for data with higher density (number of points).

The new transformation is assumed to provide both better positional and better shape fitting. Test on artificial surface data sets suggested that the accuracy of the procedure was adequate. The parameters of the transformation are seen as modelling both the non-scoliosis change and the shape change caused by scoliosis: scaling factors can be
interpreted as the dilation caused by natural growth in all three directions, while the shearing parameters can be used to depict the deformation caused by scoliosis.

There is also the option of first removing the positional shifts by a rigid transformation. Although this does not improve the closeness of the match, it appears to yield transformation parameters which are less influenced by the rotations, and this approach is recommended if the shear parameters are sought. The failure of the r.m.s. to be reduced suggests that the current transformation parameters may still be imperfect representatives of the complete deformation mechanism. New parameters may be able to model twist, stress and strain rate, and torsion while not all of the current parameters may be appropriate.

It is appreciated that more work must be done to extract medical information from the transformation parameters. In fact, the interpretation of the parameters’ numerical values, and the selection of the most appropriate shear and scale parameters, is not something the writers would claim to be able to do, but the results shown here may enable that discussion to be commenced with medical personnel.

This research is among the first using shears and scales as the transformation parameters to match the surfaces into same reference frame. This research is also the first using non-rigid transformation to match the scoliosis data for modelling and monitoring purposes. Further, this study will help to determine how to best use surface topography as a tool in scoliosis deformation monitoring. This study will also help to identify the useful surface topography parameters as there is still lack of consensus in the current literature.

Besides its utility in the current algorithm, the author believes that the shear parameters will also serve as a powerful shape matching tool and can be used to aid in many existing techniques towards effective shape analysis. This new non-rigid matching algorithm should give new insights to many problems in surface matching. Therefore, the author would like to conclude that the successful non-rigid matching algorithm can supplement the classical rigid matching algorithm for surface deformation monitoring, whilst requiring no prior knowledge about the surfaces.
Once the matching algorithm was established it was necessary to develop a system which could detect and remove any outliers, whilst having minimal effect on the matching process. A robust estimator was then incorporated into parts of the purely least squares procedure. This was achieved by incorporating Least Trimmed Squares into the non-rigid matching algorithm. Tests with this algorithm showed that it was capable of detecting differences covering up to 50% of the surfaces being matched with no decrease in the matching accuracy. The more important finding is that the differences percentage can be varied from 0% to 50% (0.0 -0.5) to suit the deformation rate of the application. Improvement of about 79% of r.m.s. values can be detected indicating LTS estimator is highly desirable to be incorporated into existing matching algorithm in detecting the surface deformation. It can be concluded that the TrLNDD matching algorithm developed here is feasible for scoliosis deformation monitoring to detect scoliotic deformities.

Therefore, it can be concluded that a successful robust non-rigid surface matching algorithm has been developed. Researchers can continue pursuing the possibilities to solve the deformation problems by designing extra new parameters that can be used to explain the deformations. This extended method requires determining more parameters concerned with shape than just the six translation and rotation parameters. The findings might not only be of interest to the practitioners from the scoliosis, other interested parties could be scientists and engineers that are trying to solve also problems where non-rigid deformation occurred on the surfaces.

The main achievements of the thesis can be summarised as follows:

i) Non-Rigid Surface Matching Algorithm

A non-rigid surface matching algorithm has been derived and developed in this research. This new matching algorithm enables non-rigid surface matching which has not been discussed in the literature. This algorithm can be applied on matching the surfaces that experience non-rigid deformations. This is particular useful in the medical applications.
ii) New Robust Matching Algorithm

The TrLNDD is able to detect local deformation covering up to 50% of the surfaces being matched. This is the highest achievable breakdown point reported in the literature. Based on the results obtain, the TrLNDD has the capability of dealing with the surfaces that contain high percentages of deformations. It is believed that the TrLNDD algorithm developed here is suitable for many other applications of detecting surface difference, including debris flow and coastal change analysis.

iii) Descriptive Parameters

This algorithm enabled the estimation of significant deformation parameters resulting from the deformation mechanism. It can calculate simultaneously the direction and magnitude of the change. It can be used to determine the spatial distribution of surface changes.

iv) Accuracy and Precision

Another very important finding of this research is the higher accuracy and precision achieved by the non-rigid surface matching algorithm. Advantages are illustrated on both synthetic and actual data. An accuracy improvement of about 20% was recorded for data with a high density of points.

v) Applicable in Medical Field

As proven in the research, this algorithm is feasible and favourable for application in scoliosis to determine the scoliotic deformities. It is believed that this algorithm can be further applied in other medical field like heart and liver motion analysis which experience non-rigid changes.
6.3 Future Work

In the future work following this research, some goals can be achieved for improving this new non-rigid surface matching algorithm. There are three directions for the future work:

i) Adding extra parameters to fully model the deformation mechanism.

Current transformation parameters, namely six-shear and three-scale, are still insufficient to represent the complete deformation mechanism. Further, comparison of deformed surfaces can be complicated due to the existence of apparent shape change. New parameters may be able to model twist, stress and strain rate, and torsion to model the change. If the parameters can be added to cover complete deformation mechanism, this algorithm can be useful in all medical application, for example to register the ventricles of the heart and brain.

ii) Further analyse the robustness of this algorithm.

In both the classical rigid surface matching algorithm and non-rigid matching algorithm, the assumption that outliers exist only on the Model Surface has been made. However, in reality, the outliers can exist on both the Reference and Model Surface. A rigorous outlier detection method is required to cancel the bias effect from outlying data on both surfaces.

iii) Apply the non-rigid surface matching approach to more applications especially in medical application.

Although this research has demonstrated the effectiveness of this non-rigid surface matching algorithm on scoliosis deformation modelling, it can equally be applied to other applications where digital surfaces experience non-rigid deformations. Besides its utility in this current application, the use of shear parameters may also serve as a powerful shape matching tool to aid many existing techniques towards effective shape analysis. In other cases of
surface deformation monitoring, such as in the registration of heart, brain and breast images and in non-medical applications in non-rigid changes resulting from earthquake and volcanic activities.
References


Grivas, T.B., Vasiliadis E. S. and et al., 2007. The effect of growth on the correlation between the spinal and rib cage deformity: implications on idiopathic scoliosis pathogenesis. [www.scoliosisjournal.com/content/2/1/11](http://www.scoliosisjournal.com/content/2/1/11) [Accessed 15 August, 2009]


149


Scoliosis Association of Australia.


Appendix A

Derivation of the Observation Equation of MICP

In Euclidean three-space, the distance between two points is defined as:

$$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

(A1)

The problem statement is to estimate the final location and orientation of the model surface $S_2(X, Y, Z)$, which satisfies the minimum condition of least squares matching with respect to reference surface $S_1(X, Y, Z)$. The functional model is:

$$S_1(X, Y, Z)^T = S_2(X, Y, Z)^T$$

(A2)

$$S_2(X, Y, Z)^T = T + mR S_{20}(X, Y, Z)^T$$

(A3)

$$
\begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix}
= 

\begin{bmatrix}
  T_x \\
  T_y \\
  T_z
\end{bmatrix} + m

\begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}

\begin{bmatrix}
  X_0 \\
  Y_0 \\
  Z_0
\end{bmatrix}
$$

(A4)

where, $S_1(X, Y, Z)^T$ is the coordinate of a point in reference surface $S_1$, $S_{20}(X, Y, Z)^T$ is the coordinate of a point in model surface $S_2$ before transformation, $S_2(X, Y, Z)^T$ is the coordinate of a point in model surface $S_2$ after transformation, $r_{ij} = R(\omega, \phi, \kappa)$ are the elements of the orthogonal rotation matrix, which are a function of the three independent rotation parameters $\omega, \phi, \kappa$ as shown below:
\[ r_{11} = \cos(\phi) \cdot \cos(\kappa); \]
\[ r_{12} = -\cos(\phi) \cdot \sin(\kappa); \]
\[ r_{13} = \sin(\phi); \]
\[ r_{21} = \sin(\omega) \cdot \sin(\phi) \cdot \cos(\kappa) + \cos(\omega) \cdot \sin(\kappa); \]
\[ r_{22} = -\sin(\omega) \cdot \sin(\phi) \cdot \sin(\kappa) + \cos(\omega) \cdot \cos(\kappa); \]
\[ r_{23} = -\sin(\omega) \cdot \cos(\phi); \]
\[ r_{31} = -\cos(\omega) \cdot \sin(\phi) \cdot \cos(\kappa) + \sin(\omega) \cdot \sin(\kappa); \]
\[ r_{32} = \cos(\omega) \cdot \sin(\phi) \cdot \sin(\kappa) + \sin(\omega) \cdot \cos(\kappa); \]
\[ r_{33} = \cos(\omega) \cdot \cos(\phi); \]

\((T_x, T_y, T_z)^T\) is the translation vector, and \(m\) is the uniform scale factor and is set to 1.

Equation A4 can be written as:

\[
\begin{align*}
X &= (r_{11}X_0 + r_{12}Y_0 + r_{13}Z_0) + T_x; \\
Y &= (r_{21}X_0 + r_{22}Y_0 + r_{23}Z_0) + T_y; \\
Z &= (r_{31}X_0 + r_{32}Y_0 + r_{33}Z_0) + T_z;
\end{align*}
\]

(A5)

In order to perform least squares estimation, equation A2 must be linearised by Taylor’s expansion, ignoring second and higher order terms.

\[
S_i(X, Y, Z) = S_{20}(X, Y, Z) + \left[ \frac{\partial S_{20}}{\partial X}(X, Y, Z) \right] dX + \left[ \frac{\partial S_{20}}{\partial Y}(X, Y, Z) \right] dY + \left[ \frac{\partial S_{20}}{\partial Z}(X, Y, Z) \right] dZ
\]

(A6)

where, subscript \(0\) indicates evaluation at the estimated match position, with:

\[
\begin{align*}
dX &= (\partial X/\partial p_i) * dpi; \\
dY &= (\partial Y/\partial p_i) * dpi; \\
dZ &= (\partial Z/\partial p_i) * dpi;
\end{align*}
\]
where, \( pi \in \{ \omega, \varphi, \kappa, T_x, T_y, T_z \} \) is the i-th transformation parameters in equation A6.

The Euclidean distance between \( S_i(X, Y, Z)^T \) and \( S_{20}(X, Y, Z)^T \) is defined as:

\[
d = \left[ (S_i(X, Y, Z)^T - S_{20}(X, Y, Z)^T)^2 \right]^{1/2}
\]

\[
= \left[ (S_i X - S_{20} X)^2 + (S_i Y - S_{20} Y)^2 + (S_i Z - S_{20} Z)^2 \right]^{1/2}
\]

(A7)

The derivative of \( S_{20}(X, Y, Z) \) with respect to \( X, Y \) and \( Z \) can be written as:

\[
S_{20}X = \frac{\partial S_{20}(X, Y, Z)}{\partial X} = \frac{S_i X - S_{20} X}{d};
\]

\[
S_{20}Y = \frac{\partial S_{20}(X, Y, Z)}{\partial Y} = \frac{S_i Y - S_{20} Y}{d};
\]

\[
S_{20}Z = \frac{\partial S_{20}(X, Y, Z)}{\partial Z} = \frac{S_i Z - S_{20} Z}{d};
\]

(A8)

where, \( d \) is the Euclidean distance between points and \( \{ S_2X, S_2Y, S_2Z \} \) are numeric first derivatives of function \( S_2 (X, Y, Z) \).

Equation A5 is linearised by Taylor’s expansion, ignoring second and higher order terms gives:

\[
X = (X_0) + (\frac{\partial X}{\partial r_{11}})_0 dr_{11} + (\frac{\partial X}{\partial r_{12}})_0 dr_{12} + (\frac{\partial X}{\partial r_{13}})_0 dr_{13} + (\frac{\partial X}{\partial T_x})_0 dT_x;
\]

\[
Y = (Y_0) + (\frac{\partial Y}{\partial r_{21}})_0 dr_{21} + (\frac{\partial Y}{\partial r_{22}})_0 dr_{22} + (\frac{\partial Y}{\partial r_{23}})_0 dr_{23} + (\frac{\partial Y}{\partial T_y})_0 dT_y;
\]

\[
Z = (Z_0) + (\frac{\partial Z}{\partial r_{31}})_0 dr_{31} + (\frac{\partial Z}{\partial r_{32}})_0 dr_{32} + (\frac{\partial Z}{\partial r_{33}})_0 dr_{33} + (\frac{\partial Z}{\partial T_z})_0 dT_z;
\]

(A9)
which can be simplified to,

\[ X = (X_0)_0 + X_0 \, dr_{11} + Y_0 \, dr_{12} + Z_0 \, dr_{13} + dT_X; \]
\[ Y = (Y_0)_0 + X_0 \, dr_{21} + Y_0 \, dr_{22} + Z_0 \, dr_{23} + dT_Y; \]
\[ Z = (Z_0)_0 + X_0 \, dr_{31} + Y_0 \, dr_{32} + Z_0 \, dr_{33} + dT_Z; \]

(A10)

Further substitution can be made by evaluating the following:

\[ dr_{11} = \frac{\partial r_{11}}{\partial \phi} \, d\phi + \frac{\partial r_{11}}{\partial \kappa} \, d\kappa \]
\[ = (-\sin(\phi) \cdot \cos(\kappa)).d\phi + (\cos(\phi) \cdot \sin(\kappa)).d\kappa \]

(A11)

\[ dr_{12} = \frac{\partial r_{12}}{\partial \phi} \, d\phi + \frac{\partial r_{12}}{\partial \kappa} \, d\kappa \]
\[ = (\sin(\phi) \cdot \sin(\kappa)).d\phi + (-\cos(\phi) \cdot \cos(\kappa)).d\kappa \]

(A12)

\[ dr_{13} = \frac{\partial r_{13}}{\partial \phi} \, d\phi \]
\[ = \cos(\phi).d\phi \]

(A13)

\[ dr_{21} = \frac{\partial r_{21}}{\partial \omega} \, d\omega + \frac{\partial r_{21}}{\partial \phi} \, d\phi + \frac{\partial r_{21}}{\partial \kappa} \, d\kappa \]
\[ = (\cos(\omega) \cdot \sin(\phi) \cdot \cos(\kappa) - \sin(\omega) \cdot \sin(\kappa)).d\omega \]
\[ + (\sin(\omega) \cdot \cos(\phi) \cdot \cos(\kappa)).d\phi \]
\[ + (-\sin(\omega) \cdot \sin(\phi) \cdot \sin(\kappa) + \cos(\omega) \cdot \cos(\kappa)).d\kappa \]

(A14)
\[ dr_{22} = \frac{\partial r_{22}}{\partial \omega} d\omega + \frac{\partial r_{22}}{\partial \phi} d\phi + \frac{\partial r_{22}}{\partial \kappa} d\kappa \]
\[ = (-\cos(\omega).\sin(\phi).\sin(\kappa) - \sin(\omega).\cos(\kappa)).d\omega \]
\[ + (-\sin(\omega).\cos(\phi).\sin(\kappa)).d\phi \]
\[ + (-\sin(\omega).\sin(\phi).\cos(\kappa) - \cos(\omega).\sin(\kappa)).d\kappa \]

(A15)

\[ dr_{23} = \frac{\partial r_{23}}{\partial \omega} d\omega + \frac{\partial r_{23}}{\partial \phi} d\phi \]
\[ = (-\cos(\omega).\cos(\phi)).d\omega \]
\[ + (\sin(\omega).\sin(\phi)).d\phi \]

(A16)

\[ dr_{31} = \frac{\partial r_{31}}{\partial \omega} d\omega + \frac{\partial r_{31}}{\partial \phi} d\phi + \frac{\partial r_{31}}{\partial \kappa} d\kappa \]
\[ = (\sin(\omega).\sin(\phi).\cos(\kappa) + \cos(\omega).\sin(\kappa)).d\omega \]
\[ + (-\cos(\omega).\cos(\phi).\cos(\kappa)).d\phi \]
\[ + (\cos(\omega).\sin(\phi).\sin(\kappa) + \sin(\omega).\cos(\kappa)).d\kappa \]

(A17)

\[ dr_{32} = \frac{\partial r_{32}}{\partial \omega} d\omega + \frac{\partial r_{32}}{\partial \phi} d\phi + \frac{\partial r_{32}}{\partial \kappa} d\kappa \]
\[ = (-\sin(\omega).\sin(\phi).\sin(\kappa) + \cos(\omega).\cos(\kappa)).d\omega \]
\[ + (\cos(\omega).\cos(\phi).\sin(\kappa)).d\phi \]
\[ + (\cos(\omega).\sin(\phi).\cos(\kappa) - \sin(\omega).\sin(\kappa)).d\kappa \]

(A18)
\[
\frac{dr_{33}}{\partial \omega}d\omega + \frac{\partial r_{33}}{\partial \phi}d\phi \\
= (-\sin(\omega).\cos(\phi))d\omega \\
+ (-\cos(\omega).\sin(\phi))d\phi
\]

(A19)

Substituting equation A11 – A19 into equation A10, gathering like terms and simplifying gives:

\[
X = (X_0)_0 + dT_x \\
+ X_0.[-r_{13}.\cos(\kappa).d\phi - r_{12}.d\kappa] \\
+ Y_0.[r_{13}.\sin(\kappa).d\phi - r_{14}.d\kappa] \\
+ Z_0.[\cos(\phi).d\phi]
\]

(A20)

\[
Y = (Y_0)_0 + dT_y \\
+ X_0.[-r_{31}.d\omega + \sin(\omega).r_{11}.d\phi + r_{22}.d\kappa] \\
+ Y_0.[r_{32}.d\omega + \sin(\omega).r_{12}.d\phi - r_{21}.d\kappa] \\
+ Z_0.[r_{33}.d\omega + \sin(\omega).r_{13}.d\phi]
\]

(A21)

\[
Z = (Z_0)_0 + dT_z \\
+ X_0.[r_{23}.d\omega - \cos(\omega).r_{11}.d\phi + r_{32}.d\kappa] \\
+ Y_0.[r_{22}.d\omega - \cos(\omega).r_{12}.d\phi - r_{21}.d\kappa] \\
+ Z_0.[r_{23}.d\omega - \cos(\omega).r_{13}.d\phi]
\]

(A22)

Substituting letters for partial derivative coefficients, and rearranging terms for \(dX, dY, dZ\), the following equations result:
\[
\begin{align*}
&dX = a_{11}d\omega + a_{12}d\varphi + a_{13}d\kappa + T_X; \\
&dY = a_{21}d\omega + a_{22}d\varphi + a_{23}d\kappa + T_Y; \\
&dZ = a_{31}d\omega + a_{32}d\varphi + a_{33}d\kappa + T_Z;
\end{align*}
\]

(A23)

To clarify the coefficients of equation A23, the partial derivative terms, which must be evaluated at the initial approximations, are as follows:

\[
\begin{align*}
&a_{11} = 0; \\
&a_{12} = -X_0 r_{13} \cos(\kappa) + Y_0 r_{13} \sin(\kappa) + Z_0 \cos(\phi); \\
&a_{13} = -X_0 r_{12} - Y_0 r_{11}; \\
&a_{21} = -X_0 r_{23} - Y_0 r_{22} - Z_0 r_{33}; \\
&a_{22} = X_0 \sin(\omega) r_{11} + Y_0 \sin(\omega) r_{12} + Z_0 \sin(\omega) r_{13}; \\
&a_{23} = X_0 r_{22} - Y_0 r_{21}; \\
&a_{31} = X_0 r_{32} + Y_0 r_{33}; \\
&a_{32} = -X_0 \cos(\omega) r_{11} - Y_0 \cos(\omega) r_{12} - Z_0 \cos(\omega) r_{13}; \\
&a_{33} = X_0 r_{33} - Y_0 r_{31};
\end{align*}
\]

Substituting equation A8 and A23, equation A6 becomes:

\[
\begin{align*}
S_1(X, Y, Z) - S_20(X, Y, Z) = d \\
&= (S_2 X a_{11} + S_2 Y a_{21} + S_2 Z a_{31})d\omega \\
&\quad + (S_2 X a_{12} + S_2 Y a_{22} + S_2 Z a_{32})d\varphi \\
&\quad + (S_2 X a_{13} + S_2 Y a_{23} + S_2 Z a_{33})d\kappa + S_2 X T_X + S_2 Y T_Y + S_2 Z T_Z
\end{align*}
\]

(A24)
Appendix B

Derivation of the Observation Equation of LNDD

As shown in Chapter 3 Section 3.5, equation 3.27 is computing the normal distance from a point of transformed model surface \( S_2 ' \) to a plane defined by the parameters, \( a, b, c, \) and \( d \). The coordinate of the point \( (X', Y', Z') \in S_2 ' \) are expressed as a function of the coordinates of a point \( (X, Y, Z) \in S_2 \) and the parameters of a rigid transformation \( \omega, \phi, \kappa, T_x, T_y \) and \( T_z \):

\[
D = \frac{|d[(m_{11}.X + m_{12}.Y + m_{13}.Z) + T_x] + b[(m_{21}.X + m_{22}.Y + m_{23}.Z) + T_y] + c[(m_{31}.X + m_{32}.Y + m_{33}.Z) + T_z] - d|}{(a^2 + b^2 + c^2)^{\frac{1}{2}}}
\]

(B1)

where,

\[
\begin{align*}
    m_{11} &= \cos(\phi).\cos(\kappa); \\
    m_{12} &= -\cos(\phi).\sin(\kappa); \\
    m_{13} &= \sin(\phi); \\
    m_{21} &= \sin(\omega).\sin(\phi).\cos(\kappa) + \cos(\omega).\sin(\kappa); \\
    m_{22} &= -\sin(\omega).\sin(\phi).\sin(\kappa) + \cos(\omega).\cos(\kappa); \\
    m_{23} &= -\sin(\omega).\cos(\phi); \\
    m_{31} &= -\cos(\omega).\sin(\phi).\cos(\kappa) + \sin(\omega).\sin(\kappa); \\
    m_{32} &= \cos(\omega).\sin(\phi).\sin(\kappa) + \sin(\omega).\cos(\kappa); \\
    m_{33} &= \cos(\omega).\cos(\phi);
\end{align*}
\]

and \( (T_x, T_y, T_z) \) is the translation vector.

Let \( \eta \) be the expression inside the absolute value bracket in equation B1, the first order derivatives of the normal distance expression with respect to the six transformation parameters are as follows:
**Derivative of D with respect to \( \omega \)**

\( a, b \) and \( c \) are constant. The derivative of \( D \) with respect to \( \omega \) can be written as:

\[
\frac{\partial D}{\partial \omega} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial \omega} = \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial \omega}
\]

\((B2)\)

where, the signum function (abbreviated \( \text{sign} \)) is the function \( \text{sign} : \mathbb{R} \rightarrow \mathbb{R} \):

\[
\text{sign}(x) = \begin{cases} 
-1 & \text{when } x < 0, \\
0 & \text{when } x = 0, \\
1 & \text{when } x > 0, 
\end{cases}
\]

\((B3)\)

Expanding expression \( \eta \):

\[
\eta = a.X.\cos(\phi).\cos(\kappa) - a.Y.\cos(\phi).\sin(\kappa) + a.Z.\sin(\phi) + a.T_x \\
+ b.X.\sin(\omega).\sin(\phi).\cos(\kappa) + b.X.\cos(\omega).\sin(\kappa) - b.Y.\sin(\omega).\sin(\phi).\sin(\kappa) \\
+ b.Y.\cos(\omega).\cos(\kappa) - b.Z.\sin(\omega).\cos(\phi) + b.T_y - c.X.\cos(\omega).\sin(\phi).\cos(\kappa) \\
+ c.X.\sin(\omega).\sin(\kappa) + c.Y.\cos(\omega).\sin(\phi).\sin(\kappa) + c.Y.\sin(\omega).\cos(\kappa) \\
+ c.Z.\cos(\omega).\cos(\phi) + c.T_z - d
\]

\((B4)\)
The derivative of $\eta$ with respect to $\omega$ is:

$$\frac{\partial \eta}{\partial \omega} = b.X.\cos(\omega).\sin(\phi).\cos(\kappa) - b.X.\sin(\omega).\sin(\kappa) - b.Y.\cos(\omega).\sin(\phi).\sin(\kappa) - b.Y.\sin(\omega).\cos(\phi) - b.Z.\cos(\omega).\cos(\phi) + c.X.\sin(\omega).\sin(\phi).\cos(\kappa) + c.X.\cos(\omega).\sin(\phi).\sin(\kappa) + c.X.\cos(\omega).\cos(\phi) + c.Y.\sin(\omega).\sin(\phi).\sin(\kappa) + c.Y.\cos(\omega).\cos(\kappa) + c.Z.\sin(\omega).\cos(\phi)$$

(B5)

Regrouping similar terms and simplifying gives:

$$\frac{\partial \eta}{\partial \omega} = -b.X.m_{31} - b.Y.m_{32} - b.Z.m_{33} + c.X.m_{21} + c.Y.m_{22} + c.Z.m_{23}$$

(B6)

The derivative of the normal distance $D$ with respect to $\omega$ is given as:

$$\frac{\partial D}{\partial \omega} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial \omega} = \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial \omega} = \frac{\text{sign}(\eta). - b.X.m_{31} - b.Y.m_{32} - b.Z.m_{33} + c.X.m_{21} + c.Y.m_{22} + c.Z.m_{23}}{\sqrt{a^2 + b^2 + c^2}}$$

(B7)
**Derivative of D with respect to \( \phi \)**

The derivative of \( \eta \) with respect to \( \phi \) is:

\[
\frac{\partial \eta}{\partial \phi} = -a.X.\sin(\phi)\cos(\kappa) + a.Y.\sin(\phi)\sin(\kappa) + a.Z.\cos(\phi) \\
+ b.X.\sin(\omega)\cos(\phi)\cos(\kappa) - b.Y.\sin(\omega)\cos(\phi)\sin(\kappa) \\
+ b.Z.\sin(\omega)\sin(\phi) - c.X.\cos(\omega)\cos(\phi)\cos(\kappa) \\
+ c.Y.\cos(\omega)\cos(\phi)\sin(\kappa) - c.Z.\cos(\omega)\sin(\phi)
\]

(B8)

After regrouping the similar terms and simplifying, the derivative of the normal distance \( D \) with respect to \( \phi \) is given as:

\[
\frac{\partial D}{\partial \phi} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial \phi} = \text{sign}(\eta) \frac{\partial \eta}{\partial \phi}
\]

\[
= \text{sign}(\eta). - a.X.m_{13}.\cos(\kappa) + a.Y.m_{13}.\sin(\kappa) + a.Z.\cos(\phi) + \\
b.X.\sin(\omega).m_{11} + b.Y.\sin(\omega).m_{12} + b.Z.\sin(\omega).m_{13} \\
= -c.X.\cos(\omega).m_{11} - c.Y.\cos(\omega).m_{12} - c.Z.\cos(\omega).m_{13} \sqrt{a^2 + b^2 + c^2}
\]

(B9)
Derivative of $D$ with respect to $\kappa$

The derivative of $\eta$ with respect to $\kappa$ is:

$$\frac{\partial \eta}{\partial \kappa} = -a.X.\cos(\phi).\sin(\kappa) - a.Y.\cos(\phi).\cos(\kappa)$$

$$- b.X.\sin(\omega).\sin(\phi).\sin(\kappa) + b.X.\cos(\omega).\cos(\kappa) - b.Y.\sin(\omega).\sin(\phi).\cos(\kappa)$$

$$- b.Y.\cos(\omega).\sin(\kappa) + c.X.\cos(\omega).\sin(\phi).\sin(\kappa)$$

$$+ c.X.\sin(\omega).\cos(\kappa) + c.Y.\cos(\omega).\sin(\phi).\cos(\kappa) - c.Y.\sin(\omega).\sin(\kappa)$$

(B10)

After regrouping the similar terms and simplifying, the derivative of the normal distance $D$ with respect to $\kappa$ is given as:

$$\frac{\partial D}{\partial \kappa} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial \kappa}$$

$$= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial \kappa}$$

$$= \frac{\text{sign}(\eta) \cdot a.X.m_{12} - a.Y.m_{11} + b.X.m_{22} - b.Y.m_{21} + c.X.m_{32} - c.Y.m_{31}}{\sqrt{a^2 + b^2 + c^2}}$$

(B11)

Derivative of $D$ with respect to $T_x$

The derivative of $\eta$ with respect to $T_x$ is:

$$\frac{\partial \eta}{\partial T_x} = a$$

(B12)
After regrouping the similar terms and simplifying, the derivative of the normal distance $D$ with respect to $T_x$ is given as:

$$
\frac{\partial D}{\partial T_x} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{\partial |\eta|}{\partial T_x} = \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{\partial \eta}{\partial T_x} = \frac{\text{sign}(\eta) a}{\sqrt{a^2 + b^2 + c^2}}
$$

(B13)

**Derivative of $D$ with respect to $T_y$**

The derivative of $\eta$ with respect to $T_y$ is:

$$
\frac{\partial \eta}{\partial T_y} = b
$$

(B14)

After regrouping the similar terms and simplifying, the derivative of the normal distance $D$ with respect to $T_y$ is given as:

$$
\frac{\partial D}{\partial T_y} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{\partial |\eta|}{\partial T_y} = \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{\partial \eta}{\partial T_y} = \frac{\text{sign}(\eta) b}{\sqrt{a^2 + b^2 + c^2}}
$$

(B15)
Derivative of $D$ with respect to $T_z$

The derivative of $\eta$ with respect to $T_z$ is:

$$\frac{\partial \eta}{\partial T_z} = c$$

(B16)

After regrouping the similar terms and simplifying, the derivative of the normal distance $D$ with respect to $T_z$ is given as:

$$\frac{\partial D}{\partial T_z} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial T_z}$$

$$= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial T_z}$$

$$= \frac{\text{sign}(\eta)c}{\sqrt{a^2 + b^2 + c^2}}$$

(B17)
Substituting letters for partial derivative coefficients, and rearranging terms, the following equations result:

\[
\begin{align*}
\partial D / \partial \omega &= \text{sign}(\eta)\left[a \cdot a_{11} + b \cdot a_{21} + c \cdot a_{31}\right] \left[1 + \left(a^2 + b^2 + c^2\right)^{1/2}\right]; \\
\partial D / \partial \varphi &= \text{sign}(\eta)\left[a \cdot a_{12} + b \cdot a_{22} + c \cdot a_{32}\right] \left[1 + \left(a^2 + b^2 + c^2\right)^{1/2}\right]; \\
\partial D / \partial \kappa &= \text{sign}(\eta)\left[a \cdot a_{13} + b \cdot a_{23} + c \cdot a_{33}\right] \left[1 + \left(a^2 + b^2 + c^2\right)^{1/2}\right]; \\
\partial D / \partial T_x &= \text{sign}(\eta) \cdot a \left[1 + \left(a^2 + b^2 + c^2\right)^{1/2}\right]; \\
\partial D / \partial T_y &= \text{sign}(\eta) \cdot b \left[1 + \left(a^2 + b^2 + c^2\right)^{1/2}\right]; \\
\partial D / \partial T_z &= \text{sign}(\eta) \cdot c \left[1 + \left(a^2 + b^2 + c^2\right)^{1/2}\right]; \\
\end{align*}
\]

(B18)

To clarify the coefficients of equation B18, the partial derivative terms, which must be evaluated at the initial approximations, are as follows:

\[
\begin{align*}
a_{11} &= 0; \\
a_{12} &= -X \cdot m_{31} - Y \cdot m_{32} - Z \cdot m_{33}; \\
a_{13} &= X \cdot m_{21} + Y \cdot m_{22} + Z \cdot m_{23}; \\
a_{21} &= -X \cdot m_{13} \cdot \cos(\kappa) + Y \cdot m_{13} \cdot \sin(\kappa) + Z \cdot \cos(\phi); \\
a_{22} &= X \cdot \sin(\omega) \cdot m_{11} + Y \cdot \sin(\omega) \cdot m_{12} + Z \cdot \sin(\omega) \cdot m_{13}; \\
a_{23} &= -X \cdot \cos(\omega) \cdot m_{11} - Y \cdot \cos(\omega) \cdot m_{12} - Z \cdot \cos(\omega) \cdot m_{13}; \\
a_{31} &= X \cdot m_{12} - Y \cdot m_{11}; \\
a_{32} &= X \cdot m_{22} - Y \cdot m_{21}; \\
a_{33} &= X \cdot m_{32} - Y \cdot m_{31};
\end{align*}
\]
Appendix C

Derivation of the Observation Equation of LNDD with Non-Rigid Transformation Parameters

As shown in Chapter 4 Section 4.3, LNDD algorithm is minimising the normal distance from a point of transformed model surface $S'_2$ to a plane defined by the parameters, $a$, $b$, $c$, and $d$ on reference surface $S_1$. The coordinate of the point $(X', Y', Z') \in S'_2$ are expressed as a function of the coordinates of a point $(X, Y, Z) \in S_2$ and the parameters of non-rigid transformation $S_X, S_Y, S_Z, Sh_{XY}, Sh_{XZ}, Sh_{YZ}, Sh_{XY}, Sh_{YX}, Sh_{YZ}$, and $Sh_{ZY}$:

$$D = \left| a \left( S_X \cdot X + Sh_{XY} \cdot S_Y \cdot Y + Sh_{YZ} \cdot S_Z \cdot Z \right) + b \left( Sh_{XY} \cdot S_X \cdot X + S_Y \cdot Y + Sh_{YX} \cdot S_Y \cdot Z \right) + c \left( Sh_{YX} \cdot S_X \cdot X + Sh_{ZY} \cdot S_Y \cdot Y + S_Z \cdot Z \right) - d \right| \left( a^2 + b^2 + c^2 \right)^{\frac{1}{2}}$$

(C1)

Let $\eta$ be the expression inside the absolute value bracket in equation C1, the first order derivatives of the normal distance expression with respect to the nine transformation parameters are as follows:

**Derivative of $D$ with respect to $S_X$**

$a$, $b$ and $c$ are constant. The derivative of $D$ with respect to $S_X$ can be written as:

$$\frac{\partial D}{\partial S_X} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial S_X} = \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial S_X}$$

(C2)
where, the signum function (abbreviated sign) is the function

\[
\text{sign}(x) = \begin{cases} 
-1 & \text{when } x < 0, \\
0 & \text{when } x = 0, \\
1 & \text{when } x > 0,
\end{cases}
\]

(C3)

Expanding expression \( \eta \):

\[
\eta = a.S_x.X + a.Sh_{xy}.S_y.Y + a.Sh_{xz}.S_z.Z \\
+ b.Sh_{xy}.S_x.X + b.S_y.Y + b.Sh_{yz}.S_z.Z \\
+ c.Sh_{xz}.S_x.X + c.Sh_{yz}.S_y.Y + S_z.Z - d
\]

(C4)

The derivative of \( \eta \) with respect to \( S_x \) is:

\[
\frac{\partial \eta}{\partial S_x} = a.X + b.Sh_{yz}.X + c.Sh_{yx}.X
\]

(C5)

The derivative of the normal distance \( D \) with respect to \( S_x \) is given as:

\[
\frac{\partial D}{\partial S_x} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial S_x} \\
= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial S_x} \\
= \frac{\text{sign}(\eta). (a.X + b.Sh_{yz}.X + c.Sh_{yx}.X)}{\sqrt{a^2 + b^2 + c^2}}
\]

(C6)
Derivative of $D$ with respect to $S_Y$

The derivative of $\eta$ with respect to $S_Y$ is:

$$
\frac{\partial \eta}{\partial S_Y} = a.\text{Sh}_{xy}.Y + b.Y + c.\text{Sh}_{yz}.Y
$$

(C7)

The derivative of the normal distance $D$ with respect to $S_Y$ is given as:

$$
\frac{\partial D}{\partial S_Y} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{\partial |\eta|}{\partial S_Y}
= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \cdot \frac{\partial \eta}{\partial S_Y}
= \frac{\text{sign}(\eta).(a.\text{Sh}_{xy}.Y + b.Y + c.\text{Sh}_{yz}.Y)}{\sqrt{a^2 + b^2 + c^2}}
$$

(C8)

Derivative of $D$ with respect to $S_Z$

The derivative of $\eta$ with respect to $S_Z$ is:

$$
\frac{\partial \eta}{\partial S_Z} = a.\text{Sh}_{xz}.Z + b.\text{Sh}_{yz}.Z + c.Z
$$

(C9)
The derivative of the normal distance $D$ with respect to $S_Z$ is given as:

$$
\frac{\partial D}{\partial S_Z} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial S_Z}
$$

$$
= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial S_Z}
$$

$$
= \frac{\text{sign}(\eta)(a\cdot Sh_{XY}.Z + b\cdot Sh_{YZ}.Z + c\cdot Z)}{\sqrt{a^2 + b^2 + c^2}}
$$

(C10)

**Derivative of $D$ with respect to $Sh_{XY}$**

The derivative of $\eta$ with respect to $Sh_{XY}$ is:

$$
\frac{\partial \eta}{\partial Sh_{XY}} = a\cdot S_Y\cdot Y
$$

(C11)

The derivative of the normal distance $D$ with respect to $Sh_{XY}$ is given as:

$$
\frac{\partial D}{\partial Sh_{XY}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial Sh_{XY}}
$$

$$
= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial Sh_{XY}}
$$

$$
= \frac{\text{sign}(\eta)(a\cdot S_Y\cdot Y)}{\sqrt{a^2 + b^2 + c^2}}
$$

(B13)
Derivative of $D$ with respect to $Sh_{XZ}$

The derivative of $\eta$ with respect to $Sh_{XZ}$ is:

$$\frac{\partial \eta}{\partial Sh_{XZ}} = a.S_{Z}.Z$$

(C13)

The derivative of the normal distance $D$ with respect to $Sh_{XZ}$ is given as:

$$\frac{\partial D}{\partial Sh_{XZ}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial Sh_{XZ}} = \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial Sh_{XZY}} = \frac{\text{sign}(\eta)(a.S_{X}.Z)}{\sqrt{a^2 + b^2 + c^2}}$$

(C14)

Derivative of $D$ with respect to $Sh_{YX}$

The derivative of $\eta$ with respect to $Sh_{YX}$ is:

$$\frac{\partial \eta}{\partial Sh_{YX}} = b.S_{X}.X$$

(C15)
The derivative of the normal distance $D$ with respect to $\text{Sh}_{YX}$ is given as:

$$\frac{\partial D}{\partial \text{Sh}_{YX}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial \text{Sh}_{YX}}$$

$$= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial \text{Sh}_{YX}}$$

$$= \frac{\text{sign}(\eta)(b.S_{Y,X})}{\sqrt{a^2 + b^2 + c^2}}$$

(C16)

**Derivative of $D$ with respect to $\text{Sh}_{YZ}$**

The derivative of $\eta$ with respect to $\text{Sh}_{YZ}$ is:

$$\frac{\partial \eta}{\partial \text{Sh}_{YZ}} = b.S_{Z,Z}$$

(C17)

The derivative of the normal distance $D$ with respect to $\text{Sh}_{YZ}$ is given as:

$$\frac{\partial D}{\partial \text{Sh}_{YZ}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial \text{Sh}_{YZ}}$$

$$= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial \text{Sh}_{YZ}}$$

$$= \frac{\text{sign}(\eta)(b.S_{Z,Z})}{\sqrt{a^2 + b^2 + c^2}}$$

(C18)
Derivative of $D$ with respect to $Sh_{ZX}$

The derivative of $\eta$ with respect to $Sh_{ZX}$ is:

$$\frac{\partial \eta}{\partial Sh_{ZX}} = c.S_x \cdot X$$

\hspace{1cm} (C19)

The derivative of the normal distance $D$ with respect to $Sh_{ZX}$ is given as:

$$\frac{\partial D}{\partial Sh_{ZX}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \left| \eta \right|}{\partial Sh_{ZX}}$$

$$= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial Sh_{ZX}}$$

$$= \frac{\text{sign}(\eta)(c.S_x \cdot X)}{\sqrt{a^2 + b^2 + c^2}}$$

\hspace{1cm} (C20)

Derivative of $D$ with respect to $Sh_{ZY}$

The derivative of $\eta$ with respect to $Sh_{ZY}$ is:

$$\frac{\partial \eta}{\partial Sh_{ZY}} = c.S_y \cdot Y$$

\hspace{1cm} (C21)
The derivative of the normal distance $D$ with respect to $S_{ZY}$ is given as:

$$\frac{\partial D}{\partial S_{ZY}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial |\eta|}{\partial S_{ZY}}$$

$$= \frac{\text{sign}(\eta)}{\sqrt{a^2 + b^2 + c^2}} \frac{\partial \eta}{\partial S_{ZY}}$$

$$= \frac{\text{sign}(\eta) \cdot (c.S_y.Y)}{\sqrt{a^2 + b^2 + c^2}}$$

(C22)

Substituting letters for partial derivative coefficients, and rearranging terms, the following equations result:

$$\frac{\partial D}{\partial S_X} = \text{sign}(\eta) \cdot (a.X + b.S_{XX}.X + c.S_{XY}.X) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_Y} = \text{sign}(\eta) \cdot (a.S_{XY}.Y + b.Y + c.S_{XY}.Y) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_Z} = \text{sign}(\eta) \cdot (a.S_{XZ}.Z + b.S_{YZ}.Z + c.Z) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_{XY}} = \text{sign}(\eta) \cdot (a.S_y.Y) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_{XZ}} = \text{sign}(\eta) \cdot (a.S_z.Z) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_{YZ}} = \text{sign}(\eta) \cdot (b.S_X.X) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_{XX}} = \text{sign}(\eta) \cdot (b.S_Z.Z) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_{XY}} = \text{sign}(\eta) \cdot (c.S_Z.X) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

$$\frac{\partial D}{\partial S_{YY}} = \text{sign}(\eta) \cdot (c.S_y.Y) / (a^2 + b^2 + c^2)^{\frac{1}{2}};$$

(C23)
Appendix D

List of Publication


