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On the transcendence degree of the differential field generated by Siegel modular forms

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§1. Statement of the results

It is a classical fact that the elliptic modular function $\lambda = (\vartheta_{10}/\vartheta_{00})^4$ satisfies an algebraic differential equation of order 3 (this goes back to Jacobi's *Fundamenta nova*), and none of lower order (cf. [Ra], [M]). In this paper, we show how these properties generalize to Siegel modular functions of arbitrary degree.

Some notations are necessary before we can state our main results. Let g be a positive integer (called indifferently degree, or genus), let k be an algebraically closed subfield of $\mathbb{C}$, and set:

$\mathfrak{H}_g =$ Siegel half space of degree g; the $\mathbb{Q}$-vector group Z_g formed by symmetric matrices of order g has dimension

$$n := \frac{g(g+1)}{2},$$

and $\mathfrak{H}_g$ is open in $Z_g(\mathbb{C})$.

$\boldsymbol{\tau} = (\tau_{jl})$ a generic point on $\mathfrak{H}_g$, so that $k(2\pi i\boldsymbol{\tau})$ can be viewed as the field of rational functions on Z_g/k.

$\Gamma =$ a congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (equivalently, a subgroup of finite index if $g > 1$). We recall that the symplectic group Sp_{2g} has dimension $\dim \mathrm{Sp}_{2g} = 2g^2 + g$.

$R_w(\Gamma, k) = k$-vector-space of k-rational modular forms of weight w (a non-negative integer) relative to Γ, i.e. holomorphic functions f on $\mathfrak{H}_g$ which satisfy

$$f(\gamma\boldsymbol{\tau}) = \det(c\boldsymbol{\tau} + d)^w f(\boldsymbol{\tau}) \quad \text{for all } \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \boldsymbol{\tau}\right) \in \Gamma \times \mathfrak{H}_g,$$

and (if $g = 1$) which are holomorphic at the cusps of Γ; such f's admit a Fourier expansion

$$f(\boldsymbol{\tau}) = \sum_{\boldsymbol{v} \in T^+} a_{\boldsymbol{v}} \exp\bigl(2\pi i \operatorname{Tr}(\boldsymbol{v}\boldsymbol{\tau})\bigr),$$

where T^+ is the set of non-negative elements of a suitable lattice in Z_g, and we require that the coefficients $a_{\boldsymbol{v}}$ all belong to k.

$R := R(\Gamma, k) =$ the graded ring $\bigoplus_{w \geqq 0} R_w(\Gamma, k)$; in the fraction field of R, quotients of modular forms of weights w_1, w_2 are called meromorphic modular forms of weight $w_1 - w_2$ (or simply, *modular functions* if $w_1 = w_2$).

$K := K(\Gamma, k) =$ the field of modular functions; for $g > 1$, the field $K \otimes_k \mathbb{C}$ identifies with the field of meromorphic functions on $\mathfrak{H}_g$ which are invariant under the action of Γ (cf. [S], §25.4). In fact, $\operatorname{Proj}(R)$ is a projective variety over k, whose field of k-rational functions identifies with K. In particular,

$$\operatorname{tr}\deg(K/k) = \frac{g(g+1)}{2} = n, \quad \operatorname{tr}\deg(R/k) = n + 1,$$

and we may write $K = k(\lambda)$, where $\lambda := \{\lambda_1, \dots, \lambda_N\}$ is a set of modular functions relative to Γ, whose first n elements $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_n\}$ are algebraically independent[1)]; in particular,

$\partial/\partial\boldsymbol{\lambda} := \{\partial/\partial\lambda_1, \dots, \partial/\partial\lambda_n\}$ is a K-basis of the space $\operatorname{Der}(K/k)$ of k-derivations of the extension K/k.

$\boldsymbol{\delta} = \{\delta_{jl}, 1 \leqq j \leqq l \leqq g\}$ where

$$\delta_{jl} = \frac{1}{2\pi i} \frac{\partial}{\partial\tau_{jl}}, \quad 1 \leqq j < l \leqq g, \quad \text{while} \quad \delta_{jj} = \frac{1}{\pi i} \frac{\partial}{\partial\tau_{jj}}, \quad 1 \leqq j \leqq g.$$

We sometimes reindex this set of partial derivations as $\boldsymbol{\delta} = \{\delta_1, \dots, \delta_n\}$, and abbreviate it by $\partial/\pi i \partial\boldsymbol{\tau}$; they form a $k(2\pi i\boldsymbol{\tau})$-basis of $\operatorname{Der}\bigl(k(2\pi i\boldsymbol{\tau})/k\bigr)$.

$M := M(\Gamma, k) = k\langle\lambda_1, \dots, \lambda_N\rangle =$ the $\boldsymbol{\delta}$-differential field generated by K, i.e. the field generated over k by the partial derivatives of all orders in the δ_{jl}, $1 \leqq j \leqq l \leqq g$, of all the elements of K, or equivalently, the field generated over K by the $\boldsymbol{\delta}$-derivatives of all orders of $\boldsymbol{\lambda}$ (recall that in characteristic 0, the derivatives of an element x algebraic over a differential field K lie in $K(x)$).

With these notations in mind, we can state:

Theorem 1. *The $\boldsymbol{\delta}$-differential field $M = k\langle\lambda\rangle$ generated by the field of modular functions $K = k(\lambda)$ is a finite extension of the field generated over K by the $\boldsymbol{\delta}$-partial derivatives of $\lambda_1, \dots, \lambda_n$ of order $\leqq 2$, and has transcendence degree*

$$\operatorname{tr}\deg(M/k) = \dim \operatorname{Sp}_{2g} = 2g^2 + g$$

[1)] As a rule, we reserve boldface Greek letters to n-tuples (of numbers, functions or derivatives).

over k. Furthermore, M and $\mathbb{C}(\boldsymbol{\tau})$ are linearly disjoint over k, hence

$$\operatorname{tr}\deg\bigl(M(2\pi i\boldsymbol{\tau})/k\bigr) = \dim(\mathrm{Sp}_{2g} \times Z_g) = \frac{1}{2}g(5g+3)$$

(*and π is transcendental over $M(2\pi i\boldsymbol{\tau})$ if $k = \mathbb{Q}^{\mathrm{alg}}$*), *while all modular forms in R are algebraic over M.*

The statement concerning M and $\mathbb{C}(\boldsymbol{\tau})$ is clear, since M embeds in the fraction field of the ring of convergent Puiseux series in $\exp(2\pi i\tau_{jl})$, $1 \leqq j \leqq l \leqq g$, with coefficients in k, which is linearly disjoint from $\mathbb{C}(\boldsymbol{\tau})$ over k. Thus, the second formula is an immediate corollary of the first one. Now, both are easily seen to be equivalent, and our strategy will consist in proving the second formula, i.e. in studying the $\partial/\pi i\partial\boldsymbol{\tau}$-differential extension $M(2\pi i\boldsymbol{\tau})$ of k.

Since an algebraic extension of a differential field of characteristic 0 is automatically a differential extension, we may assume, in order to prove Theorem 1, that Γ is contained in a principal congruence subgroup of level at least 3, so that $\Gamma\backslash\mathfrak{H}_g$ is a complex manifold, whose natural image in $\operatorname{Proj}\bigl(R(\Gamma,k)\bigr)$ is the set of complex points of a smooth quasi-projective variety $S(\Gamma)/k$ (cf. [MF], p. 190, and [P], §2). For instance, we may take for Γ the theta-group of level $(4,8)$

$$\Gamma_{4,8} = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}), \gamma \equiv \mathbf{1}_g \pmod 4, \operatorname{diag}(a^t b) \equiv \operatorname{diag}(c^t d) \equiv 0 \pmod 8\right\}.$$

By a well-known result of Igusa ([I], pp. 178, 190 and 224), the corresponding ring $R(\Gamma_{4,8},k)$ is the integral closure of the ring $k[\vartheta_{\boldsymbol{a}}\vartheta_{\boldsymbol{b}}, \boldsymbol{a} \in (\mathbb{Z}/2\mathbb{Z})^{2g}, \boldsymbol{b} \in (\mathbb{Z}/2\mathbb{Z})^{2g}]$, where

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{\tau}) = \vartheta_{(\boldsymbol{a}',\boldsymbol{a}'')}(\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\left(\pi i\left({}^t\!\left(\boldsymbol{n}+\frac{1}{2}\boldsymbol{a}'\right)\boldsymbol{\tau}\left(\boldsymbol{n}+\frac{1}{2}\boldsymbol{a}'\right) + {}^t\!\left(\boldsymbol{n}+\frac{1}{2}\boldsymbol{a}'\right)\boldsymbol{a}''\right)\right)$$

denotes the 'thetanull' modular form attached to the 2-characteristic $\boldsymbol{a} = (\boldsymbol{a}',\boldsymbol{a}'') \in (\mathbb{Z}/2\mathbb{Z})^{2g}$. In particular, we may here take $\lambda = \{\vartheta_{\boldsymbol{a}}/\vartheta_{\boldsymbol{0}}, \boldsymbol{a} \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$ as a set of generators of $K(\Gamma_{4,8},k)$. In this case, Theorem 1 can be given a more precise form, as follows.

Theorem 2. *The $\boldsymbol{\delta}$-derivatives of order $\leqq 2$ of the modular functions $\{\vartheta_{\boldsymbol{a}}/\vartheta_{\boldsymbol{0}}, \boldsymbol{a} \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$ generate over k a $\boldsymbol{\delta}$-stable field $M(\Gamma_{4,8},k)$ of transcendence degree $2g^2+g$ over k, over which $\vartheta_{\boldsymbol{0}}$ is algebraic.*

In fact, the thetanulls themselves satisfy a system of partial differential equations of the *second* (rather than third) order, which has been explicited in [Z], Theorem 1, and is discussed, amongst other differential properties, in §5 of the paper. As shown by Ohyama [O], these differential equations take a simpler form in the case $g = 2$. In §6, we derive from these results and from Theorem 2 ten explicit functions making up a transcendence basis for the corresponding field M, as well as an algebraic presentation of the $\boldsymbol{\delta}$-stable *ring* generated over $\mathbb{Q}$ by the ten non-zero degree 2 thetanulls $\vartheta_{\boldsymbol{a}}$, and their thirty first order partial logarithmic derivatives $\psi_{\boldsymbol{a},l} = \delta_l\vartheta_{\boldsymbol{a}}/\vartheta_{\boldsymbol{a}}$. See Theorem 3 of §6 for a precise statement.

The following diagram and 'legend' illustrate the proof of Theorem 1. The notation P is defined in §3, Lemma 2, and $\asymp$ means equality for the algebraic closures $\bullet^{\mathrm{alg}}$. We show that $\tilde{M}^{\mathrm{alg}} = M^{\mathrm{alg}}$ in §4. But as hinted above, the crux of the proof consists in translating the problem in terms of a 'field of periods' $F := \tilde{M}(2\pi i\boldsymbol{\tau}, 2\pi i)$ and its compositum Φ with $\mathbb{C}$, hence in terms of *linear* differential equations, as described in §2. The fact that the (algebraic) differential equations satisfied by classical modular forms are governed by the (linear) Gauss-Manin connection was made crystal clear in [Ka], Appendix 1, and we are here merely extending this view-point to forms of higher degrees.

$$\begin{tikzcd}
\Phi = \mathbb{C}\big(\lambda, \Pi(\lambda)\big) = \left[\underset{k(2\pi i)}{\otimes} \mathbb{C}\right] & & F := \tilde{M}(2\pi i, 2\pi i\boldsymbol{\tau}) \asymp \left[\underset{k}{\otimes} k(2\pi i)\right] & & M(2\pi i\boldsymbol{\tau}) \\
& & \tilde{M} := k(\lambda, P, \boldsymbol{\delta} P) \arrow[u] & \asymp & M = k\langle\lambda\rangle \arrow[u, "Z_g"'] \\
& k(2\pi i\boldsymbol{\tau}) \arrow[uul] \arrow[uur] & & & \\
& & k(P) \arrow[uu] & & K = k(\lambda) \arrow[uu] \\
& & k \arrow[u] \arrow[uul] \arrow[urr] & &
\end{tikzcd}$$

$$\mathrm{Gal}_{\partial/\partial\lambda}(\Phi/K \otimes_k \mathbb{C}) = \mathrm{Sp}_{2g} \Rightarrow \operatorname{tr\,deg}(M/K) = \operatorname{codim}_{\mathrm{Sp}_{2g}}(Z_g)$$

$$\Rightarrow \operatorname{tr\,deg}(M/k) = \dim \mathrm{Sp}_{2g}$$

Further comments on the proof of Theorem 1, and on its possible generalizations and applications, are given in Remarks 3 and 4 at the end of §4. See also §3, Remark 2 for a direct relation between periods and derivatives of modular forms *via* modular tensors.

§2. Picard-Fuchs and Picard-Vessiot theories

We first recall some well-known facts (see [D2], §2) about algebraic families of abelian varieties. Let S/k be a smooth algebraic variety over $k \subset \mathbb{C}$, with generic point σ and field of rational functions $L = k(S) = k(\sigma)$, and let $f\colon \boldsymbol{A} \to S$ be a principally polarized abelian scheme over S of relative dimension g, with generic fiber $\boldsymbol{A}_\sigma := A =$ an abelian variety over L. We write f^{an} for the analytic map deduced from f after extension of scalars to $\mathbb{C}$. Let $H := H^1_{\mathrm{dR}}(A/L)$ be the $2g$-dimensional L-vector space formed by the cohomology classes of L-rational differential forms of the second kind on A/L. The Gauss-Manin connection attached to f equips H with an integrable connection $\nabla\colon H \to H \otimes \Omega^1_{L/k}$. Choose a base point $s_0 \in S^{\mathrm{an}}$, and let L_0 be the field of meromorphic functions over a small neighbourhood $U(s_0)$ of s_0. The restriction to $U(s_0)$ of the local system formed by the relative Betti cohomology $\mathbf{R}^1 f^{\mathrm{an}}_* \mathbb{Q}$ generates over $\mathbb{C}$ the full space of horizontal vectors of the extension of ∇ to $H \otimes L_0$, and provides a representation ρ of the fundamental group $\pi_1(S^{\mathrm{an}}, s_0)$ on the $2g$-dimensional $\mathbb{Q}$-vector space $H_B = H^1_B(\boldsymbol{A}^{\mathrm{an}}_{s_0}, \mathbb{Q})$, which preserves the $2\pi i\mathbb{Q}$-valued symplectic form ψ_B induced on H_B by the principal polarisation on $\boldsymbol{A}_{s_0}$. The Gauss-Manin connection admits an extension with logarithmic singularities over a suitable compactification of S, so that ∇ is fuchsian and the Zariski closure of $\rho(\pi_1)$ in $\mathrm{Aut}_{\psi_B}(H_B \otimes \mathbb{C})$ is isomorphic to the *differential Galois group* $\mathrm{Gal}(\nabla)$ of the connection ∇.

Now, choose a basis $\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g$ of H over L, and a basis $c_1, \dots, c_{2g}$ of the relative Betti homology of $\boldsymbol{A}^{\mathrm{an}}$ above $U(s_0)$ (a more specific choice of bases will be given in §3). A basis of horizontal vectors of ∇ is then represented by the inverse of the fundamental matrix of periods and quasi-periods

$$\Pi(\sigma) = \begin{pmatrix} \int_{c_i} \omega_j \\ \int_{c_i} \eta_j \end{pmatrix} = \begin{pmatrix} \Omega_1(\sigma) & \Omega_2(\sigma) \\ \mathrm{H}_1(\sigma) & \mathrm{H}_2(\sigma) \end{pmatrix},$$

whose coefficients extend over S^{an} to multivalued meromorphic functions in σ. By definition, they generate over $L \otimes_k \mathbb{C} = \mathbb{C}(\sigma)$ a Picard-Vessiot extension

$$\Phi = \mathbb{C}\big(\sigma, \Pi(\sigma)\big).$$

In particular, Φ is stable under the partial derivatives $\boldsymbol{\partial} = \{\partial_1, \dots, \partial_{\dim S}\}$ given by a basis over L of the dual $\mathrm{Der}(L/k)$ of $\Omega^1_{L/k}$, and by the main theorem of Picard-Vessiot theory, $\operatorname{tr\,deg}\big(\Phi/\mathbb{C}(\sigma)\big) = \dim \mathrm{Gal}(\nabla)$.

Assume now that the family f is 'as generic as possible', or more precisely (cf. [D1], §4.4.14.1) that the induced morphism ϕ_f from S to one of the components of the moduli space of principally polarized abelian varieties is *dominant*. We then have:

Lemma 1 ([D1], Lemme 4.4.16). *The image of $\pi_1(S, s_0)$ under ρ has finite index in the group $\mathrm{Aut}_{\psi_B}(H_B)$ of symplectic automorphisms of H_B.*

The genericity hypothesis holds tautologically when S is the moduli scheme of principally polarized abelian varieties with a level structure of order $\geqq 3$, hence, in the notations of §1, when $S = S(\Gamma)$, endowed with the corresponding universal abelian scheme f (cf. [MF], Appendix 7, A–B). In this case, L is isomorphic to $K = K(\Gamma, k) = k(\lambda)$, and we shall write indifferently λ for σ, e.g.

$$\Pi(\sigma) = \Pi(\lambda), \quad \boldsymbol{\partial} = \frac{\partial}{\partial \boldsymbol{\lambda}} := \{\partial/\partial\lambda_1, \dots, \partial/\partial\lambda_n\},$$

while the multivalued map $\lambda \mapsto \Pi(\lambda)$ lifts (after a choice of a base point $\boldsymbol{\tau}_0$ above s_0) to a meromorphic map

$$\boldsymbol{\tau} \in \mathfrak{H}_g \mapsto \tilde{\Pi}(\boldsymbol{\tau}) := \Pi\big(\lambda(\boldsymbol{\tau})\big) = \begin{pmatrix} \tilde{\Omega}_1(\boldsymbol{\tau}) & \tilde{\Omega}_2(\boldsymbol{\tau}) \\ \tilde{\mathrm{H}}_1(\boldsymbol{\tau}) & \tilde{\mathrm{H}}_2(\boldsymbol{\tau}) \end{pmatrix}$$

on the universal covering manifold $\mathfrak{H}_g$ of $S^{\mathrm{an}} \simeq \Gamma\backslash\mathfrak{H}_g$. Lemma 1 then implies that $\mathrm{Gal}_{\partial/\partial\boldsymbol{\lambda}}(\Phi/K \otimes \mathbb{C}) = \mathrm{Gal}(\nabla) = \mathrm{Aut}_{\psi_B}(H_B \otimes \mathbb{C}) \simeq \mathrm{Sp}_{2g}(\mathbb{C})$ so that

$$\operatorname{tr\,deg}\big(\Phi/\mathbb{C}(\lambda)\big) = \dim \mathrm{Sp}_{2g},$$

hence

Proposition 1. *Assume that* $S = S(\Gamma)$, *where* Γ *is any congruence subgroup of* $\mathrm{Sp}_{2g}(\mathbb{Z})$. *Then,*

$$\operatorname{tr}\deg\bigl(\mathbb{C}\bigl(\lambda, \Pi(\lambda)\bigr)/\mathbb{C}\bigr) = \dim(\mathrm{Sp}_{2g} \times Z_g).$$

Proof. $\operatorname{tr}\deg(\Phi/\mathbb{C}) = \operatorname{tr}\deg\bigl(\Phi/\mathbb{C}(\lambda)\bigr) + \operatorname{tr}\deg\bigl(\mathbb{C}(\lambda)/\mathbb{C}\bigr) = \dim \mathrm{Sp}_{2g} + n$, which is $\dim(\mathrm{Sp}_{2g} \times Z_g)$.

Remark 1. In the sequel, the intermediate fields

$$\Phi_k := k\bigl(\lambda, \Pi(\lambda)\bigr), \quad \Psi_k := k\left(\lambda, \frac{1}{2\pi i}\Omega_1(\lambda), \frac{1}{2\pi i}\mathrm{H}_1(\lambda)\right)$$

will be used. Since ∇_∂ acts on the K-vector space H for any $\partial \in \mathrm{Der}(K/k)$, they are still $\partial/\partial\boldsymbol{\lambda}$-differential extensions of K, but even Φ_k is in general *not* a Picard-Vessiot extension. Indeed, the generalized Riemann relations (i.e. the reciprocity law for differentials of the second kind on A; cf. [B], pp. 37–38, or [D2], Proposition 1.5, in connection with the form ψ_{dR} introduced below) show that the $\partial/\partial\boldsymbol{\lambda}$-constant π lies in Φ_k, although not in K if $k = \mathbb{Q}^{\mathrm{alg}}$. In a sense, it is the field $\Psi_k(2\pi i\boldsymbol{\tau})$ which gives the required Picard-Vessiot extension, but it is not a field of periods in the usual sense: it will in general contain neither $\boldsymbol{\tau}$, nor $2\pi i$! Also, note that these Riemann relations (written in the standard bases) immediately imply that $\operatorname{tr}\deg\bigl(\Phi/\mathbb{C}(\sigma)\bigr) \leqq 2g^2 + g$ for any family $\boldsymbol{A} \to S$: once Ω_1 is chosen (at most g^2 degrees of freedom), they leave at most $g(g+1)/2$ degrees of freedom for the entries of Ω_2, at most $g(g+1)/2$ for those of H_1, and none for H_2.

§3. Periods of the first kind and their derivatives

Let $f: \boldsymbol{A} \to S$ be a principally polarized abelian scheme S as in §2. In order to prepare for a modular study of the period matrix $\Pi(\sigma)$, we shall need some well-known infinitesimal properties of the structural morphism ϕ_f. Recall that in parallel with ψ_B, the polarisation on A_σ provides a non-degenerate antisymmetric form ψ_{dR} on $H := H^1_{\mathrm{dR}}(A/L)$ with values in L, which admits as maximal isotropic subspace the L-vector space $\Omega := \Omega^1_{A/L}$ of H formed by the (cohomology classes of) differentials of the first kind on A/L, so that H/Ω is canonically isomorphic to the L-dual $\Omega^* \simeq \mathrm{Lie}(A/L)$ of Ω.

When writing $\Pi(\sigma)$, we may choose a ψ_B-symplectic basis[2)] of the Betti homology for $\{c_1, \ldots, c_{2g}\}$, and a basis of Ω over L for $\omega_1, \ldots, \omega_g$. Then, $\Omega_1(\sigma)$ is invertible, and $\boldsymbol{\tau}(\sigma) := \Omega_1(\sigma)^{-1}\Omega_2(\sigma)$ is one of the points in $\mathfrak{H}_g$ parametrizing the principally polarized abelian variety A_σ. As for the last g rows of $\Pi(\sigma)$, we describe them with the help of the Kodaira-Spencer map

$$\kappa_f: \mathrm{Der}(L/k) \to \mathrm{Hom}(\Omega, H/\Omega) \simeq (\Omega^*)^{\otimes 2}: \partial \mapsto \{\omega \mapsto \nabla_\partial(\omega) \bmod \Omega\}$$

attached to f (cf. [L], p. 157). Note that the image of κ_f is contained in $\mathrm{Sym}^2(\Omega^*)$, because ψ_{dR} is horizontal for ∇.

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2) We order this basis in such a way that $\boldsymbol{\tau} = \Omega_1^{-1}\Omega_2 \in \mathfrak{H}_g$; then, Γ acts on the $(g \times 2g)$-matrix $(\tilde{\Omega}_2(\boldsymbol{\tau})\tilde{\Omega}_1(\boldsymbol{\tau})) = \tilde{\Omega}_1(\boldsymbol{\tau})(\boldsymbol{\tau}\ \mathbf{1}_g)$ by the *transpose* of its standard representation.

Assume now that f is 'sufficiently generic' in the sense of §2. Since κ_f represents the tangent map to the morphism ϕ_f at the generic point σ of S, its rank then coincides with $g(g+1)/2$ and κ_f maps onto $\mathrm{Sym}^2(\Omega^*)$ (cf. [K], p. 169, and [P], p. 255). Therefore, there exists a partial derivation $\partial_0 \in \mathrm{Der}(L/k)$ whose image under κ_f is an isomorphism from Ω to H/Ω. In particular, the differential forms of the second kind $\{\eta_l = \nabla_{\partial_0}(\omega_l), 1 \leqq l \leqq g\}$ lift a basis of H/Ω over L. Since $\int_c \nabla_\partial(\omega) = \partial\Bigl(\int_c \omega\Bigr)$ for any $\partial \in \mathrm{Der}(L/k)$, $\omega \in H$, $c \in H_B$, we then infer that $\mathrm{H}_1(\sigma)$ (resp. $\mathrm{H}_2(\sigma)$) are the ∂_0-derivatives of $\Omega_1(\sigma)$ (resp. $\Omega_2(\sigma)$). In other words, $\Pi(\sigma)$ takes in such bases the shape

$$\Pi(\sigma) = \begin{pmatrix} \Omega_1(\sigma) & \Omega_1(\sigma)\boldsymbol{\tau}(\sigma) \\ \partial_0\Omega_1(\sigma) & \partial_0\bigl(\Omega_1(\sigma)\boldsymbol{\tau}(\sigma)\bigr) \end{pmatrix}.$$

This certainly implies that the $\boldsymbol{\partial}$-stable field $\Phi = \mathbb{C}\bigl(\sigma, \Pi(\sigma)\bigr)$ is generated over $\mathbb{C}(\sigma)$ by $\Omega_1(\sigma)$, $\boldsymbol{\tau}(\sigma)$, $\boldsymbol{\partial}\Omega_1(\sigma)$, $\boldsymbol{\partial}\boldsymbol{\tau}(\sigma)$ (recall that $\boldsymbol{\partial}$ stands for a basis of $\mathrm{Der}(L/k)$), and more precisely, that the $\boldsymbol{\partial}$-stable field Φ_k of Remark 1 can be written as

$$\Phi_k := k\bigl(\sigma, \Pi(\sigma)\bigr) = k\bigl(\sigma, \Omega_1(\sigma), \boldsymbol{\tau}(\sigma), \boldsymbol{\partial}\Omega_1(\sigma), \boldsymbol{\partial}\boldsymbol{\tau}(\sigma)\bigr).$$

Similarly, each of the columns of the fundamental matrix of solutions $\Pi(\sigma)$ generates over L a $\boldsymbol{\partial}$-stable field, and by the same argument, the $\boldsymbol{\partial}$-stable field Ψ_k of Remark 1 reads

$$\Psi_k := k\left(\sigma, \frac{1}{2\pi i}\Omega_1(\sigma), \frac{1}{2\pi i}\mathrm{H}_1(\sigma)\right) = k\left(\sigma, \frac{1}{2\pi i}\Omega_1(\sigma), \boldsymbol{\partial}\left(\frac{1}{2\pi i}\Omega_1(\sigma)\right)\right).$$

In order to compare Φ_k with the (still to be defined) fields F and $\tilde{M}$ of our diagram, we restrict from now on to the modular situation of Proposition 1, with $S = S(\Gamma)$, $\sigma = \lambda$, $L = K$, $\tilde{\Omega}_1(\boldsymbol{\tau}) = \Omega_1\bigl(\lambda(\boldsymbol{\tau})\bigr)$, etc. It will be useful (though not strictly necessary, cf. Remark 2 below) to choose an explicit basis of the K-vector space Ω. We appeal to Shimura's differentials ([S], §30) for such a specification, and to fix notations, henceforth assume that $\Gamma = \Gamma_{4,8}$, so that $S(\Gamma)$ is the moduli scheme of principally polarized abelian varieties with $(4,8)$-level structure (cf. [MF], pp. 193–195).

Consider the full set of abelian theta functions with two-characteristics $\{\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{\tau}), \boldsymbol{a} \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$, as given in §5 below. Then, $\{\boldsymbol{z} \mapsto \vartheta_{\boldsymbol{a}}(2\boldsymbol{z}, \boldsymbol{\tau}), \boldsymbol{a} \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$ defines a projective embedding of $\mathbb{C}^g/(\mathbb{Z}^g \oplus \boldsymbol{\tau}\mathbb{Z}^g)$ (cf. [I], pp. 169 and 171), whose image can be identified with the generic fiber $\boldsymbol{A}_{\lambda(\boldsymbol{\tau})}$ of the corresponding universal family. Therefore (cf. [S], Lemma 30.2), we can choose $\vartheta_{\boldsymbol{0}} = \vartheta_{(\boldsymbol{0},\boldsymbol{0})}$ and g odd theta functions $\vartheta_1, \ldots, \vartheta_g$ amongst them such that the jacobian matrix at $\boldsymbol{z} = \boldsymbol{0}$ of the map

$$\boldsymbol{z} \mapsto \left(\frac{\vartheta_j}{\vartheta_{\boldsymbol{0}}}\left(\frac{\boldsymbol{z}}{2\pi i}, \boldsymbol{\tau}\right), j = 1, \ldots, g\right)$$

is invertible. Because of parities, this jacobian reads ${}^tP(\boldsymbol{\tau})$, where

$$P(\boldsymbol{\tau}) = \frac{1}{2\pi i \vartheta_{\boldsymbol{0}}(\boldsymbol{0}, \boldsymbol{\tau})} \cdot \begin{pmatrix} \dfrac{\partial\vartheta_1}{\partial z_1}(\boldsymbol{0}, \boldsymbol{\tau}) & \cdots & \dfrac{\partial\vartheta_1}{\partial z_g}(\boldsymbol{0}, \boldsymbol{\tau}) \\ \cdots & \cdots & \cdots \\ \dfrac{\partial\vartheta_g}{\partial z_1}(\boldsymbol{0}, \boldsymbol{\tau}) & \cdots & \dfrac{\partial\vartheta_g}{\partial z_g}(\boldsymbol{0}, \boldsymbol{\tau}) \end{pmatrix}.$$

Lemma 2 ([S], Theorem 30.3 and Formula (30.2f)). *Let $\boldsymbol{\tau} \in \mathfrak{H}_g$, and let $\lambda = \lambda(\boldsymbol{\tau})$. The relation*

$$(\omega_1, \dots, \omega_g) = (dz_1, \dots, dz_g) 2\pi i\, {}^t P(\boldsymbol{\tau})$$

defines a basis of differential forms of the first kind on the abelian variety $\boldsymbol{A}_{\lambda(\boldsymbol{\tau})}$ which are rational over the field $K = k(\lambda)$, and which admit $2\pi i P(\boldsymbol{\tau})(\mathbf{1}_g\ \boldsymbol{\tau})$ as period matrix. In particular, $(1/2\pi i)\tilde{\boldsymbol{\Omega}}_1(\boldsymbol{\tau}) = P(\boldsymbol{\tau})$ in such a basis.

Recall the notations at the beginning of §1, and set further:

$\xi =$ the map from $\Phi = \mathbb{C}(\lambda, \Pi(\lambda))$ to the field of meromorphic functions on $\mathfrak{H}_g$ which lifts an element $f(\lambda) \in \Phi$ to

$$(\xi(f))(\boldsymbol{\tau}) = \tilde{f}(\boldsymbol{\tau}) := f(\lambda(\boldsymbol{\tau}));$$

relatively to the bases $\{\partial/\partial\boldsymbol{\lambda}\}$ and $\{\boldsymbol{\delta} = \partial/\pi i \partial\boldsymbol{\tau}\}$, the differential of the covering map λ corresponding to ξ is given by $(\delta_1, \dots, \delta_n) = (\partial/\partial\lambda_1, \dots, \partial/\partial\lambda_n)\, {}^t W(\boldsymbol{\tau})$, where $W(\boldsymbol{\tau})$ denotes the invertible matrix

$$W(\boldsymbol{\tau}) := \left(\frac{\partial\boldsymbol{\lambda}}{\pi i \partial\boldsymbol{\tau}}\right)(\boldsymbol{\tau}) = \begin{pmatrix} \delta_1\lambda_1 & \cdots & \delta_1\lambda_n \\ \cdots & \cdots & \cdots \\ \delta_n\lambda_1 & \cdots & \delta_n\lambda_n \end{pmatrix}.$$

$\tilde{M} := k(\lambda(\boldsymbol{\tau}), P(\boldsymbol{\tau}), \boldsymbol{\delta}P(\boldsymbol{\tau}), \boldsymbol{\delta}\lambda(\boldsymbol{\tau})) = k(\lambda(\boldsymbol{\tau}), P(\boldsymbol{\tau}), \boldsymbol{\delta}P(\boldsymbol{\tau}), W(\boldsymbol{\tau}))$ (since $\lambda_{n+1}, \dots, \lambda_N$ are algebraic over the field K). A simpler description of $\tilde{M}$ will presently be given, cf. Remark 2 below.

$F := \tilde{M}(2\pi i \boldsymbol{\tau}, 2\pi i)$.

The field F being thus defined, we can now relate it to the fields of periods Φ of §1, as follows.

Proposition 2. *Assume that $S = S(\Gamma)$, with $\Gamma = \Gamma_{4,8}$. Then, ξ induces an isomorphism from $\Phi_k = k(\lambda, \Pi(\lambda))$ onto F. Moreover, both fields F and $\tilde{M}$ are stable under the partial derivatives $\boldsymbol{\delta} = \partial/\pi i \partial\boldsymbol{\tau}$.*

Proof. We already know that

$$\xi(\Phi_k) = k\left(\lambda(\boldsymbol{\tau}), 2\pi i P(\boldsymbol{\tau}), \boldsymbol{\tau}, 2\pi i \xi\left(\frac{\partial}{\partial\boldsymbol{\lambda}} P(\lambda)\right), \xi\left(\frac{\partial}{\partial\boldsymbol{\lambda}} \boldsymbol{\tau}(\lambda)\right)\right),$$

and that it contains $2\pi i$. Now, for any $f \in \Phi$,

$$(\delta_1(\xi(f)), \dots, \delta_n(\xi(f))) = \left(\xi\left(\frac{\partial f}{\partial\lambda_1}\right), \dots, \xi\left(\frac{\partial f}{\partial\lambda_n}\right)\right) {}^t W(\boldsymbol{\tau});$$

for instance, $\pi i \xi\big((\partial/\partial\boldsymbol{\lambda})\boldsymbol{\tau}(\lambda)\big)$ can be written as the inverse of the matrix $W(\boldsymbol{\tau})$, and they have the same field of definition. Therefore, $\xi(\Phi_k)$ is generated over k by $2\pi i$, $2\pi i\boldsymbol{\tau}$, $\lambda(\boldsymbol{\tau})$, $P(\boldsymbol{\tau})$, $\boldsymbol{\delta}P(\boldsymbol{\tau})$, and $W(\boldsymbol{\tau})$, which precisely constitute a set of generators for F/k. As for the second part of Proposition 2, note that Φ_k is $\partial/\partial\boldsymbol{\lambda}$-stable, so that $\xi(\Phi_k) = F$ is $(\partial/\pi i\partial\boldsymbol{\tau})\,{}^tW(\boldsymbol{\tau})^{-1}$-stable, hence $\partial/\pi i\partial\boldsymbol{\tau}$-stable as well, since the coefficients of ${}^tW(\boldsymbol{\tau})^{-1}$ lie in F. Similarly, we know that the subfield Ψ_k of Φ_k generated over K by the first g columns of $(1/2\pi i)\Pi(\lambda)$ is stable under $\partial/\partial\boldsymbol{\lambda}$, and the same argument implies that $\tilde{M} = \xi(\Psi_k)\big(W(\boldsymbol{\tau})\big)$ too is $\partial/\pi i\partial\boldsymbol{\tau}$-stable.

Remark 2. In view of [S], Formula 30.2d (or more simply, of the monodromy action on periods, cf. Footnote[2)]), ${}^tP$ is a modular tensor relative to the standard representation ρ of $\mathrm{GL}_g(\mathbb{C})$, i.e.:

$$ {}^tP(\gamma\boldsymbol{\tau}) = (c\boldsymbol{\tau}+d)\,{}^tP(\boldsymbol{\tau}) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. $$

On the other hand, it is a well-known fact that the $\boldsymbol{\delta}$-derivatives of any modular function λ_0 can be arranged into a (meromorphic) $\mathrm{Sym}^2\rho$-modular tensor; in other words (cf. Formula (4) of §5 below):

$$ (\boldsymbol{\delta}\lambda_0)(\gamma\boldsymbol{\tau}) = (c\boldsymbol{\tau}+d)\cdot(\boldsymbol{\delta}\lambda_0)(\boldsymbol{\tau})\cdot {}^t(c\boldsymbol{\tau}+d) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. $$

Consequently, the matrix valued functions ${}^tP^{-1}\boldsymbol{\delta}\lambda_j P^{-1}$, $j = 1, \dots, n$, are invariant under Γ. Looking at their Fourier expansions, we deduce that their entries all belong to the field $K = K(\Gamma, k) = k(\lambda)$. Therefore, the last set of generators $\boldsymbol{\delta}\lambda(\boldsymbol{\tau})$ (equivalently: the entries of $W(\boldsymbol{\tau})$) occuring in the definition of $\tilde{M}$ already lies in the field $K\big(P(\boldsymbol{\tau})\big)$ generated by the first two ones, i.e. $k(\lambda, \boldsymbol{\delta}\lambda) \subset k(\lambda, P)$, and in parallel with $F = \xi(\Phi_k)$, we finally obtain

$$ \tilde{M} = k\big(\lambda(\boldsymbol{\tau}), P(\boldsymbol{\tau}), \boldsymbol{\delta}P(\boldsymbol{\tau})\big) = k\left(\lambda(\boldsymbol{\tau}), \frac{1}{2\pi i}\tilde{\Omega}_1(\boldsymbol{\tau}), \frac{1}{2\pi i}\tilde{\mathrm{H}}_1(\boldsymbol{\tau})\right) = \xi(\Psi_k) $$

as a simpler expression for the field $\tilde{M}$ of the diagram of §1.

In fact, this type of argument can be reversed (using the fact that $\boldsymbol{\delta}\lambda_1, \dots, \boldsymbol{\delta}\lambda_n$ form a *basis* of the K-vector-space of meromorphic $\mathrm{Sym}^2\rho$-modular tensors), and implies that all binomials in the entries of $P(\boldsymbol{\tau})$ belong to the field $K(\boldsymbol{\delta}\lambda)$. In particular, $k(\lambda, P) \subset \big(k(\lambda, \boldsymbol{\delta}\lambda)\big)^{\mathrm{alg}}$. We shall give a precise version of this statement in Proposition 4 of §5, but notice that the above proofs and results extend to the study of $(1/2\pi i)\tilde{\Omega}_1(\boldsymbol{\tau})$, $(1/2\pi i)\tilde{\mathrm{H}}_1(\boldsymbol{\tau})$ for any congruence subgroup Γ of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

§4. Proof of Theorems 1 and 2

We now complete the proof of Theorems 1 and 2 in four steps. Since the algebraic closure of $K(\Gamma, k)$, hence of $M(\Gamma, k)$, is independent of the group Γ, Theorem 1 is a corollary of Theorem 2 and for simplicity, we throughout assume that $\Gamma = \Gamma_{4,8}$, although this hypothesis may truly be needed only in the last step: the first three ones are valid for all Γ's, and suffice for the proof of Theorem 1.

For each $m = 0, 1, \ldots, \infty$, we denote by $K^{(m)}$ the field generated over k by the partial derivatives of order $\leqq m$ with respect to $\boldsymbol{\delta} = \partial/\pi i \partial \boldsymbol{\tau}$ of all the elements of K. Thus, $K^{(0)} = K \subset K^{(1)} = K(\boldsymbol{\delta}\lambda) \subset \cdots \subset K^{(\infty)} = M$. All these fields, as well as $\tilde{M}$, F and the ring R, are contained in the field of meromorphic functions on $\mathfrak{H}_g$.

Step 1. $R \subset (K^{(1)})^{\mathrm{alg}}$ (hence $R \subset M^{\mathrm{alg}}$).

It suffices to prove that $K^{(1)}$ contains a non-zero meromorphic modular form Δ of positive weight w, since for any modular form f of weight $w(f)$, $f^w/\Delta^{w(f)}$ will then lie in K. (Incidentally, this further implies that the logarithmic derivatives $\boldsymbol{\delta} f/f$ of f belong to $K^{(2)}$.) An explicit choice for Δ is given in Lemma 4 of §5, but here is a general construction, along the lines of Remark 2 of §3. Since the $\boldsymbol{\delta}$-derivatives of any modular function λ_0 make up a (meromorphic) $\mathrm{Sym}^2 \rho$-modular tensor $\boldsymbol{\delta}\lambda_0$ with respect to the standard representation ρ of $\mathrm{GL}_g(\mathbb{C})$, and since the representation $\Lambda^{g(g+1)/2}(\mathrm{Sym}^2 \rho)$ of $\mathrm{GL}_g(\mathbb{C})$ is isomorphic to $(\Lambda^g \rho)^{\otimes(g+1)}$, the exterior product $\Delta(\boldsymbol{\tau})$ of $\boldsymbol{\delta}\lambda_1, \ldots, \boldsymbol{\delta}\lambda_n$ is a non-zero, meromorphic modular form of weight $g+1$. Now, $\Delta(\boldsymbol{\tau})$ is given in coordinates by the determinant $\det(\partial\boldsymbol{\lambda}/\pi i \partial\boldsymbol{\tau})$ of the matrix $W(\boldsymbol{\tau})$ introduced in §3, which clearly lies in $k\big(\lambda, \boldsymbol{\delta}(\boldsymbol{\lambda})\big) = K^{(1)}$.

Step 2. $K^{(2)} \subset M \subset \tilde{M} \subset (K^{(2)})^{\mathrm{alg}}$ (hence M is finite over $K^{(2)}$).

The first inclusion is trivial. Since M is the δ-differential field generated by K, the second one follows from the stability of $\tilde{M} \supset K$ under differentiation (cf. Proposition 2). To check the last one, we appeal to Proposition 4 (i) of §5, according to which

$$\left(\frac{1}{2\pi i \vartheta_{\mathbf{0}}} \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z_j}\bigg|_{z=\mathbf{0}}\right)^2 = \frac{1}{2^{g-1}} \sum_{\boldsymbol{b} \in (\mathbb{Z}/2\mathbb{Z})^{2g}} (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''} \left(\frac{\vartheta_{\boldsymbol{a}+\boldsymbol{b}}}{\vartheta_{\mathbf{0}}}\right)^2 \left(\frac{\vartheta_{\boldsymbol{b}}}{\vartheta_{\mathbf{0}}}\right)^2 \frac{\delta_{jj}(\vartheta_{\boldsymbol{a}+\boldsymbol{b}}/\vartheta_{\mathbf{0}})}{\vartheta_{\boldsymbol{a}+\boldsymbol{b}}/\vartheta_{\mathbf{0}}}$$

for any odd theta function $\vartheta_{\boldsymbol{a}}$. Since quotient of thetanulls are modular functions for $\Gamma_{4,8}$, this implies that the entries of $P(\boldsymbol{\tau})$ are algebraic over $K^{(1)}$ (see the end of Remark 2 for an implicit argument). Both P and $\boldsymbol{\delta}P$ (and of course $\boldsymbol{\delta}\lambda$) are then algebraic over $K^{(2)}$, and $\tilde{M}$, hence M, is a finite extension of $K^{(2)}$.

Step 3. $\operatorname{tr}\deg\big(M(2\pi i\boldsymbol{\tau})/k\big) = \dim(\mathrm{Sp}_{2g} \times Z_g)$.

From the explicit expression of the Fourier expansions of its generators, we infer that $\tilde{M}$ embeds in the fraction field of the ring of convergent Puiseux series in $\exp(2\pi i \tau_{jl})$, $1 \leqq j \leqq l \leqq g$, with coefficients in k. Since this field is linearly disjoint from $\mathbb{C}(\boldsymbol{\tau})$ over k, and since M and $\tilde{M}$ have the same algebraic closure by Step 2, we deduce from Proposition 2 that

$$\begin{aligned}\operatorname{tr}\deg\big(M(2\pi i\boldsymbol{\tau})/k\big) &= \operatorname{tr}\deg\big(\tilde{M}(2\pi i\boldsymbol{\tau})/k\big) = \operatorname{tr}\deg\big(\tilde{M}(2\pi i\boldsymbol{\tau}).\mathbb{C}/\mathbb{C}\big)\\ &= \operatorname{tr}\deg(F.\mathbb{C}/\mathbb{C}) = \operatorname{tr}\deg(\Phi_k.\mathbb{C}/\mathbb{C})\\ &= \operatorname{tr}\deg(\Phi/\mathbb{C}),\end{aligned}$$

which Lemma 1 shows to be equal to $\dim(\mathrm{Sp}_{2g} \times Z_g)$.

Step 4. $M = K^{(2)}$ if $\Gamma = \Gamma_{4,8}$ (and consequently, as soon as $\Gamma \subset \Gamma_{4,8}$).

We must show that for any modular function λ_0 relative to $\Gamma_{4,8}$, the components of $\boldsymbol{\delta}^{(3)}\lambda_0$ are rational (rather than just algebraic) over the field $K^{(2)}$. In view of Igusa's theorem, λ_0 can be expressed as a rational function in the quotients $\{\vartheta_{\boldsymbol{a}}/\vartheta_{\boldsymbol{0}}, \boldsymbol{a} \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$, and Proposition 3 below implies that both $\boldsymbol{\delta}^{(2)}\lambda_0$ and $\boldsymbol{\delta}(\boldsymbol{\delta}\vartheta_{\boldsymbol{0}}/\vartheta_{\boldsymbol{0}})$ are defined over $K^{(1)}(\boldsymbol{\delta}\vartheta_{\boldsymbol{0}}/\vartheta_{\boldsymbol{0}})$. Consequently, $\boldsymbol{\delta}^{(3)}\lambda_0 \in K^{(2)}(\boldsymbol{\delta}\vartheta_{\boldsymbol{0}}/\vartheta_{\boldsymbol{0}})$. Since $\boldsymbol{\delta}\vartheta_{\boldsymbol{0}}/\vartheta_{\boldsymbol{0}} \in K^{(2)}$ by Step 1, it follows that $\boldsymbol{\delta}^{(3)}\lambda_0 \in K^{(2)}$.

We conclude this section with two remarks on Theorem 1 and its proof.

Remark 3. In the case of an arbitrary family of abelian varieties $f: \boldsymbol{A} \to S$, the algebraic group Sp_{2g} must be replaced by the *Hodge group* G_A of its generic fiber A. Although not always, an equality $\mathrm{Gal}(\nabla) = G_A \otimes \mathbb{C}$ as in Lemma 1 of §1 often holds, cf. [A1]; the above method can then be extended to the study of the corresponding field of automorphic functions. One can thus show that the differential field generated by Hilbert modular functions (relative to a totally real number field of degree g) has transcendence degree $3g$ over $\mathbb{C}$. See [Re] for some results in this direction, and for an explicit form of the corresponding differential equations.

Remark 4. Going back to the generic situation where $S = S(\Gamma)$, denote by Ψ the field of definition of the 'first periods' of A/K, i.e. the compositum of the field

$$\Psi_k = k\left(\lambda, \frac{1}{2\pi i}\Omega_1(\lambda), \frac{1}{2\pi i}\mathrm{H}_1(\lambda)\right)$$

with $\mathbb{C}$, and view the group Z_g as an algebraic subgroup of Sp_{2g} via the usual map $\{U \in Z_g\} \mapsto \left\{\begin{pmatrix} \mathbf{1}_g & U \\ 0 & \mathbf{1}_g \end{pmatrix} \in \mathrm{Sp}_{2g}\right\}$. By Lemma 1 (i.e. by Picard-Lefschetz theory), the field Ψ coincides with the subfield of the Picard-Vessiot extension $\Phi/\mathbb{C}(\lambda)$ invariant under Z_g. Combined with the relation $\xi(\Psi_k) = \tilde{M}$ from Remark 2, this provides a direct proof that $\operatorname{tr\,deg}(M/K) = \operatorname{codim}_{\mathrm{Sp}_{2g}}(Z_g)$ when $k = \mathbb{C}$.

When $k = \mathbb{Q}^{\mathrm{alg}}$, this point of view (extended to the case of Shimura varieties, as in the previous remark) may prove useful in the study of the transcendence degree of the field of periods of abelian varieties defined over $\mathbb{Q}^{\mathrm{alg}}$. In fact, the inclusions $K^{(1)} \subset k(\lambda, P) \subset (K^{(1)})^{\mathrm{alg}}$, $K^{(2)} \subset \tilde{M} = \xi(\Psi_k) \subset (K^{(2)})^{\mathrm{alg}}$ obtained in the course of our proof (cf. §3, Remark 2 and §4, Step 2), i.e.

$$k(\lambda, \boldsymbol{\delta}\lambda) \subset k\left(\lambda, \frac{1}{2\pi i}\tilde{\Omega}_1\right) \subset \left(k(\lambda, \boldsymbol{\delta}\lambda)\right)^{\mathrm{alg}},$$

$$k(\lambda, \boldsymbol{\delta}\lambda, \boldsymbol{\delta}^{(2)}\lambda) \subset k\left(\lambda, \frac{1}{2\pi i}\tilde{\Omega}_1, \frac{1}{2\pi i}\tilde{\mathrm{H}}_1\right) \subset \left(k(\lambda, \boldsymbol{\delta}\lambda, \boldsymbol{\delta}^{(2)}\lambda)\right)^{\mathrm{alg}},$$

generalize the classical fact that the quotient by $2\pi i$ of the periods and quasi-periods of an elliptic curve along a locally invariant cycle can be expressed both as values of hypergeometric functions and in terms of modular forms. See [A2] for an approach to Chudnovsky's theorem on $\{\omega/2\pi i, \eta/2\pi i\}$ based on these relations, which, in genus 1, translate into the classical formulae

$$\frac{\omega_1}{2\pi} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right) = \vartheta_{00}^2,$$

$$\frac{\eta_1}{2\pi} = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1; \lambda\right) = \frac{1}{3\vartheta_{00}^2}\left(2\vartheta_{00}^4 - \vartheta_{10}^4 - 4\frac{1}{\pi i}\frac{d}{d\tau}\operatorname{Log}(\vartheta_{00}\vartheta_{10}\vartheta_{01})\right),$$

where $\lambda = (\vartheta_{10}/\vartheta_{00})^4$ is Legendre's modular function, whose derivative satisfies

$$\frac{1}{\pi i}\frac{d\lambda}{d\tau} = \lambda\vartheta_{01}^4 = \lambda(1-\lambda)\vartheta_{00}^4.$$

§5. Miscellaneous on theta functions

In this section, we give the explicit formulae on derivatives of theta functions already used or mentioned in Steps 4, 2, and 1 of §4.

We define the (abelian) *theta function with characteristic* $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') \in \mathbb{Z}^{2g}$ by the convergent series

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{z}) := \vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{\tau}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp\left(\pi i \,{}^t\!\left(\boldsymbol{n} + \frac{1}{2}\boldsymbol{a}'\right)\boldsymbol{\tau}\left(\boldsymbol{n} + \frac{1}{2}\boldsymbol{a}'\right) + 2\pi i \,{}^t\!\left(\boldsymbol{n} + \frac{1}{2}\boldsymbol{a}'\right)\left(\boldsymbol{z} + \frac{1}{2}\boldsymbol{a}''\right)\right)$$

where $\boldsymbol{z} \in \mathbb{C}^g$ is a vector-column and $\boldsymbol{\tau} \in \mathfrak{H}_g$. The quasi-periodicity of these functions with respect to the lattice $\mathbb{Z}^g + \boldsymbol{\tau}\mathbb{Z}^g \subset \mathbb{C}^g$ allows us to consider only *reduced* characteristics $\boldsymbol{a} \in \mathfrak{R} = \{0, 1\}^{2g}$, $\#\mathfrak{R} = 2^{2g}$. In the customary fashion, we identify $\{0, 1\}$ with its image in $\mathbb{Z}/2\mathbb{Z}$, and set for all $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'')$, $\boldsymbol{b} = (\boldsymbol{b}', \boldsymbol{b}'') \in \mathfrak{R}$:

$$|\boldsymbol{a}| = {}^t\boldsymbol{a}' \cdot \boldsymbol{a}'', \quad \langle \boldsymbol{a}, \boldsymbol{b} \rangle = {}^t\boldsymbol{a}'\boldsymbol{b}'' - {}^t\boldsymbol{b}'\boldsymbol{a}'' \equiv |\boldsymbol{a} + \boldsymbol{b}| + |\boldsymbol{a}| + |\boldsymbol{b}| \ (\mathrm{mod}\, 2).$$

Thetanulls are the values of even theta functions at the point $\boldsymbol{z} = \boldsymbol{0}$. When no confusion may arise, we denote them by $\vartheta_{\boldsymbol{a}} = \vartheta_{\boldsymbol{a}}(\boldsymbol{\tau}) := \vartheta_{\boldsymbol{a}}(\boldsymbol{0}, \boldsymbol{\tau})$. Since $\vartheta_{\boldsymbol{a}}(-\boldsymbol{z}, \boldsymbol{\tau}) = (-1)^{|\boldsymbol{a}|}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{\tau})$, a theta function $\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{\tau})$ is even if and only if the number $|\boldsymbol{a}|$ is even. Hence all non-zero thetanulls are assigned to *even* characteristics from the set $\mathfrak{R}_+ = \{\boldsymbol{a} \in \mathfrak{R} : |\boldsymbol{a}| \equiv 0 \ (\mathrm{mod}\, 2)\}$, $\#\mathfrak{R}_+ = 2^{g-1}(2^g + 1)$, and trivial ones correspond to *odd* characteristics from $\mathfrak{R}_- = \mathfrak{R} \backslash \mathfrak{R}_+$. We recall (cf. [I], pp. 185 and 171) that the thetanulls are modular forms of weight $\dfrac{1}{2}$ for $\Gamma_{4,8}$.

In order to describe the partial logarithmic derivatives of the thetanulls

$$(1) \qquad \psi_{\boldsymbol{a},jl} = \psi_{\boldsymbol{a},jl}(\boldsymbol{\tau}) := \frac{\delta_{jl}\vartheta_{\boldsymbol{a}}}{\vartheta_{\boldsymbol{a}}} = \psi_{\boldsymbol{a},lj}, \quad \boldsymbol{a} \in \mathfrak{R}_+, \ j, l = 1, \ldots, g,$$

with respect to the derivations $\boldsymbol{\delta}$:

$$\delta_{jj} = \frac{1}{\pi i}\frac{\partial}{\partial \tau_{jj}}, \quad j = 1, \dots, g, \quad \delta_{jl} = \frac{1}{2\pi i}\frac{\partial}{\partial \tau_{jl}} = \delta_{lj}, \quad j, l = 1, \dots, g, j \neq l,$$

of §1, we use the following conventions. To any meromorphic function $f\colon \mathfrak{H}_g \to \mathbb{C}$, we assign a meromorphic function with values in the space of quadratic forms in $\boldsymbol{u} \in \mathbb{C}^g$ by the formula

$$\boldsymbol{\delta} f(\boldsymbol{u}) = \sum_{j,l=1}^{g} \delta_{jl} f \cdot u_j u_l.$$

Then,

$$\psi_{\boldsymbol{a}}(\boldsymbol{u}) = \frac{\boldsymbol{\delta}\vartheta_{\boldsymbol{a}}(\boldsymbol{u})}{\vartheta_{\boldsymbol{a}}} = \sum_{j,l=1}^{g} \psi_{\boldsymbol{a},jl} \cdot u_j u_l$$

is the quadratic form corresponding to the symmetric matrix $\boldsymbol{\psi}_{\boldsymbol{a}} = (\psi_{\boldsymbol{a},jl})_{j,l=1,\dots,g}$, $\boldsymbol{a} \in \mathfrak{R}_+$. To two quadratic forms ϕ, η in $\boldsymbol{u} \in \mathbb{C}^g$, we attach the quartic form $\phi \otimes \eta := \phi\eta$ given by

$$\phi\eta(\boldsymbol{u}) = \sum_{j,l,m,p=1}^{g} \phi_{jl}\eta_{mp} \cdot u_j u_l u_m u_p,$$

and when ϕ has meromorphic coefficients, we denote by $\boldsymbol{\delta}\phi$ the quartic form

$$\boldsymbol{\delta}\phi(\boldsymbol{u}) = \sum_{j,l,m,p=1}^{g} \delta_{jl}\phi_{mp} \cdot u_j u_l u_m u_p.$$

With (1) and these conventions in mind, the system of differential equations on which Step 4 of §4 is based can then be stated as an equality between *quartic* forms, as follows.

Proposition 3 ([Z], Theorem 1)**.** *For all $\boldsymbol{a} \in \mathfrak{R}_+$, the thetanulls satisfy the system of second order partial differential equations*

$$\vartheta_{\boldsymbol{a}}^4 \cdot \boldsymbol{\delta}\psi_{\boldsymbol{a}} = \frac{1}{2^{g-2}} \sum_{\boldsymbol{b} \in \mathfrak{R}_+} (-1)^{\langle \boldsymbol{a}, \boldsymbol{b} \rangle} \vartheta_{\boldsymbol{b}}^4 \cdot \psi_{\boldsymbol{b}}^2 - 2\vartheta_{\boldsymbol{a}}^4 \cdot \psi_{\boldsymbol{a}}^2, \quad \boldsymbol{a} \in \mathfrak{R}_+.$$

To allow for a comparison with the case of low degrees g in §6, consider the rings

(2) $$Q_g = \mathbb{Q}[\vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},jl}]_{\boldsymbol{a} \in \mathfrak{R}_+;\, j,l=1,\dots,g} \quad \text{and} \quad Q_g' = \mathbb{Q}[\psi_{\boldsymbol{a},jl}]_{\boldsymbol{a} \in \mathfrak{R}_+;\, j,l=1,\dots,g}.$$

Proposition 3 then implies that the *fraction field* of Q_g is stable under $\boldsymbol{\delta}$, and (in the notations of §4) that its compositum with k coincides with the $\boldsymbol{\delta}$-differential field $M(\vartheta_{\boldsymbol{0}})$ generated by $k(\vartheta_{\boldsymbol{a}})_{\boldsymbol{a} \in \mathfrak{R}_+} = K(\vartheta_{\boldsymbol{0}})$.

Contrary to the case $g = 1$, where Jacobi's well-known formula expresses the z-derivative at $z = 0$ of the unique odd theta function as a product of thetanulls, the $\boldsymbol{z}$-derivatives of odd theta functions at $\boldsymbol{z} = \boldsymbol{0}$ in higher degrees are not modular forms. However, they are algebraic over the field $K^{(1)}$ of §4, and integral over the ring Q_g. Indeed:

Proposition 4. *For all* $\boldsymbol{a} \in \mathfrak{K}_-$, $j = 1, \dots, g$, *the following equalities hold*:

(i) $$\left(\frac{1}{2\pi i \vartheta_{\mathbf{0}}} \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z_j}\bigg|_{z=\mathbf{0}}\right)^2 = \frac{1}{2^{g-1}} \sum_{\boldsymbol{b} \in \mathfrak{K}_+} (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''} \left(\frac{\vartheta_{\boldsymbol{a}+\boldsymbol{b}}}{\vartheta_{\mathbf{0}}}\right)^2 \left(\frac{\vartheta_{\boldsymbol{b}}}{\vartheta_{\mathbf{0}}}\right)^2 (\psi_{\boldsymbol{a}+\boldsymbol{b},jj} - \psi_{\mathbf{0},jj});$$

(ii) $$\left(\frac{1}{2\pi i} \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z_j}\bigg|_{z=\mathbf{0}}\right)^4 = \frac{1}{2^{g-2}} \sum_{\boldsymbol{b} \in \mathfrak{K}_+} (-1)^{|\boldsymbol{a}+\boldsymbol{b}|} \vartheta_{\boldsymbol{b}}^4 \psi_{\boldsymbol{b},jj}^2.$$

Proof. (i) We shall use the quartic Riemann relations in the following form, valid for *all* characteristics $\boldsymbol{a}, \boldsymbol{c} \in \mathfrak{K}$:

$$\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2(z)\vartheta_{\boldsymbol{a}}^2(z) = \frac{1}{2^g} \sum_{\boldsymbol{b} \in \mathfrak{K}} (-1)^{\langle \boldsymbol{a},\boldsymbol{b}\rangle} (-1)^{{}^t\boldsymbol{c}'(\boldsymbol{a}''+\boldsymbol{b}'')} \vartheta_{\boldsymbol{b}+\boldsymbol{c}}(2z)\vartheta_{\boldsymbol{b}+\boldsymbol{c}}(\mathbf{0})\vartheta_{\boldsymbol{b}}^2(\mathbf{0}) \tag{3}$$

(see [Kr], VII, §10), together with the heat equation

$$\delta_{jl}\vartheta_{\boldsymbol{a}}(z,\boldsymbol{\tau}) = \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial z_j \partial z_l} \vartheta_{\boldsymbol{a}}(z,\boldsymbol{\tau}), \quad \boldsymbol{a} \in \mathfrak{K}, j, l = 1, \dots, g$$

(see, e.g., [Kr], I, §5, Satz 13), which allows to write the Taylor expansions of *even* theta functions as

$$\vartheta_{\boldsymbol{b}}(z) = \vartheta_{\boldsymbol{b}} \cdot \left(1 + \frac{(2\pi i)^2}{2}\psi_{\boldsymbol{b}}(z)\right) + O(z^4), \quad \boldsymbol{b} \in \mathfrak{K}_+.$$

Notice that in Riemann's formula (3), only even characteristics $\boldsymbol{b}, \boldsymbol{b}+\boldsymbol{c}$ actually appear on the right-hand side. Setting $\boldsymbol{a} = \boldsymbol{c} \in \mathfrak{K}_-$ in (3), we derive

$$\vartheta_{\mathbf{0}}^2(z)\vartheta_{\boldsymbol{a}}^2(z) = \frac{1}{2^g} \sum_{\boldsymbol{b} \in \mathfrak{K}_+} (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''} \vartheta_{\boldsymbol{a}+\boldsymbol{b}}(2z)\vartheta_{\boldsymbol{a}+\boldsymbol{b}}(\mathbf{0})\vartheta_{\boldsymbol{b}}^2(\mathbf{0}).$$

Since $\boldsymbol{a}$ is odd, the Taylor expansion of the left-hand side of this formula reads

$$\vartheta_{\mathbf{0}}^2 \cdot \left({}^t z \cdot \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z}\bigg|_{z=\mathbf{0}}\right)^2 + O(z^4).$$

Equating quadratic terms, we obtain

$$\vartheta_{\mathbf{0}}^2 \cdot \left({}^t z \cdot \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z}\bigg|_{z=\mathbf{0}}\right)^2 = \frac{(2\pi i)^2}{2^{g-1}} \sum_{\boldsymbol{b} \in \mathfrak{K}_+} (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''} \vartheta_{\boldsymbol{a}+\boldsymbol{b}}^2 \vartheta_{\boldsymbol{b}}^2 \cdot \psi_{\boldsymbol{a}+\boldsymbol{b}}(z).$$

Furthermore,

$$\sum_{\boldsymbol{b} \in \mathfrak{K}_+} (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''} \vartheta_{\boldsymbol{a}+\boldsymbol{b}}^2 \vartheta_{\boldsymbol{b}}^2 = 0,$$

since no constant term appears on the left-hand side. Multiplying the latter relation by $\psi_{\mathbf{0}}(z)$ and substracting, we finally get

$$\vartheta_{\mathbf{0}}^2 \cdot \left({}^t z \cdot \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z} \bigg|_{z=\mathbf{0}} \right)^2 = \frac{(2\pi i)^2}{2^{g-1}} \sum_{\boldsymbol{b} \in \mathfrak{R}_+} (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''} \vartheta_{\boldsymbol{a}+\boldsymbol{b}}^2 \vartheta_{\boldsymbol{b}}^2 \cdot (\psi_{\boldsymbol{a}+\boldsymbol{b}} - \psi_{\mathbf{0}})(z),$$

and Proposition 4 (i) follows on considering the z_j^2-term of these quadratic forms.

(ii) Setting $\boldsymbol{a} \in \mathfrak{R}_-$, $\boldsymbol{c} = \mathbf{0}$ in Riemann's formula (3), and developing the Taylor expansions

$$\vartheta_{\boldsymbol{b}}(z) = \vartheta_{\boldsymbol{b}} \cdot \left(1 + \frac{(2\pi i)^2}{2} \psi_{\boldsymbol{b}}(z) + \frac{(2\pi i)^4}{24} (\psi_{\boldsymbol{b}}^2 + \boldsymbol{\delta}\psi_{\boldsymbol{b}})(z) \right) + O(z^6), \quad \boldsymbol{b} \in \mathfrak{R}_+,$$

of even theta functions to the fourth order (cf. [Z], §2, Lemma 2), we derive from Formula (3) the equality of quartic forms

$$\left({}^t z \cdot \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z} \bigg|_{z=\mathbf{0}} \right)^4 = \frac{(2\pi i)^4}{3 \cdot 2^{g-1}} \sum_{\boldsymbol{b} \in \mathfrak{R}_+} (-1)^{\langle \boldsymbol{a}, \boldsymbol{b} \rangle} \vartheta_{\boldsymbol{b}}^4 \cdot (\psi_{\boldsymbol{b}}^2 + \boldsymbol{\delta}\psi_{\boldsymbol{b}})(z).$$

In view of Proposition 3 and the identity

$$\sum_{\boldsymbol{b} \in \mathfrak{R}_+} (-1)^{\langle \boldsymbol{a}+\boldsymbol{c}, \boldsymbol{b} \rangle} = (-1)^{|\boldsymbol{a}+\boldsymbol{c}|} \cdot 2^{g-1}, \quad \boldsymbol{a} \in \mathfrak{R}_-, \boldsymbol{c} \in \mathfrak{R}_+$$

(cf. [Z], §3, Lemma 7), this transforms into

$$\left({}^t z \cdot \frac{\partial \vartheta_{\boldsymbol{a}}}{\partial z} \bigg|_{z=\mathbf{0}} \right)^4 = \frac{(2\pi i)^4}{2^{g-2}} \sum_{\boldsymbol{b} \in \mathfrak{R}_+} (-1)^{|\boldsymbol{a}+\boldsymbol{b}|} \vartheta_{\boldsymbol{b}}^4 \cdot \psi_{\boldsymbol{b}}^2(z), \quad \boldsymbol{a} \in \mathfrak{R}_-,$$

and Proposition 4 (ii) follows on considering the z_j^4-term of these quartic forms.

The next statement holds for all pairs $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{R}_+$ of distinct even characteristics (and a sharper result will be given in Formula (6a) below in the case of degree 2), but one instance suffices for our purpose (cf. §4, Step 1). Recall that $\boldsymbol{\psi}_{\boldsymbol{a}}$ denotes the symmetric matrix $\boldsymbol{\delta}\vartheta_{\boldsymbol{a}}/\vartheta_{\boldsymbol{a}}$ attached to the quadratic form $\psi_{\boldsymbol{a}}$.

Lemma 4. *There exists even characteristics $\boldsymbol{a}, \boldsymbol{b}$ such that* $\det(\boldsymbol{\psi}_{\boldsymbol{b}} - \boldsymbol{\psi}_{\boldsymbol{a}})$ *is a non-zero meromorphic modular form of weight* 2 *with respect to* $\Gamma_{4,8}$.

Proof. Since $\lambda_0 = \vartheta_{\boldsymbol{b}}/\vartheta_{\boldsymbol{a}}$ is a modular function,

$$(4) \qquad (\boldsymbol{\delta}\lambda_0)(\gamma\boldsymbol{\tau}) = (c\boldsymbol{\tau} + d) \cdot (\boldsymbol{\delta}\lambda_0)(\boldsymbol{\tau}) \cdot {}^t(c\boldsymbol{\tau} + d), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{4,8}$$

(see [Z], §8, Lemma 16), where $\boldsymbol{\delta}\lambda_0$ is arranged as a symmetric matrix, and we obtain that

$$\eta_{\boldsymbol{a},\boldsymbol{b}} := \det\left(\frac{\boldsymbol{\delta}\lambda_0}{\lambda_0} \right) = \det(\boldsymbol{\psi}_{\boldsymbol{b}} - \boldsymbol{\psi}_{\boldsymbol{a}}), \quad \boldsymbol{a}, \boldsymbol{b} \in \mathfrak{R}_+, \boldsymbol{a} \neq \boldsymbol{b},$$

is a meromorphic modular form of weight 2.

We now show that $\eta_{\boldsymbol{a},\boldsymbol{b}}$ is non-zero for $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') = (\mathbf{0}, \mathbf{0}) \in \mathfrak{R}_+$ and $\boldsymbol{b} = (\boldsymbol{b}', \boldsymbol{b}'') = (\mathbf{1}, \mathbf{0}) \in \mathfrak{R}_+$, where $\mathbf{1}$ is a g-vector-column with unit entries. Choosing $\boldsymbol{\tau} = \tau \mathbf{1}_g \in \mathfrak{H}_g$, where $\tau \in \mathfrak{H}_1$, we get from the definition of the theta functions

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \tau\mathbf{1}_g) = \prod_{j=1}^{g} \vartheta_{00}(z_j, \tau) \quad \text{and} \quad \vartheta_{\boldsymbol{b}}(\boldsymbol{z}, \tau\mathbf{1}_g) = \prod_{j=1}^{g} \vartheta_{10}(z_j, \tau);$$

here, $\vartheta_{00}(z, \tau)$ and $\vartheta_{10}(z, \tau)$ stand for the usual elliptic theta functions. Taking $\partial^2/\partial z_j\,\partial z_l$-derivatives and evaluating at $\boldsymbol{z} = \mathbf{0}$, we deduce from the heat equations in degrees g and 1, and from the even character of $\vartheta_{00}(z, \tau)$, that

$$\psi_{\boldsymbol{a},jl}(\tau\mathbf{1}_g) = \begin{cases} \dfrac{1}{\vartheta_{00}} \dfrac{\partial \vartheta_{00}}{\pi i \partial \tau}(0, \tau) := \psi_{00}(\tau) & \text{if } j = l, \\ 0 & \text{if } j \neq l. \end{cases}$$

Similarly, $\psi_{\boldsymbol{b},jl}(\tau\mathbf{1}_g)$ vanishes if $j \neq l$ and is otherwise equal to $\psi_{10}(\tau)$. Thus, $(\boldsymbol{\psi}_{\boldsymbol{b}} - \boldsymbol{\psi}_{\boldsymbol{a}})(\tau\mathbf{1}_g)$ is a diagonal matrix, with determinant

$$\eta_{\boldsymbol{a},\boldsymbol{b}}(\tau\mathbf{1}_g) = \det(\boldsymbol{\psi}_{\boldsymbol{b}} - \boldsymbol{\psi}_{\boldsymbol{a}})(\tau\mathbf{1}_g) = \big(\psi_{10}(\tau) - \psi_{00}(\tau)\big)^g.$$

Now, Legendre's modular function $\lambda = (\vartheta_{10}/\vartheta_{00})^4$ is not constant, so that

$$\psi_{10} - \psi_{00} = \frac{1}{4\lambda} \frac{\partial \lambda}{\pi i \partial \tau} \neq 0,$$

and $\eta_{\boldsymbol{a},\boldsymbol{b}}$ is indeed a non-zero degree g modular form.

§6. The differential ring of thetanulls in genus 2

Theorem 2 can be sharpened when $g = 1$ and $g = 2$, because in both of these cases, the *rings*

$$Q_g = \mathbb{Q}[\vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},jl}]_{\boldsymbol{a} \in \mathfrak{R}_+;\, j,l=1,\dots,g} \quad \text{and} \quad Q_g' = \mathbb{Q}[\psi_{\boldsymbol{a},jl}]_{\boldsymbol{a} \in \mathfrak{R}_+;\, j,l=1,\dots,g}$$

generated over $\mathbb{Q}$ by the thetanulls and their logarithmic derivatives (cf. §5, (1), (2)) are themselves stable under derivation.

First, we consider the well-known case $g = 1$ where we have three even characteristics, $\mathfrak{R}_+ = \{00, 01, 10\}$. In this case there exists only one parameter $\tau = \tau_{11}$, and the notations δ, $\psi_{\boldsymbol{a}}$, need no jl-indexation. The rings Q_1 and Q_1' are δ-stable, since $\delta\vartheta_{\boldsymbol{a}} = \vartheta_{\boldsymbol{a}}\psi_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathfrak{R}_+$, by definition, while

$$\begin{aligned} \delta\psi_{10} &= 2(\psi_{10}\psi_{00} + \psi_{10}\psi_{01} - \psi_{00}\psi_{01}), \\ \delta\psi_{00} &= 2(\psi_{10}\psi_{00} + \psi_{00}\psi_{01} - \psi_{10}\psi_{01}), \\ \delta\psi_{01} &= 2(\psi_{10}\psi_{01} + \psi_{00}\psi_{01} - \psi_{10}\psi_{00}) \end{aligned}$$

(this system was discovered by G. Halphen in 1881). Moreover, Q_1 is integral over Q_1', in view of the formulae

$$\vartheta_{00}^4 = 4(\psi_{10} - \psi_{01}), \quad \vartheta_{01}^4 = 4(\psi_{10} - \psi_{00}), \quad \vartheta_{10}^4 = 4(\psi_{00} - \psi_{01})$$

(see [Z], Introduction). By [Ra], [M] or Theorem 2, both rings Q_1 and Q_1' have transcendence degree 3 over $\mathbb{Q}$.

In the case $g = 2$, we use for simplicity the map $(\mathbb{Z}/2\mathbb{Z})^2 \to \{0, 1, 2, 3\}$,

$$(0,0) \mapsto 0, \quad (0,1) \mapsto 1, \quad (1,0) \mapsto 2, \quad (1,1) \mapsto 3,$$

to represent a characteristic $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ by two digits only. Then,

$$\mathfrak{R}_+ = \{00, 01, 02, 03, 10, 12, 20, 21, 30, 33\}.$$

We renumerate the entries of the matrix $\boldsymbol{\tau}$ as $\tau_1 = \tau_{11}$, $\tau_2 = \tau_{22}$, and $\tau_3 = \tau_{12}$, and proceed similarly with the derivations

$$\boldsymbol{\delta} = \left\{ \delta_1 = \frac{1}{\pi i} \frac{\partial}{\partial \tau_1}, \delta_2 = \frac{1}{\pi i} \frac{\partial}{\partial \tau_2}, \delta_3 = \frac{1}{2\pi i} \frac{\partial}{\partial \tau_3} \right\},$$

and with the $\psi_{\boldsymbol{a}}$-notation.

Theorem 3. (i) *In the case $g = 2$, the rings*

$$Q_2 = \mathbb{Q}[\vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},j}]_{\boldsymbol{a} \in \mathfrak{R}_+;\, j=1,2,3} \quad \text{and} \quad Q_2' = \mathbb{Q}[\psi_{\boldsymbol{a},j}]_{\boldsymbol{a} \in \mathfrak{R}_+;\, j=1,2,3}$$

are stable under the derivations δ_j, $j = 1, 2, 3$.

(ii) *All thetanulls are algebraic over $\mathbb{Q}(\psi_{\boldsymbol{a},j})_{\boldsymbol{a} \in \mathfrak{R}_+;\, j=1,2,3}$.*

(iii) *Both rings Q_2 and Q_2' have transcendence degree* 10 *over* $\mathbb{Q}$.

(iv) *A possible choice for ten elements of Q_2, algebraically independent over $\mathbb{Q}$, is given by*

$$\vartheta_{00}, \vartheta_{01}, \vartheta_{02}, \psi_{00,1}, \psi_{01,1}, \psi_{02,1}, \psi_{00,2}, \psi_{01,2}, \psi_{02,2}, \psi_{00,3}.$$

Proof. (i) Let $\boldsymbol{a}_1 = \boldsymbol{a}$, $\boldsymbol{a}_2 = \boldsymbol{a} + \boldsymbol{c}$, $\boldsymbol{a}_3 = \boldsymbol{a} + \boldsymbol{d}$, $\boldsymbol{a}_4 = \boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d}$ be four different even characteristics (such a collection $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is called a *Göpel system*, and there exist fifteen Göpel systems). Then

$$\text{(5a)} \quad \boldsymbol{\delta}(\psi_{\boldsymbol{a}_1} + \psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_4}) = (\psi_{\boldsymbol{a}_1} + \psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_4})^2 - 2(\psi_{\boldsymbol{a}_1}^2 + \psi_{\boldsymbol{a}_2}^2 + \psi_{\boldsymbol{a}_3}^2 + \psi_{\boldsymbol{a}_4}^2).$$

System (5a), which was obtained in [O], provides an expression for each $\delta\psi_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathfrak{R}_+$. Namely (cf. [Z], §6, Formulae (6.17)):

$$\text{(5b)} \quad \boldsymbol{\delta}\psi_{\boldsymbol{a}} = -2\psi_{\boldsymbol{a}}^2 - \frac{1}{3} \sum_{\boldsymbol{b} \in \mathfrak{R}_+} \psi_{\boldsymbol{b}}^2 - \frac{1}{6} \left(\sum_{\boldsymbol{b} \in \mathfrak{R}_+} \psi_{\boldsymbol{b}} \right)^2 + \frac{1}{4} \sum_{G \ni \boldsymbol{a}} \left(\sum_{\boldsymbol{b} \in G} \psi_{\boldsymbol{b}} \right)^2, \quad \boldsymbol{a} \in \mathfrak{R}_+,$$

where the summation $\sum\limits_{G \ni \boldsymbol{a}}$ goes over all different Göpel systems containing $\boldsymbol{a}$. The differential stability of Q_2' is clear from (5b), and that of Q_2 then follows from the defining equations (1).

(ii) This result (which is not the statement of algebraicity over $K^{(1)}$ given in Step 1 of §4) is shown in Theorem 6 of [Z]. Here is a sketch of its proof. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{R}_+$, $\boldsymbol{a} \neq \boldsymbol{b}$. As noticed in §5, Lemma 4,

$$\eta_{\boldsymbol{a},\boldsymbol{b}} = \eta_{\boldsymbol{b},\boldsymbol{a}} := \det(\boldsymbol{\psi}_{\boldsymbol{a}} - \boldsymbol{\psi}_{\boldsymbol{b}}) = (\psi_{\boldsymbol{a},1} - \psi_{\boldsymbol{b},1})(\psi_{\boldsymbol{a},2} - \psi_{\boldsymbol{b},2}) - (\psi_{\boldsymbol{a},3} - \psi_{\boldsymbol{b},3})^2$$

is a meromorphic modular form of weight 2, so that $\vartheta_{\boldsymbol{a}}^2 \vartheta_{\boldsymbol{b}}^2 \eta_{\boldsymbol{a},\boldsymbol{b}}$ is a holomorphic modular form of weight 4 with respect to $\Gamma_{4,8}$. By the sharper version of Igusa's theorem in the genus 2 case, any such form can be expressed as a polynomial in thetanulls, and indeed,

$$\eta_{\boldsymbol{a},\boldsymbol{b}} = \pm \frac{1}{2^4} \prod_{\boldsymbol{c} \in \mathfrak{R}_+} \vartheta_{\boldsymbol{c}}^2 \cdot \prod_{G \supset \{\boldsymbol{a},\boldsymbol{b}\}} \left(\prod_{\boldsymbol{d} \in G} \vartheta_{\boldsymbol{d}}^{-2} \right), \tag{6a}$$

(see [Z], §8, Lemma 20 (a), and Formula (6c) below for an explicit expression); here, the product $\prod\limits_{G \supset \{\boldsymbol{a},\boldsymbol{b}\}}$ goes over all (i.e. the two) Göpel systems containing $\boldsymbol{a}$ and $\boldsymbol{b}$, and the only term that appears in the denominator of the right-hand side of (6a) is $\vartheta_{\boldsymbol{a}}^2 \vartheta_{\boldsymbol{b}}^2$.

Multiplying formulae (6a) for fixed $\boldsymbol{a}$ over all $\boldsymbol{b} \neq \boldsymbol{a}$, and then over all different pairs $\boldsymbol{a}, \boldsymbol{b}$, we get (cf. [Z], (8.30))

$$\vartheta_{\boldsymbol{a}}^{72} = \pm 2^{-4 \cdot 27} \prod_{\boldsymbol{c} \in \mathfrak{R}_+} \vartheta_{\boldsymbol{c}}^{18} \cdot \prod_{\substack{\boldsymbol{b} \in \mathfrak{R}_+ \\ \boldsymbol{b} \neq \boldsymbol{a}}} \eta_{\boldsymbol{a},\boldsymbol{b}}^{-3} = \pm 2^{72} \prod_{\substack{\text{different} \\ \text{pairs } \boldsymbol{c},\boldsymbol{d}}} \eta_{\boldsymbol{c},\boldsymbol{d}} \cdot \prod_{\substack{\boldsymbol{b} \in \mathfrak{R}_+ \\ \boldsymbol{b} \neq \boldsymbol{a}}} \eta_{\boldsymbol{a},\boldsymbol{b}}^{-3}, \quad \boldsymbol{a} \in \mathfrak{R}_+. \tag{6b}$$

Consequently, all thetanulls are algebraic over $\mathbb{Q}(\psi_{\boldsymbol{a},j})_{\boldsymbol{a} \in \mathfrak{R}_+; j=1,2,3}$.

(iii) Recalling the notations of §4, we infer from (i) and (ii) above that M and the fraction fields of Q_2' and of Q_2 have the same algebraic closure. Since $2g^2 + g = 10$, the result immediately follows from our Theorem 2.

(iv) We shall find a maximal set of algebraically independent elements in the ring Q_2 by exhibiting thirty independent relations between its fourty generators[3].

Specializing the Riemann relations (3) used in the proof of Proposition 4 to the case $\boldsymbol{c} \in \{01, 02, 03\}$, $\boldsymbol{a} = 00$, we obtain

$$\vartheta_{00}^2 \vartheta_{01}^2 - \vartheta_{02}^2 \vartheta_{03}^2 = \vartheta_{20}^2 \vartheta_{21}^2, \quad \vartheta_{00}^2 \vartheta_{02}^2 - \vartheta_{01}^2 \vartheta_{03}^2 = \vartheta_{10}^2 \vartheta_{12}^2, \tag{3a}$$
$$\vartheta_{00}^2 \vartheta_{03}^2 - \vartheta_{01}^2 \vartheta_{02}^2 = \vartheta_{30}^2 \vartheta_{33}^2,$$

[3] We do not claim that these relations generate a full ideal of definition for Q_2.

while the case $\boldsymbol{c} = 00$ yields

$$\text{(3b)} \qquad \vartheta_{00}^4 - \vartheta_{01}^4 = \vartheta_{10}^4 + \vartheta_{33}^4, \quad \vartheta_{00}^4 - \vartheta_{02}^4 = \vartheta_{21}^4 + \vartheta_{30}^4,$$
$$\vartheta_{00}^4 - \vartheta_{03}^4 = \vartheta_{12}^4 + \vartheta_{20}^4.$$

In view of (3a) and (3b), all thetanulls can be expressed in terms of the four functions $\{\vartheta_{00}, \vartheta_{01}, \vartheta_{02}, \vartheta_{03}\}$, which must therefore be algebraically independent. We shall denote by $\mathfrak{K}_0 := \{00, 01, 02, 03\}$ the corresponding set of characteristics.

By $\boldsymbol{\delta}$-differention of (3a) and (3b), we obtain each $\psi_{\boldsymbol{a},j}$, $\boldsymbol{a} \in \mathfrak{K}_+$, $j = 1, 2, 3$, as a linear combination with coefficients in $\mathbb{Q}(\vartheta_{\boldsymbol{a}})_{\boldsymbol{a} \in \mathfrak{K}_+}$ of the functions $\psi_{\boldsymbol{a},j}$, $\boldsymbol{a} \in \mathfrak{K}_0$, $j = 1, 2, 3$. We are thus reduced to finding six independent relations linking the thirteen functions ϑ_{03} and $\psi_{\boldsymbol{a},j}$, $\boldsymbol{a} \in \mathfrak{K}_0$, $j = 1, 2, 3$, over $\mathbb{Q}(\vartheta_{00}, \vartheta_{01}, \vartheta_{02})$.

Developing the functions $\eta_{\boldsymbol{a},\boldsymbol{b}}$, $\boldsymbol{a} \neq \boldsymbol{b} \in \mathfrak{K}_0$, used in (ii) above and expliciting the right-hand side of Formula (6a), we obtain:

$$\text{(6c)} \qquad \eta_{00,01} := (\psi_{00,1} - \psi_{01,1})(\psi_{00,2} - \psi_{01,2}) - (\psi_{00,3} - \psi_{01,3})^2 = \frac{1}{16} \frac{\vartheta_{10}^2 \vartheta_{12}^2 \vartheta_{30}^2 \vartheta_{33}^2}{\vartheta_{00}^2 \vartheta_{01}^2},$$

$$\eta_{00,02} := (\psi_{00,1} - \psi_{02,1})(\psi_{00,2} - \psi_{02,2}) - (\psi_{00,3} - \psi_{02,3})^2 = \frac{1}{16} \frac{\vartheta_{20}^2 \vartheta_{21}^2 \vartheta_{30}^2 \vartheta_{33}^2}{\vartheta_{00}^2 \vartheta_{02}^2},$$

$$\eta_{01,02} := (\psi_{01,1} - \psi_{02,1})(\psi_{01,2} - \psi_{02,2}) - (\psi_{01,3} - \psi_{02,3})^2 = -\frac{1}{16} \frac{\vartheta_{10}^2 \vartheta_{12}^2 \vartheta_{20}^2 \vartheta_{21}^2}{\vartheta_{01}^2 \vartheta_{02}^2},$$

$$\eta_{00,03} := (\psi_{00,1} - \psi_{03,1})(\psi_{00,2} - \psi_{03,2}) - (\psi_{00,3} - \psi_{03,3})^2 = \frac{1}{16} \frac{\vartheta_{10}^2 \vartheta_{12}^2 \vartheta_{20}^2 \vartheta_{21}^2}{\vartheta_{00}^2 \vartheta_{03}^2},$$

$$\eta_{01,03} := (\psi_{01,1} - \psi_{03,1})(\psi_{01,2} - \psi_{03,2}) - (\psi_{01,3} - \psi_{03,3})^2 = -\frac{1}{16} \frac{\vartheta_{20}^2 \vartheta_{21}^2 \vartheta_{30}^2 \vartheta_{33}^2}{\vartheta_{01}^2 \vartheta_{03}^2},$$

$$\eta_{02,03} := (\psi_{02,1} - \psi_{03,1})(\psi_{02,2} - \psi_{03,2}) - (\psi_{02,3} - \psi_{03,3})^2 = -\frac{1}{16} \frac{\vartheta_{10}^2 \vartheta_{12}^2 \vartheta_{30}^2 \vartheta_{33}^2}{\vartheta_{02}^2 \vartheta_{03}^2}.$$

Notice that in view of relations (3a), the numerators of the right-hand terms of (6c) are polynomials in $\vartheta_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathfrak{K}_0$. For instance, the first one reads

$$\vartheta_{10}^2 \vartheta_{12}^2 \vartheta_{30}^2 \vartheta_{33}^2 = (\vartheta_{00}^2 \vartheta_{02}^2 - \vartheta_{01}^2 \vartheta_{03}^2)(\vartheta_{00}^2 \vartheta_{03}^2 - \vartheta_{01}^2 \vartheta_{02}^2) = -\vartheta_{00}^2 \vartheta_{01}^2 \vartheta_{03}^4 + \cdots.$$

Now, consider the expressions

$$\chi_1 = (\psi_{00,3} - \psi_{01,3})^2, \quad \chi_2 = (\psi_{00,3} - \psi_{02,3})^2, \quad \chi_3 = (\psi_{02,3} - \psi_{01,3})^2.$$

They formally satisfy

$$\chi_1^2 + \chi_2^2 + \chi_3^2 - 2\chi_1\chi_2 - 2\chi_2\chi_3 - 2\chi_3\chi_1 = 0.$$

Substituting into this relation the expressions for the χ's given by the first three lines of (6c) and developing the numerators of the right-hand terms of (6c) as above, we obtain a polynomial relation $\mathfrak{R}_0$ between

$$\vartheta_{03}, \psi_{00,1}, \psi_{00,2}, \psi_{01,1}, \psi_{01,2}, \psi_{02,1}, \psi_{02,2}$$

with coefficients in $\mathbb{Q}(\vartheta_{00}, \vartheta_{01}, \vartheta_{02})$. The coefficient of ϑ_{03}^8 in $\mathfrak{R}_0$, viewed as a polynomial in ϑ_{03}, is the non-zero constant $-3 \cdot \dfrac{1}{16^2}$, and we deduce that ϑ_{03} *is algebraic over the field*

$$\text{(7)} \qquad \mathbb{Q}(\vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},j}; \boldsymbol{a} \in \mathfrak{K}', j = 1, 2) \quad \text{where } \mathfrak{K}' := \{00, 01, 02\} = \mathfrak{K}_0 \backslash \{03\}.$$

By (6c), the four functions $\psi_{\boldsymbol{a},3}$, $\boldsymbol{a} \in \mathfrak{K}_0$, are then algebraic over the field generated by anyone of them, say $\psi_{00,3}$, over the field (7).

We conclude the proof of Theorem 3 (iv) by finding (in a similar, though more subtle, way) two independent relations $\mathfrak{R}_1, \mathfrak{R}_2$ linking $\psi_{03,1}, \psi_{03,2}$ to the nine generators of the field (7). Consider the expressions

$$\text{(8a)} \quad \begin{aligned} \phi_0 &= (\psi_{03,1} - \psi_{01,1})(\psi_{01,1} - \psi_{02,1})\eta_{02,03} + (\psi_{02,1} - \psi_{03,1})(\psi_{01,1} - \psi_{02,1})\eta_{03,01} \\ &\quad + (\psi_{02,1} - \psi_{03,1})(\psi_{03,1} - \psi_{01,1})\eta_{01,02}, \\ \phi_1 &= (\psi_{03,1} - \psi_{00,1})(\psi_{00,1} - \psi_{02,1})\eta_{02,03} + (\psi_{02,1} - \psi_{03,1})(\psi_{00,1} - \psi_{02,1})\eta_{03,00} \\ &\quad + (\psi_{02,1} - \psi_{03,1})(\psi_{03,1} - \psi_{00,1})\eta_{00,02}, \\ \phi_2 &= (\psi_{03,1} - \psi_{00,1})(\psi_{00,1} - \psi_{01,1})\eta_{01,03} + (\psi_{01,1} - \psi_{03,1})(\psi_{00,1} - \psi_{01,1})\eta_{03,00} \\ &\quad + (\psi_{01,1} - \psi_{03,1})(\psi_{03,1} - \psi_{00,1})\eta_{00,01}, \\ \phi_3 &= (\psi_{02,1} - \psi_{00,1})(\psi_{00,1} - \psi_{01,1})\eta_{01,02} + (\psi_{01,1} - \psi_{02,1})(\psi_{00,1} - \psi_{01,1})\eta_{02,00} \\ &\quad + (\psi_{01,1} - \psi_{02,1})(\psi_{02,1} - \psi_{00,1})\eta_{00,01}. \end{aligned}$$

When the η's are developed as in the middle terms of (6c), a computation shows that they formally satisfy the relation

$$\text{(8b)} \quad \left((\phi_0 + \phi_1 + \phi_2 + \phi_3)^2 - 2(\phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2)\right)^2 - 64\phi_0\phi_1\phi_2\phi_3 = 0.$$

Substituting into (8a) the expressions for the η's given by the right-hand terms of (6c), we deduce from (8b) a polynomial relation $\mathfrak{R}_1$ between

$$\psi_{00,1}, \psi_{01,1}, \psi_{02,1}, \psi_{03,1}$$

with coefficients in $\mathbb{Q}(\vartheta_{\boldsymbol{a}})_{\boldsymbol{a} \in \mathfrak{K}_0}$. Now, the coefficient of $\psi_{03,1}^8$ in $\mathfrak{R}_1$, viewed as a polynomial in $\psi_{03,1}$, is

$$\left((\eta_{00,01} + \eta_{00,02} + \eta_{01,02})^2 - 2(\eta_{00,01}^2 + \eta_{00,02}^2 + \eta_{01,02}^2)\right)^2,$$

and we can check that this expression is not identically 0 by evaluating it at $\boldsymbol{\tau} = \tau\mathbf{1}_2$ with $\tau \in \mathfrak{H}_1$, as in the proof of Lemma 4: one finds that it reduces to the non-zero genus 1 modular form $\eta_{01,02}^4(\tau\mathbf{1}_2) = \frac{1}{16^4}\vartheta_{10}^{32}(\tau)$. We thus deduce from $\mathfrak{R}_1$ that $\psi_{03,1}$ is algebraic over the field generated by $\vartheta_{\boldsymbol{a}}$, $\psi_{\boldsymbol{a},1}$, $\boldsymbol{a} \in \mathfrak{K}'$, and ϑ_{03}, hence over the field (7), in view of the algebraicity of ϑ_{03}. By the same argument, $\psi_{03,2}$ is algebraic over the field generated by $\vartheta_{\boldsymbol{a}}$, $\psi_{\boldsymbol{a},2}$, $\boldsymbol{a} \in \mathfrak{K}'$, and ϑ_{03}, hence over the field (7). Therefore, the ten functions

$$\vartheta_{00}, \vartheta_{01}, \vartheta_{02}, \psi_{00,1}, \psi_{01,1}, \psi_{02,1}, \psi_{00,2}, \psi_{01,2}, \psi_{02,2}, \psi_{00,3}$$

are algebraically independent over $\mathbb{Q}$, and the proof of Theorem 3 is completed.

Remark 5. In the same vein, it can be proved in the case $g = 3$ that the *fraction field* of the ring

$$Q_3' = \mathbb{Q}[\psi_{\boldsymbol{a},jl}]_{\boldsymbol{a} \in \mathfrak{K}_+;\, j,l=1,2,3}$$

is $\boldsymbol{\delta}$-stable, and that all thetanulls are algebraic over it. This follows from [Z], Theorems 4 and 6. Theorem 2 of the present paper then implies that Q_3' has transcendence degree 21 over $\mathbb{Q}$.

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