Sparse Anti-magic Squares and Vertex-magic Labelings of Bipartite Graphs

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April 12, 2006

Abstract

A sparse anti-magic square is an $n \times n$ array whose non-zero entries are the consecutive integers $1, \cdots, m$ for some $m \leq n^2$ and whose row-sums and column-sums form a set of consecutive integers. We derive some basic properties of these arrays and provide constructions for several infinite families of them. Our main interest in these arrays is their application to constructing vertex-magic labelings for bipartite graphs.

Keywords: anti-magic square, magic labelling

1 Introduction

Magic squares and their various generalisations have been objects of interest for many centuries and in many cultures. From time to time they have been found useful in some application to another area of mathematics. One such application was described in [6] where the notion of a *vertex-magic total labeling* (VMTL) for a graph was introduced. For a simple graph G with v vertices and e edges, this is a one-to-one assignment of the labels $1, 2, \cdots, v+e$ to the vertices and edges of G so that the sum of labels of any vertex and its incident edges is a constant. In [1] the authors introduced a generalisation of the magic square called a *sparse semi-magic square* - the constant row and column-sums are still required but the square may now contain zero entries. Apart from their intrinsic interest, our main reason for studying these arrays was their application to the construction of VMTLs for bipartite graphs.

During that work we realised that a related technique for constructing VMTLs could be devised if we had an array with an *anti-magic* property, i.e. the row-sums and column-sums were all different. We define a *sparse anti-magic square* to be a square array of order n whose non-zero entries are the consecutive numbers $1, \cdots, m$ (for some $m \leq n^2$) and whose row-sums and column-sums constitute a set of consecutive integers. The construction of VMTLs using sparse semi-magic squares described in [1] did not make any use of the diagonals of the square (whence the "semi"); neither will the construction described here, so our definition does not require the diagonal sums to be considered. Nevertheless some of our constructions can be modified to include the 2 diagonal sums in the set of consecutive integers, if desired. We call such an array a *sparse totally anti-magic square.*

In this paper we investigate some basic properties of these sparse anti-magic squares and sparse totally anti-magic squares and provide constructions for various infinite families of them. We then show how to use these arrays in the construction of vertex-magic total labelings for a large family of bipartite graphs.

Vertex magic total labelings for various classes of graphs have been studied in [2, 3, 4, 7, 8] among others and a recent account of magic labelings is given by Wallis in [9]. A good up-to-date account of anti-magic squares is given in [5].

2 Sparse Anti-Magic (SAM) and Sparse Totally Anti-Magic (STAM) Squares

Let S be any $n \times n$ array of non-negative integers, not necessarily distinct, with the property that all of its row-sums and column-sums are distinct. Let $nd - r$ be the number of positive entries in S where $d \leq n$ and $r < n$. Then S contains $n^2 - (nd - r)$ zeroes. We will call d the *density* of the array; it is the minimum number of rows required to accommodate the positive entries if they were packed as densely as possible. The *deficiency* r of the array is the number of cells by which the array fails to achieve that ceiling. In a traditional anti-magic square there are no zero entries so, of course, $d = n$ and $r = 0$. At the other extreme, we can ask for the minimum possible number of non-zero entries. If $n \geq 3$, then $d \geq 2$ since otherwise the row and column-sums could not be distinct. The lower bound is given by the following:

Theorem 1 *Let S be any $n \times n$ array $(n > 2)$ of non-negative integers with row and column-sums all distinct. If $d = 2$, then*

(i) $r \leq (n+2)/2$ *if S contains a row or column of* $0s$.

(ii) $r \leq n/2$ *if S does not contain a row or column of* $0s$.

Proof. First, suppose that one row (column) consists entirely of 0s. There are $2n - r$ distinct positive integers in the square and $n - 1$ rows contain at least one positive integer, hence at most $n - r + 1$ rows contain two or more positive integers and so at least $r - 2$ rows contain a single positive integer. Similarly, at most $n - r$ columns contain 2 or more integers and at least r columns contain a single integer. Each integer in a single-integer row must have a column-mate otherwise two of the row/column-sums would be the same. Hence at least $r - 2$ columns contain 2 or more integers. Let c be the number of columns containing 2 or more integers. Then we have $n - r \geq c \geq r - 2$, so that $\frac{1}{2}(n+2) \geq r$ as required.

If no row or column can consist entirely of 0s, then a similar argument shows that $r \leq \frac{1}{2}n$. ■

The upper bounds on r can in fact be met, as shown by the following squares.

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

We say that a square is *regular* if all of its rows and columns contain the same number of positive entries, and *totally regular* if its rows, columns, and diagonals contain the same number of positive entries. An *almost regular* square has exactly one row and one column containing one fewer positive entries than the rest. If we have a regular square in which the entry 1 appears once only, then by subtracting 1 from all positive entries we can obtain an almost regular square which still retains the magic property. In such a case we say that the almost regular square is *derived* from the regular one.

Definition *A sparse anti-magic square $SAM(n, d, r)$ is an $n \times n$ array in which the non-zero entries are the integers $1, \cdots, nd - r$ and the row-sums and column-sums constitute a set of consecutive integers.*

Definition *A sparse totally anti-magic square $STAM(n, d, r)$ is an $n \times n$ array in which the non-zero entries are the integers $1, \cdots, nd - r$ and the* *row-sums, column-sums and two principal diagonal sums constitute a set of consecutive integers.*

It should be clear from these definitions that a SAM is not necessarily a STAM and vice-versa. However it will be shown later that a square can be both a SAM and a STAM and that in fact there are infinite families of such squares.

For a SAM, it follows immediately from summing the row-sums and column-sums and then summing the entries of the square that

$$n(2a + 2n - 1) = (nd - r)(nd - r + 1) \tag{1}$$

for some integer a. If n is odd then the left side of this identity must also be odd whereas the right side must be even, which of course is impossible. So we have the following:

Theorem 2 *A sparse anti-magic square has even order.*

On the other hand STAMs of odd order can exist as shown by the following example of a $STAM(11, 2, 6)$ (where the empty cells represent zero entries and the diagonals are marked with * for convenience).

1								**10**		*	11
	*	**15**							*	**2**	17
		*						*	**16**		16
			*	**4**	**14**		*				18
				*	**5**	*					5
		6			*						6
				*		*			**7**		7
	12		*			**8**	*				20
		*	**13**				**9**	*			22
11	**3**								*		14
*										*	0
12	15	21	13	4	19	8	9	10	23	2	1\3/

Theorem 1 gives a lower bound on the number of positive entries in a SAM or a STAM. We show that this lower bound cannot be met for STAMs of odd order except for a few small cases.

Theorem 3 *There is no* $STAM(2m + 1, 2, m + 1)$ *for* $m > 5$.

Proof. Since one row (or column) has a sum of zero and the row, column and diagonal sums comprise a set of consecutive integers, these sums will form the set $\{0, 1, \cdots, 4m+3\}$. Let x_1 and x_2 be the sums of the diagonals. Then the row and column-sums add to $(2m+2)(4m+3) - x_1 - x_2$. The positive integers used to fill the matrix are $1, \cdots, 3m+1$ and each of these integers takes part in both a row and a column sum. So the rows and columns also sum to $(3m+1)(3m+2)$. Hence: $(3m+1)(3m+2) = (2m+2)(4m+3) - x_1 - x_2$. So: $x_1 + x_2 = -m^2 + 5m + 4$. But we must have $x_1 + x_2 \geq 3$, hence $0 \geq m^2 - 5m - 1$ and since m is an integer, $5 \geq m$. The result follows. ∎

For each odd order from 3 to 11 we have constructed a STAM attaining this lower bound. The $STAM(11, 2, 6)$ shown above is one such example.

The lower bounds on r given in Theorem 1 also apply to any SAM. In Section 3.3, we show how to construct an infinite family of SAMs not containing a row or column of 0s which meet the lower bound of $r = n/2$. The order 2 SAM shown above has a column of 0s and meets the lower bound $r = (n+2)/2$. It is easy to show that no SAM of higher order does so.

Theorem 4 *No SAM of order $n > 2$ can contain a row or column of 0s.*

Proof. Substituting $a = 0$ and $d = 2$ into Equation (1) yields

$$n(2n-1) = (2n-r)(2n-r+1).$$

Regarding this as a quadratic in r and solving, we find

$$r = \frac{1}{2}\left[(4n+1) - \sqrt{8n^2 - 4n + 1}\right]$$

and since $r \leq (n+2)/2$ by Theorem 1, we get

$$(4n+1) - \sqrt{8n^2 - 4n + 1} \leq n + 2.$$

Simplifying gives $n \leq 2$. ∎

2.1 Restrictions on the value of r for a SAM

One of the obvious questions to study is what relationships might exist among n, d and r. The theorems contain relations that basically result from applying equation 1.

Theorem 5 *For a $SAM(2m, d, r)$, if $r = 0$ or 1 then d is odd.*

Proof. If $n = 2m$ and $r = 0$, then from equation 1 we get

$$(2md)(2md+1) = 2m(2a+4m-1) \text{ so that}$$
$$d(2md+1) = (2a+4m-1)$$

for some positive integer a. Since the RHS is odd and $(2md+1)$ is odd it follows that d is odd also, i.e. a regular SAM of even order must be of odd density. If $n = 2m$ and $r = 1$ then we have

$$(2md-1)(2md) = 2m(2k+4m-1)$$

which gives $d(2md-1) = 2a+4m-1$. Hence d is odd again. ■

Theorem 6 *For a $SAM(2^m, d, r)$ to exist we must have $r = 0$ or 1.*

Proof. The positive entries are $1, \cdots, 2^m d - r$ so we have

$$(2^m d - r)(2^m d - r + 1) = 2^m(2a + 2.2^m - 1)$$

Hence $2^m | r(r-1)$ which implies $r = 0$ or $r - 1 = 0$ since r and $r - 1$ are relatively prime and $r < 2^m$. Thus $r = 0$ or $r = 1$ as required. ■

We note that the above proof is equally valid for any prime power $n = p^m$, but odd order squares are already impossible by Theorem 1.

Next we consider the case of $n = 2p^m$ where p is an odd prime, which is a little more complicated. Equation 1 now gives us

$$(2p^m d - r)(2p^m d - r + 1) = 2p^m(2a + 4p^m - 1).$$

Since p^m divides the LHS it follows that either $p^m | r$ or $p^m | (r-1)$ and since $r < 2p^m$ it follows that $r = p^m, r = p^m + 1, r = 0$, or $r = 1$.

Suppose $r = p^m$. Then we have

$$(2p^m d - p^m)(2p^m d - p^m + 1) = 2p^m(2a + 4p^m - 1)$$

whence

$$4p^m d(d-1) + 2d + p^m + 1 = 4(a + 2p^m).$$

In order for the equality to hold, we need $4|(2d + p^m + 1)$ or equivalently $2|(d+\frac{1}{2}(p^m+1))$. Hence d and $\frac{1}{2}(p^m+1)$ must be of the same parity. $\frac{1}{2}(p^m+1)$ will be even if $p = 3 \bmod 4$ and m is odd, and will be odd otherwise. Hence r can take the value p^m only if:

(i) d is even, $p = 3 \bmod 4$ and m is odd
(ii) d is odd and either $p = 1 \bmod 4$ or m is even

Suppose $r = p^m + 1$. Then we have

$$\begin{aligned}(2p^m d - (p^m + 1))(2p^m d - (p^m + 1) + 1) &= (2p^m d - p^m - 1)\, p^m \,(2d - 1) \\ &= 2p^m (2a + 4p^m - 1)\end{aligned}$$

or

$$(2p^m d - p^m - 1)\,(2d - 1) = 4a + 8p^m - 2.$$

From this we see that $4|(p^m - 2d + 3)$ or equivalently $2|\frac{1}{2}(p^m - 1) - d$. Hence $\frac{1}{2}(p^m - 1)$ and d must be of the same parity. $\frac{1}{2}(p^m - 1)$ will be odd if $p = 3 \bmod 4$ and m is odd, and will be even otherwise. Hence r can take the value $p^n + 1$ only if:
(i) d is odd, $p = 3 \bmod 4$ and m is odd
(ii) d is even and either $p = 1 \bmod 4$ or m is even

We summarise these results as follows:

Theorem 7 *If p is prime, then for a $SAM(2p^m, d, r)$ to exist we must have one of the following:*

d	$p \bmod 4$	m	r
odd	3	*odd*	$0, 1, p^m + 1$
odd	3	*even*	$0, 1, p^m$
even	3	*even*	$p^m + 1^*$
even	3	*odd*	p^m
odd	1	*odd*	$0, 1, p^m$
odd	1	*even*	$0, 1, p^m$
even	1	*even*	$p^m + 1^*$
even	1	*odd*	$p^m + 1^*$

The cases in the table marked with $*$ are not possible when $d = 2$; this can be shown by the following argument. If a row or column of a $SAM(2p^m, 2, r)$ consisted entirely of 0s, then the row/column-sums must comprise the set $0, 1, \cdots, 4p^m - 1$. This would mean that

$$(4p^m - r)\,(4p^m - r + 1) = \frac{1}{2}\,(4p^m)\,(4p^m - 1)\,.$$

Substituting $r = p^m + 1$ we obtain $p^m = 0$ or 1, both of which are impossible. If there is no row or column of 0s, then Theorem 1 requires that $r \leq p^m$, so $r = p^m + 1$ is not possible in this case either.

The obvious (difficult) question now is whether the conditions described in the theorems are sufficient as well as necessary for the existence of a SAM of the given order. We don't know the complete answer yet. In the next section we provide constructions for infinite families of SAMs for various combinations of the parameters.

To conclude this section we look in detail at the special case of $n = 6$. Here, at least, we can answer this question. If d is even then $r = 3$, if d is odd then $r = 0, 1$ or 4. This yields the following permissible combinations of parameters:

$SAM(6,2,3)$, $SAM(6,3,0)$, $SAM(6,3,1)$, $SAM(6,5,0)$, $SAM(6,5,1)$
$SAM(6,4,3)$, $SAM(6,6,3)$, $SAM(6,3,4)$, $SAM(6,5,4)$.

The first five of these can be constructed using constructions found later in this paper. We have found examples of each of the last four cases and they are shown below, proving that all permissible cases can be realised.

$SAM(6,3,4)$

10	0	1	9	0	0	**20**
2	3	0	0	14	0	**19**
0	6	0	0	0	12	**18**
0	4	13	0	0	0	**17**
0	0	0	5	0	11	**16**
0	0	0	7	8	0	**15**
12	**13**	**14**	**21**	**22**	**23**	

$SAM(6,5,4)$

1	14	0	16	8	22	**61**
13	15	0	9	23	0	**60**
19	12	3	4	21	0	**59**
0	11	17	7	5	18	**58**
0	2	25	0	6	24	**57**
20	0	10	26	0	0	**56**
53	**54**	**55**	**62**	**63**	**64**	

$SAM(6,4,3)$

1	2	0	9	19	10	**41**
0	0	0	8	11	21	**40**
12	17	3	0	7	0	**39**
0	15	18	5	0	0	**38**
4	0	0	20	0	13	**37**
16	0	14	0	6	0	**36**
33	**34**	**35**	**42**	**43**	**44**	

$SAM(6,6,3)$

12	2	15	32	29	6	**96**
24	0	22	17	7	25	**95**
31	14	3	0	16	30	**94**
1	33	10	28	21	0	**93**
18	26	20	4	5	19	**92**
13	23	27	9	11	8	**91**
99	**98**	**97**	**90**	**89**	**88**	

3 Constructing families of SAMs or STAMs

3.1 Constructing a $\mathbf{SAM(2m,3,0)}$

Theorem 5 guarantees that if a sparse anti-magic square has $r = 0$ or $r = 1$ then it must have odd density; so we were naturally interested in whether we could always construct such squares. In other words, do there exist a $SAM(2m, 2u+1, 0)$ and a $SAM(2m, 2u+1, 1)$ for all m and u? The answer is affirmative; we have found constructions for the $r = 0$ case and these constructions produce regular squares. We can then construct from them the derived squares which will have $r = 1$. The case $d = 3$ is straightforward so we deal with it first. In addition, an extension of the method used here will be required in the next section to deal with the general case of $3u + 1$.

It is convenient to partition the $2m \times 2m$ array M into four $m \times m$ blocks. The construction follows these steps:

(i) Fill the cells on the main diagonal of M with the integers $1, \cdots, 2m$ in order, starting from the upper left.

(ii) Fill the cells of the main diagonal of the upper right block with the integers $2m + 1, \cdots, 3m$, starting from the lower right.

(iii) Fill the cells of the main diagonal of the lower left block with the integers $3m + 1, \cdots, 4m$, starting from the lower right.

(iv) In the upper left block, select any transversal that doesn't intersect the main diagonal and place the integers $4m + 1$ to $5m$ in these cells in any order. For example, we could fill the cell $(m, 1)$ with $4m + 1$ and then fill the cells $(i, m - i)$ with $4m + 1 + i$ for $i = 1, \cdots, m - 1$.

(v) In the lower right block, select any transversal that doesn't intersect the main diagonal and place the integers $5m + 1$ to $6m$ in these cells in any order. For example, we could fill the cell $(2m, m)$ with $5m + 1$ and then fill the cells $(m + i, m - i)$ with $5m + 1 + i$ for $i = 1, \cdots, m - 1$.

(vi) Fill the remaining empty cells with zeroes.

We show that the array M produced by this construction has the required properties:

Theorem 8 *A regular $SAM(2m, 3, 0)$ exists for all $m \geq 2$*

Proof. Let M be an array produced by the construction above. After steps (i), (ii) and (iii) it is clear that all the column-sums will be $4m + 1$, the upper m rows of M will have sums of $3m + 1$ and the lower m rows will have sums of $5m + 1$. Inserting the entries in the upper left block in step (iv) produces sums of $\{8m + 2, \cdots, 9m + 1\}$ for the first m columns and $\{7m + 2, \cdots, 8m + 1\}$ for the first m rows. Inserting the entries in the lower right block in step (v) produces sums $\{9m + 2, \cdots, 10m + 1\}$ for the last m columns and $\{10m + 2, \cdots, 11m + 1\}$ for the last m rows. In other words the row/column-sums form the set of consecutive integers $7m + 2, \cdots, 11m + 1$ as required. Clearly there are the three non-zero entries in each row and column, so the square is regular with $d = 3$. ■

Corollary 1 *A* $SAM(2m, 3, 1)$ *exists for all* $m \geq 2$

Proof. The array $M = (M_{i,j})$ has a single entry of 1, so the derived square B defined by $B_{i,j} = M_{i,j} - 1$ will still be anti-magic. There are three positive entries in each row and column of M, so B will have row-sums and column-sums each 3 less than those of M and will have one less positive entry than M. Therefore B will be a $SAM(2m, 3, 1)$. ■

Notice that as m increases, we have considerable latitude for the placement of entries in steps (iv) and (v), so this construction actually produces a large number of squares that are different in the sense that none can be obtained from another by row or column permutation.

Two examples of $SAM(6, 3, 0)$ constructed this way appear below:

$$\begin{bmatrix} 1 & 14 & 0 & 9 & 0 & 0 \\ 0 & 2 & 15 & 0 & 8 & 0 \\ 13 & 0 & 3 & 0 & 0 & 7 \\ 12 & 0 & 0 & 4 & 17 & 0 \\ 0 & 11 & 0 & 0 & 5 & 18 \\ 0 & 0 & 10 & 16 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 15 & 9 & 0 & 0 \\ 13 & 2 & 0 & 0 & 8 & 0 \\ 0 & 14 & 3 & 0 & 0 & 7 \\ 12 & 0 & 0 & 4 & 0 & 18 \\ 0 & 11 & 0 & 16 & 5 & 0 \\ 0 & 0 & 10 & 0 & 17 & 6 \end{bmatrix}$$

Since these SAMs are regular, in each case we can subtract 1 from each positive entry to give the derived $SAM(6, 3, 1)$ as described in the Corollary.

3.2 Constructing a $\mathbf{SAM(2m, 2u+1, 0)}$

To construct a $SAM(2m, 2u+1, 0)$ we will use an extension of the method described above. The strategy in the previous construction was to pair integers in steps (i)-(iii) so that the column-sums are all equal and the row-sums comprise two equal sets, before adding the final $2m$ elements to satisfy the anti-magic property. We elaborate on this same strategy. The key observation for generalising steps (i) to (v) is that we will be entering elements on various transversals of each $m \times m$ block, in increasing order downward on some transversals and decreasing order downward on others. The easiest way to ensure that there are as many transversals as possible available is to choose some latin square of order m to determine their arrangement.

We choose some arbitrary latin squares A and B of order m and construct a matrix of order $2m$ as a 2×2 array of order m blocks in the following pattern:

$$\begin{bmatrix} A & B \\ A & B \end{bmatrix}$$

We will label the entries of A as $a_1, \cdots, a_m$ and of B as $b_1, \cdots, b_m$ and will refer to the arrangement of 4 blocks as the *template* for the SAM.

We are going to construct a SAM

$$\begin{bmatrix} C & S \\ T & D \end{bmatrix}$$

using the integers $\{1, \cdots, (2u+1)2m\}$. Let $J = \{0, \cdots, 4u+1\}$ and partition J as $J = J_C \cup J_S \cup J_T \cup J_D$ (a subset for each quadrant) with the properties that

(i) $|J_C| = |J_D| = u$,

(ii) $|J_S| = |J_T| = u + 1$,

(iii) $|\sum J_C - \sum J_D| = 1$,

(iv) $|\sum J_S - \sum J_T| = 2$.

There are a number of ways in which we could carry out this partition. One transparent way is as follows: for u odd:

$$J_C = \{0\} \cup \left\{6 + 8q, 13 + 8q : q = 0, \cdots, \frac{1}{2}(u-3)\right\}$$
$$J_D = \{1\} \cup \left\{7 + 8q, 12 + 8q : q = 0, \cdots, \frac{1}{2}(u-3)\right\}$$
$$J_S = \{2, 4\} \cup \left\{8 + 8q, 11 + 8q : q = 0, \cdots, \frac{1}{2}(u-3)\right\}$$
$$J_T = \{3, 5\} \cup \left\{9 + 8q, 10 + 8q : q = 0, \cdots, \frac{1}{2}(u-3)\right\}$$

(where the right-hand set in the union is empty if $u < 3$), and for u even:

$$J_C = \{2, 3\} \cup \left\{10 + 8q, 17 + 8q : q = 0, \cdots, \frac{1}{2}(u-4)\right\}$$
$$J_D = \{0, 4\} \cup \left\{11 + 8q, 16 + 8q : q = 0, \cdots, \frac{1}{2}(u-4)\right\}$$
$$J_S = \{1, 7, 9\} \cup \left\{12 + 8q, 15 + 8q : q = 0, \cdots, \frac{1}{2}(u-4)\right\}$$
$$J_T = \{5, 6, 8\} \cup \left\{13 + 8q, 14 + 8q : q = 0, \cdots, \frac{1}{2}(u-4)\right\}$$

(where the right-hand set in the union is empty if $u < 4$). In both cases the entries in the right-hand set of the four unions add to the same sum, so it is easy to see that conditions (i) to (iv) are satisfied.

We now use the template to construct the SAM. We will let s_r indicate the rth element in J_S and similarly for c_r, t_r and d_r.

1. For $r = 1$ to u :
   (i) If $B(i,j) = r$ then let $S(i,j) = i + ms_r$.
   (ii) If $A(i,j) = r$ then let $C(i,j) = (m+1-i) + mc_r$.
   (iii) If $A(i,j) = r$ then let $T(i,j) = i + mt_r$.
   (iv) If $B(i,j) = r$ then let $D(i,j) = (m+1-i) + md_r$.
2. If $B(i,j) = u+1$ then let $S(i,j) = i + ms_{u+1}$.
3. If $A(i,j) = u+1$ then let $S(i,j) = i + mt_{u+1}$.
4. All other entries of S, C, T, D are 0s.

Theorem 9 *A $SAM(2m,d,0)$ exists for every $m \geq 3$ and all odd d with $3 \leq d < 2m$.*

Proof. After step 1 above, the row-sums will all be equal and the column-sums will form two sets, the sums in each set being equal. Steps 2 and 3 add one new entry from a consecutive set to each row and column in such a way that the row and column-sums are now consecutive. ■

As before, we can form the derived square of the $SAM(2m,d,0)$ to get a square with deficiency 1:

Corollary 2 *A $SAM(2m,d,1)$ exists for every $m \geq 3$ and all odd d with $3 \leq d < 2m$*

We illustrate the construction with an example of a $SAM(10,7,0)$. Here we have $m = 5$ and $u = 3$. The partition is

$$\begin{aligned} J_C &= \{0,6,13\} \\ J_D &= \{1,7,12\} \\ J_S &= \{2,4,8,11\} \\ J_T &= \{3,5,9,10\}\,. \end{aligned}$$

and for this example we will choose $A = B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$ so the template will be:

$$\left[\begin{array}{ccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\
5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 \\
4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\
3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 \\
2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 \\
\hline
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\
5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 \\
4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\
3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 \\
2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1
\end{array}\right]$$

Each of the positive entries in the SAM can be expressed as $f(i) + mq_r$, where the values of $f(i)$ and $q_x (\in \{s_x, t_x, c_x, d_x\})$ depend on the quadrant. The construction assigns the $(f(i), q_r)$ as follows:

$$\begin{bmatrix}
(5,0) & (5,6) & (5,13) & - & - & (1,2) & (1,4) & (1,8) & (1,11) & - \\
- & (4,0) & (4,6) & (4,13) & - & - & (2,2) & (2,4) & (2,8) & (2,11) \\
- & - & (3,0) & (3,6) & (3,13) & (3,11) & - & (3,2) & (3,4) & (3,8) \\
(2,13) & - & - & (2,0) & (2,6) & (4,8) & (4,11) & - & (4,2) & (4,4) \\
(1,6) & (1,13) & - & - & (1,0) & (5,4) & (5,8) & (5,11) & - & (5,2) \\
(1,3) & (1,5) & (1,9) & (1,10) & - & (5,1) & (5,7) & (5,12) & - & - \\
- & (2,3) & (2,5) & (2,9) & (2,10) & - & (4,1) & (4,7) & (4,12) & - \\
(3,10) & - & (3,3) & (3,5) & (3,9) & - & - & (3,1) & (3,7) & (3,12) \\
(4,9) & (4,10) & - & (4,3) & (4,5) & (2,12) & - & - & (2,1) & (2,7) \\
(5,5) & (5,9) & (5,10) & - & (5,3) & (1,7) & (1,12) & - & - & (1,1)
\end{bmatrix}$$

which yields the desired square

$$\begin{bmatrix}
5 & 35 & 70 & 0 & 0 & 11 & 21 & 41 & 56 & 0 \\
0 & 4 & 34 & 69 & 0 & 0 & 12 & 22 & 42 & 57 \\
0 & 0 & 3 & 33 & 68 & 58 & 0 & 13 & 23 & 43 \\
67 & 0 & 0 & 2 & 32 & 44 & 59 & 0 & 14 & 24 \\
31 & 66 & 0 & 0 & 1 & 25 & 45 & 60 & 0 & 15 \\
16 & 26 & 46 & 51 & 0 & 10 & 40 & 65 & 0 & 0 \\
0 & 17 & 27 & 47 & 52 & 0 & 9 & 39 & 64 & 0 \\
53 & 0 & 18 & 28 & 48 & 0 & 0 & 8 & 38 & 63 \\
49 & 54 & 0 & 19 & 29 & 62 & 0 & 0 & 7 & 37 \\
30 & 50 & 55 & 0 & 20 & 36 & 61 & 0 & 0 & 6
\end{bmatrix}$$

with row-sums $\{239, 240, 241, 242, 243, 254, 255, 256, 257\}$ and column-sums $\{251, 252, 253, 249, 250, 246, 247, 248, 244, 245\}$.

We conclude this section by noting that templates other than the basic one described above can also be used. For example, we have found other (more complicated) templates constructed from a set of 3 mutually orthogonal latin squares.

Because of the amount of choice involved at several stages of this construction, we can actually produce a large family of squares with the given parameters.

3.3 Constructing a $\mathbf{SAM(8t+6, 2, 4t+3)}$ and $\mathbf{STAM(8t+6, 2, 4t+3)}$

As noted in Theorem 1, the SAM with no non-zero rows or columns having the minimum possible number of positive entries has density 2 and deficiency $\frac{1}{2}n$. We were naturally interested whether arrays exist which achieve this bound. From Theorem 7, the case $n = 2p^m$ and density 2 with maximum r is only feasible for $p = 3 \bmod 4$ and m odd. For such a combination, $n = 6 \bmod 8$. It turns out that if $n = 8t + 6$ then we can always construct a $SAM(8t + 6, 2, 4t + 3)$ even where $4t + 3$ is not a prime power. This construction is also interesting in that it provides an infinite family of arrays which are of minimum density and are simultaneously SAMs and STAMs.

For $SAM(8t + 6, 2, 4t + 3)$ with $t \geq 1$ we use the integers $1, \cdots, 12t + 9$ to fill the array. We need

$$(12t + 9)\,(12t + 10) = (8t + 6)\,(2a + 16t + 11)$$

for some integer a. This gives us $a = t + 2$ with the row/column-sums in the range $t + 2, \cdots, 17t + 13$.

We will call the array S and as usual $S(i, j)$ indicates the entry in row i and column j of S. All the positive entries of S will lie on the three diagonals (i, i), $(i, i - 1)$, $(i, i + t)$. For ease of description we identify 4 blocks of S which contain all the positive entries and have the property that each row and column of S intersects exactly one block, i.e., no two blocks share a common row or common column. This means that the row/column-sums for S will be the union of the sets of row/column-sums for the four blocks.

Block	rows	columns
I	$1, \cdots, t+3$	$1, \cdots, t+2$
II	$t+4, \cdots, 5t+5$	$t+3, \cdots, 5t+4$
III	$5t+6, \cdots, 6t+6$	$5t+5, \cdots, 7t+6$
IV	$6t+7, \cdots, 8t+6$	$7t+7, \cdots, 8t+6$

We assign the integers $1, \cdots, 12t+9$ to the four blocks as follows:

Block 1	$S(i,i) = 9t+6+i$	$i = 1, \cdots, t+2$
	$S(i, i-1) = i-1$	$i = 2, \cdots, t+3$

which uses the labels $\{1, \cdots, t+2\} \cup \{9t+7, \cdots, 10t+8\}$. These entries produce row-sums $\{9t+2i+5 : i = 2, \cdots, t+2\} \cup \{9t+7\}$ and column-sums $\{9t+2i+6 : i = 2, \cdots, t+2\} \cup \{t+2\}$.

Block 2	$S(i,i) = \frac{1}{2}(19t+14+i)$	$i = t+4, \cdots, 5t+4$
	$S(i, i-1) = i-1$	$i = t+4, \cdots, 5t+5$

which uses the labels $\{10t+9, \cdots, 12t+9\} \cup \{t+3, \cdots, 5t+4\}$. These entries produce row-sums $\{\frac{1}{2}(19t+12+3i) : t+4, t+6, \cdots, 5t+4\} \cup \{t+4, t+6, \cdots, 5t+4\}$ and column-sums $\{\frac{1}{2}(19t+14+3i) : t+4, t+6, \cdots, 5t+4\}$ $\cup$ $\{t+3, t+5, \cdots, 5t+3\}$

Block 3	$S(i, i-1) = i-1$	$i = 5t+6, \cdots, 6t+6$
	$S(i, i+t) = 2i-6-4t$	$i = 5t+6, \cdots, 6t+6$

which uses the labels $\{5t+6, \cdots, 6t+5\} \cup \{6t+6, \cdots, 8t+6\}$. These entries produce row-sums $\{3i-7-4t : i = 5t+6, \cdots, 6t+6\}$ and column-sums $\{5t+5, \cdots, 6t+5\} \cup \{6t+6, 6t+8, \cdots, 8t+6\}$.

Block 4	$S(i,i) = i+t$	$i = 7t+7, \cdots, 8t+6$
	$S(i, i+t) = 2i-7-6t$	$i = 6t+7, \cdots, 8t+6$

which uses the labels $\{8t+7, \cdots, 9t+6\} \cup \{6t+7, 6t+9, \cdots, 8t+5\}$. These entries produce row-sums $\{8t+7, \cdots, 9t+6\} \cup \{6t+7, 6t+9, \cdots, 8t+5\}$ and column-sums $\{3i-7-5t : i = 7t+7, \cdots, 8t+6\} \cup \{16t+14, 16t+17, \cdots, 19t+11\}$.

Theorem 10 *A $SAM(8t+6,2,4t+3)$ exists for all $t \geq 0$.*

Proof. It is now an easy matter to check that the sets of row-sums and column-sums from the four blocks constitute the complete set $\{t+2,\cdots,17t+13\}$ providing the necessary SAM. The construction above works for all $t>0$ and we need to provide an example for $t=0$ to complete the proof. An example of $SAM(6,2,3)$ (which is simultaneously a $STAM(6,2,3)$ appears below. ■

*	7				*	**7**
	1			*	8	**9**
		*	*		2	**2**
		9	*	3		**12**
	*	4		*		**4**
5			6		*	**11**
5	**8**	**13**	**6**	**3**	**10**	**14/1**\

To illustrate the above construction, we show the square produced when $t=2$, namely a $SAM(22,2,11)$, which is the smallest order that clearly shows the pattern used in the blocks of the array

25																					
1	26																				
	2	27																			
		3	28																		
			4																		
				5	29																
					6																
						7	30														
							8														
								9	31												
									10												
										11	32										
											12										
												13	33								
													14								
														15			18				
															16			20			
																17			22		
																				19	
																					21
																				23	
																					24

Corollary 3 *A $STAM(8t+6, 2, 4t+3)$ exists for all $t \geq 0$.*

Proof. We can obtain a $STAM$ by a suitable permutation of the columns of the SAM S created above. Clearly, permuting the columns does not change either the row-sums or the column-sums. We will describe a permutation that moves the integer $t+1$ onto the back diagonal and two integers totalling $17t+14$ onto the main diagonal. These diagonal sums "bookend" the range $\{t+2, \cdots, 17t+13\}$ of row and column-sums in order to satisfy the requirements of a STAM. We accomplish this in a number of steps:

1. Move columns 2 to $t+2$ in Block I to between Blocks III and Block IV.

2. We then swap the first $2t$ columns of Block II and the last $2t$ columns of Block II.

3. Reverse the order of the entries on the diagonals of Block IV

4. In Block IV, permute the rows so that the entries in column j appear in rows $2j-8t-6$ and $2j-8t-7$ without changing the order of the entries within the column.

The net effect of the first three steps is:

Block 1	$f(i,i) = 9t+6+i$	$i = 1$
	$f(i,i+6t+4) = 9t+6+i$	$i = 2,\cdots,t+2$
	$f(i,i-1) = i-1$	$i = 2$
	$f(i,i+6t+3) = i-1$	$i = 3,\cdots,t+3$

Block 2	$f(i,i+t+1) = \frac{1}{2}(19t+14+i)$	$i = t+4,t+6,\cdots,3t+2$
	$f(i,i-t-1) = \frac{1}{2}(19t+14+i)$	$i = 3t+6,3t+8,\cdots,5t+4$
	$f(i,i-3t-3) = \frac{1}{2}(19t+14+i)$	$i = 3t+4$
	$f(i,i+t) = i-1$	$i = t+4,\cdots,3t+3$
	$f(i,i-t-2) = i-1$	$i = 3t+4,3t+5$
	$f(i,i-3t-4) = i-1$	$i = 3t+6,\cdots,5t+5$

Block 3	$f(i,i-t-2) = i-1$	$i = 5t+6,\cdots,6t+6$
	$f(i,i-1) = 2i-4t-6$	$i = 5t+6,\cdots,6t+6$

Block 4	$f(i,i) = 16t+13-i$	$i = 7t+7,\cdots,8t+6$
	$f(i,i+t) = 20t+19-2i$	$i = 6t+7,\cdots,7t+6$

The 4th step ensures that the only positive entry in Block 4 which occurs on the principal diagonal is in position $(8t+6,8t+6)$.

It can be seen that once step 4 is completed, the only positive entries appearing on the principal diagonal are $9t+7$ at position $(1,1)$ and $8t+7$ at position $(8t+6,8t+6)$. These entries sum to $17t+14$. Similarly, the only positive entry on the back diagonal is $t+1$ at position $(t+2,7t+5)$. Thus the set of row, column and diagonal sums is $t+1,...,17t+14$ as required for a $STAM$. ■

We illustrate the construction with the STAM created from the $SAM(22,2,11)$ shown above (the diagonals are highlighted with * for convenience):

25																					*
1	*																26			*	
		*															2	27	*		
			*															3	28		
				*													*		4		
					*		5	29								*					
						*		6							*						
							*		7	30				*							
								*		8			*								
					9	31			*			*									
						10				*	*										
	11	32								*	*										
		12							*			*									
			13	33				*					*								
				14			*							*							
						*					15			18	*						
					*							16			20	*					
				*									17			22	*				
			*															*		21	
		*																	*	24	
	*																			*	19
*																					23

4 Application of SAMs to Vertex Magic Labelings

While SAMs and STAMs are interesting in their own right, they provide a means of constructing vertex magic total labelings for many bipartite graphs. It will be recalled that a VMTL of a graph G with v vertices and e edges is an assignment λ of the integers $1, \cdots, v+e$ to the vertices and edges of G in such a way that for each vertex, if we sum of the label on a vertex and the labels on its incident edges we obtain the same constant. More precisely, there exists a constant k such that $\lambda(x) + \sum_{y \in N(x)} \lambda(xy) = k$ for all $x \in V(G)$.

In this section we consider bipartite graphs with partite sets of equal order - the vertex sets will be $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$. We can describe a VMTL for such a bipartite graph by giving its *label adjacency matrix* - an order $n+1$ matrix L whose entries are the labels; specifically

$$L(i,j) = \left\{ \begin{array}{ll} \lambda(x_i y_j) & x_i \text{ adjacent to } y_j, \quad i,j \leq n \\ 0 & x_i \text{ not adjacent to } y_j, \quad i,j \leq n \\ \lambda(x_i) & j = n+1, \quad i \leq n \\ \lambda(y_j) & i = n+1, \quad j \leq n \\ 0 & i = j = n+1 \end{array} \right\}$$

This is an $n \times n$ matrix containing the edge labels and bordered on the right side and bottom by a column and row containing the vertex labels. In creating a $SAM(n,d,r)$, we use the integers $1, \cdots, nd-r$ and we will construct a label adjacency matrix L beginning with the SAM as the $n \times n$ matrix of edge labels. As mentioned earlier, $(nd-r)(nd-r+1) = n(2a + 2n - 1)$ where a is the smallest of the row/column-sums. It follows that the row and column-sums are the consecutive numbers $a, \cdots, a+2n-1$ where

$$a = \frac{1}{2}\left(\frac{r(r-1)}{n} + n(d^2-2) - d(2r-1) + 1\right).$$

We now use the integers $nd-r+1, \cdots, nd-r+2n$ to fill the border cells, i. e. as labels for the vertices, assigning $nd-r+i$ to the cell whose corresponding row-sum or column-sum is $a+2n-i$. This will result in all of the rows and columns of L (except for the last row and column) having the same sum $a+2n+nd-r$. Now the entries in row j or column j represents the sum of the labels associated with vertex x_j or y_j. Since these sums are constant and the entries are the consecutive numbers $1, \cdots, v+e$, we have a VMTL for the graph whose adjacencies are determined by the distribution of the positive entries in the SAM.

Conversely, given a labeled bipartite graph with equal sized parts having the largest labels on the vertices, its label adjacency matrix will clearly be a SAM. So we can sum up this discussion in the following theorem:

Theorem 11 *A spanning subgraph G of $K_{m,m}$ has a VMTL in which the largest labels appear on the vertices if and only if there exists a SAM(m,d,r) whose zero entries correspond to the zero entries in an adjacency matrix for G.*

Figure 1 shows the labeled graph obtained from the $SAM(4,3,1)$ and the corresponding label adjacency matrix L shown below.

$$S = \begin{bmatrix} 0 & 8 & 5 & 0 \\ 9 & 1 & 0 & 4 \\ 7 & 0 & 2 & 10 \\ 0 & 6 & 11 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 8 & 5 & 0 & 19 \\ 9 & 1 & 0 & 4 & 18 \\ 7 & 0 & 2 & 10 & 13 \\ 0 & 6 & 11 & 3 & 12 \\ 16 & 17 & 14 & 15 & 0 \end{bmatrix}$$

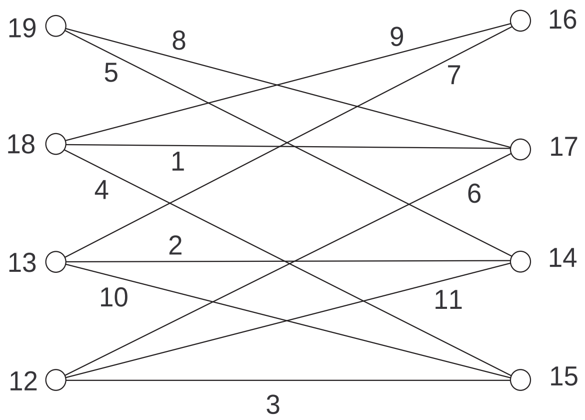

Figure 1: Labeling a subgraph of $K_{4,4}$ using a $SAM(4,3,1)$

We note that if the SAM has a row or column of 0s, then the vertex corresponding to this row or column will be an isolated vertex in the graph. According to Theorem 2, none other than the $SAM(2,2,2)$ has this property. Thus the only vertex-magic graph with an isolated vertex and having the largest labels on the vertices is the graph corresponding to this SAM, namely $P_3 \cup P_1$.

It is worth emphasising that where there are non-isomorphic SAMs with the same parameters, these correspond to different magic labelings of the same graph.

Because of the correspondence described above, each of the theorems in the previous sections now gives us information about the existence or non-existence of labeled bipartite graphs. We sum up some of these results as follows:

Theorem 12 *There exists a vertex-magic spanning subgraph G of $K_{m,m}$ with the largest labels on the vertices in these cases:*

1. *m is even and G is $r-$regular with r odd*
2. *m is even, the degree sequence of G is $\{r^{m-1}, r-1\}$*
3. *$m \equiv 6 \bmod 8$ and $e = \frac{3m}{2}$.*

Proof. These follow from Theorems 9 and 10 and Corollary 2. ■

The particular construction we used to create the $SAM(8t+6,2,4t+3)$ in Sec.3.3 corresponds to an interesting subgraph of $K_{m,m}$. The fact that there are never more than two entries in any row or column indicates that the graph will be a union of paths. In particular, the first block produces a single path, the second block produces a union of 4-vertex paths, the last 2 blocks produce a union of 3-vertex paths. More precisely, the graph is $P_{2t+5} \cup (2t+1)P_4 \cup (2t+1)P_3$. It would be interesting to know if there were a different construction for a $SAM(8t+6, 2, 4t+3)$ in order to label a different minimal spanning forest of $K_{m,m}$.

As a contrast to the above theorem, we have the following negative results.

Theorem 13 *No VMTL with the largest labels on the vertices exists for a spanning subgraph G of $K_{m,m}$ if*

1. *G has fewer than $\frac{3m}{2}$ edges*
2. *m is odd*
3. *m is even, G is r-regular with r even*
4. *$m = 2^t$ and $e \neq dm$ or $dm - 1$*

Proof. These follow from Theorems 1, 2, and 4 respectively. ■

The constructions we produced in Section 3 have just scratched the surface. What is the complete set of parameters for which sparse anti-magic squares will exist? For these parameters, what constructions (perhaps based on completely different techniques) exist? In particular we have not systematically investigated non-regular squares at all. Our impression is that it is almost certain that many anti-magic squares will exist for any combination of parameters not explicitly ruled out by a small collection of algebraic conditions such as those presented in Section 2.

In graph terms, we have shown that there is a large number of spanning subgraphs of $K_{m,m}$ that possess vertex magic total labelings, many of them possessing many different labelings.

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