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Note on edge irregular reflexive labelings of graphs

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Abstract

For a graph G, an edge labeling $f_e : E(G) \to \{1, 2, \dots, k_e\}$ and a vertex labeling $f_v : V(G) \to \{0, 2, 4, \dots, 2k_v\}$ are called total k-labeling, where $k = \max\{k_e, 2k_v\}$. The total k-labeling is called an *edge irregular reflexive k-labeling* of the graph G, if for every two different edges xy and $x'y'$ of G, one has

$$wt(xy) = f_v(x) + f_e(xy) + f_v(y) \neq wt(x'y') = f_v(x') + f_e(x'y') + f_v(y').$$

The minimum k for which the graph G has an edge irregular reflexive k-labeling is called the *reflexive edge strength* of G.

In this paper we determine the exact value of the reflexive edge strength for cycles, Cartesian product of two cycles and for join graphs of the path and cycle with $2K_2$.

Keywords: Edge irregular reflexive labeling; Reflexive edge strength; Cycles; Cartesian product of cycles

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V(G) \cup E(G)$ then we call the labeling *total labeling*. Thus, for an edge k-labeling $\varphi : E(G) \to \{1, 2, \dots, k\}$ the associated weight of a vertex is $w_\varphi(x) = \sum \varphi(xy)$, where the sum is over all vertices y adjacent to x.

Chartrand et al. in [1] introduced edge k-labeling φ of a graph G such that $w_\varphi(x) \neq w_\varphi(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. An excellent

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survey on the irregularity strength is Lehel [2]. For recent results see papers by Amar and Togni [3], Dimitz et al. [4], Gyárfás [5] and Nierhoff [6].

Motivated by these papers Bača et al. [7] introduced the concept of *edge irregular total k-labeling* as a labeling of the vertices and edges of G, $f : V \cup E \to \{1, 2, \dots, k\}$, such that the edge-weights $wt(xy) = f(x) + f(xy) + f(y)$ are different for all edges, i.e., $wt(xy) \neq wt(x'y')$ for all edges $xy, x'y' \in E(G)$ with $xy \neq x'y'$. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of the graph G, tes(G). Some results on the total edge irregularity strength can be found in [8–10,14] and [11].

Given an edge labeling $f_e : E(G) \to \{1, 2, \dots, k_e\}$ and a vertex labeling $f_v : V(G) \to \{0, 2, \dots, 2k_v\}$, then labeling f defined by $f(x) = f_v(x)$ if $x \in V(G)$ and $f(x) = f_e(x)$ if $x \in E(G)$ is a total k-labeling where $k = \max\{k_e, 2k_v\}$. The total k-labeling f is called an *edge irregular reflexive k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G one has $wt(xy) = f_v(x) + f_e(xy) + f_v(y) \neq wt(x'y') = f_v(x') + f_e(x'y') + f_v(y')$. The smallest value of k for which such labeling exists is called the *reflexive edge strength* of the graph G and is denoted by res(G). The concept of the edge irregular reflexive k-labeling was introduced by Ryan, Munasinghe and Tanna in [12].

In this paper we determine the exact value of the reflexive edge strength for cycles C_n, Cartesian product of the cycle C_n and the cycle C_3, and for join graphs of the path P_n and cycle C_n with $2K_2$.

Let us recall the following lemma proved in [12].

Lemma 1 (*[12]*)**.** *For every graph G,*

$$\mathrm{res}(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil & \text{if } |E(G)| \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1 & \text{if } |E(G)| \equiv 2, 3 \pmod 6. \end{cases}$$

The lower bound for res(G) follows from the fact that the minimal edge weight under an edge irregular reflexive labeling is 1 and the minimum of the maximal edge weights, that is $|E(G)|$, can be achieved only as the sum of 3 numbers from whose at least two are even.

1. Cycles and Cartesian product of cycles

First we will deal with edge irregular reflexive labeling of cycles.

Theorem 1. *For every positive integer n, $n \geq 3$*

$$\mathrm{res}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n \not\equiv 2, 3 \pmod 6, \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2, 3 \pmod 6. \end{cases}$$

Proof. Let $C_n = (x_1, x_2, \dots, x_n, x_1)$ be a cycle. It follows from Lemma 1 that

$$\mathrm{res}(C_n) \geq \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n \not\equiv 2, 3 \pmod 6, \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2, 3 \pmod 6. \end{cases}$$

Now we distinguish two cases.

Case 1: $n \equiv 3 \pmod 6$. From the lower bound we get res$(C_3) \geq 2$ and the corresponding edge irregular reflexive 2-labeling of C_3 is illustrated in Fig. 1.

For $n \geq 9$ we define the total $(n/3 + 1)$-labeling f of C_n in the following way

$$\begin{aligned} f(x_i) &= 2\left(\lceil \tfrac{i+1}{3} \rceil - 1\right) & i &= 1, 2, \dots, \tfrac{n+3}{2}, \\ f(x_{n-i+1}) &= 2\lceil \tfrac{i-1}{3} \rceil & i &= 1, 2, \dots, \tfrac{n-3}{2}, \\ f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) &= 2\lfloor \tfrac{n}{6} \rfloor & & \\ f(x_i x_{i+1}) &= 2\lceil \tfrac{i}{3} \rceil - 1 & i &= 1, 2, \dots, \tfrac{n+1}{2}, \\ f(x_{n-i} x_{n-i+1}) &= 2\lceil \tfrac{i+1}{3} \rceil & i &= 1, 2, \dots, \tfrac{n-5}{2}, \\ f(x_n x_1) &= 2. & & \end{aligned}$$

The vertices of C_n are labeled with even numbers.

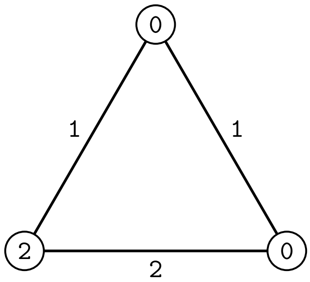

Fig. 1. The edge irregular reflexive 2-labeling of C_3.

The edge weights of the edges in C_n under the labeling f are the following. For $i = 1, 2, \dots, \frac{n+1}{2}$

$$\begin{aligned} wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 2\big(\lceil \tfrac{i+1}{3} \rceil - 1\big) + \big(2\lceil \tfrac{i}{3} \rceil - 1\big) \\ &\quad + 2\big(\lceil \tfrac{i+2}{3} \rceil - 1\big) = 2\big(\lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil + \lceil \tfrac{i+2}{3} \rceil\big) - 5 \\ &= 2(i+2) - 5 = 2i - 1. \end{aligned}$$

Thus, the corresponding edge weights are $1, 3, \dots, n$. Also

$$\begin{aligned} wt_f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) &= f(x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) + f(x_{\frac{n+5}{2}}) \\ &= 2\left(\left\lceil \frac{\frac{n+3}{2}+1}{3} \right\rceil - 1\right) + 2\lfloor \tfrac{n}{6} \rfloor + 2\left(\left\lceil \frac{\frac{n-3}{2}-1}{3} \right\rceil\right) \\ &= 2\left(\left\lceil \tfrac{n+5}{6} \right\rceil + \lfloor \tfrac{n}{6} \rfloor + \left\lceil \tfrac{n-5}{6} \right\rceil\right) - 2 = n - 1. \end{aligned}$$

For $i = 1, 2, \dots, (n-5)/2$

$$\begin{aligned} wt_f(x_{n-i} x_{n-i+1}) &= f(x_{n-i}) + f(x_{n-i} x_{n-i+1}) + f(x_{n-i+1}) \\ &= 2\lceil \tfrac{i}{3} \rceil + 2\lceil \tfrac{i+1}{3} \rceil + 2\lceil \tfrac{i-1}{3} \rceil = 2\big(\lceil \tfrac{i-1}{3} \rceil + \lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil\big) \\ &= 2i + 2. \end{aligned}$$

Thus, these edge weights are $4, 6, \dots, n-3$. Moreover,

$$wt_f(x_n x_1) = f(x_n) + f(x_n x_1) + f(x_1) = 0 + 2 + 0 = 2.$$

Thus the edge weights are distinct numbers from the set $\{1, 2, \dots, n\}$.

Case 2: $n \not\equiv 3 \pmod 6$. Define a total labeling f of C_n such that

$$\begin{aligned} f(x_i) &= 2\left(\lceil \tfrac{i+1}{3} \rceil - 1\right) && i = 1, 2, \dots, \lceil \tfrac{n}{2} \rceil, \\ f(x_{n-i+1}) &= 2\lceil \tfrac{i-1}{3} \rceil && i = 1, 2, \dots, \lfloor \tfrac{n}{2} \rfloor, \\ f(x_i x_{i+1}) &= 2\lceil \tfrac{i}{3} \rceil - 1 && i = 1, 2, \dots, \lceil \tfrac{n}{2} \rceil, \\ f(x_{n-i} x_{n-i+1}) &= 2\lceil \tfrac{i+1}{3} \rceil && i = 1, 2, \dots, \lfloor \tfrac{n}{2} \rfloor - 1, \\ f(x_n x_1) &= 2. \end{aligned}$$

Evidently the vertices of C_n are labeled with even numbers and the used labels are at most $\lceil n/3 \rceil$ if $n \not\equiv 2 \pmod 6$ or they are at most $(\lceil n/3 \rceil + 1)$ if $n \equiv 2 \pmod 6$.

The edge weights of the edges in C_n under the labeling f are the following.

For $i = 1, 2, \dots, \lceil n/2 \rceil - 1$

$$\begin{aligned} wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 2\big(\lceil \tfrac{i+1}{3} \rceil - 1\big) + \big(2\lceil \tfrac{i}{3} \rceil - 1\big) \\ &\quad + 2\big(\lceil \tfrac{i+2}{3} \rceil - 1\big) = 2\big(\lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil + \lceil \tfrac{i+2}{3} \rceil\big) - 5 \\ &= 2(i+2) - 5 = 2i - 1. \end{aligned}$$

Thus, for n even the edge weights are $1, 3, \ldots, n-3$ and for n odd the edge weights are $1, 3, \ldots, n-2$.

$$\begin{aligned} wt_f(x_{\lceil \frac{n}{2}\rceil} x_{\lceil \frac{n}{2}\rceil+1}) &= f(x_{\lceil \frac{n}{2}\rceil}) + f(x_{\lceil \frac{n}{2}\rceil} x_{\lceil \frac{n}{2}\rceil+1}) + f(x_{\lceil \frac{n}{2}\rceil+1}) \\ &= f(x_{\lceil \frac{n}{2}\rceil}) + f(x_{\lceil \frac{n}{2}\rceil} x_{\lceil \frac{n}{2}\rceil+1}) + f(x_{n-\lfloor \frac{n}{2}\rfloor+1}) \\ &= 2\left(\left\lceil \frac{\lceil \frac{n}{2}\rceil+1}{3}\right\rceil - 1\right) + \left(2\left\lceil \frac{\lceil \frac{n}{2}\rceil}{3}\right\rceil - 1\right) + 2\left(\left\lceil \frac{\lfloor \frac{n}{2}\rfloor-1}{3}\right\rceil\right) \\ &= 2\left(\left\lceil \frac{\lceil \frac{n}{2}\rceil-1}{3}\right\rceil + \left\lceil \frac{\lceil \frac{n}{2}\rceil}{3}\right\rceil + \left\lceil \frac{\lfloor \frac{n}{2}\rfloor+1}{3}\right\rceil\right) - 3 \\ &= 2\left(\lceil \tfrac{n}{2}\rceil + 1\right) - 3 = 2\lceil \tfrac{n}{2}\rceil - 1, \end{aligned}$$

which is equal to $(n-1)$ for n even and is equal to n for n odd.

For $i = 1, 2, \ldots, \lfloor n/2 \rfloor - 1$

$$\begin{aligned} wt_f(x_{n-i}x_{n-i+1}) &= f(x_{n-i}) + f(x_{n-i}x_{n-i+1}) + f(x_{n-i+1}) = 2\lceil \tfrac{i}{3}\rceil + 2\lceil \tfrac{i+1}{3}\rceil \\ &\quad + 2\lceil \tfrac{i-1}{3}\rceil = 2\big(\lceil \tfrac{i-1}{3}\rceil + \lceil \tfrac{i}{3}\rceil + \lceil \tfrac{i+1}{3}\rceil\big) = 2i + 2. \end{aligned}$$

Thus for n even, these edge weights are $4, 6, \ldots, n$ and for n odd these edge weights are $4, 6, \ldots, n-1$. Moreover,

$$wt_f(x_n x_1) = f(x_n) + f(x_n x_1) + f(x_1) = 0 + 2 + 0 = 2.$$

Combining the previous facts we get that weights of the edges are distinct numbers from the set $\{1, 2, \ldots, n\}$. This completes the proof. □

In the next theorem we give the exact values of reflexive edge strength of Cartesian product of a cycle C_n and C_3.

Theorem 2. *For every positive integer n, $n \geq 3$*

$$\text{res}(C_n \Box C_3) = 2n.$$

Proof. Let

$$\begin{aligned} V(C_n \Box C_3) &= \{x_i, y_i, z_i : i = 1, 2, \ldots, n\}, \\ E(C_n \Box C_3) &= \{x_i y_i, x_i z_i, y_i z_i, x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1} : i = 1, 2, \ldots, n\}, \end{aligned}$$

where indices are taken modulo n.

It follows from Lemma 1 that $\text{res}(C_n \Box C_3) \geq 2n$.

Now we distinguish two cases according to the parity of n.

Case 1: n is even. Define a total $2n$-labeling f as follows.

$$\begin{aligned} f(x_i) &= 0 && i = 1, 2, \ldots, n, \\ f(y_i) &= n && i = 1, 2, \ldots, n, \\ f(z_i) &= 2n && i = 1, 2, \ldots, n, \\ f(x_i x_{i+1}) &= i && i = 1, 2, \ldots, n-1, \\ f(x_n x_1) &= n, && \\ f(y_i y_{i+1}) &= n + i && i = 1, 2, \ldots, n-1, \\ f(y_n y_1) &= 2n, && \\ f(z_i z_{i+1}) &= n + i && i = 1, 2, \ldots, n-1, \\ f(z_n z_1) &= 2n, && \\ f(x_i y_i) &= i && i = 1, 2, \ldots, n, \\ f(y_i z_i) &= n + i && i = 1, 2, \ldots, n, \\ f(x_i z_i) &= i && i = 1, 2, \ldots, n. \end{aligned}$$

Evidently f is a $2n$-labeling. Now we calculate the edge weights.

$$\begin{aligned}
wt_f(x_ix_{i+1}) &= f(x_i)+f(x_ix_{i+1})+f(x_{i+1}) = 0+i+0 = i\\
&\text{for } i=1,2,\dots,n-1,\\
wt_f(x_1x_n) &= f(x_1)+f(x_1x_n)+f(x_n) = 0+n+0 = n,\\
wt_f(x_iy_i) &= f(x_i)+f(x_iy_i)+f(y_i) = 0+i+n = n+i\\
&\text{for } i=1,2,\dots,n,\\
wt_f(x_iz_i) &= f(x_i)+f(x_iz_i)+f(z_i) = 0+i+2n = 2n+i\\
&\text{for } i=1,2,\dots,n,\\
wt_f(y_iy_{i+1}) &= f(y_i)+f(y_iy_{i+1})+f(y_{i+1}) = n+(n+i)+n = 3n+i\\
&\text{for } i=1,2,\dots,n,\\
wt_f(y_iz_i) &= f(y_i)+f(y_iz_i)+f(z_i) = n+(n+i)+2n = 4n+i\\
&\text{for } i=1,2,\dots,n,\\
wt_f(z_iz_{i+1}) &= f(z_i)+f(z_iz_{i+1})+f(z_{i+1}) = 2n+(n+i)+2n = 5n+i\\
&\text{for } i=1,2,\dots,n.
\end{aligned}$$

Thus the set of edge weights is $\{1,2,\dots,6n\}$.

Case 2: n is odd. Define a total $2n$-labeling f in the following way.

$$\begin{aligned}
f(x_i) &= 0 && i=1,2,\dots,n,\\
f(y_i) &= n+1 && i=1,2,\dots,n-1,\\
f(y_n) &= n-1, &&\\
f(z_i) &= 2n && i=1,2,\dots,n,\\
f(x_ix_{i+1}) &= i && i=1,2,\dots,n-1,\\
f(x_nx_1) &= n, &&\\
f(y_iy_{i+1}) &= n+i && i=1,2,\dots,n-2,\\
f(y_ny_1) &= n+1, &&\\
f(y_{n-1}y_n) &= n+2, &&\\
f(z_iz_{i+1}) &= n+i && i=1,2,\dots,n-1,\\
f(z_1z_n) &= 2n, &&\\
f(x_iy_i) &= i && i=1,2,\dots,n-1,\\
f(x_ny_n) &= 2, &&\\
f(y_iz_i) &= n+i && i=1,2,\dots,n-1,\\
f(y_nz_n) &= n+2, &&\\
f(x_iz_i) &= i && i=1,2,\dots,n.
\end{aligned}$$

Also in this case the vertices are labeled with even numbers and the labels are at most $2n$. For the edge weights we have

$$\begin{aligned}
wt_f(x_ix_{i+1}) &= f(x_i)+f(x_ix_{i+1})+f(x_{i+1}) = 0+i+0 = i\\
&\text{for } i=1,2,\dots,n-1,\\
wt_f(x_1x_n) &= f(x_1)+f(x_1x_n)+f(x_n) = 0+n+0 = n,\\
wt_f(x_iy_i) &= f(x_i)+f(x_iy_i)+f(y_i) = 0+i+(n+1) = n+i+1\\
&\text{for } i=1,2,\dots,n-1,\\
wt_f(x_ny_n) &= f(x_n)+f(x_ny_n)+f(y_n) = 0+2+(n-1) = n+1,\\
wt_f(x_iz_i) &= f(x_i)+f(x_iz_i)+f(z_i) = 0+i+2n = 2n+i\\
&\text{for } i=1,2,\dots,n,
\end{aligned}$$

$$\begin{aligned}
wt_f(y_i y_{i+1}) &= f(y_i) + f(y_i y_{i+1}) + f(y_{i+1}) = (n+1) + (n+i) + (n+1) \\
&= 3n + i + 2 \\
&\qquad \text{for } i = 1, 2, \dots, n-2, \\
wt_f(y_{n-1} y_n) &= f(y_{n-1}) + f(y_{n-1} y_n) + f(y_n) = (n+1) + (n+2) + (n-1) \\
&= 3n + 2, \\
wt_f(y_n y_1) &= f(y_n) + f(y_n y_1) + f(y_1) = (n-1) + (n+1) + (n+1) \\
&= 3n + 1, \\
wt_f(y_i z_i) &= f(y_i) + f(y_i z_i) + f(z_i) = (n+1) + (n+i) + 2n = 4n + i + 1 \\
&\qquad \text{for } i = 1, 2, \dots, n-1, \\
wt_f(y_n z_n) &= f(y_n) + f(y_n z_n) + f(z_n) = (n-1) + (n+2) + 2n = 4n + 1, \\
wt_f(z_i z_{i+1}) &= f(z_i) + f(z_i z_{i+1}) + f(z_{i+1}) = 2n + (n+i) + 2n = 5n + i \\
&\qquad \text{for } i = 1, 2, \dots, n.
\end{aligned}$$

Hence the edge weights are distinct numbers from the set $\{1, 2, \dots, 6n\}$. □

2. Join of graphs

The join $G \oplus H$ of the disjoint graphs G and H is the graph $G \cup H$ together with all the edges joining vertices of $V(G)$ and vertices of $V(H)$.

The join of a cycle C_n, $n \geq 3$, and a complete graph K_1 is a graph known as a wheel W_n. The join of a path P_n, $n \geq 2$, and a complete graph K_1 is called a fan F_n. Tanna et al. [13] have proved that for $n \geq 3$,

$$\operatorname{res}(W_n) = \begin{cases} 4 & \text{if } n = 3, \\ \lceil \frac{2n}{3} \rceil & \text{if } n \equiv 0, 2 \pmod 3 \text{ and } n \geq 5, \\ \lceil \frac{2n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3 \end{cases}$$

and for $n \geq 3$,

$$\operatorname{res}(F_n) = \begin{cases} 3 & \text{if } n = 3, \\ 4 & \text{if } n = 4, \\ \lceil \frac{2n}{3} \rceil & \text{if } n \geq 5. \end{cases}$$

In the next two theorems we will deal with the join of a path or a cycle with $2K_1$.

Theorem 3. *For every positive integer n, $n \geq 2$*

$$\operatorname{res}(P_n \oplus (2K_1)) = \begin{cases} 3 & \text{if } n = 2, \\ n+1 & \text{if } n \text{ is odd, } n \geq 3, \\ n & \text{if } n \text{ is even, } n \geq 4. \end{cases}$$

Proof. Let

$$V(P_n \oplus (2K_1)) = \{x_i : i = 1, 2, \dots, n\} \cup \{y, z\},$$
$$E(P_n \oplus (2K_1)) = \{x_i x_{i+1} : i = 1, 2, \dots, n-1\} \cup \{y x_i, z x_i : i = 1, 2, \dots, n\}.$$

It follows from Lemma 1 that

$$\operatorname{res}(P_n \oplus (2K_1)) \geq \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

However, it is easy to see that $\operatorname{res}(P_2 \oplus (2K_1)) \geq 3$. The corresponding 3-labeling for $P_2 \oplus (2K_1)$ is illustrated in Fig. 2.

For $n \geq 3$ we distinguish two cases according to the parity of n.

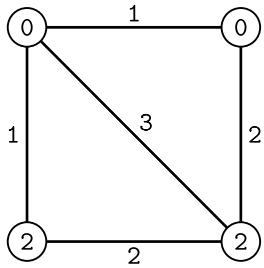

Fig. 2. The edge irregular reflexive 3-labeling of $P_2 \oplus (2K_1)$.

Case 1: n is even. Then define a total n-labeling f as follows.

$$
\begin{aligned}
f(x_i) &= 0 && i = 1, 2, \dots, \tfrac{n}{2},\\
f(x_{\frac{n}{2}+1}) &= n - 2, &&\\
f(x_i) &= n && i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n,\\
f(y) &= 0, &&\\
f(z) &= n, &&\\
f(x_i x_{i+1}) &= \tfrac{n}{2} + i && i = 1, 2, \dots, \tfrac{n}{2} - 1,\\
f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) &= 2, &&\\
f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) &= 3, &&\\
f(x_i x_{i+1}) &= i - \tfrac{n}{2} && i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n - 1,\\
f(y x_i) &= i && i = 1, 2, \dots, \tfrac{n}{2},\\
f(y x_{\frac{n}{2}+1}) &= 3, &&\\
f(y x_i) &= i - \tfrac{n}{2} && i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n,\\
f(z x_i) &= \tfrac{n}{2} + i && i = 1, 2, \dots, \tfrac{n}{2},\\
f(z x_{\frac{n}{2}+1}) &= \tfrac{n}{2} + 2, &&\\
f(z x_i) &= i - 1 && i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n.
\end{aligned}
$$

For the edge weights we get

$$
\begin{aligned}
wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 0 + (\tfrac{n}{2} + i) + 0 = \tfrac{n}{2} + i\\
&\qquad \text{for } i = 1, 2, \dots, \tfrac{n}{2} - 1,\\
wt_f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) &= f(x_{\frac{n}{2}}) + f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = 0 + 2 + (n - 2) = n,\\
wt_f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) &= f(x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = (n - 2) + 3 + n\\
&= 2n + 1,\\
wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = n + (i - \tfrac{n}{2}) + n = \tfrac{3n}{2} + i\\
&\qquad \text{for } i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n - 1,\\
wt_f(y x_i) &= f(y) + f(y x_i) + f(x_i) = 0 + i + 0 = i\\
&\qquad \text{for } i = 1, 2, \dots, \tfrac{n}{2},\\
wt_f(y x_{\frac{n}{2}+1}) &= f(y) + f(y x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = 0 + 3 + (n - 2) = n + 1,\\
wt_f(y x_i) &= f(y) + f(y x_i) + f(x_i) = 0 + (i - \tfrac{n}{2}) + n = \tfrac{n}{2} + i\\
&\qquad \text{for } i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n,
\end{aligned}
$$

$$wt_f(zx_i) = f(z) + f(zx_i) + f(x_i) = n + (\tfrac{n}{2} + i) + 0 = \tfrac{3n}{2} + i \quad \text{for } i = 1, 2, \dots, \tfrac{n}{2},$$
$$wt_f(zx_{\frac{n}{2}+1}) = f(z) + f(zx_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = n + (\tfrac{n}{2} + 2) + (n-2) = \tfrac{5n}{2},$$
$$wt_f(zx_i) = f(z) + f(zx_i) + f(x_i) = n + (i-1) + n = 2n + i - 1 \quad \text{for } i = \tfrac{n}{2}+2, \tfrac{n}{2}+3, \dots, n.$$

Thus the set of edge weights is $\{1, 2, \dots, 3n-1\}$.

Case 2: n is odd. Define a total $(n+1)$-labeling f in the following way.

$$f(x_i) = 0 \qquad i = 1, 2, \dots, \tfrac{n+1}{2},$$
$$f(x_{\frac{n+3}{2}}) = n-1,$$
$$f(x_i) = n+1 \qquad i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n,$$
$$f(y) = 0,$$
$$f(z) = n+1,$$
$$f(x_i x_{i+1}) = \tfrac{n+1}{2} + i \qquad i = 1, 2, \dots, \tfrac{n-1}{2},$$
$$f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) = 2,$$
$$f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) = 2,$$
$$f(x_i x_{i+1}) = i - \tfrac{n+3}{2} \qquad i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n-1,$$
$$f(yx_i) = i \qquad i = 1, 2, \dots, \tfrac{n+1}{2},$$
$$f(yx_{\frac{n+3}{2}}) = 3,$$
$$f(yx_i) = i - \tfrac{n+1}{2} \qquad i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n,$$
$$f(zx_i) = \tfrac{n-1}{2} + i \qquad i = 1, 2, \dots, \tfrac{n+1}{2},$$
$$f(zx_{\frac{n+3}{2}}) = \tfrac{n+1}{2},$$
$$f(zx_i) = i - 3 \qquad i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n.$$

Evidently, the vertices are labeled with even numbers and the label of every element is at most $n+1$.

Now we will calculate the weights of the edges.

$$wt_f(x_i x_{i+1}) = f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 0 + (\tfrac{n+1}{2} + i) + 0 = \tfrac{n+1}{2} + i \quad \text{for } i = 1, 2, \dots, \tfrac{n-1}{2},$$
$$wt_f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) = f(x_{\frac{n+1}{2}}) + f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = 0 + 2 + (n-1) = n+1,$$
$$wt_f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) = f(x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) + f(x_{\frac{n+5}{2}}) = (n-1) + 2 + (n+1) = 2n+2,$$
$$wt_f(x_i x_{i+1}) = f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = (n+1) + (i - \tfrac{n+3}{2}) + (n+1) = \tfrac{3n+1}{2} + i \quad \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n-1,$$
$$wt_f(yx_i) = f(y) + f(yx_i) + f(x_i) = 0 + i + 0 = i \quad \text{for } i = 1, 2, \dots, \tfrac{n+1}{2},$$
$$wt_f(yx_{\frac{n+3}{2}}) = f(y) + f(yx_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = 0 + 3 + (n-1) = n+2,$$
$$wt_f(yx_i) = f(y) + f(yx_i) + f(x_i) = 0 + (i - \tfrac{n+1}{2}) + (n+1) = \tfrac{n+1}{2} + i \quad \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n,$$

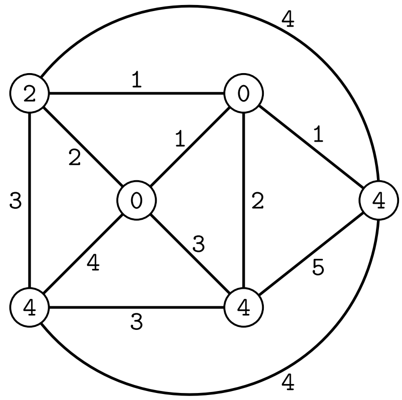

Fig. 3. The edge irregular reflexive 5-labeling of $C_4 \oplus (2K_1)$.

$$
\begin{aligned}
wt_f(zx_i) &= f(z) + f(zx_i) + f(x_i) = (n+1) + (\tfrac{n-1}{2} + i) + 0 \\
&= \tfrac{3n+1}{2} + i \\
&\quad \text{for } i = 1, 2, \dots, \tfrac{n+1}{2}, \\
wt_f(zx_{\frac{n+3}{2}}) &= f(z) + f(zx_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = (n+1) + \tfrac{n+1}{2} + (n-1) \\
&= \tfrac{5n+1}{2}, \\
wt_f(zx_i) &= f(z) + f(zx_i) + f(x_i) = (n+1) + (i-3) + (n+1) \\
&= 2n + i - 1 \\
&\quad \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n.
\end{aligned}
$$

It is easy to check that the edge weights are distinct consecutive integers $\{1, 2, \dots, 3n-1\}$. This concludes the proof. □

Theorem 4. *For every positive integer n, $n \geq 3$*

$$
\operatorname{res}(C_n \oplus (2K_1)) = \begin{cases} 5 & \text{if } n = 4, \\ n+1 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}. \end{cases}
$$

Proof. Let

$$
V(C_n \oplus (2K_1)) = \{x_i : i = 1, 2, \dots, n\} \cup \{y, z\},
$$
$$
E(C_n \oplus (2K_1)) = \{x_i x_{i+1}, yx_i, zx_i : i = 1, 2, \dots, n\},
$$

where the indices are taken modulo n. It follows from Lemma 1 that

$$
\operatorname{res}(C_n \oplus (2K_1)) \geq \begin{cases} n & \text{if } n \text{ is even, } n \geq 6, \\ n+1 & \text{if } n \text{ is odd.} \end{cases}
$$

Let us consider two cases according to the parity of n.

Case 1: n is even. It is easy to see that $\operatorname{res}(C_4 \oplus (2K_1)) \geq 5$. The corresponding 5-labeling for $C_4 \oplus (2K_1)$ is illustrated in Fig. 3.

For $n \geq 6$ we define n-labeling f of $C_n \oplus (2K_1)$ such that

$$\begin{aligned}
f(x_i) &= 0 && i = 1, 2, \dots, \tfrac{n}{2} - 1,\\
f(x_{\frac{n}{2}}) &= n - 4,\\
f(x_{\frac{n}{2}+1}) &= n - 2,\\
f(x_{\frac{n}{2}+2}) &= n - 2,\\
f(x_i) &= n && i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n,\\
f(y) &= 0,\\
f(z) &= n,\\
f(x_i x_{i+1}) &= \tfrac{n}{2} + i - 1 && i = 1, 2, \dots, \tfrac{n}{2} - 2,\\
f(x_{\frac{n}{2}-1} x_{\frac{n}{2}}) &= 2,\\
f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) &= 6,\\
f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) &= 5,\\
f(x_{\frac{n}{2}+2} x_{\frac{n}{2}+3}) &= 4,\\
f(x_i x_{i+1}) &= i - \tfrac{n}{2} && i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n - 1,\\
f(x_n x_1) &= n - 1,\\
f(y x_i) &= i && i = 1, 2, \dots, \tfrac{n}{2} - 1,\\
f(y x_{\frac{n}{2}}) &= 3,\\
f(y x_{\frac{n}{2}+1}) &= 2,\\
f(y x_{\frac{n}{2}+2}) &= 3,\\
f(y x_i) &= i - \tfrac{n}{2} - 1 && i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n,\\
f(z x_i) &= \tfrac{n}{2} + i - 1 && i = 1, 2, \dots, \tfrac{n}{2} - 1,\\
f(z x_{\frac{n}{2}}) &= \tfrac{n}{2} + 4,\\
f(z x_{\frac{n}{2}+1}) &= \tfrac{n}{2} + 3,\\
f(z x_{\frac{n}{2}+2}) &= \tfrac{n}{2} + 4,\\
f(z x_i) &= i && i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n.
\end{aligned}$$

For the edge weights we get

$$\begin{aligned}
wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 0 + (\tfrac{n}{2} + i - 1) + 0\\
&= \tfrac{n}{2} + i - 1\\
&\qquad \text{for } i = 1, 2, \dots, \tfrac{n}{2} - 2,\\
wt_f(x_{\frac{n}{2}-1} x_{\frac{n}{2}}) &= f(x_{\frac{n}{2}-1}) + f(x_{\frac{n}{2}-1} x_{\frac{n}{2}}) + f(x_{\frac{n}{2}}) = 0 + 2 + (n - 4) = n - 2,\\
wt_f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) &= f(x_{\frac{n}{2}}) + f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = (n - 4) + 6 + (n - 2)\\
&= 2n,\\
wt_f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) &= f(x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = (n - 2) + 5\\
&\quad + (n - 2) = 2n + 1,\\
wt_f(x_{\frac{n}{2}+2} x_{\frac{n}{2}+3}) &= f(x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2} x_{\frac{n}{2}+3}) + f(x_{\frac{n}{2}+3}) = (n - 2) + 4 + n\\
&= 2n + 2,\\
wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = n + (i - \tfrac{n}{2}) + n = \tfrac{3n}{2} + i\\
&\qquad \text{for } i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n - 1,
\end{aligned}$$

$$\begin{aligned}
wt_f(x_nx_1) &= f(x_n) + f(x_nx_1) + f(x_1) = n + (n-1) + 0 = 2n-1,\\
wt_f(yx_i) &= f(y) + f(yx_i) + f(x_i) = 0 + i + 0 = i\\
&\quad \text{for } i = 1, 2, \dots, \tfrac{n}{2}-1,\\
wt_f(yx_{\frac{n}{2}}) &= f(y) + f(yx_{\frac{n}{2}}) + f(x_{\frac{n}{2}}) = 0 + 3 + (n-4) = n-1,\\
wt_f(yx_{\frac{n}{2}+1}) &= f(y) + f(yx_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = 0 + 2 + (n-2) = n,\\
wt_f(yx_{\frac{n}{2}+2}) &= f(y) + f(yx_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = 0 + 3 + (n-2) = n+1,\\
wt_f(yx_i) &= f(y) + f(yx_i) + f(x_i) = 0 + (i - \tfrac{n}{2} - 1) + n = \tfrac{n}{2} + i - 1\\
&\quad \text{for } i = \tfrac{n}{2}+3, \tfrac{n}{2}+4, \dots, n,\\
wt_f(zx_i) &= f(z) + f(zx_i) + f(x_i) = n + (\tfrac{n}{2} + i - 1) + 0 = \tfrac{3n}{2} + i - 1\\
&\quad \text{for } i = 1, 2, \dots, \tfrac{n}{2}-1,\\
wt_f(zx_{\frac{n}{2}}) &= f(z) + f(zx_{\frac{n}{2}}) + f(x_{\frac{n}{2}}) = n + (\tfrac{n}{2} + 4) + (n-4) = \tfrac{5n}{2},\\
wt_f(zx_{\frac{n}{2}+1}) &= f(z) + f(zx_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = n + (\tfrac{n}{2} + 3) + (n-2)\\
&= \tfrac{5n}{2} + 1,\\
wt_f(zx_{\frac{n}{2}+2}) &= f(z) + f(zx_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = n + (\tfrac{n}{2} + 4) + (n-2)\\
&= \tfrac{5n}{2} + 2,\\
wt_f(zx_i) &= f(z) + f(zx_i) + f(x_i) = n + i + n = 2n + i\\
&\quad \text{for } i = \tfrac{n}{2}+3, \tfrac{n}{2}+4, \dots, n.
\end{aligned}$$

It is easy to get that the edge weights are $\{1, 2, \dots, 3n\}$.

Case 2: n is odd. Define a total $(n+1)$-labeling f as follows.

$$\begin{aligned}
f(x_i) &= 0 && i = 1, 2, \dots, \tfrac{n+1}{2},\\
f(x_{\frac{n+3}{2}}) &= n-1, &&\\
f(x_i) &= n+1 && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n,\\
f(y) &= 0, &&\\
f(z) &= n+1, &&\\
f(x_ix_{i+1}) &= \tfrac{n+1}{2} + i && i = 1, 2, \dots, \tfrac{n-1}{2},\\
f(x_{\frac{n+1}{2}}x_{\frac{n+3}{2}}) &= 2, &&\\
f(x_{\frac{n+3}{2}}x_{\frac{n+5}{2}}) &= 3, &&\\
f(x_ix_{i+1}) &= i - \tfrac{n+1}{2} && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n-1,\\
f(x_nx_1) &= n+1, &&\\
f(yx_i) &= i && i = 1, 2, \dots, \tfrac{n+1}{2},\\
f(yx_{\frac{n+3}{2}}) &= 3, &&\\
f(yx_i) &= i - \tfrac{n+1}{2} && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n,\\
f(zx_i) &= \tfrac{n-1}{2} + i && i = 1, 2, \dots, \tfrac{n+1}{2},\\
f(zx_{\frac{n+3}{2}}) &= \tfrac{n+3}{2}, &&\\
f(zx_i) &= i - 2 && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n.
\end{aligned}$$

Thus the vertices are labeled with even numbers 0, $n-1$ or $n+1$.

For the edge weights we get the following.

$$\begin{aligned}
wt_f(x_ix_{i+1}) &= f(x_i) + f(x_ix_{i+1}) + f(x_{i+1}) = 0 + (\tfrac{n+1}{2} + i) + 0 = \tfrac{n+1}{2} + i\\
&\quad \text{for } i = 1, 2, \dots, \tfrac{n-1}{2},
\end{aligned}$$

$$
\begin{aligned}
wt_f(x_{\frac{n+1}{2}}x_{\frac{n+3}{2}}) &= f(x_{\frac{n+1}{2}}) + f(x_{\frac{n+1}{2}}x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = 0 + 2 + (n-1) \\
&= n+1, \\
wt_f(x_{\frac{n+3}{2}}x_{\frac{n+5}{2}}) &= f(x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}x_{\frac{n+5}{2}}) + f(x_{\frac{n+5}{2}}) = (n-1) + 3 \\
&\quad + (n+1) = 2n+3, \\
wt_f(x_i x_{i+1}) &= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = (n+1) + (i - \tfrac{n+1}{2}) \\
&\quad + (n+1) = \tfrac{3n+3}{2} + i \\
&\qquad \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n-1, \\
wt_f(x_n x_1) &= f(x_n) + f(x_n x_1) + f(x_1) = (n+1) + (n+1) + 0 \\
&= 2n+2, \\
wt_f(y x_i) &= f(y) + f(y x_i) + f(x_i) = 0 + i + 0 = i \\
&\qquad \text{for } i = 1, 2, \dots, \tfrac{n+1}{2}, \\
wt_f(y x_{\frac{n+3}{2}}) &= f(y) + f(y x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = 0 + 3 + (n-1) = n+2, \\
wt_f(y x_i) &= f(y) + f(y x_i) + f(x_i) = 0 + (i - \tfrac{n+1}{2}) + (n+1) = \tfrac{n+1}{2} + i \\
&\qquad \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n, \\
wt_f(z x_i) &= f(z) + f(z x_i) + f(x_i) = (n+1) + (\tfrac{n-1}{2} + i) + 0 \\
&= \tfrac{3n+1}{2} + i \\
&\qquad \text{for } i = 1, 2, \dots, \tfrac{n+1}{2}, \\
wt_f(z x_{\frac{n+3}{2}}) &= f(z) + f(z x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = (n+1) + \tfrac{n+3}{2} + (n-1) \\
&= \tfrac{5n+3}{2}, \\
wt_f(z x_i) &= f(z) + f(z x_i) + f(x_i) = (n+1) + (i-2) + (n+1) \\
&= 2n + i \\
&\qquad \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n.
\end{aligned}
$$

Evidently, the edge weights are distinct numbers from the set $\{1, 2, \dots, 3n\}$. □

3. Conclusion

In this paper we determined the precise value of the reflexive edge strength for cycles C_n, $n \geq 3$, for the Cartesian product $C_n \square C_3$, $n \geq 3$, and for join graphs $P_n \oplus (2K_1)$, $n \geq 2$, and $C_n \oplus (2K_1)$, $n \geq 3$.

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