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ON THE POWER DOMINATION NUMBER OF DE BRUIJN AND KAUTZ DIGRAPHS

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Abstract. Let $G = (V, A)$ be a directed graph without parallel arcs, and let $S \subseteq V$ be a set of vertices. Let the sequence $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$ be defined as follows: S_1 is obtained from S_0 by adding all out-neighbors of vertices in S_0. For $k \geqslant 2$, S_k is obtained from S_{k-1} by adding all vertices w such that for some vertex $v \in S_{k-1}$, w is the unique out-neighbor of v in $V \setminus S_{k-1}$. We set $M(S) = S_0 \cup S_1 \cup \cdots$, and call S a *power dominating set* for G if $M(S) = V(G)$. The minimum cardinality of such a set is called the *power domination number* of G. In this paper, we determine the power domination numbers of de Bruijn and Kautz digraphs.

1. Introduction

Let $G = (V, A)$ be a directed graph. For a vertex $i \in V$ let $N^{\text{in}}(i)$ and $N^{\text{out}}(i)$ denote its in- and out-neighborhood, respectively, i.e.,

$$N^{\text{in}}(i) = \{j \in V \ : \ (j, i) \in A\}, \qquad N^{\text{out}}(i) = \{j \in V \ : \ (i, j) \in A\}.$$

For a node set S, we use the corresponding notation

$$N^{\text{in}}(S) = \bigcup_{i \in S} N^{\text{in}}(i), \qquad N^{\text{out}}(S) = \bigcup_{i \in S} N^{\text{out}}(i).$$

Let G be a directed graph and S a subset of its vertices. Then we denote the set monitored by S with $M(S)$ and define it as $M(S) = S_0 \cup S_1 \cup \cdots$ where the sequence $S_0, S_1, \ldots$ of vertex sets is defined by $S_0 = S$, $S_1 = N^{\text{out}}(S)$, and

$$S_k = S_{k-1} \cup \left\{w \ : \ \{w\} = N^{\text{out}}(v) \cap (V \setminus S_{k-1}) \ \text{ for some } v \in S_{k-1}\right\}.$$

A set S is called a *power dominating set* of G if $M(S) = V(G)$ and the minimum cardinality of such a set is called the *power domination number* denoted as $\gamma_p(G)$.

The undirected version of the power domination problem was introduced in [11]. The problem was inspired by a problem in electric power systems concerning the placements of phasor measurement units. The directed version of the power domination problem was introduced as a natural extension in [1] where a linear time algorithm was presented for digraphs whose underlying undirected graph has bounded treewidth. Good literature reviews on the power domination problem can be found in [7, 8, 18]

A closely related concept is zero forcing which was introduced for undirected graphs by the *AIM Minimum Rank – Special Graphs Work Group* in [2] as a tool to bound the minimum rank of matrices associated with the graph G. This notion was extended to digraphs in [4] with the same motivation. For a red/blue coloring of the vertex set of a digraph G consider the following color-change rule: a red vertex w is converted to blue if it is the only red out-neighbor of some vertex u. We say u forces w and denote this by $u \to w$. A vertex set $S \subseteq V$ is called *zero-forcing* if, starting with the vertices in S blue and the vertices in the complement $V \setminus S$ red, all the vertices can be converted to blue by repeatedly applying the color-change rule. The minimum cardinality of

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a zero-forcing set for the digraph G is called the *zero-forcing number* of G, denoted by $Z(G)$. Since its introduction the zero-forcing number has been studied for its own sake as an interesting graph invariant [3, 5, 6, 10, 16]. In [12], the *propagation time* of a graph is introduced as the number of steps it takes for a zero forcing set to turn the entire graph blue. Physicists have independently studied the zero forcing parameter, referring to it as the *graph infection number*, in conjunction with the control of quantum systems [17].

Recently, Dong *et al.* (2015) [9] investigated the domination number of generalized de Bruijn and Kautz digraphs. Kuo *et al.*(2015) [15] gave an upper bound for power domination in undirected de Bruijn and Kautz graphs. In this paper we study the directed versions, i.e., the zero forcing number and power domination number of de Bruijn and Kautz digraphs. Due to their attractive connectivity features these digraphs have been widely studied as a topology for interconnection networks [13], and some generalizations of these digraphs were proposed [14].

Section 2 contains some notation and precise statements of our main result. In Section 3 we determine the power domination number and zero forcing number for de Bruijn digraphs. In Section 4 we determine the power domination number and zero forcing number for Kautz digraphs.

2. Notations and main result

We give an interpretation of the power domination problem and zero forcing problem as a set cover problem. We call a vertex set W *strongly critical* if there is no vertex in G which has exactly one out neighbor in W. We call a vertex set W *weakly critical* if there is no vertex outside W which has exactly one out-neighbor in W. If W is strongly (weakly) critical, but no proper subset of W is strongly (weakly) critical, then we call W *minimal strongly (weakly) critical.*

Note that a vertex set S is a zero forcing set if and only if $S \cap W \neq \emptyset$ for every strongly critical set $W \subseteq V$. Similarly, S is a power dominating set if and only if $N^{\mathrm{out}}(S) \cap W \neq \emptyset$ for every weakly critical set $W \subseteq V$, and therefore

$$\begin{aligned}
Z(G) &= \min\{|S| \ :\ S \cap W \neq \emptyset \text{ for every strongly critical set } W \subseteq V\},\\
\gamma_p(G) &= \min\left\{|S| \ :\ (S \cup N_G^{\mathrm{out}}(S)) \cap W \neq \emptyset \text{ for every weakly critical set } W \subseteq V\right\}.
\end{aligned}$$

For an integer $d \geqslant 2$, let $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$ denote the cyclic group of order d. The de Bruijn digraph, denoted $\mathrm{B}(d,n)$, with parameters $d \geqslant 2$ and $n \geqslant 2$ is defined to be the graph $G = (V, A)$ with vertex set V and arcs set A where

$$\begin{aligned}
V &= \mathbb{Z}_d^n = \{(a_1, \dots, a_n) \ :\ a_i \in \mathbb{Z}_d \text{ for } i = 1, \dots, n\},\\
A &= \{((a_1, a_2, \dots, a_n), (a_2, \dots, a_n, b)) \ :\ (a_1, a_2, \dots, a_n) \in V,\ b \in \mathbb{Z}_d\}.
\end{aligned}$$

The Kautz digraph, denoted $\mathrm{K}(d,n)$, with parameters $d \geqslant 2$ and $n \geqslant 2$ is defined to be the graph $G = (V, A)$ with vertex set V and arcs set A where

$$\begin{aligned}
V &= \{(a_1, \dots, a_n) \ :\ a_i \in \mathbb{Z}_{d+1},\ a_i \neq a_{i+1}\}\\
A &= \{((a_1, a_2, \dots, a_n), (a_2, \dots, a_n, b)) \ :\ (a_1, a_2, \dots, a_n) \in V,\ b \in \mathbb{Z}_{d+1} \setminus \{a_n\}\}.
\end{aligned}$$

Our main results are the following theorems.

Theorem 1. *Let G be a de Bruijn digraph with parameters $d, n \geqslant 2$. Then the zero forcing number and power domination number of G are $(d-1)d^{n-1}$ and $(d-1)d^{n-2}$, respectively.*

Theorem 2. *Let G be a Kautz digraph with parameters $d \geqslant 2$ and $n \geqslant 3$. Then, the zero forcing number and power domination number of G are $(d-1)(d+1)d^{n-2}$ and $(d-1)(d+1)d^{n-3}$, respectively.*

3. The power domination number of de Bruijn digraphs

In this section we prove Theorem 1. Let us define the sets

$$X(a_1,\dots,a_{n-1}) = \{(a_1,\dots,a_{n-1},\alpha)\ :\ \alpha\in\mathbb{Z}_d\}$$

which partition the vertex set V into d^{n-1} sets of size d. Furthermore, $N^{\text{out}}(v) = X(a_1,\dots,a_{n-1})$ for every vertex v of the form $(\alpha,a_1,a_2,\dots,a_{n-1})$.

Lemma 1. *Let G be a de Bruijn digraph with parameters d and n. Then $Z(G)\geqslant(d-1)d^{n-1}$.*

Proof. Every 2-element subset of each of the sets $X(a_1,\dots,a_{n-1})$ is strongly critical, and therefore, any zero forcing set S needs to intersect $X(a_1,\dots,a_{n-1})$ in at least $d-1$ elements, and the result follows. □

Lemma 2. *Let G be a de Bruijn digraph with parameters d and n. Then $Z(G)\leqslant(d-1)d^{n-1}$.*

Proof. Consider the vertex set $S=\{(a_1,\dots,a_{n-1},a_n)\in V\ :\ a_1\neq a_n\}$. To show that S is a zero forcing set, it is sufficient to verify that each vertex $v=(a_1,\dots,a_{n-1},a_n)$ is either in S or is the unique out-neighbor in $V\setminus S$ for some vertex w. If $a_1\neq a_n$, then $v\in S$. If $a_1=a_n$, then for any vertex of the form $w=(\beta,a_1,\dots,a_{n-1})$, v is the only neighbor of w in $V\setminus S$. □

Lemmas 1 and 2 imply the first statement of Theorem 1. In order to prove the second part of this theorem we recall that $S\subseteq V$ is a power dominating set if and only if $S\cup N^{\text{out}}(S)$ intersects every weakly critical set. In particular, it is necessary that $|(S\cup N^{\text{out}}(S))\cap X(a_1,\dots,a_{n-1})|\geqslant d-1$ for every $(a_1,\dots,a_{d-1})\in\mathbb{Z}_d^{n-1}$.

Lemma 3. *Let G be a de Bruijn graph with parameters d and n. Then every power dominating set has size at least $(d-1)d^{n-2}$.*

Proof. Let S be a power-dominating set, suppose $|S|<(d-1)d^{n-2}$ and set $Z=S\cup N^{\text{out}}(S)$. We have

$$(Z\setminus S)\cap X(a_1,\dots,a_{n-1})\neq\emptyset\implies X(a_1,\dots,a_{n-1})\subseteq Z.$$

For $k=0,1,\dots,d$, we set $\alpha_k=\#\{(a_1,\dots,a_{n-1})\ :\ |S\cap X(a_1,\dots,a_{n-1})|=k\}$, and get

$$|S| = \alpha_1+2\alpha_2+\dots+(d-1)\alpha_{d-1}+d\alpha_d.$$

Now let $I_0=\{(a_1,\dots,a_{n-1})\ :\ X(a_1,\dots,a_{n-1})\subseteq Z\}$. Then

$$|I_0|\leqslant|S|+\alpha_d=\alpha_1+2\alpha_2+\dots+(d-1)\alpha_{d-1}+(d+1)\alpha_d.$$

For $(a_1,\dots,a_{n-1})\notin I_0$ we must have $|Z\cap X(a_1,\dots,a_{n-1})|=d-1$, and this implies that $|S\cap X(a_1,\dots,a_{n-1})|=d-1$. We conclude $|I_0|+\alpha_{d-1}\geqslant d^{n-1}$. Therefore

$$\alpha_1+2\alpha_2+\dots+(d-2)\alpha_{d-2}+d\alpha_{d-1}+(d+1)\alpha_d\geqslant d^{n-1},$$

and together with $|S|<(d-1)d^{n-2}$ this yields

$$\alpha_{d-1}+\alpha_d>d^{n-1}-(d-1)d^{n-2}=d^{n-2}.$$

But then $|S|\geqslant(d-1)(\alpha_{d-1}+\alpha_d)>(d-1)d^{n-2}$, which is the required contradiction. □

We define a set $S\subseteq V$ by

$$S=\begin{cases}\{(0,1),(0,2),\dots,(0,d-1)\} & \text{if } n=2,\\ \{(a_1,a_2,a_3)\in V\ :\ a_2=a_1,\ a_3\neq a_1\} & \text{if } n=3,\\ \{(a_1,\dots,a_n)\in V\ :\ a_{n-1}=a_1+a_{n-2},\ a_n\neq a_1+a_2+a_{n-2}\} & \text{if } n\geqslant 4.\end{cases}\tag{1}$$

Note that $|S|=(d-1)d^{n-2}$. The construction of the set S defined in (1) can be visualized by arranging the vertices of G in a $d^2\times d^{n-2}$-array where the rows are indexed by pairs (a_{n-1},a_n) and the columns are indexed by $(n-2)$-tuples $(a_1,\dots,a_{n-2})$. Then column $(a_1,\dots,a_{n-2})$ is the the union of the d sets $X(a_1, \ldots, a_{n-2}, a_{n-1})$ over $a_{n-1} \in \mathbb{Z}_d$, and the set S contains $d-1$ elements from each column. More precisely, the intersection of S with column $(a_1, \ldots, a_{n-2})$ is

$$X(a_1, \ldots, a_{n-2}, a_1 + a_{n-2}) \setminus \{(a_1, \ldots, a_{n-2}, a_1 + a_{n-2}, a_1 + a_2 + a_{n-2})\}.$$

In Figure 1 this is illustrated for two columns with $d = 5$ and $n = 7$.

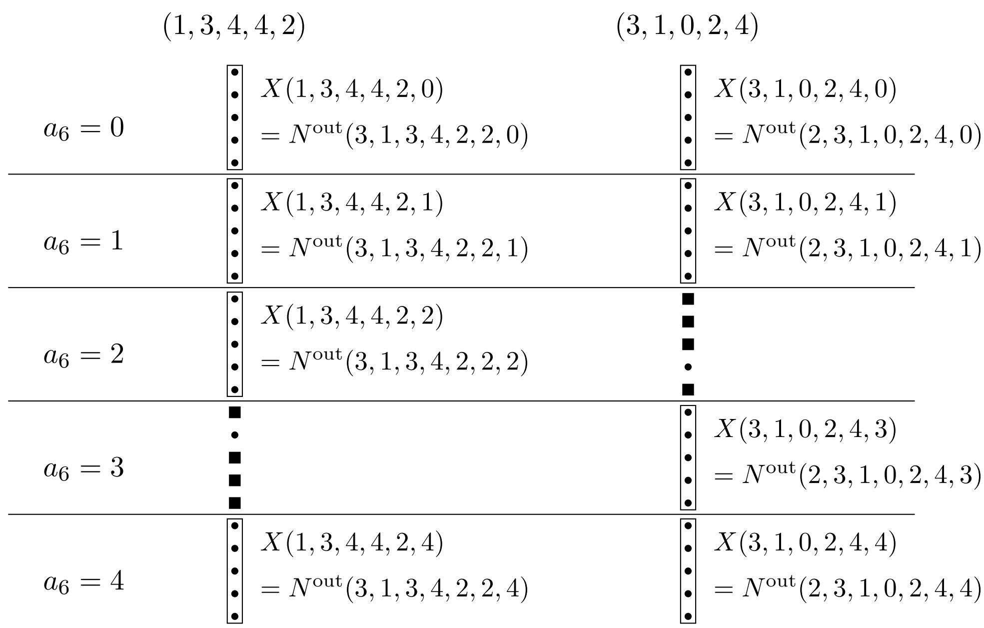

FIGURE 1. Illustration of the construction of the power dominating set S for $d = 5$ and $n = 7$. For the two columns $(a_1, \ldots, a_5) = (1, 3, 4, 4, 2)$ and $(a_1, \ldots, a_5) = (3, 1, 0, 2, 4)$ we show the elements of S (black squares), and we indicate for the sets $X(a_1, \ldots, a_6)$ (enclosed by rectangles) the elements of S having them as their out-neighbourhood.

Lemma 4. *The set S defined in (1) is a power dominating set for G.*

Proof. For $Z = S \cup N^{\text{out}}(S)$ it is sufficient to show that $|Z \cap X(a_1, \ldots, a_{n-1})| \geqslant d - 1$ for every $(a_1, \ldots, a_{n-1})$. We provide the full argument for $n \geqslant 4$ (the cases $n = 2$ and $n = 3$ are easy to check).

Case 1.: If $a_{n-1} = a_1 + a_{n-2}$, then by (1),

$$S \cap X(a_1, \ldots, a_{n-1}) = \{(a_1, \ldots, a_n) \ : \ a_n \in \mathbb{Z}_d \setminus \{a_1 + a_2 + a_{n-2}\}\},$$

hence $|Z \cap X(a_1, \ldots, a_{n-1})| \geqslant |S \cap X(a_1, \ldots, a_{n-1})| = d - 1$.

Case 2.: If $a_{n-1} \neq a_1 + a_{n-2}$, then $X(a_1, \ldots, a_{n-1}) \subseteq Z$ because

$$X(a_1, \ldots, a_{n-1}) = N^{\text{out}}((a_{n-2} - a_{n-3}, a_1, a_2, \ldots, a_{n-1}))$$

and $(a_{n-2} - a_{n-3}, a_1, a_2, \ldots, a_{n-1}) \in S$. $\square$

The second part of Theorem 1 follows from Lemmas 3 and 4.

4. The power domination number of Kautz digraphs

In this section we prove Theorem 2. Let us define the sets

$$X(a_1,\dots,a_{n-1})=\{(a_1,\dots,a_{n-1},a_n)\ :\ a_n\in\mathbb{Z}_{d+1}\setminus\{a_{n-1}\}\}$$

for $(a_1,\dots,a_{n-1})\in\mathbb{Z}_{d+1}^{n-1}$ with $a_i\neq a_{i+1}$ for all i. These sets partition the vertex set V into $(d+1)d^{n-2}$ sets of size d. Furthermore, $N^{\text{out}}(v)=X(a_1,\dots,a_{n-1})$ for every vertex v of the form $(a_0,a_1,a_2,\dots,a_{n-1})$.

Lemma 5. *Let G be a Kautz digraph with parameters $d,n\geqslant 2$. Then $Z(G)\geqslant(d-1)(d+1)d^{n-2}$.*

Proof. Every 2-element subset of each of the sets $X(a_1,\dots,a_{n-1})$ is strongly critical, and therefore, any zero forcing set S needs to intersect $X(a_1,\dots,a_{n-1})$ in at least $d-1$ elements, and the result follows. □

Lemma 6. *Let G be a Kautz digraph with parameters $d,n\geqslant 2$. Then $Z(G)\leqslant(d-1)(d+1)d^{n-2}$.*

Proof. Consider the vertex set

$$S=\begin{cases}\{(a_1,a_2)\in V\ :\ a_2\neq a_1+1\} & \text{if } n=2,\\ \{(a_1,\dots,a_n)\in V\ :\ a_n\neq a_{n-2}\} & \text{if } n\geqslant 3.\end{cases}$$

We have $|S|=(d-1)(d+1)d^{n-2}$, and to show that S is a zero forcing set, it is sufficient to verify that each vertex $v=(a_1,\dots,a_{n-1},a_n)$ is either in S or is the unique out-neighbor in $V\setminus S$ for some vertex w.

- **Case $n=2$.:** If $a_2\neq a_1+1$ then $v\in s$. If $a_2=a_1+1$ then for any vertex of the form $w=(\beta,a_1)$, v is the only neighbor of w in $V\setminus S$.
- **Case $n\geqslant 3$.:** If $a_n\neq a_{n-2}$, then $v\in S$. If $a_n=a_{n-2}$, then for any vertex of the form $w=(\beta,a_1,\dots,a_{n-1})$, v is the only neighbor of w in $V\setminus S$. □

Lemmas 5 and 6 imply the first statement of Theorem 2.

Lemma 7. *Let G be a Kautz digraph with parameters $d\geqslant 2$ and $n\geqslant 3$. Then, every power dominating set has size at least $(d-1)(d+1)d^{n-3}$.*

Proof. Let S be a power-dominating set, suppose $|S|<(d-1)(d+1)d^{n-3}$ and set $Z=S\cup N^{\text{out}}(S)$. We have

$$(Z\setminus S)\cap X(a_1,\dots,a_{n-1})\neq\emptyset\implies X(a_1,\dots,a_{n-1})\subseteq Z.$$

For $k=0,1,\dots,d$, we set $\alpha_k=\#\{(a_1,\dots,a_{n-1})\ :\ |S\cap X(a_1,\dots,a_{n-1})|=k\}$, and get

$$|S|=\alpha_1+2\alpha_2+\dots+(d-1)\alpha_{d-1}+d\alpha_d.$$

Now let $I_0=\{(a_1,\dots,a_{n-1})\ :\ X(a_1,\dots,a_{n-1})\subseteq Z\}$. Clearly,

$$|I_0|\leqslant|S|+\alpha_d=\alpha_1+2\alpha_2+\dots+(d-1)\alpha_{d-1}+(d+1)\alpha_d.$$

For $(a_1,\dots,a_{n-1})\notin I_0$ we must have $|Z\cap X(a_1,\dots,a_{n-1})|=d-1$ because Z intersects every weakly critical set. This implies that $|S\cap X(a_1,\dots,a_{n-1})|=d-1$, and we conclude $|I_0|+\alpha_{d-1}\geqslant(d+1)d^{n-2}$. Therefore

$$\alpha_1+2\alpha_2+\dots+(d-2)\alpha_{d-2}+d\alpha_{d-1}+(d+1)\alpha_d\geqslant(d+1)d^{n-2},$$

and together with $|S|<(d-1)(d+1)d^{n-3}$ this yields

$$\alpha_{d-1}+\alpha_d>(d+1)d^{n-2}-(d-1)(d+1)d^{n-3}=(d+1)d^{n-3}.$$

But then $|S|\geqslant(d-1)(\alpha_{d-1}+\alpha_d)>(d-1)(d+1)d^{n-3}$, which is the required contradiction. □

We define a set $S \subseteq V$ by
(2)

$$S = \begin{cases} \{(0,1),(0,2),\dots,(0,d)\} & \text{if } n=2,\\ \{(a_1,a_2,a_3)\in V\ :\ a_2=a_1+1,\ a_3\neq a_1+2\} & \text{if } n=3,\\ \{(a_1,a_2,a_3,a_4)\in V\ :\ a_3=a_1,\ a_4\neq a_2\} & \text{if } n=4,\\ \{(a_1,\dots,a_n)\in V\ :\ ((a_{n-2},a_{n-1})=(a_1,a_2)\wedge a_n\neq a_3)\vee(a_{n-1}=a_1\wedge a_n\neq a_2)\} & \text{if } n\geqslant 5.\end{cases}$$

Lemma 8. $|S| = \begin{cases} d & \textit{if } n=2,\\ (d-1)(d+1)d^{n-3} & \textit{if } n\geqslant 3.\end{cases}$

Proof. For $n \leqslant 4$ this is easy to check. For $n \geqslant 5$ we proceed by the following argument. We consider the partition $S = S_1 \cup S_2$ where

$$S_1 = \{(a_1,\dots,a_n)\in S\ :\ a_{n-3}=a_1\}, \qquad S_2 = \{(a_1,\dots,a_n)\in S\ :\ a_{n-3}\neq a_1\}.$$

Let s_k be the number of words $a_1\dots a_k$ over the alphabet $\mathbb{Z}_{d+1}$ which satisfy $a_k = a_1$ and $a_i \neq a_{i+1}$ for all $i \in \{1,\dots,k-1\}$. Then $s_2 = 0$ and $s_k = (d+1)d^{k-2} - s_{k-1}$ for $k \geqslant 3$. It follows by induction on k that $s_k = d^{k-1} - (-1)^k d$. Every vector $(a_1,\dots,a_{n-3}) \in \mathbb{Z}_{d+1}^{n-3}$ with $a_i \neq a_{i+1}$ and $a_{n-3} = a_1$ can be extended to an element of S_1 by choosing $a_{n-2} \in \mathbb{Z}_{d+1}\setminus\{a_1\}$, $a_{n-1} = a_1$ and $a_n \in \mathbb{Z}_{d+1}\setminus\{a_1,a_2\}$, hence

$$|S_1| = s_{n-3}d(d-1) = \left(d^{n-4} - (-1)^{n-3}d\right)d(d-1).$$

If $a_{n-3} \neq a_1$ then we can choose $(a_{n-2},a_{n-1}) = (a_1,a_2)$ and $a_n \in \mathbb{Z}_{d+1}\setminus\{a_2,a_3\}$, or $a_{n-2} \in \mathbb{Z}_{d+1}\setminus\{a_1,a_{n-3}\}$, $a_{n-1} = a_1$ and $a_n = \mathbb{Z}_{d+1}\setminus\{a_1,a_2\}$, hence

$$\begin{aligned}|S_2| &= \left[(d+1)d^{n-4} - s_{n-3}\right]\left[(d-1)+(d-1)^2\right]\\ &= \left[(d+1)d^{n-4} - d^{n-4} + (-1)^{n-3}d\right]d(d-1)\\ &= \left[d^{n-3} + (-1)^{n-3}d\right]d(d-1).\end{aligned}$$

Finally,

$$|S| = |S_1| + |S_2| = d(d-1)\left[d^{n-4} - (-1)^{n-3}d + d^{n-3} + (-1)^{n-3}d\right] = (d+1)(d-1)d^{n-3}. \qquad \square$$

Lemma 9. *The set S defined in (2) is a power dominating set for $G = \mathrm{K}(d,n)$.*

Proof. For $Z = S \cup N^{\mathrm{out}}(S)$ it is sufficient to show that $|Z \cap X(a_1,\dots,a_{n-1})| \geqslant d-1$ for every $(a_1,\dots,a_{n-1})$. We provide the full argument for $n \geqslant 5$ (the cases $n=2$, $n=3$ and $n=4$ are easy to check).

Case 1.: If $a_{n-2} = a_1$ and $a_{n-1} = a_2$ then

$$|S \cap X(a_1,\dots,a_{n-1})| = |\{(a_1,\dots,a_n)\ :\ a_n \in \mathbb{Z}_{d+1}\setminus\{a_2,a_3\}\}| = d-1,$$

and the claim follows from $Z \supseteq S$.

Case 2.: If $a_{n-2} = a_1$ and $a_{n-1} \neq a_2$, then $X(a_1,\dots,a_{n-1}) \subseteq Z$ because

$$X(a_1,\dots,a_{n-1}) = N^{\mathrm{out}}((a_{n-3},a_1,a_2,\dots,a_{n-1}))$$

and $(a_{n-3},a_1,a_2,\dots,a_{n-1}) \in S$.

Case 3.: If $a_{n-2} \neq a_1$ and $a_{n-1} = a_2$, then $X(a_1,\dots,a_{n-1}) \subseteq Z$ because

$$X(a_1,\dots,a_{n-1}) = N^{\mathrm{out}}((a_{n-2},a_1,a_2,\dots,a_{n-1}))$$

and $(a_{n-2},a_1,a_2,\dots,a_{n-1}) \in S$.

Case 4.: If $a_{n-2} \neq a_1$ and $a_{n-1} = a_1$ then

$$|S \cap X(a_1,\dots,a_{n-1})| = |\{(a_1,\dots,a_n)\ :\ a_n \in \mathbb{Z}_{d+1}\setminus\{a_1,a_2\}\}| = d-1,$$

and the claim follows from $Z \supseteq S$.

Case 5.: If $a_{n-2} \neq a_1$ and $a_{n-1} \notin \{a_1, a_2\}$, then $X(a_1, \ldots, a_{n-1}) \subseteq Z$ because

$$X(a_1, \ldots, a_{n-1}) = N^{\text{out}}((a_{n-2}, a_1, a_2, \ldots, a_{n-1}))$$

and $(a_{n-2}, a_1, a_2, \ldots, a_{n-1}) \in S$. □

The second part of Theorem 2 follows from Lemmas 7, 8 and 9.

5. Conclusion

In this paper, we have determined the zero forcing number and power domination number of de Bruijn and Kautz digraphs. There are many variants of de Bruijn and Kautz digraphs introduced and studied over the years, one of them being generalized de Bruijn digraphs $GB(d, n)$ and generalised Kautz digraphs $GK(d, n)$ which can be defined as follows:

$$\begin{aligned} V(GB(d,n)) &= \{0, 1, \ldots, n-1\}, \\ A(GB(d,n)) &= \{(x,y) \ : \ y \equiv dx + i \pmod n, \ 0 \leqslant i \leqslant d-1\}, \\ V(GK(d,n)) &= \{0, 1, \ldots, n-1\}, \\ A(GK(d,n)) &= \{(x,y) \ ; \ y \equiv -dx - i \pmod n, \ 1 \leqslant i \leqslant d\}. \end{aligned}$$

We leave it as an open problem to determine the zero forcing number and power domination number of generalised de Bruijn and Kautz digraphs.

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