Sequencing Partial Steiner Triple Systems

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Abstract

A partial Steiner triple system of order n is sequenceable if there is a sequence of length n of its distinct points such that no proper segment of the sequence is a union of point-disjoint blocks. We prove that if a partial Steiner triple system has at most three point-disjoint blocks, then it is sequenceable.

Keywords: partial Steiner triple system, sequenceable.

1 Introduction

An approach using posets for the problem of determining whether abelian groups are strongly sequenceable has been introduced in [1]. The approach involves extracting a poset from a combinatorial problem and working with the poset. The purpose of

this paper is to apply the poset approach to partial Steiner triple systems. Following are several definitions to establish the landscape.

Given a sequence $\pi = s_1, s_2, \ldots, s_n$, a *segment* is a subsequence of consecutive entries of π. We do not use the term 'interval' because that term has a different meaning in the context of posets. If the segment has t terms, it sometimes is called a *t-segment*. A segment is *proper* if it is neither empty nor π itself.

1.1 Definition. Let $\mathcal{P}$ be a poset on a ground set Ω, where the elements of $\mathcal{P}$ are subsets of Ω. If there is a sequence

$$\pi = s_1, s_2, \ldots, s_t$$

of the distinct elements of Ω such that no proper segment of π belongs to $\mathcal{P}$, then the poset is *sequenceable.*

When we say that a segment belongs or does not belong to $\mathcal{P}$, we of course mean the subset of elements comprising the segment. Note that all of Ω is allowed to be in the poset.

By a *partial Steiner triple system* we mean a collection of edge-disjoint subgraphs isomorphic to K_3 whose union is the edge set of a graph X. For simplicity we refer to the K_3 subgraphs in the collection as *blocks*. If the graph is complete, a partial Steiner triple system is, in fact, a Steiner triple system. The *order* of a partial Steiner triple system is the order of the graph X, that is, the number of vertices and we use the notation PST(n) for a partial Steiner triple system of order n.

1.2 Definition. Let $\mathcal{T}$ be a partial Steiner triple system of order n. The *associated poset* $\mathcal{P}(\mathcal{T})$ is the poset with ground set $V(X)$ whose elements are any subsets which can be partitioned into vertex-disjoint blocks.

Note that the blocks themselves are minimal elements of the poset $\mathcal{P}(\mathcal{T})$. Also note that the cardinalities of the elements of $\mathcal{P}(\mathcal{T})$ are multiples of 3.

1.3 Definition. A partial Steiner triple system $\mathcal{T}$ is *sequenceable* if the associated poset $\mathcal{P}(\mathcal{T})$ is sequenceable.

Distinct blocks are edge-disjoint by definition so that involving the word disjoint for this situation is redundant. Consequently, and this is a point of emphasis, when we say two blocks are disjoint, we mean they are vertex-disjoint.

A sequence of the distinct elements of a poset that satisfies the definition of sequenceability is said to be *admissible*. In accordance with this, a proper segment of a sequence is *admissible* if it contains no segments (including itself) that belong to the poset. A segment that is not admissible is *inadmissible*. Thus, given a PST(n), we shall be looking for a sequence of the distinct elements of the vertex set $V(X)$ such that all proper segments are admissible, that is, no proper segment of

the sequence has a partition into disjoint blocks. One common technique is to make small perturbations of a given sequence that eliminate all inadmissible segments.

We write blocks using square brackets in order to distinguish them from arbitrary subsets of vertices of cardinality three. We use V to denote $V(X)$ throughout the rest of the paper unless confusion would result.

2 Universality

When one first learns of this problem, the immediate thought is "Wouldn't it be nice if every partial Steiner triple system is sequenceable." Alas, this is not the case as we soon shall see, but some questions arise and we present them in this short section.

2.1 Theorem. *The cyclic Steiner triple system of order 13 is not sequenceable.*

PROOF. The cyclic Steiner triple system of order 13 admits the permutation $\rho = (0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12)$ as an automorphism. The blocks are obtained by the action of the group $\langle\rho\rangle$ on the two blocks $[0,1,4]$ and $[0,2,7]$. Then the four blocks $[0,2,7], [1,3,8], [5,6,9]$ and $[4,10,12]$ are disjoint and miss the vertex 11. Hence, for any vertex i there is an appropriate power of ρ that maps the preceding four disjoint blocks to four disjoint blocks missing i. Therefore, every sequence of the distinct vertices has two inadmissible 12-segments, namely, those obtained by deleting the first vertex and the last vertex. ■

The preceding theorem has a straightforward proof and certainly destroys any hopes that all partial Steiner triple systems are sequenceable. Several questions immediately arise and here are a few we consider in the remainder of this paper.

Question 1. Is the cyclic Steiner triple system of order 13 the smallest non-sequenceable PST(n)?

Question 2. Is there a nice construction for producing non-sequenceable partial Steiner triple systems of other orders?

Question 3. Are there conditions on the collection of blocks so that a given PST(n) is sequenceable?

3 Disjoint Blocks

One feature of the PST(13) used for the proof of Theorem 2.1 is that there are four mutually disjoint blocks. In this section, we explore Question 3 above from the standpoint of the number of mutually disjoint blocks. First, we present a general proof strategy which becomes useful as the number of disjoint blocks increases.

The PST(n) is given in terms of one, two or three disjoint blocks and additional vertices not belonging to the given blocks. If there is one block, it is denoted B_1. If there are two disjoint blocks, they are designated B_1 and B_2. Three disjoint blocks are denoted B_1, B_2 and B_3. The set of additional vertices is denoted L in all cases.

At the beginning none of the vertices of the PST(n) are labelled. An initial labelling is described under the restrictions that the vertices of B_1 are labelled 1, 2 and 3; the vertices of B_2 are labelled 4, 5 and 6; the vertices of B_3 are labelled 7, 8 and 9; and the vertices of L are labelled with lower case letters a, b, c and so on. After an initial labelling is in place, vertices may be relabelled so that a given sequence becomes admissible. The sequence may be given ahead of time or described at some point later in the process.

The next easy result looks at the case that the PST(n) has no two disjoint blocks.

3.1 Lemma. *A* PST(n) *in which any two blocks have a vertex in common is sequenceable.*

PROOF. Note that a sequence is inadmissible if and only if there is an inadmissible 3-segment. This makes checking the sequences straightforward.

If there are no blocks, then every sequence of the ground set is admissible. So arbitrarily choose a block to be B_1 and arbitrary label its vertices with 1, 2 and 3. If there are no additional blocks, then every sequence of the distinct elements is admissible when $V = \{1, 2, 3\}$. If there is at least one additional element a but no additional blocks, then every sequence beginning $1, 2, a, 3$ is admissible.

If there is at least one additional block, then by relabelling, if necessary, we may assume the block is $[1, a, b]$. If there are no additional elements, then $1, 2, a, b, 3$ is an admissible sequence. If there is at least one additional element c, then every sequence beginning $1, c, 2, 3, a, b$ is admissible. ■

We now move to two disjoint blocks which means that 6-segments may be inadmissible. The next result provides useful information about 6-sets.

3.2 Lemma. *Let A be a 6-subset of vertices in a* PST(n). *If A has a partition into two blocks A_1 and A_2, then the following are true:*

(1) *The partition is unique;*

(2) *A contains no blocks other than A_1 and A_2; and*

(3) *A set obtained by replacing any element of A with another element from V does not have a partition into two blocks.*

PROOF. Given that A partitions into A_1 and A_2, any other 3-subset of A must intersect one of A_1 or A_2 in two elements. This implies the other 3-subset cannot be a block, thereby, proving (1) and (2). Part (3) then follows because replacing a single element of A by an element not in A, leaves either A_1 or A_2 intact. The latter would then have to be in any partition but the complement then intersects the original block whose element is removed in two elements so it cannot be a block. ■

3.3 Lemma. *A* PST(n) *with at most two disjoint blocks is sequenceable.*

PROOF. Note that the only possible inadmissible segments in this case are 3-segments and 6-segments. If there are no disjoint blocks, then PST(n) is sequenceable by Lemma 3.1. So let $B_1 = [1, 2, 3]$ and $B_2 = [4, 5, 6]$ be two disjoint blocks that have had their elements arbitrarily labelled. If $|V| = 6$, there are no additional blocks and $1, 2, 4, 5, 3, 6$ is an admissible sequence.

If $|V| = 7$, let a be the additional vertex. Additional blocks must include a and such a block involves an edge in the bipartite graph $K_{3,3}$ with parts $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Thus the maximum number of additional blocks is three and corresponds to a perfect matching in $K_{3,3}$. We may assume the additional blocks are a subset of $\{[1, 4, a], [2, 5, a], [3, 6, a]\}$ by suitably relabelling the elements of $\{1, 2, 3\}$ and $\{4, 5, 6\}$. An admissible sequence for all possibilities is $1, 2, 4, a, 5, 3, 6$.

When $|V| = 8$, let a, b be the additional vertices. Because we have freedom of choosing which block is B_1 and of labelling the vertices in the two blocks, we may make the following assumptions. If a, b belong to a block, we may assume the block is $[1, a, b]$. If $3, a$ belong to a block, we may assume it is $[3, 6, a]$. Hence, the sequence $1, 2, 4, 3, a, 5, 6, b$ has no 3-segments that are blocks. It is easy to verify that all three of the 6-segments are admissible.

Now let $|V| > 8$ and $V' = V \setminus \{1, 2, 3, 4, 5, 6\}$. There are no blocks contained in V' as this would give three mutually disjoint blocks contrary to the hypothesis. Choose $A \subseteq V'$ so that $|A| = 3$. We are free to label the vertices of the two blocks $[1, 2, 3]$ and $[4, 5, 6]$ so that neither $[5, u, v]$ nor $[6, u, v]$ are blocks for any $u, v \in A$. We also may label the elements of A with a, b, c so that $[3, 5, a]$ and $[3, a, b]$ are not blocks.

Let π be any sequence beginning $1, 2, 4, 3, 5, a, 6, b, c$. It is easy to verify that all 3-segments are admissible. No 6-segment disjoint from $1, 2, 4, 3$ is inadmissible as this would give three mutually disjoint blocks. The three 6-segments $1, 2, 4, 3, 5, a$; $2, 4, 3, 5, a, 6$ and $4, 3, 5, a, 6, b$ are easily seen to be admissible. The only remaining possible inadmissible 6-segment is $3, 5, a, 6, b, c$ but it cannot be inadmissible because $[5, u, v]$ and $[6, u, v]$ are not blocks for any $u, v \in A$. This completes the proof. ■

The preceding two results arise from somewhat restrictive conditions. A natural question is whether the order of the PST(n) is bounded. It is easy to see that the order is not bounded. The friendship graph is obtained by taking m 3-cycles and amalgamating them at a common vertex. The resulting graph is a PST($2m + 1$) and every two blocks intersect in a single vertex.

The friendship graph provides an obvious recipe for obtaining partial Steiner triple systems of arbitrarily large order such that there are k disjoint blocks but not $k + 1$. Start with k vertex-disjoint friendship graphs $G_1, G_2, \ldots, G_k$ so that there at least two blocks in each G_i. Then amalgamate G_i and G_{i+1} at a vertex of valency 2 for $i = 1, 2, \ldots, k - 1$ making sure the block you use from G_{i+1} for the amalgamation of G_i and G_{i+1} is different from the block of G_{i+1} you use for G_{i+1} and G_{i+2}. It is easy to see that there are k disjoint blocks, and there cannot be $k + 1$ disjoint blocks by the pigeon-hole principle.

4 Three Disjoint Blocks

In the previous sections, we have shown that a partial Steiner triple system with at most two disjoint blocks is sequenceable, and that there exists a non-sequenceable partial Steiner triple system with four disjoint blocks. This makes three disjoint blocks a focal point and, as we are about to see, the resolution for the situation is lengthy. We break up the argument into several lemmas depending on the order of the PST(n).

Reiterating the notation of the proof strategy described earlier, there are three disjoint blocks denoted B_1, B_2, B_3 and the set L of remaining vertices.

4.1 Lemma. *A partial Steiner triple system of order 9 having three disjoint blocks is sequenceable.*

PROOF. The three disjoint blocks are arbitrarily labelled B_1, B_2, B_3. Arbitrarily label the elements of B_i with $3i-2, 3i-1, 3i$ for $i = 1, 2, 3$. Consider the sequence $1, 2, 4, 3, 5, 7, 6, 8, 9$. It is easy to see that the only possible inadmissible 3-segment is 3,5,7. (Note that if $[3, 5, 7]$ is a block, that makes all the 6-segments inadmissible.)

If 3,5,7 is admissible, then every 6-segment is admissible by Lemma 3.2. This implies that the sequence is admissible. If $[3, 5, 7]$ is a block, then interchange the labels 2 and 3. The sequence $1, 2, 4, 3, 5, 7, 6, 8, 9$ is now admissible. ■

Some additional information is required when extra vertices are added. Let $\mathcal{B}$ be the set of blocks of $\mathcal{T}$, where the vertex set V has cardinality n. The maximum number of blocks is given by

$$|\mathcal{B}| = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor & \text{if } n \equiv 0, 1, 2, 3, 4 \pmod 6 \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1 & \text{if } n \equiv 5 \pmod 6 \end{cases},$$

(see [4, 5, 6]). This bound is called the *Johnson-Schönheim bound* in [2]. A proof can be found in [5] or the text [3].

With respect to a given partial triple system PST(n), we say that a subset $M \subset V$ is *bad* if $V \setminus M$ can be partitioned into three disjoint blocks; otherwise, we say the set is a *good* set. We say that three disjoint blocks $A, B,$ and C *realize* a bad set M if $A \cup B \cup C = V \setminus M$. Notice that if M is bad, then $|M| = n - 9$. If $M = \{a\}$ is a singleton set, we say that a is a *bad* point or *good* point as appropriate.

4.2 Lemma. *A partial Steiner triple system of order* 10 *has at most four bad points.*

PROOF. Let $\mathcal{T}$ be a partial triple system of order 10 and assume there are five bad points. The Johnson-Schönheim bound implies that $\mathcal{T}$ has at most 13 blocks. Hence, by the pigeonhole principle, there must be a block A and a pair of bad points v, w, such that $A \cup B \cup C = V \setminus \{v\}$ and $A \cup D \cup E = V \setminus \{w\}$ for some blocks B, C, D, E.

Assume without loss of generality that $v \in D$ and $w \in B$. Then $E \subset B \cup C$, which implies $E \in \{B, C\}$ by Lemma 3.2. But then by counting points we see that $|B \cap D| = 2$, and this is a contradiction.

Therefore, there are at most four bad points completing the proof. ■

4.3 Lemma. *A* PST(10) *with three disjoint blocks is sequenceable.*

PROOF. Let $|V| = 10$ and label the single point of L with a. We know there are at most four bad points from Lemma 4.2. Because a itself is a bad point, there are at least six good points among the remaining nine points. If there is a block with fewer than two good points, label this block B_3; otherwise, arbitrarily label a block as B_3. Thus, we know the other two blocks each have at least two good points.

Label an arbitrary point of B_3 with 9 and label another block B_1 so that $[9, x, a]$ is not a block for $x \in B_1$. Arbitrarily label the points of B_1 with 1, 2, 3 so that the bad point (if there are any) has label 3. This means that both 1 and 2 are good points. In the block B_2, label the points with 4, 5, 6 and in the block B_3 label the other two points with 7 and 8 so that none of $[3, 6, 8], [3, 6, 9]$ or $[3, 6, a]$ are blocks. Finally, from the good points 1 and 2, relabel the points, if necessary, so that $[2, 6, 9]$ is not a block.

We claim the sequence $1, 4, 5, 7, 6, 8, 9, 3, a, 2$ is admissible. All of the 3-segments are admissible because the labelling is chosen so that $[9, 3, a]$ is not a block. The two 9-segments are admissible because 1 and 2 are good points.

The only two possible inadmissible 6-segments are $7, 6, 8, 3, 9, a$ and $6, 8, 3, 9, a, 2$. The first is not inadmissible because Lemma 3.2 implies that $[3, 6, a]$ is a block if it is inadmissible and the latter is not the case. The only possible partitions of the second 6-segment into two blocks are $[2, 8, a] \cup [3, 6, 9]$ and $[3, 8, a] \cup [2, 6, 9]$, but neither $[3, 6, 9]$ nor $[2, 6, 9]$ are blocks and this completes the verification that $\mathcal{T}$ is sequenceable when there are three disjoint blocks and $|V| = 10$. ■

Now that we are considering three disjoint blocks, some sets of cardinality nine have partitions into three blocks. This type of set is more complicated than the sets of cardinality six that have partitions into two blocks. Nevertheless, it would be useful if we could find a result for nine points that is analogous to the powerful Lemma 3.2 for six points. Towards that end we introduce some notation.

If A_1 and A_2 are disjoint blocks in a PST(n), let K_{A_1,A_2} denote the complete bipartite graph of order 6 with parts A_1 and A_2. Let A be a set of cardinality 9 containing both A_1 and A_2 as subsets, and let $A_3 = A \setminus (A_1 \cup A_2)$.

If A_3 also is a block, then A_1, A_2, A_3 is a partition of A into three blocks. On the other hand, if A_3 is not a block, then there can be no partition of A having either A_1 or A_2 as a part because of Lemma 3.2. Thus, given any partition of A into three blocks in the latter situation, every block in such a partition must intersect each A_1 and A_2 in a single point. We may interpret these intersections as a perfect matching in K_{A_1,A_2} and say that the partition *induces* the perfect matching.

4.4 Lemma. *Let A_1 and A_2 be disjoint blocks in a partial Steiner triple system $\mathcal{T}$ and let $V' = V \setminus (A_1 \cup A_2)$. The following three properties hold in $\mathcal{T}$.*

(1). *If $\{\alpha, \beta, \gamma, x\}$ is a set of cardinality 4 in V' containing no block with x as a member, then at most two sets of cardinality 9 containing x, A_1 and A_2 plus two members from $\{\alpha, \beta, \gamma\}$ have a partition into blocks.*

(2). *If $\{\alpha, \beta, \gamma, x, y\}$ is a set of cardinality 5 in V' containing no block with both x and y as members, then to within labelling there is a unique way that the three sets of cardinality 9 containing x, y, A_1, A_2 plus one element from $\{\alpha, \beta, \gamma\}$ have a partition into blocks.*

(3). *Let A be a 9-set with a partition into three blocks A_1, A_2 and A_3 and let A' be any 9-set formed by replacing a point of A_3 with a point not in A. If either A_1 or A_2 must appear in a partition of A' into three blocks, then A' has no partition into three blocks.*

PROOF. Consider (1) first. There are three sets of the form described in the statement and note that none of them have a partition with either A_1 or A_2 as a part because Lemma 3.2 would force a block containing x and two elements from $\{\alpha, \beta, \gamma\}$. Assume each of the three sets has a partition into three blocks which then induces three perfect matchings in K_{A_1,A_2}. We leave it up to the reader to verify that the three perfect matchings are mutually edge-disjoint.

Label an edge of K_{A_1,A_2} with the other point in the block containing the edge. We may label the points of A_1 with u_1, u_2, u_3 and the points of A_2 with v_1, v_2, v_3 so that the three edges with label x are u_1v_1, u_2v_2 and u_3v_3. The remaining edges form a 6-cycle.

In the partition with the block $[x, u_3, v_3]$, the edges for the other two parts must be u_1v_2 and u_2v_1 because the three edges form a 3-matching. Without loss of generality we may assume u_1v_2 is labelled α and u_2v_1 is labelled β. The partition with the block $[x, u_2, v_2]$ forces the other two edges to be u_1v_3 and u_3v_1 with one of them labelled γ. Suppose it is u_1v_3. Then the edge u_3v_1 must be labelled α. This means that the partition containing the block $[x, u_1, v_1]$ must involve the points β and γ but the edge u_2v_3 cannot receive either of these two labels. This contradiction proves (1).

For part (2), it is easy to see that there is no partition of any of the sets of the hypothesised form into blocks with either A_1 or A_2 as parts. So if all three have partitions, then there are three edge-disjoint perfect matchings arising again but this time there is a perfect matching with edges labelled x and a perfect matching whose edges are labelled y. The union of these two perfect matchings is a 6-cycle and we may label points so that the 6-cycle is $u_1v_1u_2v_2u_3v_3$ with u_1v_1 labelled x (which determines the other labels). The remaining edges are u_1v_2 labelled α, u_2v_3 labelled β and u_3v_1 labelled γ. The actual partition is determined uniquely from this labelling. This proves (2)

If A_1 must be in a partition of A', then A_2 must be a block as well by Lemma 3.2. This would force the final block to contain two points of A_3 which is not possible. Part (3) then follows. ■

4.5 Lemma. *If $\mathcal{T}$ is a partial Steiner triple system of order 11 having three disjoint blocks, then $\mathcal{T}$ is sequenceable.*

PROOF. Let $|V| = 11$ and arbitrarily label the points of L with a and b. There are at least six good points for the residual partial triple system on $V \setminus \{a\}$ and at least six good points points for the residual partial triple system on $V \setminus \{b\}$. Thus, there is a point that is good for both residual triple systems because the points are coming from a set of cardinality 9. Label the block containing such a point B_3 and label the point with 9. Note that this implies that $\{9, b\}$ is a good set for V.

We know there is another block containing two points that are good points for $V \setminus \{b\}$ because there are at least six altogether. Label such a block B_1 and label two good points arbitrarily with 1 and 2 (thereby labelling 3). Arbitrarily label the points of B_2 with 4, 5 and 6, and the remaining two points of B_3 with 7 and 8.

Consider the sequence $\pi = b, 1, 2, 4, 3, 5, 7, 6, 8, a, 9$. Because 1 is a good point for $V\setminus\{b\}$, the 9-segment $2, 4, 3, 5, 7, 6, 8, a, 9$ cannot be partitioned into three blocks. Because 9 is a good point for both $V \setminus \{a\}$ and $V \setminus \{b\}$, neither of the other two 9-segments can be partitioned into three blocks.

The only possible remaining inadmissible segments are the two 3-segments $3, 5, 7$ and $6, 8, a$, and the 6-segment $3, 5, 7, 6, 8, a$. First, let the 6-segment be admissible. If both $3, 5, 7$ and $6, 8, a$ are admissible, then π is admissible. So assume that $[3, 5, 7]$ is a block. If $[5, 8, a]$ is not a block, then interchanging the labels of 5 and 6 makes π admissible. If $[5, 8, a]$ is a block, then interchange the labels of 7 and 8 and we again have the resulting π admissible. We do a similar thing if we start with $[6, 8, a]$ being a block.

Second, let $3, 5, 7, 6, 8, a$ have a partition into two blocks. Then interchange the labels of 4 and 5 giving us an admissible segment $3, 5, 7, 6, 8, a$ by Lemma 3.2. We are back at the preceding situation and this completes the proof. ■

The next lemma sets the stage for the completion of the verification for three vertex-disjoint blocks. Of course, it does force us to concentrate on partial Steiner triple system of order 12.

4.6 Lemma. *Let $\mathcal{T}$ be a partial Steiner triple system of order 12 with three vertex-disjoint blocks but not four vertex-disjoint blocks, and let*

$$[1, 2, 3], [4, 5, 6], [7, 8, 9], \{a, b, c\}$$

be a partition of V into three blocks and three points. If there is an admissible sequence of the form $1, 2, 4, 3, 5, 7, 6, 8, a, 9, b, c$, then all partial Steiner triple systems of order at least 13 having at most three vertex-disjoint blocks are sequenceble.

PROOF. Because of Lemmas 3.1 and 3.3, only partial Steiner triple systems with three vertex-disjoint blocks, but not four, need be considered. Consider a partial Steiner triple system $\mathcal{T}'$ of order at least 13. Choose a residual partial Steiner triple system consisting of the three blocks $[1, 2, 3], [4, 5, 6], [7, 8, 9]$ and three other arbitrary points a, b, c. Let $1, 2, 4, 3, 5, 7, 6, 8, a, 9, b, c$ be an admissible sequence for the residual partial Steiner triple system.

Form a new sequence π by adding an arbitrary permutation of the remaining points $V \setminus \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c\}$ to the end of the given sequence. We claim that π is an admissible sequence for the partial Steiner triple system $\mathcal{T}'$.

The sequence π has no inadmissible 3-segments because such a segment would have to start with b or further along the sequence π. However, this produces four disjoint blocks in $\mathcal{T}'$ because none of $\{1, 2, 3, \ldots, 9\}$ have been used. Any inadmissible 6-segment would have to begin with 8 or further along the sequence. This again would produce four disjoint blocks. A similar argument works for 9-segments because none of $\{1, 2, 3\}$ would be involved. Therefore, π is admissible for $\mathcal{T}'$ and the result follows. ■

5 Order Twelve

This section deals with partial Steiner triple systems of order 12 having three vertex-disjoint blocks but not four. This is a separate section because the proof that they are sequenceable is lengthy and intricate. We continue to use the notation that there are three disjoint blocks which will at some point be labelled B_1, B_2 and B_3 and a set L of three additional points. The points of B_1, B_2 and B_3 will be labelled with $\{1, 2, 3\}, \{4, 5, 6\}$ and $\{7, 8, 9\}$, respectively, and the points of L will be labelled a, b, c.

Throughout this proof we work to prove that there is a labelling of the points so that the sequence

$$\pi = 1, 2, 4, 3, 5, 7, 6, 8, a, 9, b, c$$

is admissible. This is done by providing an initial labelling of the blocks as B_1, B_2 and B_3 and this labelling does not change. The elements within the blocks will be given an initial labelling following the guidelines of the preceding paragraph and various relabellings may take place as we work through the proof.

We now provide an outline of the proof comprising this section. The given sequence π contains four 3-segments, three 6-segments and four 9-segments which may be inadmissible. They are:

- 3-segments: $3, 5, 7$; $6, 8, a$; $a, 9, b$; and $9, b, c$;
- 6-segments: $3, 5, 7, 6, 8, a$; $7, 6, 8, a, 9, b$; and $6, 8, a, 9, b, c$; and
- 9-segments: all four of them.

The first result below gives a flexible choice for a particular 6-segment and this 6-segment is of interest because it contains two of the possible inadmissible 3-segments. Thus, if the 6-segment is admissible, all the 3-segments contained in it are admissible.

We then move to considering 9-segments. Of course, the 9-segments are the most complicated because they may have more than one partition into blocks, whereas, an inadmissible 6-segment has a unique partition into blocks and a 3-segment either is or is not a block itself. We give two results that provide a way of guaranteeing that none of the four 9-segments have partitions into three disjoint blocks and leave only a couple of potentially problematic blocks. To eliminate the problematical blocks, we consider a variety of cases.

5.1 Lemma. *Arbitrarily label the blocks B_1, B_2 and B_3. If we label an arbitrary point of L with a and an arbitrary point of B_1 with 3, then we may label the points of B_2 and B_3 so that the segment $3, 5, 7, 6, 8, a$ is admissible, where there are at least two choices for the point to be labelled 9.*

PROOF. First label arbitrary points of L and B_1 with a and 3, respectively. By Lemma 3.2, at most one of the sets $\{a, 3, 4, 5, 6, x, y\}$, over the three choices of $\{x, y\} \subset B_3$, has a partition into two blocks. That gives at least two choices for the point labelled 9 so that $\{a, 3, 4, 5, 6, 7, 8\}$ has no partition into two blocks.

Consider the sequence $\pi' = 3, 5, 7, 6, 8, a$ for a choice that has no partition of the 6-set. The only possible inadmissible 3-segments contained in π' are $3, 5, 7$ and $6, 8, a$. If both are admissible, then π' is admissible and we are done. If $[3, 5, 7]$ is a block and $[5, 8, a]$ is not a block, then interchanging the labels of 5 and 6 results in π' being properly admissible. If both $[3, 5, 7]$ and $[5, 8, a]$ are blocks, then interchanging the labels of 5 and 6, and 7 and 8 results in π' being admissible (noting that $[3, 6, 7]$ is not a block prior to relabelling because $[5, 8, a]$ is a block and $3, 5, 7, 6, 8, a$ has no partition into two blocks). A similar argument works when $[6, 8, a]$ is a block. This completes the proof. ■

Recall that a set A of cardinality 3 is a good set if $V \setminus A$ has no partition into three blocks. We are interested in the complements of the four 9-segments, that is, $\{1, 2, 4\}, \{1, 2, c\}, \{b, c, 1\}$ and $\{b, c, 9\}$ as we require all four of them to be good. If there is an $\alpha \in B_i$ such that $\{b, c, \alpha\}$ is a good set, we say that α is an *a-replacement*. It is then clear what b- and c-replacements are.

Our next goal is to show there is a choice of a for which B_1, B_2 and B_3 all have at least two a-replacements. The next result achieves the goal.

5.2 Lemma. *There is a point in L which may be labelled a such that each B_i, $1 \leq i \leq 3$, has at least two a-replacements.*

PROOF. If the result is not true, then for each $x \in L$ there is a B_i such that two of the numbers in B_i are not x-replacements. Without loss of generality we label the points of B_1 with 1, 2, 3 so that neither 2 nor 3 are b-replacements. This means that both

$$\{1, 2, 4, 5, 6, 7, 8, 9, b\} \text{ and } \{1, 3, 4, 5, 6, 7, 8, 9, b\}$$

have partitions into three blocks. It is easy to see that neither B_2 nor B_3 can be parts in either partition. So the two partitions generate perfect matchings in K_{B_2,B_3} and it is easy to check that the two perfect matchings are edge-disjoint.

We may assume the partition of $\{1, 2, 4, 5, 6, 7, 8, 9, b\}$ is $[1, 5, 8], [2, 6, 9]$ and $[4, 7, b]$ without loss of generality. A partition of $\{1, 3, 4, 5, 6, 7, 8, 9, b\}$ is determined once the block containing b is known. If the block is $[5, 9, b]$, then the other two blocks are forced to be $[1, 6, 7]$ and $[3, 4, 8]$, whereas, if the block is $[6, 8, b]$, then the remaining two blocks are $[1, 4, 9]$ and $[2, 5, 7]$. However, if we relabel the vertices

for the latter situation using the permutation $(2\ 3)(6\ 4\ 5)(8\ 7\ 9)$, we obtain the same set of blocks as the first situation. Thus, we may assume that

$$[4,7,b],[5,9,b],[1,5,8],[1,6,7],[2,6,9] \text{ and } [3,4,8]$$

are blocks realizing the edges $47, 48, 58, 59, 67$ and 69 in the union of the two perfect matchings.

We now wish to show that if 3 also is not an x-replacement, where $x \in \{a, c\}$, then 1 and 2 are x-replacements. So let 3 also not be an x-replacement. This means that $\{1, 2, 4, 5, 6, 7, 8, 9, x\}$ has a partition into blocks. Look at the block containing 1. It cannot be $[1, 5, 8]$ as this would force $[4, 7, x]$ to be a block which is impossible. It cannot be $[1, 6, 7]$ because this prevents 5 from belonging to a block. Hence, the block must be $[1, 4, 9]$ which admits $[2, 5, 7]$ and $[6, 8, x]$ as the only completion to a partition.

The three partitions use all of the edges of K_{B_2,B_3} in blocks. We now need to show that both 1 and 2 are x-replacements. The set $\{2, 3, 4, 5, 6, 7, 8, 9, x\}$ has no partition because the only two blocks involving 2 are $[2, 5, 7]$ and $[2, 6, 9]$. The first one forces $[3, 4, 9]$ to be a block and the second leaves no block to contain x. Hence, 1 is an x-replacement. Then $\{1, 3, 4, 5, 6, 7, 8, 9, x\}$ also has no partition because the only block containing 3 is $[3, 4, 8]$ and the only block containing x is $[6, 8, x]$. Therefore, both 1 and 2 are x-replacements.

The preceding argument slightly modified also works with 2 and 3 interchanging roles. We now are able to conclude that B_1 contains at least two x-replacements for all $x \in \{a, c\}$. If both 2 and 3 are x-replacements, the conclusion follows, whereas, if one of 2 or 3 is not an x-replacement, then the remaining two numbers in B_1 are x-replacements validating the conclusion. Therefore, the only way the conclusion of the lemma fails is if there are two numbers in B_2 that fail to be x-replacements for $x \in \{a, c\}$ and two numbers in B_3 that fail to be y-replacements, where $\{x, y\} = \{a, c\}$. We now show this cannot happen.

There are cases to consider and we work through one case to illustrate how the proof works. All we know is the existence of the six blocks earlier in the proof. We examine B_2 starting with 5 with $x \in \{a, c\}$. Look at the edges incident with 6 for possible partitions of $\{1, 2, 3, 4, 6, 7, 8, 9, x\}$. The edge 67 belongs to the block $[1, 6, 7]$ and its presence forces the block $[3, 4, 8]$ to be there as well. Then $[2, 9, x]$ would have to be a block but the block containing 29 is $[2, 6, 9]$.

In a similar fashion one can show that there is no partition involving the edges 68 and 69. This implies the set $\{1, 2, 3, 4, 6, 7, 8, 9, x\}$ has no partition into blocks which means that 5 is an x-replacement.

Examining the set $\{1, 2, 3, 5, 6, 7, 8, 9, x\}$ leads to the only possible partition being $[1, 5, 8], [2, 6, 9]$ and $[3, 7, x]$. There are several possible partitions of $\{1, 2, 3, 4, 5, 7, 8, 9, x\}$. We must take each of them and pair it with the preceding partition of $\{1, 2, 3, 5, 6, 7, 8, 9, x\}$ and show that the two of them imply that at least two numbers from B_3 are y-replacements as this would give two or more numbers from each B_i, $i = 1, 2, 3$, as y-replacements.

One possible partition of $\{1, 2, 3, 4, 5, 7, 8, 9, x\}$ is $[1, 4, 9], [2, 8, x]$ and $[3, 5, 7]$. Combining this partition with that for $\{1, 2, 3, 5, 6, 7, 8, 9, x\}$ we find that both 7 and

8 are y-replacements as required. The other possible partition for $\{1,2,3,4,5,7,8,9,x\}$ is $[1,9,x],[2,5,7],[3,4,8]$. It turns out that 7 and 9 are y-replacements and that completes the proof. ■

Let's take a moment to see the current status of our progress. As there are two choices for the element of B_3 to be labelled 9 for Lemmas 5.1 and 5.2, there is at least one element of B_3 that can be labelled 9 satisfying both lemmas. This makes many, but not all, of the problematic segments admissible.

5.3 Lemma. *Suppose there is a labelling for which $\{1,2,c\}$ and $\{1,2,4\}$ are good sets, and both Lemmas 5.1 and 5.2 hold. If none of the following 3-sets are blocks, then π is admissible: $\{6,a,b\},\{9,a,b\},\{9,b,c\}$ and $\{6,8,b\}$ when $\{9,a,c\}$ is a block.*

PROOF. Because both $\{1,2,4\}$ and $\{1,2,c\}$ are good sets, the 9-segments ending with b and c do not have partitions into three blocks. Because of Lemma 5.2, either 1 or 2 is an a-replacement so we may choose to label them so that 1 is an a-replacement and the 9-segment beginning with 2 has no partition into three disjoint blocks.

Similarly, we have two choices for the point labelled 9 for Lemmas 5.1 and 5.2. This gives us that the remaining 9-segment has no partition into three disjoint blocks and that the segment $3,5,7,6,8,a$ is admissible. The latter implies that $\{3,5,7\}$ and $\{6,8,a\}$ are not blocks.

Thus, the segments that may be inadmissible are $a,9,b$; $9,b,c$; $7,6,8,a,9,b$; and $6,8,a,9,b,c$. The first two are admissible by hypothesis. The segment $7,6,8,a,9,b$ is inadmissible only if $[6,8,b]$ is a block and it is not. The only possible partition of the 6-segment $6,8,a,9,b,c$ is into the blocks $[6,8,b]$ and $[9,a,c]$ which is not the case by hypothesis. ■

We now have set the stage for the proof of the main theorem. The primary idea is to obtain a labelling in accordance with Lemma 5.3 and do so in a way that avoids the blocks listed in the lemma. There are cases depending on the distribution of blocks composed of two points from L and one point from a B_i. Such a block is called a *tiple* and the point from the B_i is called the *tip*.

5.4 Theorem. *A partial Steiner triple system $\mathcal{T}$ with at most three disjoint blocks is sequenceable.*

PROOF. Lemmas 3.1 and 3.3 take care of the cases that $\mathcal{T}$ has at most two disjoint blocks. If there are three disjoint blocks, but not four, then Lemmas 4.1, 4.3 and 4.5 take care of the cases that $\mathcal{T}$ has order 9, 10 or 11. Finally, Lemma 4.6 tells us that if we can find an admissible sequence of the form $\pi = 1,2,4,3,5,7,6,8,a,9,b,c$ when $\mathcal{T}$ has order 12, then the result holds for all larger orders. The rest of the proof consists of arguments verifying there is a labelling so that π is admissible.

Case 1. There are three tiples all of whose tips are in the same block B_i. The first labelling step is giving label B_2 to the block containing the tips of the three

tiples, and arbitrarily labelling the other two blocks B_1 and B_3. Label the point of L satisfying Lemma 5.2 with a, and arbitrarily label the other two points b and c. By the pigeonhole principle, there is a point in B_3 we label 9 so that both Lemma 5.1 applies and it is an a-replacement. Label the point in B_2 with 4 so that $[4, a, b]$ is a block.

Claim 1. At most one of the sets $\{1, 2, c\}, \{1, 3, c\}$ and $\{2, 3, c\}$ is bad for any labelling of the points of B_1. Suppose that $\{\alpha, 4, 5, 6, 7, 8, 9, a, b\}$, $\alpha \in \{1, 2, 3\}$, has a partition into three blocks. Neither B_2 nor B_3 are parts in the partition as this would imply that $[\alpha, a, b]$ is a block. The partition generates a perfect matching in K_{B_2,B_3} with one edge labelled α, one labelled a and the other labelled b.

If there is a partition of $\{\beta, 4, 5, 6, 7, 8, 9, a, b\}$, $\beta \neq \alpha$, then the perfect matching generated is edge-disjoint from the first perfect matching by part (3) of Lemma 4.4. It has an edge labelled β incident with 4 leaving only one choice for the edge labelled a. This leaves no choices for an edge to be labelled b and the claim follows.

Claim 2. At most one of the sets $\{1, 2, 4\}, \{1, 3, 4\}$ and $\{2, 3, 4\}$ is bad under any labelling of the points of B_1. Supposing that $\{\alpha, 5, 6, 7, 8, 9, a, b, c\}$, $\alpha \in \{1, 2, 3\}$, has a partition into blocks, it is easy to verify that none of the parts may be a tiple or B_3. We may label the remaining two points of B_2 with 5 and 6 so that $[5, a, c]$ and $[6, b, c]$ are the other two tiples. Then B_3 and $B' = [5, a, c]$ are disjoint blocks so that a partition generates a perfect matching in $K_{B_3,B'}$. This forces one of the blocks to contain $\{6, a\}$. But this is the same for any $\alpha \in \{1, 2, 3\}$ and part (3) of Lemma 4.4 then implies none of the other two possibilities have a partition into blocks.

From the two claims we conclude that there is a labelling of B_1 so that both $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good sets. There are no tiples with tips in B_3 so that the only possible problem 3-set in Lemma 5.3 is $\{6, a, b\}$. However, the relabelling done in Lemma 5.1 does not affect 4 so that $[4, a, b]$ still is a block which means $\{6, a, b\}$ is not a block. Thus, π is admissible for this case.

Case 2. There are two tiples, but not three, whose tips are in the same B_i. Label the block containing two tips B_2. Label the point of L satisfying Lemma 5.2 with a, and do not label the other two points of L at this time. There may or may not be a third tiple. It there is, label the block containing its tip B_1.

If the point labelled a is in both blocks with tips in B_2, then label points so that $[5, a, b]$ and $[6, a, c]$ are blocks which, of course, means that when the third tiple exists it contains b and c. Claim 1 of the preceding proof still holds so that two of the sets $\{1, 2, c\}, \{1, 3, c\}$ and $\{2, 3, c\}$ are good. Claim 2 also remains valid except that we are using $[5, a, b]$ and $[6, a, c]$ in place of $[5, a, c]$ and $[6, b, c]$, respectively.

Note that $[6, 8, a]$ cannot be a block because $[6, a, c]$ is a block. Thus, when we apply the relabelling of Lemma 5.1 to make the segment $3, 5, 7, 6, 8, a$ admissible, the labels of 5 and 6 do not change so that $[5, a, b]$ remains a block so that $\{6, a, b\}$ is not. Lemma 5.3 implies that π is admissible.

The preceding case was for a belonging to two tiples whose tips are in the same block which is again labelled B_2. Now let a belong to just one of these two tiples. This subcase is the most intricate portion of the proof. Label the other points of

L so that b is the point belonging to both tiples with tips in B_2. Label the points of B_2 so that $[4, a, b]$ and $[6, b, c]$ are blocks. If there is a third tiple label the block containing its tip B_1 and arbitrarily label the points of B_1. Claim 1 still is valid via essentially the same argument.

If at least two of the sets $\{1, 2, 4\}, \{1, 3, 4\}$ and $\{2, 3, 4\}$, are good, then we complete the argument as before. If this is not the case we need a variation. If all three of the sets $\{1, 2, 5\}, \{1, 3, 5\}$ and $\{2, 3, 5\}$ are good, we may assume that $\{1, 2, 5\}, \{1, 2, c\}, \{1, 3, 5\}$ and $\{1, 3, c\}$ are good as Claim 1 still holds. First consider $\{1, 2, 5\}$ and $\{1, 2, c\}$ and choose 1 as the a-replacement by relabelling if necessary. It is straightforward to verify that the sequence $\pi' = 1, 2, 5, 3, 4, 7, 6, 8, a, 9, b, c$ is admissible when the segment $3, 4, 7, 6, 8, a$ is admissible.

When the segment $3, 4, 7, 6, 8, a$ is inadmissible, the relabelling we are allowed to carry out is interchanging 7 and 8. This then makes π' admissible except in the following situations: 1) Both $[3, 4, 7]$ and $[6, 8, a]$ are blocks; 2) both $[3, 4, 8]$ and $[6, 7, a]$ are blocks; 3) both $[3, 4, 7]$ and $[6, 7, a]$ are blocks; and 4) both $[3, 4, 8]$ and $[6, 8, a]$ are blocks.

We then turn to the good sets $\{1, 3, 5\}$ and $\{1, 3, c\}$. Suppose that 1 and 2 did not switch labels in the preceding step to obtain 1 as an a-replacement. Then 1 still is an a-replacement here. Now interchange the labels of 2 and 3. The segment $3, 4, 7, 6, 8, a$ now is admissible because either $[6, 7, a]$ or $[6, 8, a]$ still is a block in all four of the problematic block scenarios of the preceding paragraph, but $[2, 4, 7]$ and $[2, 4, 8]$ are not blocks. Thus, π' is now admissible.

If 1 and 2 did switch labels in the first step because 1 is not an a-replacement, then $\{1, 3, c\}$ and $\{1, 3, 5\}$ become $\{2, 3, c\}$ and $\{2, 3, 5\}$, respectively. Then 3 is the a-replacement and the rest of the argument carries through in the obvious way.

To complete this troublesome subcase, we start at the point that $[4, a, b]$ and $[6, b, c]$ are blocks and rebuild other assumptions. It is easy to show that at least one of the sets $\{1, 2, 4\}, \{1, 3, 4\}$ and $\{2, 3, 4\}$ is good and from above, if two of them are good, we are done. Hence, we may assume that both $\{1, 2, 4\}$ and $\{1, 3, 4\}$ are bad. One of them must involve $[7, 8, 9]$ as a block so that we may assume the partition consists of the blocks $[7, 8, 9]$, $[6, b, c]$ and $[2, 5, a]$. The other partition is forced to have $a, 6$ in the same block and we assume that the blocks are $[6, 7, a], [5, 8, b]$ and $[3, 9, c]$. Note that this labels B_3.

Now consider the three sets $\{1, 2, 5\}, \{1, 3, 5\}$ and $\{2, 3, 5\}$. If the 9-set $\{\beta, 4, 6, 7, 8, 9, a, b, c\}$, $\beta \in \{1, 2, 3\}$, has a partition into blocks, then $[7, 8, 9]$ cannot be one of the parts as both $[4, a, b]$ and $[6, b, c]$ are contained in the remaining six points. Thus, each part in a partition must be a block intersecting one point of $\{7, 8, 9\}$, one point of $[4, a, b]$ and one point of $[6, b, c]$. In particular, this forces a to be in a block with 6, that is, the block $[6, 7, a]$. The other blocks must be $[4, 8, c]$ and $[\beta, 9, b]$.

This implies there is a partition for at most one $\beta \in \{1, 2, 3\}$. The case for no partitions already has been done so we may assume exactly two of the sets $\{1, 2, 5\}, \{1, 3, 5\}$ and $\{2, 3, 5\}$ are good.

If $[\beta, 4, 7]$ is not a block, then $\beta, 4, 7, 6, 8, a$ is admissible. Furthermore, if $\beta, 4, 7, 6, 9, a$ is inadmissible, then $[\beta, 4, 9]$ is a block and $\beta, 4, 8, 6, 9, a$ is admissible. Thus, after relabelling $1, 2, 4, 3, 5, 7, 6, 8, a, 9, b, c$ is an admissible sequence.

Hence, we will assume that $[\beta, 4, 7]$ does form a block, but we will use the label γ instead of β to keep using β as an unlabelled element in our arguments.

Assume that $[\gamma, 4, 7]$ is a block for some $\gamma \in \{1, 2, 3\}$. Look at the sets $\{1, 2, 6\}, \{1, 3, 6\}$ and $\{2, 3, 6\}$. Assume $\{\beta, 4, 5, 7, 8, 9, a, b, c\}$ can be partitioned into three blocks for some $\beta \in \{1, 2, 3\}$. As $[4, a, b]$ and $[4, 8, c]$ are blocks, no block of the partition may contain $\{\beta, 4\}$ as it would leave the three letters for the remaining two blocks, but there are no tips in $\{5, 7, 8, 9\}$. Because $[2, 5, a]$ and $[5, b, 8]$ are blocks, $[4, 8, c]$ cannot be a block of the partition as this would force either a block containing the pair $\{7, 9\}$, or one of 7 and 9 to be a tip. Therefore, any partition of $\{\beta, 4, 5, 7, 8, 9, a, b, c\}$ must use $[4, a, b]$ which forces $[7, 8, 9]$ and $[\beta, 5, c]$ to be the remaining parts. We conclude that at least two of $\{1, 2, 6\}, \{1, 3, 6\}$ and $\{2, 3, 6\}$ are good sets.

Consider the segment $\alpha, 4, 8, 5, \delta, a$ for any $\alpha \in \{1, 2, 3\}$ and $\delta \in \{7, 9\}$. The set $\{\alpha, 4, 8\}$ is not a block as $[4, 8, c]$ is a block. Similarly, $\{5, \delta, a\}$ is not a block because $[2, 5, a]$ is a block. Finally, neither $\{\alpha, 5, 8\}$ nor $\{5, 8, a\}$ form a block because $[5, 8, b]$ is a block. Therefore, the segment is admissible. This yields two choices for 9 so that we may relabel and obtain an admissible sequence of the correct form.

Case 3. There is at least one tiple and no two with tips in the same B_i. We are assuming there are one, two or three tiples. First label a point of L with a to accommodate Lemmas 5.1 and 5.2. If a belongs to no tiple, then there is only one tiple and we arbitrarily label the remaining points in L with b and c, and the points in B_2 so that $[5, b, c]$ is the tiple. Arbitrarily label the remaining two blocks B_1 and B_3.

We look at 9-sets of the form $\{\alpha, 4, 5, 6, 7, 8, 9, a, b\}$, where $\alpha \in B_1$, and the labels for B_3 are not yet chosen. As before, when a set of this form has a partition into three blocks, it must generate a perfect matching in K_{B_2,B_3}. The pair $5, b$ appears in the block $[5, b, c]$ so that no perfect matching has an edge labelled b incident with the point 5. This means there are at most two perfect matchings so that there is at least one set of the form under discussion which does not have a partition into three blocks.

Now consider the situation that the following two 9-sets

$$\{\alpha, 4, 5, 6, 7, 8, 9, a, b\} \text{ and } \{\beta, 4, 5, 6, 7, 8, 9, a, b\},$$

where $\alpha, \beta \in B_1$ and $\alpha \neq \beta$, have partitions into three blocks. Without loss of generality, we may assume B_3 is labelled so that the first perfect matching has the edge 47 labelled α, the edge 58 labelled a and the edge 69 labelled b. In the second perfect matching, the label b must be on the edge 48 which forces the edge 67 to be labelled a, thereby, leaving the edge 59 to be labelled β.

Given the edges as labelled in the preceding paragraph, we then have that $\{\alpha, \beta, c\}$ is a good set. If we show that $\{\gamma, 5, 6, 7, 8, 9, a, b, c\}$ has no partition into blocks, where $B_1 = \{\alpha, \beta, \gamma\}$, then $\{\alpha, \beta, 4\}$ also is a good set. The block B_3 cannot be a part in a partition as this would force $[5, b, c]$ to be a part which, in turn, would imply that $[\gamma, 6, a]$ is a block. This is not possible because $[6, 7, a]$ is a block.

Thus, any partition of the 9-set must generate a perfect matching in $K_{B_3,B}$, where $B = [5, b, c]$. The points a and 5 must be in the same block because neither

a, b nor a, c are in a block. So the block is $[5, 8, a]$. Lemma 3.2 forces $[6, 9, b]$ to be a block leaving $\gamma, 4, 7$. However, $[\alpha, 4, 7]$ is a block and the 9-set under discussion has no partition into three blocks.

We now label B_1 so that both $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good sets and 1 is an a-replacement. Now that 3 has been labelled we label 9 so that it is an a-replacement. Lemma 5.3 now implies that π is admissible because none of the problematic 3-sets are blocks.

If there is only one tiple and it contains a, then we label the other point of the tiple in L with c and choose B_2 as we just did with the tiple being $[5, a, c]$. The proof now goes through the same with a playing the role of b in the preceding proof.

If there are two tiples and a belongs to just one of them, then do the obvious labelling so that $[5, b, c]$ is one tiple and $[\alpha, a, c]$ is the other with $\alpha \in B_1$. The preceding argument still works because the tiple $[\alpha, a, c]$ restricts the perfect matchings even more

We must be more careful when a belongs to both tiples because there will be a tiple containing both a and b. What we do in this case after labelling a is to label the given blocks so that one tip is in B_2 and the other is in B_3. Label the remaining points of L and a point of B_2 so that $[4, a, b]$ is a block which implies that a and c are the points of the tiple whose tip lies in B_3.

It takes a little work to show there is a labelling such that both $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good sets. If all three of the sets of the form $\{\alpha, \beta, c\}$, $\alpha, \beta \in B_1$, are good, then use the fact that at least one set of the form $\{\alpha, \beta, 4\}$ is good to establish a labelling of B_1 that works.

If not all three are good, it is straightforward to show that at least two of the sets of the form $\{\alpha, \beta, c\}$ are good. Then assuming that the 9-set $\{\alpha, 4, 5, 6, 7, 8, 9, a, b\}$ has a partition into three disjoint blocks, we can show there are at least two good sets of the form $\{\alpha, \beta, 4\}$, $\alpha, \beta \in B_1$. The key is to show there is no partition of a 9-set of the form $\{\alpha, 5, 6, 7, 8, 9, a, b, c\}$ with $[x, a, c]$ as a part for $x \in B_3$. Then work with the perfect matchings arising in $K_{B_3,L}$ for possible partitions.

Hence, we label the points of B_1 so that $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good sets, and choose 9 so that Lemma 5.1 is valid. The sequence π is now admissible from Lemma 5.3 because the tiple $[4, a, b]$ does not change in any relabelling.

When there are three tiples, we choose a in the usual way and arbitrarily label the other two elements of L with b and c. We label B_2 so that it contains the tip in the tiple containing a, b. We label 4 so that $[4, a, b]$ is the tiple. We then label B_1 for the tip of the tiple containing b, c and B_3 for the tiple containing a, c. This puts more restrictions on the perfect matchings arising from partitions of certain 9-sets so the argument for the preceding case still works.

This completes the proof of Case 3.

Case 4. There are no tiples. Because there are no tiples, Lemma 5.3 means we need only find a labelling so that $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good sets. To do this we work with the distribution of the perfect matching edges. If there is a point $x \in L$ such that there are three edges in K_{B_i,B_j} labelled x, then there are no edges labelled

x incident with a point of B_k, $L = \{i, j, k\}$. So label x with a and make $B_2 = B_k$ and arbitrarily label B_1 and B_3. There are no edges labelled with a between B_2 and B_3 so that all sets of the forms $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good. Lemma 5.3 then implies that π is admissible.

Because the set B_i, B_j, B_k contain nine points, there are at most four edges with any of the labels coming from points of L. The preceding paragraph tells us we may assume the distribution between the three pairs of those three blocks is at most 1, 1, 2. Thus, there must be at least two points $x, y \in L$ such that for some pair B_i, B_j both x and y have at most one edge labelled with x and y, respectively, between the two blocks. Hence, if we label the two blocks B_2, B_3 and the points 1 and 2, then both $\{1, 2, c\}$ and $\{1, 2, 4\}$ are good sets. Lemma 5.3 then shows that π is admissible. ■

6 Further Considerations

We have seen that not all partial Steiner triple systems are sequenceable and there may be a connection between how many disjoint blocks such a system possesses and sequenceability. The following encapsulates the thought in a question.

Define a function T as follows. For each positive integer k, let $T(k)$ be the smallest integer so that if $\mathcal{T}$ is any partial Steiner triple system with k disjoint blocks, but not $k+1$, and at least $3k+T(k)$ vertices, then $\mathcal{T}$ is sequenceable. Let's see that this is a well-defined function.

Suppose that $\mathcal{T}$ has k disjoint blocks, but not $k+1$ disjoint blocks, and has order at least $15k-5$. Call the points of k disjoint blocks $U = \{u_1, u_2, \ldots, u_{3k}\}$ and let the remaining points be $V = \{v_1, v_2, \ldots, v_{12k-5}, \ldots\}$. Define a sequence by starting with $u_1, u_2, u_3, \ldots, u_{3k}$ and then between u_i and u_{i+1}, $1 \leq i \leq 3k-1$, insert five arbitrary points from V. Any points left over from V are tacked on at the end. Call the resulting sequence σ. It is clear that no segment of length $3m$, $m > 1$, can have a partition into m blocks because there are at most $m-1$ points from U. This means at least one of the blocks must contain only points from V which is a contradiction.

Segments of length 3 may be blocks but they may be eliminated by simple interchanges. For example, if $[v_{5i}, u_i, v_{5i+1}]$ is a block, then interchange v_{5i-1} and v_{5i} and the new segment is not a block. There are several types but all are easily modified to finally obtain an admissible sequence.

In this paper we have proven that $T(1) = T(2) = T(3) = 0$ and $T(4) > 0$. An obvious problem is: what can be said about the function T?

References

[1] B. Alspach and G. Liversidge, On strongly sequenceable abelian groups, *Art Discrete Appl. Math.*, to appear.

[2] R. F. Bailey and A.C. Burgess, Generalized packing designs, *Discrete Math.* **313** (2013), 1167–1190.

[3] C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, Oxford, 1999.

[4] S. M. Johnson. A new upper bound for error-correcting codes, *IEEE Trans. Inform. Theory* **8** (1962), 203–207.

[5] J. Schönheim, On coverings, *Pacific J. Math* **14** (1964), 1405–1411.

[6] J. Schönheim, On maximal systems of k-tuples, *Studia Sci. Math. Hungar* **1** (1966), 363–368.

[7] J. Spencer, Maximal consistent families of triples, *J. Combin. Theory* **5** (1968), 1–8.