Bounding the gap between the McCormick relaxation and the convex hull for bilinear functions

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Abstract

We investigate how well the graph of a bilinear function $b : [0,1]^n \to \mathbb{R}$ can be approximated by its McCormick relaxation. In particular, we are interested in the smallest number c such that the difference between the concave upper bounding and convex lower bounding functions obtained from the McCormick relaxation approach is at most c times the difference between the concave and convex envelopes. Answering a question of Luedtke, Namazifar and Linderoth, we show that this factor c cannot be bounded by a constant independent of n. More precisely, we show that for a random bilinear function b we have asymptotically almost surely $c \geqslant \sqrt{n}/4$. On the other hand, we prove that $c \leqslant 600\sqrt{n}$, which improves the linear upper bound proved by Luedtke, Namazifar and Linderoth. In addition, we present an alternative proof for a result of Misener, Smadbeck and Floudas characterizing functions b for which the McCormick relaxation is equal to the convex hull.

An important technique in global optimization is the construction of convex envelopes for nonconvex functions over convex sets (see for instance [9]), and consequently, there has been a lot of work on such envelopes of special classes of functions [1, 5, 15, 17, 19]. Many modern global optimization solvers [2, 16, 18] follow a general approach, proposed by McCormick [11], that is based on a linear relaxation for bilinear terms. Luedtke, Namazifar, and Linderoth [10] proved a number of statements about the strength of the resulting relaxations for multilinear functions. In this note we extend their results on bilinear functions. In particular, we characterize the bilinear functions for which the McCormick relaxation describes the convex hull, we improve the upper bound on this approximation ratio, and we prove that our new bound is asymptotically tight, thus providing a negative answer to a question from [10].

Consider a bilinear function $b : [0,1]^n \to \mathbb{R}$ given by

$$b(\boldsymbol{x}) = \sum_{ij \in E} a_{ij} x_i x_j$$

with coefficients $a_{ij} \in \mathbb{R}$, where $G = (V, E)$ is an undirected graph with vertex set $V = \{1, \ldots, n\}$, and we write ij for $\{i, j\}$. The graph of b is the set

$$B = \{(\boldsymbol{x}, z) \in [0,1]^n \times \mathbb{R} \ : \ z = b(\boldsymbol{x})\},$$

and we are interested in relaxations of the convex hull of B, which can be characterized as (see [15])

$$\operatorname{conv}(B) = \left\{(\boldsymbol{x}, z) \in [0,1]^n \times \mathbb{R} \ : \ \exists \boldsymbol{\lambda} \in \Delta_{2^n} \text{ with } \boldsymbol{x} = \sum_{k=1}^{2^n} \lambda_k \boldsymbol{x}^k, \ z = \sum_{k=1}^{2^n} \lambda_k b(\boldsymbol{x}^k)\right\},$$

where $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{2^n}$ are the vertices of $[0,1]^n$ and $\Delta_{2^n} = \{\boldsymbol{\lambda} \in [0,1]^{2^n} \ : \ \sum_{k=1}^{2^n} \lambda_k = 1\}$ is the $(2^n - 1)$-simplex. The McCormick relaxation [11] approximates B by introducing for each bilinear term $x_i x_j$ a new variable y_{ij} together with the constraints $0 \leqslant y_{ij} \leqslant x_i$, $y_{ij} \leqslant x_j$ and $y_{ij} \geqslant x_i + x_j - 1$. More precisely, we define two convex polytopes $P = P(G) \subseteq \mathbb{R}^{n+|E|}$ and $Q = Q(b) \subseteq \mathbb{R}^{n+1}$:

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) \in [0,1]^n \times [0,1]^{|E|} \ : \ y_{ij} \leqslant x_i, \ y_{ij} \leqslant x_j, \ y_{ij} \geqslant x_i + x_j - 1 \text{ for all } ij \in E\}, \text{ and}$$

$$Q = \left\{(\boldsymbol{x}, z) \in [0,1]^n \times \mathbb{R} \ : \ \exists \boldsymbol{y} \in [0,1]^{|E|} \text{ with } (\boldsymbol{x}, \boldsymbol{y}) \in P \text{ and } z = \sum_{ij \in E} a_{ij} y_{ij}\right\}.$$

1 Main results

We have $\operatorname{conv}(B) \subseteq Q$ and it is natural to ask how well Q approximates $\operatorname{conv}(B)$. Following the notation from [10] we denote the concave and convex envelopes of the graph of b by $\operatorname{cav}[b]$ and $\operatorname{vex}[b]$, respectively, and the corresponding upper and lower McCormick envelopes by $\operatorname{mcu}[b]$ and $\operatorname{mcl}[b]$, respectively. These envelopes are functions from $[0,1]^n$ to $\mathbb{R}$ defined by

$$\operatorname{cav}[b](\boldsymbol{x}) = \max\{z \ : \ (\boldsymbol{x}, z) \in \operatorname{conv}(B)\}, \qquad \operatorname{vex}[b](\boldsymbol{x}) = \min\{z \ : \ (\boldsymbol{x}, z) \in \operatorname{conv}(B)\},$$
$$\operatorname{mcu}[b](\boldsymbol{x}) = \max\{z \ : \ (\boldsymbol{x}, z) \in Q\}, \qquad \operatorname{mcl}[b](\boldsymbol{x}) = \min\{z \ : \ (\boldsymbol{x}, z) \in Q\}.$$

We call the corresponding differences *convex hull gap*, denoted by $\operatorname{chgap}[b]$, and *McCormick gap*, denoted by $\operatorname{mcgap}[b]$, respectively. In other words,

$$\operatorname{chgap}[b](\boldsymbol{x}) = \operatorname{cav}[b](\boldsymbol{x}) - \operatorname{vex}[b](\boldsymbol{x}) \qquad \text{and} \qquad \operatorname{mcgap}[b](\boldsymbol{x}) = \operatorname{mcu}[b](\boldsymbol{x}) - \operatorname{mcl}[b](\boldsymbol{x}).$$

Our measure for the quality of Q as an approximation of $\operatorname{conv}(B)$ is the number

$$c^*(b) = \inf\{c \in \mathbb{R} \ : \ \operatorname{mcgap}[b](\boldsymbol{x}) \leqslant c \operatorname{chgap}[b](\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in [0,1]^n\}.$$

In [10] it is proved that under the condition that all nonzero coefficients are positive we have

$$c^*(b) \leqslant 2 - \frac{1}{\lceil \chi(G)/2 \rceil},$$

where $\chi(G)$ is the chromatic number of the graph G. For arbitrary coefficients, the much weaker bound $c^*(b) \leqslant n$ is established, and it is left as an open question if $c^*(b)$ can be bounded by a constant independent of n in the general case. We provide a negative answer to this question by proving the following theorem.

Theorem 1. *Let $G = (V, E)$ be the complete graph on the vertex set $V = \{1, \ldots, n\}$, and let $b(\boldsymbol{x}) = \sum_{ij \in E} a_{ij} x_i x_j$ where the coefficients a_{ij} are chosen independently and uniformly at random from $\{1, -1\}$. For $\boldsymbol{x} = (1/2, 1/2, \ldots, 1/2)$ we have*

$$\lim_{n\to\infty} \boldsymbol{P}\left(\operatorname{mcgap}[b](\boldsymbol{x}) \geqslant \frac{\sqrt{n}}{4} \operatorname{chgap}[b](\boldsymbol{x})\right) = 1.$$

Moreover, we show that $\sqrt{n}$ is the correct leading term for the asymptotics.

Theorem 2. *For every bilinear function $b : [0,1]^n \to \mathbb{R}$, and every $\boldsymbol{x} \in [0,1]^n$,*

$$\operatorname{mcgap}[b](\boldsymbol{x}) \leqslant 600\sqrt{n} \operatorname{chgap}[b](\boldsymbol{x}).$$

In order to prove Theorem 2 we establish the following discrepancy result which might be of independent interest.

Theorem 3. *Let $G = (V, E)$ be the complete graph on the vertex set $V = \{1, \ldots, n\}$, and let $\boldsymbol{a} = (a_{ij}) \in \mathbb{R}^{n(n-1)/2}$ be a vector of edge weights. There exists a set $U \subseteq V$ such that*

$$\left| \sum_{ij \in \delta(U)} a_{ij} \right| \geqslant \frac{1}{600\sqrt{n}} \sum_{ij \in E} |a_{ij}|$$

where the sum on the LHS is over the set $\delta(U) \subset E$ of edges with exactly one vertex in U.

Finally, we give a characterization of the functions b with with $Q = \operatorname{conv}(B)$. Let us call an edge $ij \in E$ positive if $a_{ij} > 0$ and negative if $a_{ij} < 0$. Without loss of generality we assume that $a_{ij} \neq 0$ for all $ij \in E$, so every edge is either positive or negative. The following theorem is a direct consequence of Theorem 3.10 in [12] which states that the McCormick inequalities are sufficient to describe the convex envelope of the graph of b if and only if the number of positive edges in every cycle is even. In order to capture the concave envelope as well we just need to ensure that every cycle also contains an even number of negative edges.

Theorem 4. *We have $Q = \operatorname{conv}(B)$ if and only if every cycle in G has an even number of positive edges and an even number of negative edges.*

As a consequence, we can have $Q = \operatorname{conv}(B)$ only if G is bipartite. Moreover, if G is a forest then $Q = \operatorname{conv}(B)$ for every choice of the coefficients a_{ij}, but as soon as G contains a cycle we can write down coefficients a_{ij} such that $Q \neq \operatorname{conv}(B)$.

Our proofs are based on the following ideas from [10]. For a vector $\boldsymbol{x} \in [0,1]^n$, let $T_f = T_f(\boldsymbol{x}) \subseteq V$ be the set of indices of fractional values, i.e., $T_f = \{i \in V \ : \ 0 < x_i < 1\}$. The proof of $\operatorname{mcgap}[b](\boldsymbol{x}) \leqslant n \operatorname{chgap}[b](\boldsymbol{x})$ for all $\boldsymbol{x} \in H$ in [10] proceeds in 3 steps.

1. $\boldsymbol{x} \in \{0, 1/2, 1\}^n \implies \operatorname{mcgap}[b](\boldsymbol{x}) = \dfrac{1}{2} \displaystyle\sum_{ij \in E,\, i,j \in T_f} |a_{ij}|.$

2. $\boldsymbol{x} \in \{0, 1/2, 1\}^n \implies \operatorname{chgap}[b](\boldsymbol{x}) \geqslant \dfrac{1}{2|T_f|} \displaystyle\sum_{ij \in E,\, i,j \in T_f} |a_{ij}|.$

3. The function $c\,\mathrm{chgap}[b](\boldsymbol{x}) - \mathrm{mcgap}[b](\boldsymbol{x})$ is minimized at some $\boldsymbol{x} \in \{0, 1/2, 1\}^n$.

We will show that the argument for step 2 can be modified to provide a lower bound for $\mathrm{chgap}[b](\boldsymbol{x})$ in terms of the difference between the maximum and the minimum cut in the subgraph of G induced by T_f. Theorem 1 then follows by applying the Chernoff inequality, Theorem 3, and consequently Theorem 2, is proved using probabilistic arguments that have been developed in the context of studying the discrepancy of graphs [3, 6, 7], and Theorem 4 is a consequence of the observation that the difference between the maximum and the minimum cut is equal to the sum of the absolute values of all weights if and only if the sets of positive and negative edges form two cuts of the graph.

2 Proofs of the theorems

2.1 Characterizing the convex hull gap in terms of cuts

Let $G = (V, E)$ be a graph with vertex set $V = [n]$. We use the following notation from [10].

- For $X \subseteq V$, $\gamma(X)$ is the set of edges with both vertices in X.
- For $X \subseteq V$, $\delta(X)$ is the set of edges with exactly one vertex in X.
- For $X, Y \subseteq V$ with $X \cap Y = \emptyset$, $\delta(X, Y)$ is the set of edges with one vertex in X and one vertex in Y.
- For $i \in V$, $\mathcal{S}_i$ is the collection of vertex sets that contain i, i.e., $\mathcal{S}_i = \{W \subseteq V \ :\ i \in W\}$.
- For $Z \subseteq E$, we put $a(Z) = \sum_{ij \in Z} a_{ij}$.

We denote the maximum and the minimum weight of a cut in the subgraph induced by $X \subseteq V$ with $\mu^+(X)$ and $\mu^-(X)$, i.e.,

$$\mu^+(X) = \max\left\{\sum_{ij\in\delta(U_1,U_2)} a_{ij}\ :\ U_1 \cup U_2 = X,\ U_1 \cap U_2 = \emptyset\right\},$$

$$\mu^-(X) = \min\left\{\sum_{ij\in\delta(U_1,U_2)} a_{ij}\ :\ U_1 \cup U_2 = X,\ U_1 \cap U_2 = \emptyset\right\}.$$

We identify $\{0, 1\}^n$ with the power set of V in the natural way: $\boldsymbol{x} \in \{0, 1\}^n$ is identified with the set $\{i\ :\ x_i = 1\}$. We start by establishing that the upper bound for $\mathrm{chgap}[b](\boldsymbol{x})$ in terms of cuts in induced subgraphs of G, proved in [10] (Lemma 3.10), is tight.

Lemma 1. *Let $\boldsymbol{x} \in \{0, 1/2, 1\}^n$ and put $T_1 = \{i \in V\ :\ x_i = 1\}$ and $T_f = \{i \in V\ :\ x_i = 1/2\}$. Then*

$$\mathrm{vex}[b](\boldsymbol{x}) = a(\gamma(T_1)) + \frac{1}{2}a(\delta(T_1, T_f)) + \frac{1}{2}a(\gamma(T_f)) - \frac{1}{2}\mu^+(T_f), \tag{1}$$

$$\mathrm{cav}[b](\boldsymbol{x}) = a(\gamma(T_1)) + \frac{1}{2}a(\delta(T_1, T_f)) + \frac{1}{2}a(\gamma(T_f)) - \frac{1}{2}\mu^-(T_f), \tag{2}$$

$$\mathrm{chgap}[b](\boldsymbol{x}) = \frac{1}{2}\left(\mu^+(T_f) - \mu^-(T_f)\right). \tag{3}$$

Proof. We start by writing $\mathrm{vex}[b](\boldsymbol{x})$ as follows:

$$\mathrm{vex}[b](\boldsymbol{x}) = \min\left\{\sum_{X\subseteq T_f}\lambda_X a(\gamma(X\cup T_1))\ :\ \sum_{X\subseteq T_f}\lambda_X = 1,\ \sum_{X\in\mathcal{S}_i}\lambda_X = 1/2\ \forall i\in T_f,\ \boldsymbol{\lambda}\geqslant\mathbf{0}\right\}.$$

Now

$$a(\gamma(X\cup T_1)) = a(\gamma(T_1)) + a(\delta(T_1,X)) + a(\gamma(X)),$$

and, for any $\boldsymbol{\lambda}$ satisfying $\sum_{X\in\mathcal{S}_i}\lambda_X = 1/2$ for all $i\in T_f$, we have that

$$\sum_{X\subseteq T_f}\lambda_X a(\delta(T_1,X)) = \sum_{X\subseteq T_f}\lambda_X\sum_{i\in X}\sum_{j\in T_1,\, ij\in E} a_{ij} = \sum_{j\in T_1}\sum_{i\in T_f,\, ij\in E}\sum_{X\in\mathcal{S}_i}\lambda_X a_{ij} = \frac{1}{2}a(\delta(T_1,T_f)).$$

Thus

$$\mathrm{vex}[b](\boldsymbol{x}) = a(\gamma(T_1)) + \frac{1}{2}a(\delta(T_1,T_f)) + M,$$

where

$$M = \min\left\{\sum_{X\subseteq T_f}\lambda_X a(\gamma(X))\ :\ \sum_{X\subseteq T_f}\lambda_X = 1,\ \sum_{X\in\mathcal{S}_i}\lambda_X = 1/2\ \forall i\in T_f,\ \boldsymbol{\lambda}\geqslant\mathbf{0}\right\}.$$

As in the proof of Lemma 3.10 in [10], we can set $\lambda_{U_1} = \lambda_{U_2} = 1/2$ for a maximum cut (U_1,U_2) in the subgraph induced by T_f, which yields

$$M \leqslant \frac{1}{2}\left[a(\gamma(U_1)) + a(\gamma(U_2))\right] = \frac{1}{2}\left[a(\gamma(T_f)) - \mu^+(T_f)\right].$$

In order to prove that this bound is tight, we look at the dual

$$M = \max\left\{y + \frac{1}{2}\sum_{i\in T_f} z_i\ :\ y + \sum_{i\in X} z_i \leqslant a(\gamma(X))\ \forall X\subseteq T_f\right\}.$$

Setting $y = -\mu^+(T_f)/2$ and

$$z_i = \frac{1}{2}\sum_{j\in T_f\,:\,ij\in E} a_{ij} \qquad \text{for } i\in T_f$$

we get a feasible solution, because for every $X\subseteq T_f$ we have

$$y + \sum_{i\in X} z_i = -\frac{1}{2}\mu^+(T_f) + \frac{1}{2}a(\delta(X,T_f\setminus X)) + a(\gamma(X)) \leqslant a(\gamma(X)).$$

Since the objective value

$$y + \frac{1}{2}\sum_{i\in T_f} z_i = -\frac{1}{2}\mu^+(T_f) + \frac{1}{4}\sum_{i\in T_f}\sum_{j\in T_f\,:\,ij\in E} a_{ij} = \frac{1}{2}\left(a(T_f) - \mu^+(T_f)\right)$$

is equal to the upper bound for M we have proved that M is equal to this value, and this concludes the proof of (1). For (2) we use the same method to get

$$\operatorname{cav}[b](\boldsymbol{x}) = a(\gamma(T_1)) + \frac{1}{2}a(\delta(T_1,T_f)) + M',$$

where M' is characterized by

$$\begin{aligned} M' &= \max\left\{\sum_{X\subseteq T_f}\lambda_X a(\gamma(X)) \ :\ \sum_{X\subseteq T_f}\lambda_X = 1,\ \sum_{X\in\mathcal{S}_i}\lambda_X = \frac{1}{2}\ \forall i\in T_f,\ \boldsymbol{\lambda}\geqslant\mathbf{0}\right\} \\ &= \min\left\{y + \frac{1}{2}\sum_{i\in T_f} z_i \ :\ y + \sum_{i\in X} z_i \geqslant a(\gamma(X))\ \forall X\subseteq T_f\right\}. \end{aligned}$$

Taking a minimum cut (U_1', U_2') we get a primal solution $\lambda_{U_1'} = \lambda_{U_2'} = 1/2$ and a corresponding dual solution $y = -\mu^-(T_f)/2$,

$$z_i = \frac{1}{2}\sum_{j\in T_f\,:\,ij\in E} a_{ij} \qquad \text{for } i\in T_f.$$

Finally, (3) follows by taking the difference of (1) and (2). □

By Lemma 3.9 from [10], we have $\operatorname{mcgap}[b](\boldsymbol{x}) = \frac{1}{2}\sum_{ij\in\gamma(T_f)}|a_{ij}|$ for all $\boldsymbol{x}\in\{0,1/2,1\}^n$, and using the convexity argument from the proof of Theorem 3.12 in [10] we get the following corollary.

Corollary 1. *Let c be a number such that $\sum_{ij\in\gamma(X)}|a_{ij}| \leqslant c\,(\mu^+(X) - \mu^-(X))$ for all $X\subseteq V$. Then for all $\boldsymbol{x}\in[0,1]^n$, $\operatorname{mcgap}[b](\boldsymbol{x}) \leqslant c\operatorname{chgap}[b](\boldsymbol{x})$.* □

2.2 The lower bound

Proof of Theorem 1. Let $G = (V,E)$ be the complete graph on the vertex set $V = \{1,\dots,n\}$ and consider the bilinear function is

$$b(\boldsymbol{x}) = \sum_{ij\in E} a_{ij}x_ix_j$$

where the coefficients a_{ij} are randomly chosen from $\{1,-1\}$ (independently and uniformly). Using the Chernoff inequality and the fact that $\delta(U_1,U_2)\leqslant n^2/4$ for every $\operatorname{cut}(U_1,U_2)$, we have that,

$$\mathbf{P}\left(\left|\sum_{ij\in\delta(U_1,U_2)} a_{ij}\right| > 0.6n^{3/2}\right) < 2e^{-0.72n}.$$

Taking the union bound over all 2^{n-1} cuts gives

$$\mathbf{P}\left(-0.6n^{3/2} \leqslant \sum_{ij\in\delta(U_1,U_2)} a_{ij} \leqslant 0.6n^{3/2} \text{ for all cuts } (U_1,U_2)\right) \geqslant 1 - 2^n e^{-0.72n},$$

which tends to 1 as $n \to \infty$. So

$$\lim_{n\to\infty} \mathbf{P}\left(\mu^+(V) - \mu^-(V) \leqslant 1.2n^{3/2}\right) = 1,$$

and consequently, for $\boldsymbol{x} = (1/2, 1/2, \ldots, 1/2)$, with probability tending to 1 as $n \to \infty$,

$$\mathrm{mcgap}[b](\boldsymbol{x}) = \frac{|E|}{2} = \frac{n(n-1)}{4} > \frac{\sqrt{n}}{4} 0.6n^{3/2} \geqslant \frac{\sqrt{n}}{4}\,\mathrm{chgap}[b](\boldsymbol{x}). \qquad \square$$

Theorem 1 ensures that there are many functions with a large ratio between the McCormick gap and the convex hull gap. Next we construct an explicit example for every n. We define a bilinear function $b : [0,1]^n \to \mathbb{R}$ as follows. Let $k = \lceil \log_2(n) \rceil$. With vertex $i \in V = \{1, \ldots, n\}$ we associate the vector $\boldsymbol{i} = (i_1, \ldots, i_k) \in \{0,1\}^k$ of the digits of $i-1$ in binary representation, i.e., $i - 1 = i_1 2^0 + i_2 2^1 + \cdots + i_k 2^{k-1}$, and we put $a_{ij} = (-1)^{\langle \boldsymbol{i}, \boldsymbol{j} \rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product, $\langle \boldsymbol{i}, \boldsymbol{j} \rangle = i_1 j_1 + \cdots + i_k j_k$. The following lemma is a standard discrepancy result (see for instance Chapter 10 in [4]), but for convenience we include the short proof.

Lemma 2. *We have $\mu^+(V) \leqslant (n^{3/2})/\sqrt{2}$ and $\mu^-(V) \geqslant -(n^{3/2})/\sqrt{2}$.*

Proof. Let H be the $2^k \times 2^k$ matrix with rows and columns indexed by binary strings of length k with $H_{ij} = (-1)^{\langle i,j \rangle}$. Then H is a Hadamard matrix, i.e., $H^T H = 2^k I$ where I is the identity matrix of size $2^k \times 2^k$. Therefore, $\|H\boldsymbol{v}\|_2 \leqslant 2^{k/2} \|\boldsymbol{v}\|_2$ for every $\boldsymbol{v}$. The vertices in V correspond to the first n rows and columns of H, and therefore we can identify a subset $U \subseteq V$ with a vector $\boldsymbol{u} \in \{0,1\}^{2^k}$. For a cut $(U, V \setminus U)$, let $\boldsymbol{w}$ be the vector corresponding to $V \setminus U$. We can bound the weight of this cut by

$$\left| \sum_{ij \in \delta(U)} a_{ij} \right| = \left| \sum_{i \in U} \sum_{j \in V\setminus U} (-1)^{\langle \boldsymbol{i}, \boldsymbol{j} \rangle} \right| = \left| \boldsymbol{u}^T H \boldsymbol{w} \right| \leqslant \|\boldsymbol{u}\|_2 \|H\boldsymbol{w}\|_2 \leqslant 2^{k/2} \|\boldsymbol{u}\|_2 \|\boldsymbol{w}\|_2.$$

Now $(u_1 + \cdots + u_{2^k}) + (w_1 + \cdots + w_{2^k}) = n$, and the AM-GM inequality yields

$$\begin{aligned} \|\boldsymbol{u}\|_2 \|\boldsymbol{w}\|_2 = \sqrt{\left(u_1^2 + \cdots + u_{2^k}^2\right)\left(w_1^2 + \cdots + w_{2^k}^2\right)} &\leqslant \frac{\left(u_1^2 + \cdots + u_{2^k}^2\right) + \left(w_1^2 + \cdots + w_{2^k}^2\right)}{2} \\ &= \frac{(u_1 + \cdots + u_{2^k}) + (w_1 + \cdots + w_{2^k})}{2} = n/2 \end{aligned}$$

Consequently,

$$\left| \sum_{ij \in \delta(U)} a_{ij} \right|^2 \leqslant 2^k n^2/4 \leqslant n^3/2. \qquad \square$$

From Lemmas 1 and 2 it follows that $\mathrm{chgap}[b](1/2, \ldots, 1/2) \leqslant \frac{n^{3/2}}{\sqrt{2}}$, and therefore

$$\mathrm{mcgap}[b](1/2, \ldots, 1/2) = \frac{n(n-1)}{4} \geqslant \frac{\sqrt{2}}{4}\left(\sqrt{n} - \frac{1}{\sqrt{n}}\right) \mathrm{chgap}[b](1/2, \ldots, 1/2).$$

So for $n \geqslant 18$ we have $\mathrm{mcgap}[b](1/2, \ldots, 1/2) \geqslant \frac{\sqrt{n}}{3}\,\mathrm{chgap}[b](1/2, \ldots, 1/2)$.

2.3 The upper bound

The unit weight case of Theorem 3 has been proved in [7], and here we extend this argument to the general case. We start with a partition $V = L \cup R$ such that

$$\sum_{ij \in \delta(L,R)} |a_{ij}| \geqslant \frac{1}{2} \sum_{ij \in E} |a_{ij}|. \tag{4}$$

To see why such a partition exists, consider any random partition of vertices into two subsets, where with equal probability each vertex is assigned to any one of the subsets. Taking the edge weights to be $|a_{ij}|$, the expected value of the resulting cut is $\frac{1}{2}\sum_{ij\in E}|a_{ij}|$. Therefore, there exists a specific partition $V = L \cup R$ which satisfies (4).

Now we choose a random subset $S \subseteq L$ ($\mathbf{P}(i \in S) = 1/2$ for every $i \in L$ and these events are independent).

Lemma 3. *For every $j \in R$,*

$$\boldsymbol{P}\left(\left|\sum_{i\in S} a_{ij}\right| \geqslant \frac{1}{4}\left(\sum_{i\in L} a_{ij}^2\right)^{1/2}\right) \geqslant \frac{1}{24}.$$

Proof. Fix $j \in R$, and let X_i for $i \in L$ be the random variable defined by $X_i = 1$ if $i \in S$ and $X_i = -1$ if $i \notin S$, so that

$$\sum_{i\in S} a_{ij} = \frac{1}{2}\sum_{i\in L} a_{ij} + \frac{1}{2}\sum_{i\in L} a_{ij}X_i.$$

For $Z = \left(\sum_{i\in L} a_{ij}X_i\right)^2$, we have $\mathbf{E}(Z) = \sum_{i\in L} a_{ij}^2$, and therefore

$$\mathbf{P}\left(Z \geqslant \frac{1}{2}\sum_{i\in L} a_{ij}^2\right) \geqslant \frac{1}{4}\frac{\left(\sum_{i\in L} a_{ij}^2\right)^2}{\mathbf{E}(Z^2)} \tag{5}$$

by the Paley-Zygmund inequality. From the Khintchine inequality with the Haagerup bounds [8, 13] it follows that

$$\mathbf{E}(Z^2) = \mathbf{E}\left(\left(\sum_{i\in L} a_{ij}X_i\right)^4\right) \leqslant 3\left(\sum_{i\in L} a_{ij}^2\right)^2,$$

hence (5) implies

$$\mathbf{P}\left(\sum_{i\in L} a_{ij}X_i \geqslant \frac{1}{\sqrt{2}}\left(\sum_{i\in L} a_{ij}^2\right)^{1/2}\right) = \mathbf{P}\left(\sum_{i\in L} a_{ij}X_i \leqslant -\frac{1}{\sqrt{2}}\left(\sum_{i\in L} a_{ij}^2\right)^{1/2}\right)$$
$$= \frac{1}{2}\mathbf{P}\left(Z \geqslant \frac{1}{2}\sum_{i\in L} a_{ij}^2\right) \geqslant \frac{1}{24}.$$

This gives the implications

$$\sum_{i\in L} a_{ij} \geqslant 0 \implies \mathbf{P}\left(\sum_{i\in S} a_{ij} \geqslant \frac{1}{2\sqrt{2}}\left(\sum_{i\in L} a_{ij}^2\right)^{1/2}\right) \geqslant \frac{1}{24},$$
$$\sum_{i\in L} a_{ij} \leqslant 0 \implies \mathbf{P}\left(\sum_{i\in S} a_{ij} \leqslant -\frac{1}{2\sqrt{2}}\left(\sum_{i\in L} a_{ij}^2\right)^{1/2}\right) \geqslant \frac{1}{24},$$

and thus concludes the proof of the lemma (using $1/4 < 1/(2\sqrt{2})$). □

From Lemma 3 and Cauchy-Schwarz we obtain

$$\mathbf{E}\left(\sum_{j\in R}\left|\sum_{i\in S} a_{ij}\right|\right) \geqslant \frac{1}{96}\sum_{j\in R}\left(\sum_{i\in L} a_{ij}^2\right)^{1/2} \geqslant \frac{1}{96}\sum_{j\in R}\left(\frac{1}{|L|^{1/2}}\sum_{i\in L}|a_{ij}|\right)$$
$$\geqslant \frac{1}{96\sqrt{n}}\sum_{i\in L}\sum_{j\in R}|a_{ij}| \geqslant \frac{1}{200\sqrt{n}}\sum_{ij\in E}|a_{ij}|,$$

where the last inequality follows from (4). This implies that there exists a set $S \subseteq L$ with

$$\sum_{j\in R}\left|\sum_{i\in S} a_{ij}\right| \geqslant \frac{1}{200\sqrt{n}}\sum_{ij\in E}|a_{ij}|. \tag{6}$$

Fix such a set S and define the sets

$$R_+ = \left\{j \in R \;:\; \sum_{i\in S} a_{ij} \geqslant 0\right\}, \qquad R_- = \left\{j \in R \;:\; \sum_{i\in S} a_{ij} < 0\right\}.$$

Then

$$\sum_{j\in R}\left|\sum_{i\in S} a_{ij}\right| = \sum_{j\in R_+}\sum_{i\in S} a_{ij} - \sum_{j\in R_-}\sum_{i\in S} a_{ij},$$

and it follows from (6) that

$$\max\left\{\sum_{j\in R_+}\sum_{i\in S} a_{ij},\ -\sum_{j\in R_-}\sum_{i\in S} a_{ij}\right\} \geqslant \frac{1}{400\sqrt{n}}\sum_{ij\in E}|a_{ij}|.$$

Without loss of generality, we assume that the maximum is obtained by the first term, i.e.,

$$\sum_{j\in R_+}\sum_{i\in S} a_{ij} \geqslant \frac{1}{400\sqrt{n}}\sum_{ij\in E}|a_{ij}|.$$

We conclude the proof of Theorem 3 as suggested in [14]. Let $W = V \setminus (S \cup R_+)$ and distinguish three cases.

Case 1. If $\sum_{ij\in\delta(S,W)} a_{ij} \geqslant -\frac{1}{1200\sqrt{n}}\sum_{ij\in E}|a_{ij}|$ then we can take $U=S$:

$$\sum_{ij\in\delta(S)} a_{ij} = \sum_{ij\in\delta(S,R_+)} a_{ij} + \sum_{ij\in\delta(S,W)} a_{ij} \geqslant \left(\frac{1}{400}-\frac{1}{1200}\right)\frac{1}{\sqrt{n}}\sum_{ij\in E}|a_{ij}|.$$

Case 2. If $\sum_{ij\in\delta(R_+,W)} a_{ij} \geqslant -\frac{1}{1200\sqrt{n}}\sum_{ij\in E}|a_{ij}|$ then we can take $U=R_+$:

$$\sum_{ij\in\delta(R_+)} a_{ij} = \sum_{ij\in\delta(S,R_+)} a_{ij} + \sum_{ij\in\delta(R_+,W)} a_{ij} \geqslant \left(\frac{1}{400}-\frac{1}{1200}\right)\frac{1}{\sqrt{n}}\sum_{ij\in E}|a_{ij}|.$$

Case 3. If $\max\left\{\sum_{ij\in\delta(S,W)} a_{ij},\ \sum_{ij\in\delta(R_+,W)} a_{ij}\right\} < -\frac{1}{1200\sqrt{n}}\sum_{ij\in E}|a_{ij}|$ then we can take $U = S\cup R_+$:

$$\sum_{ij\in\delta(S\cup R_+)} a_{ij} = \sum_{ij\in\delta(S,W)} a_{ij} + \sum_{ij\in\delta(R_+,W)} a_{ij} < -\frac{1}{600\sqrt{n}}\sum_{ij\in E}|a_{ij}|.$$

Proof of Theorem 2. Applying Theorem 3 to the subgraph induced by a vertex set $X\subseteq V$, yields

$$\mu^+(X)-\mu^-(X) \geqslant \frac{1}{600\sqrt{|X|}}\sum_{ij\in\gamma(X)}|a_{ij}| \geqslant \frac{1}{600\sqrt{n}}\sum_{ij\in\gamma(X)}|a_{ij}|,$$

and now Corollary 1 implies the statement of the theorem. □

2.4 Characterization of equality

As mentioned in Section 1, Theorem 4 is a direct consequence of Theorem 3.10 in [12]. We include the following short proof in order to show how this result can be derived from the correspondence between the convex hull gap and the range of cut weights in the graph G.

Proof of Theorem 4. Suppose that every cycle in G has an even number of positive edges and an even number of negative edges. Now let $X\subseteq V$ be any vertex set. We introduce two equivalence relations, $\sim_1$ and $\sim_2$, on X. For the first, we put $i\sim_1 j$ if G contains a path between i and j consisting of positive edges. Similarly, we put $i\sim_2 j$ if G contains a path between i and j consisting of negative edges. Let G_1 and G_2 be the quotient graphs, i.e., the vertices of G_k $(k=1,2)$ are the equivalence classes for $\sim_k$ and there is an edge between two classes $[i]$ and $[j]$ in G_k if there is an edge in G between any element of $[i]$ and any element of $[j]$. Note that the edges in G_1 correspond to negative edges of G, and the edges in G_2 correspond to positive edges of G. If every cycle in G contains an even number of positive and negative edges, then G_1 and G_2 are bipartite. The partition of G_1 induces a partition $X=U_1\cup U_2$ such that $\delta(U_1,U_2)$ is the set of negative edges in $\gamma(X)$, and the partition of

G_2 induces a partition $X = U_1' \cup U_2'$ such that $\delta(U_1', U_2')$ is the set of positive edges in $\gamma(X)$. Consequently, $\mu^+(X) - \mu^-(X) = \sum_{i\in\gamma(X)} |a_{ij}|$, and, since $X \subseteq V$ was chosen arbitrarily, it follows, by Corollary 1, that $\mathrm{mcgap}[b](\boldsymbol{x}) = \mathrm{chgap}[b](\boldsymbol{x})$ for all $\boldsymbol{x} \in [0,1]^n$.

Conversely, suppose that there exists a cycle that has an odd number of negative edges. Then any cut of G that contains all negative edges in the graph, i.e., that contains the set $E^- = \{ij \in E \ : \ a_{ij} < 0\}$, must contain at least one positive edge. This implies $\mu^-(V) > \sum_{ij\in E^-} a_{ij}$. So

$$\mu^+(V) - \mu^-(V) < \sum_{ij\in E} |a_{ij}|\,,$$

and consequently, by Lemma 1, $\mathrm{chgap}[b](1/2,\dots,1/2) < \mathrm{mcgap}[b](1/2,\dots,1/2)$. The argument for a cycle with an odd number of positive edges is similar. □

Theorem 4 implies that for functions without negative coefficients we have $Q = \mathrm{conv}(B)$ if and only if G is bipartite, where the "if"-part of this equivalence follows from Theorem 3.10 in [10]. In contrast, without restricting the signs of the coefficients bipartiteness does not help. The probabilistic argument in the proof of Theorem 1 also works for the complete bipartite graph with equal parts and yields that in this setting almost all functions b with coefficients in $\{1,-1\}$ have $\mathrm{mcgap}[b](\boldsymbol{x}) \geqslant \frac{\sqrt{n}}{8}\,\mathrm{chgap}[b](\boldsymbol{x})$ for $\boldsymbol{x} = (1/2,\dots,1/2)$.

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