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degree of Doctor of Philosophy

Graph Labeling Techniques

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Statements

Statement of Originality

The thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to the final version of my thesis being made available worldwide when deposited in the University's Digital Repository, subject to the provisions of the Copyright Act 1968.

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The candidate has six papers, of which he is a joint author, embodied in the thesis. As principal supervisor I can attest that the candidate's contribution, in all the cases were about 75% of the final article.

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Dr. JOE RYAN

List of Publications

Substantial sections of the following chapters are based upon the following articles submitted to the nominated journals;

1. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Edge Irregular Reflexive Labeling of Prisms and Wheels, *Electronic Notes in Discrete Mathematics (Elsevier)*, 2016.
2. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Bača, M., Vertex Irregular Reflexive Labeling of Prisms and Wheels, *Electronic Journal of Graph Theory and Applications*, 2017.
3. Bača, M., Irfan, M., Ryan, J., Semaničová-Feňovčíková, A., **Tanna, D.**, Note on Reflexive Edge Irregular Labelings of Graphs, *AKCE International Journal of Graphs and Combinatorics*, 2017.
4. Bača, M., Irfan, M., Ryan, J., Semaničová-Feňovčíková, A., **Tanna, D.**, On Reflexive Edge Irregular Labelings for the Generalized Friendship Graphs, *Turkish Journal of Mathematics*, 2017.
5. Dafik, Slamin, **Tanna, D.**, Semaničová-Feňovčíková, A., Bača, M., Constructions of H-antimagic Graphs Using Smaller Edge-antimagic Graphs, *Ars Combinatoria*, 2017.
6. Ryan, J., Munushinge, B., **Tanna, D.**, Reflexive Irregular Labelings, *Preprint*, 2014.

The following works are in still progress at the time of writing thesis and so it is not included here;

1. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Vertex Irregular Reflexive Labeling of Circulant Graphs, *Preprint*, (2017).
2. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Vertex Irregular Reflexive Labeling of Ladders and Generalised Petersen Graph, *Preprint*, (2017)

Dedication

To my loving wife, Zenit, my whole family and to the memory of my supervisor, Mirka Miller.

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Abstract

We give some background to the labeling schemes like graceful, harmonious, magic, antimagic and irregular total labelings. Followed by this we give some preliminary results and open problems in these schemes. We will introduce a new branch of irregular total labeling, irregular reflexive labeling. This new labeling technique has few variations on vertices labels from irregular total labeling. They are,

- The vertices labels are non negative even integers.
- The vertex label 0 is permissible, representing the vertex without loop.

The vertex (edge) irregular reflexive labeling is a total irregular labeling with above conditions on vertices labeling such that the vertices (edges) weights are distinct. The idea is to use minimum possible labels for vertices (edges) and thus keeping the reflexive vertex (edge) strength as low as possible.

We believe that this new technique is closer in concept to the original irregular labeling as proposed by Chartrand et al., since the vertex labels are also being used to represent edges(loops).

Again the objective is to minimize the total strength by using the smallest vertices/edges labels. We will give edge and vertex irregular reflexive strengths for many graphs such as paths, cycles, stars, complete graphs, prisms, wheels, baskets, friendship graphs, join of graphs and generalised friendship graphs and present labeling techniques for these graphs.

We also describe edge covering, H -edge covering, H -magic and H -antimagic graphs and prove some theorems based on these concepts. Many results have been established for construction of H -antimagic labelings of graphs we will use the partitions of a set of integers with determined differences, the upper bound of the difference d if the graph G^H is super $(a, d) - H$ -antimagic, establishment of connection between H -antimagic labelings and edge-antimagic total labelings. We have also posed some open problems.

Finally we address why study of graph labeling is important by explaining some applications of graph labeling and give some open problems and conjectures.