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Graph Labeling Techniques

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Statements

Statement of Originality

The thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to the final version of my thesis being made available worldwide when deposited in the University's Digital Repository, subject to the provisions of the Copyright Act 1968.

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I hereby certify that the work embodied in this thesis contains published papers of which I am a joint author. I have included as a part of the thesis a written statement, endorsed by my supervisor, attesting to my contribution to the joint publication.

Signed: ______________________________

DUSHYANT TANNA

The candidate has six papers, of which he is a joint author, embodied in the thesis. As principal supervisor I can attest that the candidate's contribution, in all the cases were about 75% of the final article.

Signed: ______________________________

Dr. JOE RYAN

List of Publications

Substantial sections of the following chapters are based upon the following articles submitted to the nominated journals;

1. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Edge Irregular Reflexive Labeling of Prisms and Wheels, *Electronic Notes in Discrete Mathematics (Elsevier)*, 2016.

2. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Bača, M., Vertex Irregular Reflexive Labeling of Prisms and Wheels, *Electronic Journal of Graph Theory and Applications*, 2017.

3. Bača, M., Irfan, M., Ryan, J., Semaničová-Feňovčíková, A., **Tanna, D.**, Note on Reflexive Edge Irregular Labelings of Graphs, *AKCE International Journal of Graphs and Combinatorics*, 2017.

4. Bača, M., Irfan, M., Ryan, J., Semaničová-Feňovčíková, A., **Tanna, D.**, On Reflexive Edge Irregular Labelings for the Generalized Friendship Graphs, *Turkish Journal of Mathematics*, 2017.

5. Dafik, Slamin, **Tanna, D.**, Semaničová-Feňovčíková, A., Bača, M., Constructions of H-antimagic Graphs Using Smaller Edge-antimagic Graphs, *Ars Combinatoria*, 2017.

6. Ryan, J., Munushinge, B., **Tanna, D.**, Reflexive Irregular Labelings, *Preprint*, 2014.

The following works are in still progress at the time of writing thesis and so it is not included here;

1. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Vertex Irregular Reflexive Labeling of Circulant Graphs, *Preprint*, (2017).

2. **Tanna, D.**, Ryan, J., Semaničová-Feňovčíková, A., Vertex Irregular Reflexive Labeling of Ladders and Generalised Petersen Graph, *Preprint*, (2017)

Dedication

To my loving wife, Zenit, my whole family and to the memory of my supervisor, Mirka Miller.

Acknowledgements

First and foremost I would like to acknowledge my sincere gratitude towards my supervisors, Emeritus Professor Mirka Miller, Dr. Joe Ryan, Dr. Thomas Kalinowski, Laureate Professor Jonathan Borwein and A/Professor Andrea Semaničová-Feňovčíková, who have supported me abundantly with their encouragement, guidance and patience.

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My debt of gratitude to Joe for always being there for me and making my vision turn to reality. Joe has always helped, motivated and guided me through out my candidature. I appreciate all his contribution in terms of time and efforts that he has invested to make my Ph.D. fruitful and productive. His rigorous guidance and ever-friendly nature has helped to earn better insights towards mathematical problem solving. Your encouragements and unfailing support were crucial in overcoming the setbacks and hurdles from start to completion of Ph.D. at University of Newcastle.

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I would like to bestow my gratitude to the University of Newcastle for providing scholarships (UNIPRS and UNRSC 50-50) to pursue my dream. I am thankful to International Leadership Experience and Development (ILEAD) Plus Program for the generous support for participating in various conferences. I acknowledge my gratitude to the Faculty of Sciences for the Conference scholarship. My sincere thanks to the School of Mathematical and Physical Sciences, all the staff, both academic and administrative staff who created a healthy and comfortable environment for me to pursue my research. Many thanks to all of them who helped through the completion of this endeavor with their support. Special thanks to Vicki Gumbleton for giving me the opportunity for tutoring.

I convey my gratitude to my wife, Zenit. She has always been by my side when I needed her the most with her endless support. She is the pillar of my ambition and all hard work. Thanks to her that I am able to complete my Ph.D. She is the reason for my success. Big thanks to my in-laws, mom, dad, brothers, sisters and whole family. Their encouragement and support helped me in completion of my Ph.D. Last but not least, I thank God Almighty for blessing me with this opportunity to pursue my research and for giving me sound body and mind to complete this successfully.

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Abstract

We give some background to the labeling schemes like graceful, harmonious, magic, antimagic and irregular total labelings. Followed by this we give some preliminary results and open problems in these schemes. We will introduce a new branch of irregular total labeling, irregular reflexive labeling. This new labeling technique has few variations on vertices labels from irregular total labeling. They are,

- The vertices labels are non negative even integers.
- The vertex label 0 is permissible, representing the vertex without loop.

The vertex (edge) irregular reflexive labeling is a total irregular labeling with above conditions on vertices labeling such that the vertices (edges) weights are distinct. The idea is to use minimum possible labels for vertices (edges) and thus keeping the reflexive vertex (edge) strength as low as possible.

We believe that this new technique is closer in concept to the original irregular labeling as proposed by Chartrand et al., since the vertex labels are also being used to represent edges(loops).

Again the objective is to minimize the total strength by using the smallest vertices/edges labels. We will give edge and vertex irregular reflexive strengths for many graphs such as paths, cycles, stars, complete graphs, prisms, wheels, baskets, friendship graphs, join of graphs and generalised friendship graphs and present labeling techniques for these graphs.

We also describe edge covering, H-edge covering, H-magic and H-antimagic graphs and prove some theorems based on these concepts. Many results have been established for construction of H-antimagic labelings of graphs we will use the partitions of a set of integers with determined differences, the upper bound of the difference d if the graph G^H is super $(a, d) - H$-antimagic, establishment of connection between H-antimagic labelings and edge-antimagic total labelings. We have also posed some open problems.

Finally we address why study of graph labeling is important by explaining some applications of graph labeling and give some open problems and conjectures.

Chapter 1

Introduction

1.1 Overview

Magic labeling is considered to be the oldest labeling techniques. Many researcher have introduced labeling schemes to generalize the idea of a magic square. A *magic square* of order n is $n \times n$ array of integers $1, 2, \ldots, n^2$ such that the sum of numbers along any row, column and main diagonals is a fixed constant that equals to $n(n^2 + 1)/2$. In 1963, Sedláček [102] pointed out the correspondence between a magic square of order n and magic labeling of a complete bipartite graph $K_{n,n}$. He discovered that if we label every edge $u_i v_j$ of $K_{n,n}$ with the number from i^{th} row and j^{th} column of the magic square of order n, we obtain the *supermagic* labeling of $K_{n,n}$. Sedláček [102] in 1963 defined a graph to be magic if it has an edge labeling, with the range of the real numbers so that the sum of the edge labels around any vertex is always the same, a constant, independent of the choice of a vertex.

Ringel, [97], in 1963 posed a conjecture that for any tree T with n vertices, the complete graph K_{2n-1} can be decomposed into $2n - 1$ trees isomorphic to T. Rosa, [99], introduced β-labeling in 1967. Rosa conjectured that if it was possible that every tree was labeled with β-labeling then Ringel's conjecture holds.

Rosa, [99], called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy between vertices x and y is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

Later in 1972 Golomb, [50] called β-labeling as graceful labeling which is a popular term now. Rosa posed a conjecture, Graceful Tree Conjecture which claims that "every tree is graceful". Rosa showed that if every tree is graceful then Ringel's conjecture holds. Since then, researchers have been trying to prove Ringel's conjecture through the Graceful Tree Conjecture. So far no one is able to solve this conjecture for trees in general but there has been several classes of trees for which this conjecture holds true and as a result there are various types of trees, which are defined in Section 2.1.

Harmonious labeling, naturally arose in the study by Graham and Sloane, [51] of modular versions of additive bases problems stemming from error-correcting codes in 1980. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q, $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$ such that when each edge xy between vertices x and y is assigned the label $(f(x)+f(y))$ $(mod\ q)$, the resulting edge labels are distinct. They proved that almost all graphs are not harmonious, [51].

Antimagic labeling is opposite to magic labeling. Here the weights of the elements of graph have to be pairwise distinct. Antimagic labeling was first introduced by Hartsfield and Ringel [54] in 1990. They called a graph G with q edges to be antimagic if there is a bijective function f from edges of G to the set $\{1, 2, \ldots, q\}$ such that all vertex weights are pairwise distinct.

Hartsfield and Ringel [54] conjectured that all graphs except K_2 are antimagic. Many researcher have tried to prove this conjecture. Using probabilistic method and techniques of analytical number theory, Alon et al., [7] showed that this conjecture is true for all graphs having minimum degree $\Omega(\log|V(G)|)$. They also proved that if G is a graph with $|V(G)| \geq 4$ and maximum degree $\Delta(G) = |V(G)| - 2$ then G is antimagic. Hartsfield and Ringel proved that many classes of graphs are antimagic. In [54], it is proved that paths P_n, $n \geq 3$, cycles C_n, wheels W_n and complete graphs K_n, $n \geq 3$ are antimagic. In this thesis, we will discuss different types of antimagic labeling and present results in a new variation, H-antimagic.

Irregular labeling was introduced by Chartrand et al., in [33]. They posed the following problem, "assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, that is, the label sums (weights) at each vertex are distinct. What is the minimum value of the largest label over all such irregular assignments?"Such a labeling is known as irregular labeling and the minimum value of largest label over all possible irregular labeling is known as irregularity strength.

Bača et al., [20] defined vertex (edge) irregular total k-labeling of a graph G. Here the vertices and edges both are labelled from integers {1,2,...,k} so that the vertex (edge) weights are pairwise distinct.

We present here a new type of irregular total labeling, known as vertex (edge) irregular reflexive labeling. This new labeling technique has few variation on vertices labels from irregular total labeling. They are,

- The vertex labels have be non negative even integers.
- The vertex label 0 is permissible, representing a vertex without a loop.

The vertex (edge) irregular reflexive labeling is a total irregular labeling with the above condition on vertex labeling such that the vertices (edges) weights are distinct. The idea is to use minimum possible labels for vertices (edges) and thus keeping the reflexive vertex (edge) strength as low as possible.

We believe that this new technique is closer in concept to the original irregular labeling as proposed by Chartrand et al., since the vertex labels are also being used to represent edges(loops).

Again the objective is to minimize the reflexive strength total weight by using the smallest maximum vertex/edge labels. We will give results for edge and vertex irregular reflexive labelings for many graphs such as paths, cycles, stars, complete graphs, prisms, wheels, baskets, friendship graphs, join of graphs and generalised friendship graphs.

We also describe edge covering, H-edge covering, H-magic and H-antimagic graphs and prove some theorems based on these concepts. Some results have been established for construction of H-antimagic labelings of graphs. We will use the partitions of a set of integers with determined differences, the upper bound of the difference d if the graph G^H is super $(a, d)-H$-antimagic. We have also established the connection between H-antimagic labelings and edge-antimagic total labelings.

Graph labeling is a rich area. There are other labeling techniques like radio labeling, sum labeling, prime labeling, cordial labeling, filicitous labeling and many more. But all these techniques must necessary fall beyond the scope of this thesis. For details study about labeling techniques, refer [47].

Finally, we address why study of graph labeling is important by explaining some applications of graph labeling and finish with some open problems and conjectures and suggestions for future research endeavours.

1.2 Research Objectives

Overall research aims of study are to investigate graph labelings and their applications. In particular:

- Study various graph labeling schemes.
- Extend existing methods and develop new techniques for investigating graph labelings.
- Find more applications of existing graph labelings.
- Motivated by real life applications, design new graph labelings.

1.3 Outline of the Thesis

This thesis is structured as follows.

Chapter 2 on page 15: Basic Concepts and Literature Review This chapter contains the necessary and useful concepts from graph theory and an extended literature review of various dimensions needed for the thesis.

Chapter 3 on page 35: Edge Irregular Reflexive Labeling In this chapter we will introduce the edge irregular reflexive labeling and we will present results for reflexive edge strength of some graphs.

Chapter 4 on page 67: Vertex Irregular Reflexive Labeling In this chapter we will introduce vertex irregular reflexive labeling and we will show results for reflexive vertex strength of some graphs.

Chapter 5 on page 91: H-antimagic Labeling In this chapter we will discuss partitions of integers with determined differences and we will obtain some new results.

Chapter 6 on page 103: Applications of Graph Labelings In this chapter we will explain why study of graph labeling is important, pose some open problems and conjectures and discuss possible future directions.

In this thesis, all original results are indicated by the symbol $\diamondsuit$.

Chapter 2

Basic Concepts and Literature Review

2.1 Basic Graph Theory

In this section, we introduce the terminology, definitions and notations that will be used throughout this thesis.

A *graph* G is an ordered triplet $(V(G), E(G), \psi_G)$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. The graph is generally denoted by $G = G(V, E)$. If the vertex set $V(G)$ and edge set $E(G)$ are finite sets than we call graph G a *finite graph*. Otherwise G is called an *infinite graph*.

In pictorial form the vertices are usually represented by points on a plane and edges are represented by lines, connecting two vertices. A graph which can be drawn on a plane in such a way that its edges intersect only at their end vertices is called a *planar graph*.

If two vertices x and y are connected by one or more edges than these vertices are said to be *adjacent* vertices. It is often convenient to write adjacent vertices as $x \sim y$. If an edge connects with two vertices then that edge is said to be *incident* with those vertices and vice versa. The *order* p of a graph is the number of vertices in a graph and the *size* q of a graph G is the number of edges in a graph. The graph of order p and size q is also denoted by (p, q)-graph. In Figure 2.1, the order of the graph is 7 and the size of the graph is 11.

An edge connecting a vertex to itself is called a *loop*, while a *multiple edge* in a graph means more than one edge exists between a pair of vertices. In Figure 2.2, e_1 is a loop while e_{10} and e_{11} are multiple edges. A graph is *simple* if it has no loops and multiple edges. A graph that contains multiple edges is called a *multigraph*, while a *pseudograph* may contain loops as well as multiple edges. Figure 2.2 gives an example of a pseudograph. Figure 2.1 presents a simple graph.

The *degree* $\deg(v)$ of a vertex v in G is the number of edges incident with v, each loop is counted as two edges when calculating the degree of a vertex.

We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of vertices of G. A vertex with degree 1 is called a *pendant vertex* and a vertex with degree 0 is called an *isolated vertex*. If a graph G has vertices $v_1, v_2, \ldots, v_n$, the monotonic non increasing sequence $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is called the *degree sequence* of G. A graph is *k-regular* if $\deg(v) = k$ for all vertices in G; a *regular graph* is one that is k-regular for some k. Figure 2.3 shows a 2-regular graph.

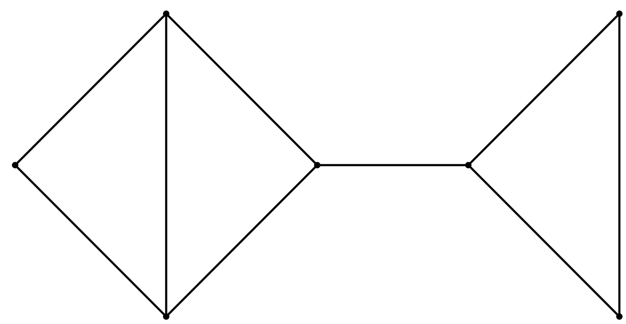

Figure 2.1: A simple graph of order 7 and size 11

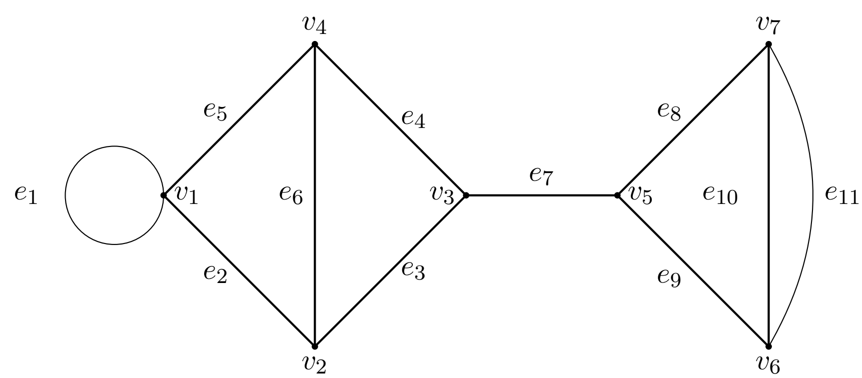

Figure 2.2: Pseudograph

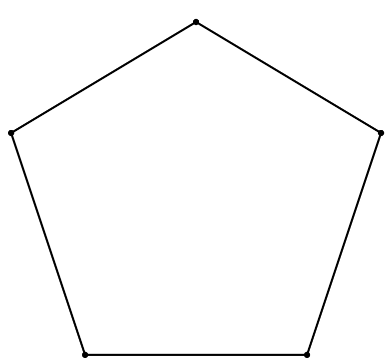

Figure 2.3: A 2-regular graph

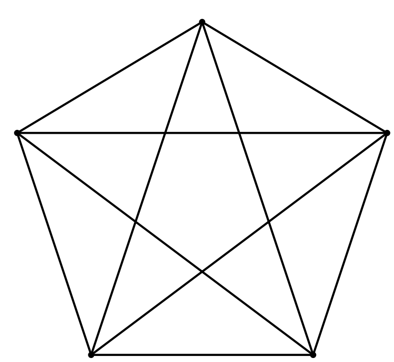

Figure 2.4: Complete graph K_5

Two graphs G and H are said to be *isomorphic*, written as $G \cong H$, if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that for the incidence function $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; for edge e and vertices u, v then such a pair (θ, ϕ) of mappings is called an *isomorphism* between G and H.

A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. A complete graph with p vertices is denoted by K_p. Figure 2.4 depicts the complete graph K_5.

A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end vertex in X and other end vertex in Y. Such a partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

Figure 2.5 presents a bipartite graph and Figure 2.6 gives the complete bipartite graph $K_{3,5}$.

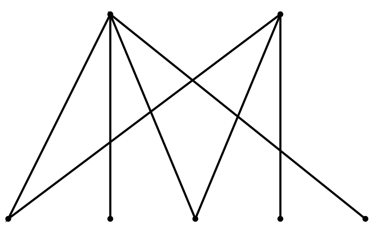

Figure 2.5: A bipartite graph

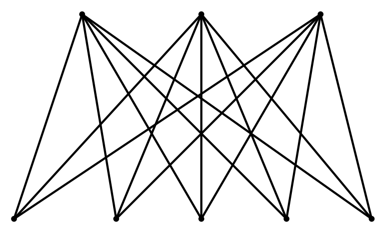

Figure 2.6: Complete bipartite graph $K_{3,5}$

A graph H is a *subgraph* of G, written as $H \subseteq G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and the incidence function ψ_H is the restriction of the incidence function ψ_G to $E(H)$. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call H a *proper subgraph* of G. If H is a subgraph of G, G is a *supergraph* of H. A *spanning subgraph* (or *spanning supergraph*) of G is a subgraph (or supergraph) H with $V(H) = V(G)$.

We say that two graphs G_1 and G_2 are *disjoint* if they have no vertex in common, and *edge-disjoint* if they have no edge in common. The *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \Box G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$, in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$.

A *walk* in G is a finite non-null sequence $W = v_0e_1v_1e_2v_2 \dots e_kv_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the end vertices of e_i are v_{i-1} and v_i. The vertices v_0 and v_k are called the *origin* and *terminus* of W, respectively and $v_1, v_2, \dots, v_{k-1}$ are *internal vertices*. The integer k is called the *length* of W. Given two walks $W_1 = v_0e_1v_1e_2v_2 \dots e_kv_k$ and $W_2 = v_ke_{k+1}v_{k+1} \dots e_nv_n$, *concatenation* of W_1 and W_2, denoted by W_1W_2, is a walk of the form $v_0e_1v_1e_2v_2 \dots e_kv_ke_{k+1}v_{k+1} \dots e_nv_n$. If the edges $e_1, e_2, \dots, e_k$ of a walk W are distinct, W is called a *trail*. If, in addition, the vertices $v_0, v_1, \dots, v_k$ are distinct, W is called a *path*. Path with n vertices is denoted by P_n.

A walk is *closed* if it has positive length and its origin and terminus are the same. A cycle is a closed path that is a path combined with the edge v_k, v_1. A cycle of length k is called a k-cycle and is denoted by C_k. A cycle C_k is *odd* if k is odd and it is *even* if k is even. A 3-cycle is often called a *triangle*. A *tour* in G is a closed walk that traverses each edge of G at least once. An *Euler tour* is a tour which traverses each edge exactly once. A graph is said to be *Eulerian* if it contains an Euler tour.

Two vertices u and v of G are said to be *connected* if there is a (u, v)-path in G. If the vertex set V can be partitioned into subsets $V_1, V_2, \ldots, V_n$ such that two vertices u and v are connected if and only if both u and v belong to the same set V_i then the subgraphs $G[V_1], G[V_2], \ldots, G[V_n]$ are called the *components* of G. If G has only one component, G is *connected*, otherwise G is *disconnected*. We denote the number of components of G by $\omega(G)$.

An *acyclic graph* is a graph that contains no cycle. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph. Each component of a forest is a tree. A *spanning tree* of G is a spanning subgraph of G that is a tree. A *rooted tree* is a tree having a distinguished vertex r called the root such that each edge is implicitly directed away from the root. A *leaf* in a tree is a vertex of degree 1. A *caterpillar* is a tree with the property that the removal of its pendant vertices give a path. In other words, a caterpillar is described as a graph obtained by attaching any number of leaves to the vertices of a path. A rooted tree consisting of k branches, where the i^{th} branch is a path of length i, is called an *olive tree*. The *star graph* S_n of order n, sometimes simply known as an n-star is a tree with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1. The star graph S_n is isomorphic to the complete bipartite graph $K_{1,n-1}$. An (n, k)-*banana tree* is obtained by connecting one leaf of each of n copies of an k-star graph with a single root vertex v that is not present in any of the starts. A *firecracker* is a graph obtained from the concatenation of stars by linking one leaf from each star to a leaf of another star by an edge. Many of these trees were originated to find a solution to the famous *graceful tree conjecture*, defined in 2.2.1.

A *cut-edge* of G is an edge e such that deletion of an edge e leads to an increase in number of components in G, that is, $\omega(G - e) > \omega(G)$. If H is a subgraph of G, the *complement* of H in G, denoted by, $\overline{H}(G)$, is the subgraph $G - E(H)$. A vertex v of G is a *cut-vertex* if $E(G)$ can be partitioned into two nonempty subsets E_1 and E_2 such that subgraph graphs induced by E_1 and E_2, that is, $G[E_1]$ and $G[E_2]$, have just the vertex v in common. A connected graph that has no cut-vertex is called a *block*.

A *wheel graph* W_n is obtained by joining every vertex of a cycle C_n to a further vertex called *center*. The other vertices of wheel W_n are called *rim vertices*. The *helm graph* H_n is the graph obtained from a wheel W_n joining a pendant edge to each rim vertex of W_n. A *web graph* $\mathcal{W}_n$ is the graph obtained from helm H_n by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. A *flower* $\mathcal{F}_n$ is a graph obtained from a helm H_n by joining each pendant vertex to the central vertex of the helm. The *ladder* L_n is a graph obtained by the Cartesian product of path P_2 and path P_n, $n \geq 2$, and the *prism* D_n is the graph obtained by the Cartesian product of path P_2 and cycle C_n. Thus, $L_n \cong P_2 \square P_n$ and $D_n \cong P_2 \square C_n$. A *friendship graph* f_n is a graph which consists of n triangles with a common vertex. The *fan graph* F_n is a graph obtained by joining all vertices of P_n, $n \geq 2$, to a further vertex called the center. Alternatively, for $n \geq 3$, the fan graph F_n can be obtained from the wheel W_n by deleting one edge joining any two rim vertices.

Figure 2.7 shows a wheel W_4, Figure 2.8 depicts a helm H_4 and Figure 2.9 presents a web $\mathcal{W}_4$. Figure 2.10 shows a fan graph F_7, Figure 2.11 represents a ladder L_5 and Figure 2.12 depicts a friendship graph f_6.

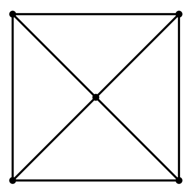

Figure 2.7: Wheel W_4

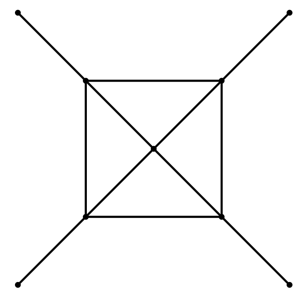

Figure 2.8: Helm H_4

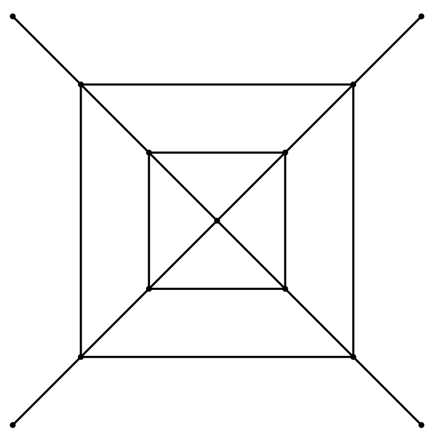

Figure 2.9: Web graph $\mathcal{W}_4$

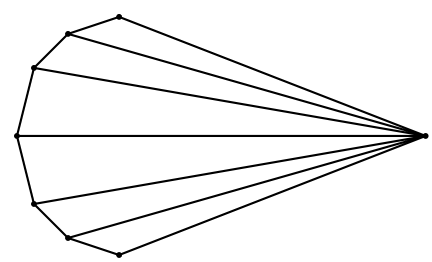

Figure 2.10: Fan graph F_7

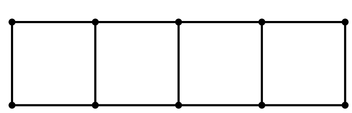

Figure 2.11: Ladder L_5

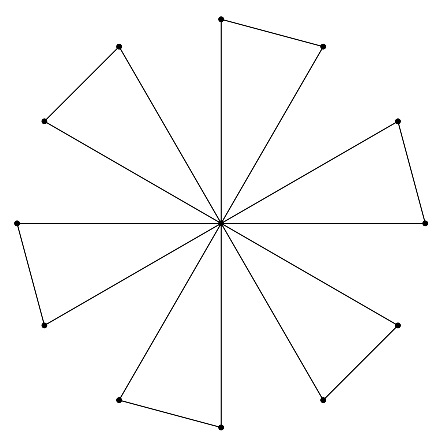

Figure 2.12: Friendship graph f_6

2.2 An Introduction to Graph Labelings with Examples

Any field of investigation becomes interesting when it gives rise to the number of problems that pose challenges to our mind for their eventual solutions. The problems arising from the study of a variety of labeling techniques of the elements of a graph or of any discrete structure form one such potential area of challenge. Graph labeling problems are really not of recent origin, for example, coloring of the vertices arose in connection with the well known Four Color Theorem, which remained for a long time known by the name Four Color Conjecture and took more than 150 years for its solution in 1976. Here, we will describe some labeling techniques.

Recall that a labeling of a graph is a mapping (function) that carries graph elements to numbers (usually non-negative integers). The most common choice for the domain are the vertex set (vertex labeling), the edge set (edge labeling) or the set of all vertices and edges (total labeling). Other domains are also possible.

In many cases, it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of the element. For example, the weight of the vertex v if it exists under total labeling is the sum of the label of vertex v and edge labels of all edges incident with v and the weight of an edge uv under total labeling is the sum of the edge label if it exists and labels of its end vertices.

Before we begin the study of labeling schemes, let us first define some useful concepts from algebra.

Definition 2.2.1. *A* group $(G, \cdot)$ *is a nonempty set G together with a binary operation $\cdot$ on G such that the following conditions hold:*

- *Closure: For all $a, b \in G$ the element $a \cdot b$ is a uniquely defined element of G.*
- *Associativity: For all $a, b, c \in G$, we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*
- *Identity: There exists an identity element $e \in G$ such that $e \cdot a = a$ and $a \cdot e = a$ for all $a \in G$.*
- *Inverses: For each $a \in G$ there exists an inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = e$ and $a^{-1} \cdot a = e$.*

Definition 2.2.2. *Let $(G_1, \cdot)$ and $(G_2, *)$ be groups and f a function from G_1 into G_2. Then f is called a* homomorphism *of G_1 into G_2 if for all $a, b \in G_1$, $f(a \cdot b) = f(a) * f(b)$.*

Definition 2.2.3. *A homomorphism f of a group G_1 into a group G_2 is called an* isomorphism *of G_1 onto G_2 if f is bijective. In this case, we write $G_1 \cong G_2$ and say that G_1 and G_2 are isomorphic. An isomorphism of a group G_1 onto G_1 is called an* automorphism.

Graceful Labeling

This labeling scheme was introduced by Rosa, [99] who called it a *β-valuation*, later named by Golomb, [50] as a *graceful labeling.* The origin of a β-valuation was due to the efforts to solve the famous conjecture by Ringel that "the complete graph K_{2n+1} can be decomposed into $2n+1$ subgraphs, each isomorphic to a given tree with n edges". Rosa defined another labeling which was known as an *α-valuation* with some additional properties to β-valuation. *α-valuation* is a graceful labeling such that for each edge $e = (u, v) \in E$, one of the labels $f(u), f(v) \leq k$ and the other is $> k$, for some fixed k, refer [99]. In the same paper, Rosa showed that if G is a (p, q)-graph and has α-valuation then for every natural number q, the complete graph with $2q+1$ vertices can be decomposed into copies of G in such a way that the automorphism group of decomposition contains $\mathbb{Z}_q$ itself.

Definition 2.2.4. *A function g is called a* graceful labeling *of a graph G if $g : V(G) \to \{0, 1, \ldots, |E(G)|\}$ is injective and the induced function $g^* : E(G) \to \{1, 2, \ldots, |E(G)|\}$, defined as*

$$g^*(uv) = |g(u) - g(v)|$$

for every $uv \in E(G)$, is bijective. A graph which admits a graceful labeling is called a graceful graph.

Figure 2.13 demonstrates a graceful labeling of wheel W_4. Here and in whole document, the weights will be given in circles.

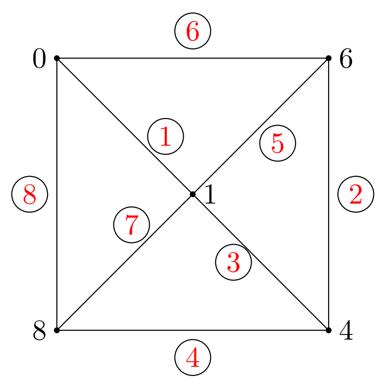

Figure 2.13: Graceful labeling of wheel W_4

Ringel and Kotzig posed below conjecture, which is still open.

Conjecture 2.2.1. [97] *All trees are graceful.*

This was the pioneer to major research development in graceful labeling. Using computer search, Aldred and McKay [5] proved that trees with at most 27 vertices are graceful. This result was improved by Fang [40] who proved that trees with at most 35 vertices are graceful. In 1976, Bermond and Sotteau [29] proved that rooted trees in which all the vertices at the same distance from the root and have the same degree (symmetrical trees) are graceful. In 1982, Huang et al., [58] proved that trees with at most 4 end vertices are graceful. Presently there are many types of trees which are known to be graceful including paths, caterpillars, olive trees, banana trees, firecrackers and many more. Rosa [99] in 1967 proved that any Eulerian graph with number of edges congruent 1 or 2 modulo 4 is not graceful. In 1972, Golomb [50] proved that complete graphs K_n are not graceful for $n \geq 5$. In 1979 Frucht et al., [46, 56] proved that all wheels are graceful. Recently in 2016, Adamaszek et al., [1] proved that trees with maximum degree $O(n/\log n)$ are graceful.

Truszczynski [119] studied unicyclic graphs and proved several classes of such graphs are graceful. He conjectured that all unicyclic graphs except C_n, where $n \equiv 1$ or $2 \pmod 4$, are graceful. In 1984, Ayel and Favaron [11] published a paper proving that helms are graceful. Kang et al., [71] proved that all web graphs are graceful. Seoud and Youssef, [105] established that all flowers are graceful. Delorme et al., [35] proved that cycles with a chord are graceful.

Graceful labeling is still an open area of research and while the conjecture is still open, many types of trees have been shown to be graceful.

Harmonious Labeling

Harmonious labeling was introduced by Graham and Sloane [51] in 1980, during the study of additive basis in number theory. Harmonious labeling has many applications in communication networks. The following example demonstrates the utility of harmonious labeling.

Consider a network which transmits signals with a criteria that every station must communicate with some other of that network via some signal. The total bandwidth, say e, is divided among all the connection channels. Every station x is assigned some number $f(x)$, the label of x. When two channels x and y communicate they use the flow $f(x) + f(y)$. If the harmonious labeling f exists for such a network then it is assured that each channel is assigned a unique link, since harmonious labeling is a bijective function.

Definition 2.2.5. *Let G be a graph of order p. A function h is called a* harmonious labeling *of G if $h : V(G) \to \mathbb{Z}_q$, where $\mathbb{Z}_q$ is the additive group, is an injective function and the induced function $h^* : E(G) \to \mathbb{Z}_q$ defined by*

$$h^*(uv) = (h(u) + h(v)) \pmod q$$

is bijective. A graph admitting a harmonious labeling is called a harmonious graph.

A harmonious labeling of the friendship graph f_n may be regarded as a modular generalization of the Langford-Skolem problem, [93], which states that, is it possible to partition the set $P = \{1, 2, \ldots, 2n\}$ in n pairs (a_i, b_i) such that the set of differences $b_i - a_i = i$, for $i = 1, 2, \ldots, n$. This problem has attracted many researchers and there has been significant progress in solving this problem, see [86]. In order to obtain the harmonious labeling of friendship graph f_n, we will label the vertices of the triangles with $(0, i, n + a_i)$, for $i = 1, 2, \ldots, n$.

There are more applications of harmonious labeling, like embedding of graphs in the plane and modular versions of many combinatorial problems, see [51].

Figure 2.14 demonstrates a harmonious labeling of wheel W_4.

It was proved by Graham and Sloane [51] that in case of trees exactly one vertex label has to be repeated for tree to be harmonious and this repeated vertex can be any element of $\mathbb{Z}_q$. Graham and Sloane [51] conjectured that all trees are harmonious, which is still an open

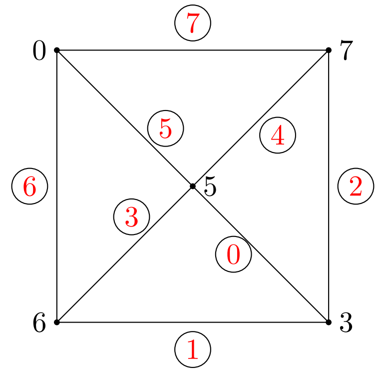

Figure 2.14: Harmonious labeling of wheel W_4

problem. Aldred and McKay [5] proved that trees with at most 26 vertices are harmonious. In 2012, this result was improved by Fang [41]. Fang proved that trees with at most 31 vertices are harmonious. Graham and Sloane [51] proved that cycles C_n, $n \geq 3$, are harmonious if and only if n is odd. There are several families of graphs which have been proved to be harmonious. In [51] Graham and Sloane proved that ladders except L_2 are harmonious. There are several results for existence of harmonious labeling of prisms. In [48, 51, 70] was proved that the generalized prisms $C_m \square P_n$ are harmonious if n is odd. Further, generalized prisms $C_m \square P_n$ are harmonious if $n = 2$, $m \neq 4$ and also if $m = 4$ and $n \geq 3$. Petersen graph is another graph which is harmonious, see [51]. Complete bipartite graphs $K_{m,n}$ are harmonious if and only if either $m = 1$ or $n = 1$, [51]. Gnanajothi [49] has shown that web graphs with odd cycles are harmonious. Gnanajothi and Liu [49, 77] proved that helms are harmonious. For more results on harmonious labeling, refer [47].

Magic Labelings

Many authors have introduced labeling schemes that generalize the idea of a magic square. A *magic square* of order n is $n \times n$ array of integers $1, 2, \ldots, n^2$ such that the sum of numbers along any row, column and main diagonals is a fixed constant that equals to $n(n^2 + 1)/2$. In 1963, Sedláček [102] pointed out the correspondence between a magic square of order n and magic labeling of a complete bipartite graph $K_{n,n}$. He found out that if we label every edge $u_i v_j$ of $K_{n,n}$ with the number from i^{th} row and j^{th} column of the magic square of order n, we obtain the *supermagic* labeling of $K_{n,n}$. Sedláček [102] in 1963 defined a graph to be magic if it has an edge labeling, with the range of the real numbers so that the sum of the edge labels around any vertex is always the same, a constant, independent of the choice of a vertex. More precisely, see the following definition.

Definition 2.2.6. *A function f is called a* magic labeling *of G if $f : E(G) \to \mathbb{Z}^+$, is an injective function and the induced function has the property that for every vertex $v \in V(G)$ the associated* *vertex weight equals to a constant independent of the choice of a vertex, that is,*

$$w_f(v) = \sum_{uv \in E(G)} f(uv) = \lambda,$$

where λ is a constant. A magic labeling is called supermagic *if the set of all the labels of the edges consists of consecutive integers. A graph admitting a magic (supermagic) labeling is called a* magic (supermagic) graph*.*

Figure 2.15 demonstrates a supermagic labeling of wheel W_4 with constant $\lambda = 24$.

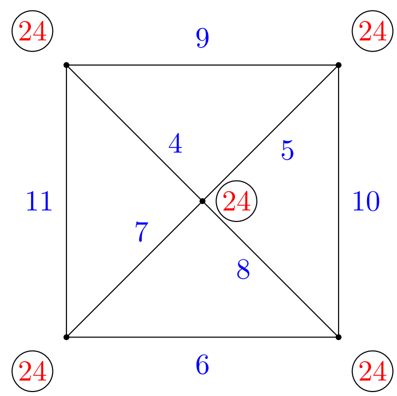

Figure 2.15: Supermagic labeling of wheel W_4

Some authors refer to a graph a supermagic if its edges, in a magic labeling can be labeled with numbers $\{1, 2, \ldots, |E(G|\}$, see [54, 106]. Note that for regular graphs these definitions are equivalent, see [62].

Some sufficient conditions for the existence of magic graphs are established in [14, 90, 103, 110]. A characterization of regular magic graphs was given by Doob in [37]. There were independently published two different characterizations of all magic graphs, see Jeurissen, [66] and Jezný and Trenkler in [68]. To this time only some special classes of supermagic graphs have been characterized. In [111], Stewart characterized supermagic complete graphs and the characterization of supermagic regular complete multipartite graphs and supermagic cubes is given by Ivančo in [62].

Now we will introduce other types of magic labelings where not only the edges but also the vertices of a graph are labeled.

Definition 2.2.7. *A function f is called a* vertex-magic total labeling *of G if $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)|+|E(G)|\}$ is a bijective function with the property that for every vertex $v \in V(G)$ the associated vertex weight equals to a constant independent of the choice of vertex, that is.,*

$$wt_f(v) = f(v) + \sum_{uv \in E(G)} f(uv) = k,$$

where k is a constant, known as a magic constant*. If the vertices are labeled with the smallest* *possible labels the vertex-magic total labeling is called* super vertex-magic total labeling*. A graph admitting a (super) vertex-magic total labeling is called a* (super) vertex-magic total graph*.*

Figure 2.16 gives an example of vertex-magic total labeling of wheel W_4 with the magic constant $k = 29$.

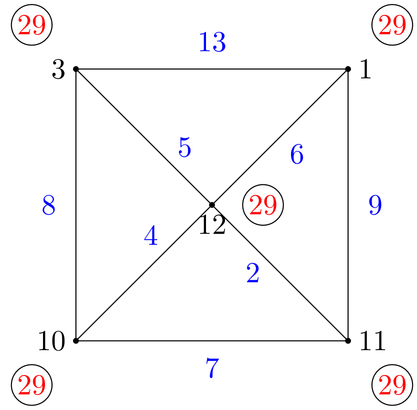

Figure 2.16: Vertex-magic total labeling of wheel W_4

The definition of a vertex-magic total labeling was introduced by MacDougall et al., in [80] and [81]. MacDougall [78] conjectured that all regular graphs other than K_2 and $2K_3$ are vertex-magic total.

Definition 2.2.8. *A function f is called an* edge-magic total labeling *of G if $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)|+|E(G)|\}$ is a bijective function with the property that for every edge $uv \in E(G)$ the associated edge weight equals to a constant independent of the choice of edge, that is,*

$$wt_f(uv) = f(u) + f(uv) + f(v) = k,$$

where k is a constant, known as a magic constant*. If the vertices are labeled with the smallest possible labels the edge-magic total labeling is called a* super edge-magic total labeling*. A graph admitting a (super) edge-magic total labeling is called a* (super) edge-magic total graph*.*

Figure 2.17 gives an example of super edge-magic total labeling of cycle C_5 with the magic constant $k = 14$.

Edge-magic total labeling was introduced by Kotzig and Rosa [73, 74] under the name *magic valuation*. In [73] Kotzig and Rosa proved that complete bipartite graphs $K_{m,n}$ are edge-magic total for all m and n, and cycles C_n are edge-magic total for all $n \geq 3$. Kotzig and Rosa [74] proved that complete graphs K_n are edge-magic total if and only if $n = 1, 2, 3, 5$ or 6. Wallis et al., [123] described edge-magic total labelings of K_n, $n = 1, 2, 3, 5$ and 6, for all possible values of the magic constant k.

One of the variations of magic labeling is a *bimagic labeling*.

Babujee [12, 13] introduced the idea of a bimagic labeling, in which there are two constants, say k_1 and k_2, such that all weights under this labeling equal to one or to other of those two

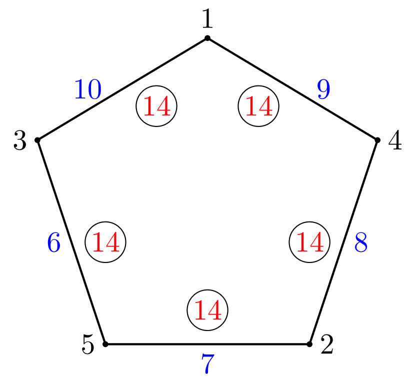

Figure 2.17: Super edge-magic total labeling of cycle C_5

constants. For example, we would define an *edge-bimagic total labeling* f of G to be a total labeling such that the edge weights $wt_f(xy) = f(x) + f(xy) + f(y)$ equals either k_1 or k_2 for every edge $xy \in E(G)$. It is interesting to consider the following two cases.

- All edges but one have the common weight. Such a labeling is said to be *almost magic*.
- The number of edges with one weight differs at most by 1 to the number of edges with the other weight. In this case the labeling is known as *equitable bimagic*.

In [82] Marr et al., proved that when $n \equiv 3 \pmod 4$, the wheel W_n has both an equitable bimagic labeling and an almost magic labeling.

For more information about magic or bimagic type labelings see Gallian's dynamic survey of graph labeling [47] or [122].

Antimagic Labelings

In the previous section we introduced the definitions of some magic type labelings. These labelings have one common property, that the weights of the considered elements are all the same. Now we will deal the case when all the weights of the considered elements are pairwise distinct. Such labeling is called an antimagic type labeling.

Antimagic labeling was first introduced by Hartsfield and Ringel [54] in 1990.

Definition 2.2.9. *A function f is called an* antimagic labeling *of G if $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ is a bijective function with the property that all the vertex weights are pairwise distinct. A graph admitting an antimagic labeling is called an* antimagic graph*.*

Figure 2.18 gives an example of an antimagic labeling of wheel W_4.

Hartsfield and Ringel [54] gave the following conjecture.

Conjecture 2.2.2. [54] *All graphs except K_2 are antimagic.*

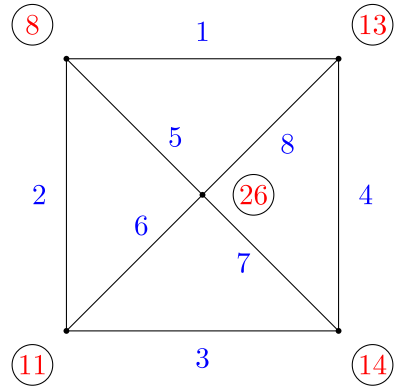

Figure 2.18: Antimagic labeling of wheel W_4

This conjecture is still open. Using probabilistic method and techniques of analytical number theory, Alon et al., [7] showed that this conjecture is true for all graphs having minimum degree $\Omega(\log |V(G)|)$. They also proved that if G is a graph with $|V(G)| \geq 4$ and maximum degree $\Delta(G) = |V(G)| - 2$ then G is antimagic. Hartsfield and Ringel proved that many classes of graphs are antimagic. In [54], it is proved that paths P_n, $n \geq 3$, cycles C_n, wheels W_n and complete graphs K_n, $n \geq 3$, are antimagic. In 1999, there has been a progress in Hartsfield - Ringel's conjecture by Alon [6]. Alon proved that all dense graphs, the graphs with number of edges close to $|V(G)|^2$, are antimagic.

Miller et al., [88] have contributed by introducing a new antimagic scheme for different types of graphs. They applied finite combinatorics methods to find antimagic labeling for graphs and using these techniques they proved that for a given degree sequence for a tree or a forest, they can provide an antimagic tree or forest with that degree sequence.

Further they proved that all trees and forests have edge-antimagic vertex labeling, that is, it is possible to label their vertices such that all edge weights are distinct. Moreover, they also proved that the labeling is super.

A total labeling f is said to be edge-antimagic total (EAT) labeling if all edge-weights are pairwise distinct. Similarly f is said to be vertex-antimagic total (VAT) labeling if all vertex-weights are pairwise distinct. In [87], Miller et al., proved that all graphs have VAT labelings. They also proved in the same paper that for every graph there exist VAT labeling which are super, repus (that is the vertex labels are $\{q+1, q+2, \ldots, q+p\}$) and neither super nor repus. The same is for EAT graphs. In [87], Miller et al., proved that all graphs have EAT labelings.

Thus as this problem for VAT graphs is completely solved, it has sense to give some restriction on vertex weights. We will get the concept of (a, d)-VAT labeling defined by Bača et al., in [15] that the weights are not only distinct, but also form arithmetic sequence with a difference d.

Another avenue of research is to consider that as all graphs have VAT labelings and also EAT labelings, do there exist graphs possessing a labeling that is simultaneously VAT and EAT? We get the concept of totally antimagic total labeling defined by Bača et al., [25] which is a natural extension of the concept of a totally magic labeling defined by Exoo et al., in [39].

Bača et al., [15] defined the concept of an (a, d)-vertex-antimagic total labeling.

Definition 2.2.10. *A function f is called an* (a, d)-vertex-antimagic total labeling *of G, denoted by (a, d)-VAT, if $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ is a bijective function with the property that the set of vertex weights is*

$$\{wt_f(v) : v \in V(G)\} = \{a, a + d, \ldots, a + (|V(G)| - 1)d\},$$

where $a > 0$ and $d \geq 0$ are two fixed integers. If the vertices are labeled with the smallest possible labels, an (a, d)-vertex-antimagic total labeling is called a super (a, d)-vertex-antimagic total labeling. *A graph admitting a (super) (a, d)-vertex-antimagic total labeling is called a* (super) (a, d)-vertex-antimagic total graph.

Figure 2.19 gives an example of $(25, 1)$-vertex-antimagic total labeling of wheel W_4.

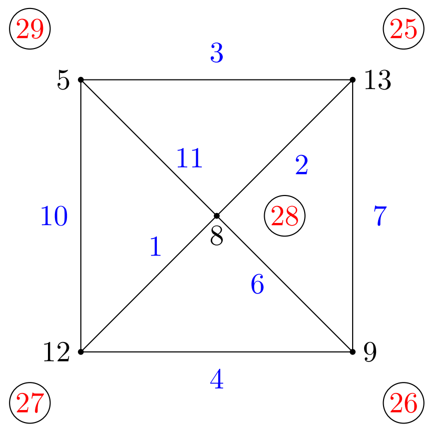

Figure 2.19: A (25,1)-vertex-antimagic total labeling of wheel W_4

Analogously, the natural extensions of the edge-magic total labeling and the super edge-magic total labeling are the (a, d)-edge-antimagic total and super (a, d)-edge-antimagic total labelings, respectively. The definition of (a, d)-edge-antimagic total labeling was introduced by Simanjuntak et al., in [108].

Definition 2.2.11. *A function f is called an* (a, d)-edge-antimagic total labeling *of G, denoted by (a, d)-EAT, if $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ is a bijective function with the property that the set of edge weights is*

$$\{wt_f(e) : e \in E(G)\} = \{a, a + d, \ldots, a + (|E(G)| - 1)d\},$$

where $a > 0$ and $d \geq 0$ are two fixed integers. If the vertices are labeled with the smallest possible labels the (a, d)-edge-antimagic total labeling is called super (a, d)-edge-antimagic total. *A graph admitting a (super) (a, d)-edge-antimagic total labeling is called a* (super) (a, d)-edge-antimagic total graph. *If $d = 0$ then f is called an* edge-magic total labeling.

Figure 2.20 gives an example of super $(10, 2)$-edge-antimagic total labeling of cycle C_5.

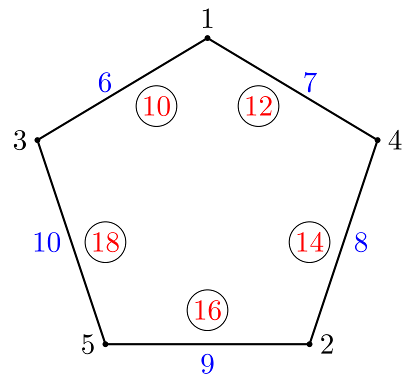

Figure 2.20: Super (10, 2)-edge-antimagic total labeling of cycle C_5

The generalization of the definition of edge-magic total and edge-antimagic total labelings is the concept of H-magic, H-antimagic labelings, respectively.

Let G be a finite simple graph. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. Then it is said that G admits an $(H_1, H_2, \ldots, H_t)$*-(edge) covering*. If every H_i is isomorphic to a given graph H, then G admits an H*-covering*. Note that in this case every subgraph isomorphic to H must be in the H-covering.

Definition 2.2.12. *Suppose that a graph G admits an H-covering. A function f is called an* (a, d)-H-antimagic total labeling *of G if $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ is a bijective function with the property that for all subgraphs H' of G isomorphic to H, the H-weights constitute an arithmetic progression*

$$a, a + d, \ldots, a + (t - 1)d,$$

where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. For the subgraph H under the total labeling f the associated H-weight is defined as

$$wt_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e).$$

If the vertices are labeled with the smallest possible labels the (a, d)-H-antimagic total labeling is called a super (a, d)-H-antimagic total labeling*. A graph admitting a (super) (a, d)-H-antimagic labeling is called a* (super) (a, d)-H-antimagic total graph.

Note that for $d = 0$ the considered labeling is called an H*-(super)magic total labeling*.

In [104], Semaničová-Feňovčíková et al., proved that wheels are cycle antimagic. Figure 2.21 illustrates super $(39, 1)$-C_3-antimagic total labeling of wheel W_4, the C_3-weights are depicted in circles.

The H-(super)magic total labelings were first studied by Gutiérrez and Lladó in [52]. They considered star-(super)magic and path-(super)magic labelings of some connected graphs and

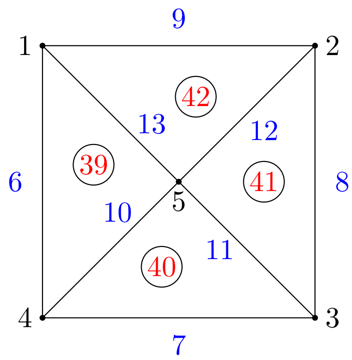

Figure 2.21: Super (39, 1)-C_3-antimagic total labeling of wheel W_4

proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Maryati et al., [83] gave P_h-(super)magic labelings of some trees such as shrubs, subdivision of shrubs and banana tree graphs. Lladó and Moragas [75] investigated C_n-(super)magic graphs and proved that wheels, windmills, books and prisms are C_h-magic for some h. Some results on C_n-supermagic labelings of several classes of graphs can be found in [91]. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [67]. Inayah et al., [59] gave a connection between graceful trees and antimagic H-decomposition of complete graphs. Maryati et al., [84] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h.

Motivated by H-(super)magic labelings, Inayah et al., [60] introduced the (a, d)-H-antimagic labeling. In [61] they investigated the super (a, d)-H-antimagic labelings for some shackles of a connected graph H. In [104] was proved that wheels are super $(a, 1)$-C_k-antimagic for every $k = 3, 4, \dots, n-1, n+1$.

For more information about antimagic type labelings see [24, 47].

Irregular Labelings

The following problem was proposed by Chartrand et al., in [33]. Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, that is, the label sums (weights) at each vertex are distinct. What is the minimum value of the largest label over all such irregular assignments? This leads to the concept of irregular labeling.

Definition 2.2.13. *A function ψ is called an* irregular assignment *of G if $\psi : E(G) \to \{1, 2, \dots, k\}$ has the property that the associated vertex weights are pairwise distinct, that is,*

$$w_\psi(u) \neq w_\psi(v)$$

for all vertices $u, v \in V(G)$, $u \neq v$. The weight of a vertex $v \in V(G)$ is

$$w_\psi(v) = \sum_{uv \in E(G)} \psi(uv),$$

where the sum is over all vertices u adjacent to v. The irregularity strength $\mathrm{s}(G)$ *of a graph G is defined as the minimum k for which G has an irregular assignment using labels at most k.*

The irregularity strength $\mathrm{s}(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph is a tough task, see [8, 36, 42, 53, 64, 65].

Definition 2.2.14. *A function ρ is called an* edge irregular k-labeling *of G if $\rho : V(G) \to \{1, 2, \dots, k\}$ has the property that the associated edge weights are pairwise distinct, that is,*

$$w_\rho(uv) \neq w_\rho(u'v')$$

for every two different edges uv and $u'v'$. The weight of an edge $uv \in E(G)$ is

$$w_\rho(uv) = \rho(u) + \rho(v).$$

The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength *of G, denoted by* $\mathrm{es}(G)$.

The notion of the edge irregularity strength was defined by Ahmad et al., [3]. They determined the exact value of the edge irregularity strength for paths, stars, double stars and Cartesian product of two paths and there is also given a lower bound for $\mathrm{es}(G)$.

Bača et al., [20] defined the edge irregular total k-labeling.

Definition 2.2.15. *A function φ is called an* edge irregular total k-labeling *of G if $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ has the property that the associated edge weights are pairwise distinct, that is,*

$$wt_\varphi(uv) \neq wt_\varphi(u'v')$$

for every two different edges uv and $u'v'$. The weight of an edge $uv \in E(G)$ is

$$wt_\varphi(uv) = \varphi(u) + \varphi(v) + \varphi(uv).$$

The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength *of G, denoted by* $\mathrm{tes}(G)$.

In [20], the bound for $\mathrm{tes}(G)$ is given. For a graph G, $\lceil (|E(G)| + 2)/3 \rceil \leq \mathrm{tes}(G) \leq |E(G)|$. They also gave tes for paths and cycles as, for $n > 1$, $\mathrm{tes}(P_n) = \mathrm{tes}(C_n) = \lceil (n+2)/3 \rceil$. Moreover they proved that for stars $K_{1,n}$, $n \geq 1$ have the same edge and vertex total irregularity strength, that is, $\mathrm{tes}(K_{1,n}) = \mathrm{tvs}(K_{1,n}) = \lceil (n+1)/2 \rceil$. They also proved that for trees T with n pendant vertices and no vertex of degree 2 it holds $\lceil (n+1)/2 \rceil \leq \mathrm{tvs}(T) \leq n$ and the

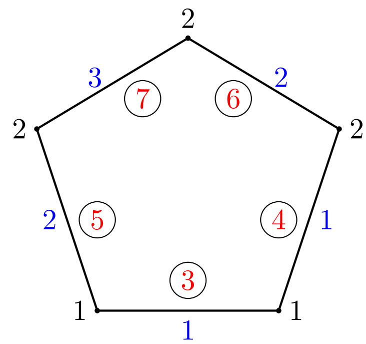

Figure 2.22: Edge irregular total labeling of cycle C_5

total irregular strength for complete graphs is $\text{tvs}(K_n) = 2$. Jendroľ et al., [64] proved that $\text{tes}(K_p) = \lceil (p^2 - p + 4)/6 \rceil$ for any $p \geq 6$.

In Figure 2.22, edge irregular total labeling of C_5 is shown.

Bača et al., [20] defined a vertex irregular total k-labeling of a graph G.

Definition 2.2.16. *A function φ is called a* vertex irregular total k-labeling *of G if $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ has the property that the associated vertex weights are different for all vertices, that is,*

$$wt_\varphi(u) \neq wt_\varphi(u)$$

for all vertices $u, v \in V(G)$, $u \neq v$. The weight of a vertex $v \in V(G)$ is

$$wt_\varphi(v) = \varphi(v) + \sum_{uv \in E(G)} \varphi(uv),$$

where the sum is over all vertices u adjacent to v. The minimum k for which the graph G has an vertex irregular total k-labeling is called the total vertex irregularity strength *of G, denoted by* $\text{tvs}(G)$.

Note that irregularity strength $\text{s}(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. However, the total vertex irregularity strength $\text{tvs}(G)$ is defined for every graph G. Thus for graphs with no component of order at most 2, $\text{tvs}(G) \leq \text{s}(G)$.

Ahmed et al., [4] described the vertex irregularity strength of helms H_n, they proved $\text{tvs}(H_n) = \lceil (n+1)/2 \rceil$, for $n \geq 4$. Let $f_{m,n}$ be a generalized friendship graph, where m is the number of cycles and n is the order of cycle. For $n = 3$ it has been shown in [124] that $tvs(f_{m,n}) = \lceil (2m+2)/3 \rceil$.

In [20], Bača et al., determined an exact value of the total vertex irregularity strength for the prism D_n, $n \geq 3$, as $\text{tvs}(D_n) = \lceil (2n+3)/4 \rceil$. They also proved that $\text{tvs}(C_n) = \lceil (n+2)/3 \rceil$.

Wijaya and Slamin [124] found the exact value of the total vertex irregularity strength for the wheels, fans, sun graphs and friendship graphs as follows. For wheel graphs W_n, $n \geq 3$,

$\text{tvs}(W_n) = \lceil (n+3)/4 \rceil$. For fan graphs F_n, $n \geq 3$, $\text{tvs}(F_n) = \lceil (n+2)/4 \rceil$. For friendship graphs f_n, $n \geq 3$, $\text{tvs}(f_n) = \lceil (2n+2)/3 \rceil$.

In Figure 2.23, vertex irregular total labeling of C_5 is shown.

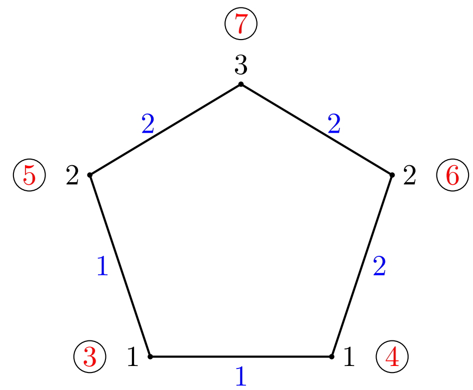

Figure 2.23: Vertex irregular total labeling of cycle C_5

In [100], Ryan et al., introduced the concept of irregular reflexive labeling in 2014, in which the vertex labels are interpreted as (double) the number of loops at a vertex. This new labeling provides two differences from irregular total labeling as follows:

- The labels of vertices have to be even number considering the fact that each loop contributes 2 to the degree of a vertex.
- Vertex label 0 is permissible which represents loopless vertex.

The irregular reflexive labeling is defined as below.

Definition 2.2.17. *For a graph G, we define labelings $\rho_e : E(G) \to \{1, 2, \dots k_e\}$ and $\rho_v : V(G) \to \{0, 2, \dots, 2k_v\}$. Let $\rho = \rho_e \cup \rho_v$ and $k = \max\{k_e, 2k_v\}$.*

The labeling ρ is said to be an edge irregular reflexive k-labeling *if distinct edges e and f have distinct weights, that is,*

$$wt_\rho(e) \neq wt_\rho(f).$$

Similarly, the labeling ρ is said to be a vertex irregular reflexive k-labeling *if distinct vertices u and v have distinct weights, that is,*

$$wt_\rho(u) \neq wt_\rho(v).$$

The smallest value of k for which such labelings exist is called the reflexive edge strength *of the graph G (resp.* reflexive vertex strength *of G),* $\text{res}(G)$ *(resp.* $\text{rvs}(G)$*).*

The subsequent chapters explain irregular reflexive labeling in details and several theorems have been proved.

Chapter 3

Edge Irregular Reflexive Labeling

3.1 Introduction

Regular graphs have been an area of interest almost as long as graphs have been studied. However, as a consequence of the Handshaking Lemma, no simple graphs can be completely irregular. That is, no simple graph can have each vertex bearing a distinct degree. Multigraphs however can display this property. In [33] authors asked, "In a loopless multigraph, determine the fewest parallel edges required to ensure that all vertices have distinct degree". For convenience, the problem was recast in terms of a labeled simple graph with the edge labels representing the number of parallel edges. The degree of a vertex was then determined by adding the labels of the edges incident to that vertex. Then the problem became,

"Assign positive values to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular. What is the minimum value of the largest label over all such irregular assignments?" The minimum value of the largest label is known as the *irregularity strength* of a graph.

Recognising the problem of finding irregularity strength as a labeling problem, Bača et al., in [20], considered total labelings on graphs that is, labeling both edges and vertices. This expanded the concept of weight which could now be measured, not only at vertices, but also at edges. In [20] they introduced the total vertex irregularity strength of a graph G, denoted by $\text{tvs}(G)$, and the total edge irregularity strength of G, denoted by $\text{tes}(G)$, as being the minimum maximum label so that the vertex weights (resp. edge weights) were pairwise distinct, for definitions see Subsection 2.2. Work of authors in [20] has inspired further work in this field such as [9, 63, 94, 107].

The concept of *irregular reflexive multigraphs* originated in [100] as a natural consequence of irregular multigraphs by allowing loops. Following the initiative of [33] and rewording the problem as a graph labeling exercise, *irregular reflexive labeling* include vertex labels representing degrees contributed by the loops. The weight of a vertex v, denoted by $wt(v)$, is now determined by adding the incident edge labels and the label of v.

Previously in [20] authors proposed an *irregular total labeling* in which the vertices were labeled by positive integers. The difference between this idea and irregular reflexive labeling is threefold:

1. The concept is consistent with the genesis of the problem by considering multigraphs with loops.
2. The vertex label must be even non negative integers, representing the fact that each loop contributes 2 to the vertex degree.
3. Vertex label 0 is permissible as representing a loopless vertex.

As in the case of irregular total labelings, this new scheme allows to consider not just vertex weights but also edge weights. An edge weight is the sum of the edge label and the labels of the vertices incident to the edge. Thus we are able to propose *vertex irregular reflexive labelings* and *edge irregular reflexive labelings.*

We present here some basic results concerning irregular reflexive labeling and provide irregular reflexive edge and vertex strengths for some families of graphs including stars, paths, cycles, complete graphs, prisms, wheel, join of graphs and generalised friendship graphs. Henceforth, reflexive edge strength and reflexive vertex strength will be abbreviated as res and rvs respectively.

We recall the definition of irregular reflexive labeling.

Definition 3.1.1. *For a graph G, we define labelings $\rho_e : E(G) \to \{1, 2, \ldots k_e\}$ and $\rho_v : V(G) \to \{0, 2, \ldots, 2k_v\}$. Let $\rho = \rho_e \cup \rho_v$ and $k = \max\{k_e, 2k_v\}$.*

The labeling ρ is said to be an edge irregular reflexive k-labeling *if distinct edges e and f have distinct weights, that is,*

$$wt_\rho(e) \neq wt_\rho(f).$$

Similarly, the labeling ρ is said to be a vertex irregular reflexive k-labeling *if distinct vertices u and v have distinct weights, that is,*

$$wt_\rho(u) \neq wt_\rho(v).$$

The smallest value of k for which such labelings exist is called the reflexive edge strength *of the graph G (resp.* reflexive vertex strength *of G),* res(G) *(resp.* rvs(G)*).*

In Section 3.2, we will consider edge irregular reflexive labelings and give examples of res(G) for some well known graphs. Due to the similarity of the labeling schemes, many of the theorems and proofs presented here will be similar to those given in [20] however respective strengths (on the same graph) may be different. For example, res$(K_5) = 4$ while in [20] it was shown that tes$(K_5) = 5$.

3.2 Edge Irregular Reflexive Labeling

We begin the study of edge irregular reflexive labelings by showing that all graphs can bear an edge irregular reflexive labeling and so every graph has a reflexive edge strength.

$\diamondsuit$ **Lemma 3.2.1.** [100] *For every graph G,*

$$\mathrm{res}(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil & \text{if } |E(G)| \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1 & \text{if } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

Proof. The lower bound for $\mathrm{res}(G)$ follows from the fact that the minimal edge weight under a edge irregular reflexive labeling is 1 and the minimum of the maximal edge weights, that is $|E(G)|$, can be achieved only as the sum of 3 numbers from which at least two are even. $\square$

$\diamondsuit$ **Theorem 3.2.2.** [100] *Let G be a simple graph, then*

$$\left\lceil \frac{|E(G)|}{3} \right\rceil \leq \mathrm{res}(G) \leq |E(G)|.$$

Proof. To get the upper bound we label each vertex of G with label 0 and the edges of G are labeled consecutively with labels $1, 2, \ldots, |E(G)|$. This labeling ensures that $wt(e) \neq wt(f)$ for any two distinct edges e and f of G.

Let ρ be an optimal labeling with respect to the $\mathrm{res}(G)$. Then the heaviest edge e of G has weight $wt(e) \geq |E(G)|$. This weight is the sum of three labels. So one label is at least $|E(G)|/3$. $\square$

We now consider the edge irregular reflexive labeling for some standard classes of graphs. The first case demonstrates that the lower bound of Theorem 3.2.2 is tight.

$\diamondsuit$ **Theorem 3.2.3.** [100] *The reflexive edge strength for the star $K_{1,n}$ is*

$$\mathrm{res}(K_{1,n}) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 2 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. The graph $K_{1,n}$ has n edges with the smallest possible edge weight being 1 and the largest being at least n. If we label the central vertex 0 then the least maximum label must be $\lceil n/2 \rceil$ when $n \not\equiv 2 \pmod 4$. For $n \equiv 2 \pmod 4$, $\lceil n/2 \rceil$ is an odd number and so cannot be a vertex label. In this case we can increase the vertex label to $n/2+1$ and reduce the corresponding edge label, or reduce the vertex label by 1 which would require increasing the corresponding edge label to $n/2+1$. So we have,

$$\mathrm{res}(K_{1,n}) \geq \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 2 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

The following labeling scheme provides the equality in the above bounds, thus proving the theorem.

Label the central vertex 0. Order the pendant vertices $v_1, v_2, \ldots, v_n$ and corresponding incident edges $e_1, e_2, \ldots, e_n$. Label $\rho(v_1) = 0, \rho(e_1) = 1, \rho(v_2) = 0, \rho(e_2) = 2$. For $i \geq 3$ label,

$$\begin{array}{lll} \rho(v_i) = \frac{i}{2}, & \rho(e_i) = \frac{i}{2} & \text{if } i \equiv 0 \pmod 4, \\ \rho(v_i) = \left\lfloor \frac{i}{2} \right\rfloor, & \rho(e_i) = \left\lceil \frac{i}{2} \right\rceil & \text{if } i \equiv 1 \pmod 4, \\ \rho(v_i) = \frac{i}{2} - 1, & \rho(e_i) = \frac{i}{2} + 1 & \text{if } i \equiv 2 \pmod 4, \\ \rho(v_i) = \left\lceil \frac{i}{2} \right\rceil, & \rho(e_i) = \left\lfloor \frac{i}{2} \right\rfloor & \text{if } i \equiv 3 \pmod 4. \end{array}$$

It is very easy to observe that the weights of edges will fall between $1, 2, \ldots, n$. □

Figure 3.1 and Figure 3.2 provides edge irregular reflexive labeling of $K_{1,6}$ and $K_{1,5}$ respectively.

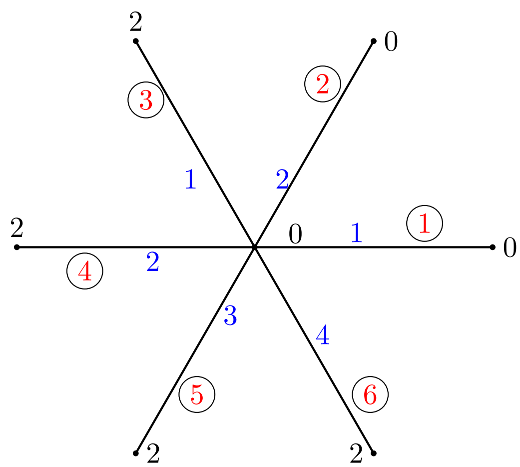

Figure 3.1: Edge irregular reflexive labeling of $K_{1,6}$

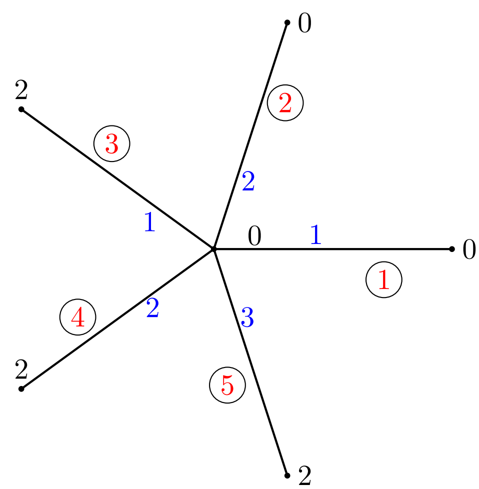

Figure 3.2: Edge irregular reflexive labeling of $K_{1,5}$

The next theorem provides the reflexive edge strength of the path P_n on n vertices.

♢ **Theorem 3.2.4.** [100] *The reflexive edge strength for the path P_n is*

$$\operatorname{res}(P_n) = \begin{cases} \frac{n+1}{3} & \text{if } n \equiv 2 \pmod 6, \\ 2\left\lfloor \frac{n+3}{6} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof. Clearly the smallest maximum edge weight for the path P_n is $n-1$, the number of edges. The following labeling scheme produces such a weight and provides the smallest maximum label by allocating labels as close as possible to one third the weight subject to the constraint on the vertices being labeled with even numbers.

Number the vertices and edges of the path from one of the leaves, beginning with 1 so that the edge numbered e_1 is incident with vertices v_1 and v_2. Define a reflexive labeling ϕ as follows;

$\phi(v_1) = 0, \phi(e_1) = 1, \phi(v_2) = 0, \phi(e_2) = 2, \phi(v_3) = 0, \phi(e_3) = 1, \phi(v_4) = 2, \phi(e_4) = 2, \phi(v_5) = 0, \phi(e_5) = 3, \phi(v_6) = 2$. For the remaining vertices and edges, label as follows;

$$
\begin{array}{lll}
\phi(v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 1, & \phi(e_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 1 & \text{if } i \equiv 0 \pmod 6, \\
\phi(v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 1, & \phi(e_i) = \left\lfloor \frac{i+3}{3} \right\rfloor & \text{if } i \equiv 1 \pmod 6, \\
\phi(v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 1, & \phi(e_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 1 & \text{if } i \equiv 2 \pmod 6, \\
\phi(v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor, & \phi(e_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 3 & \text{if } i \equiv 3 \pmod 6, \\
\phi(v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor, & \phi(e_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 2 & \text{if } i \equiv 4 \pmod 6, \\
\phi(v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor, & \phi(e_i) = \left\lfloor \frac{i+3}{3} \right\rfloor - 1 & \text{if } i \equiv 5 \pmod 6.
\end{array}
$$

It is a simple matter to check that $\phi(v_i) + \phi(e_i) + \phi(v_{i+1}) = i$ for all $i \geq 6$ so that for each edge, $wt(e_i) = i$. ◻

Figure 3.3 and Figure 3.4 provide edge irregular reflexive labelings of P_7 and P_8 respectively.

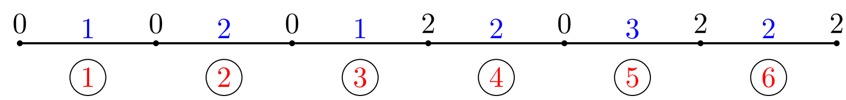

Figure 3.3: Edge irregular reflexive labeling of P_7

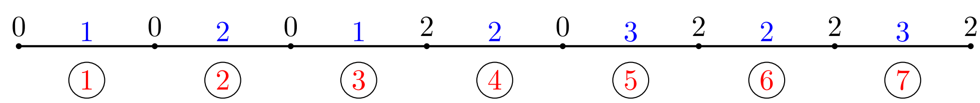

Figure 3.4: Edge irregular reflexive labeling of P_8

♢ **Theorem 3.2.5.** [18] *For every positive integer n, $n \geq 3$*

$$
\operatorname{res}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n \not\equiv 2,3 \pmod 6, \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2,3 \pmod 6. \end{cases}
$$

Proof. Let us denote the vertex set and the edge set of the cycle C_n, $n \geq 3$ such that $V(C_n) = \{x_i : i = 1, 2, \ldots, n\}$, $E(C_n) = \{x_i x_{i+1} : i = 1, 2, \ldots, n\}$, where indices are taken modulo n.

As the cycle C_n has n edges, immediately from Lemma 3.2.1 we get that

$$
\operatorname{res}(C_n) \geq \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n \not\equiv 2,3 \pmod 6, \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2,3 \pmod 6. \end{cases}
$$

Now we distinguish two subcases.

Case 1. Let $n \equiv 3 \pmod 6$. If $n = 3$, from the lower bound we get $\operatorname{res}(C_3) \geq 2$ and the edge irregular reflexive labeling of C_3, as illustrated shown equality in Figure 3.5.

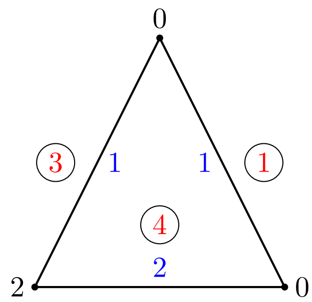

Figure 3.5: Edge irregular reflexive labeling of C_3

For $n \geq 9$ we define the total $(n/3+1)$-labeling f of C_n in the following way

$$
\begin{aligned}
f(x_i) &= 2\left(\lceil \tfrac{i+1}{3} \rceil - 1\right) && i = 1, 2, \dots, \tfrac{n+3}{2},\\
f(x_{n-i+1}) &= 2\lceil \tfrac{i-1}{3} \rceil && i = 1, 2, \dots, \tfrac{n-3}{2},\\
f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) &= 2\lfloor \tfrac{n}{6} \rfloor, &&\\
f(x_i x_{i+1}) &= 2\lceil \tfrac{i}{3} \rceil - 1 && i = 1, 2, \dots, \tfrac{n+1}{2},\\
f(x_{n-i} x_{n-i+1}) &= 2\lceil \tfrac{i+1}{3} \rceil && i = 1, 2, \dots, \tfrac{n-5}{2},\\
f(x_n x_1) &= 2.
\end{aligned}
$$

The vertices of C_n are labeled with even numbers.

The edge weights of the edges in C_n under the labeling f are the following. For $i = 1, 2, \dots, (n+1)/2$ is

$$
\begin{aligned}
wt_f(x_i x_{i+1}) =& f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 2\left(\lceil \tfrac{i+1}{3} \rceil - 1\right) + \left(2\lceil \tfrac{i}{3} \rceil - 1\right) + 2\left(\lceil \tfrac{i+2}{3} \rceil - 1\right)\\
=& 2\left(\lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil + \lceil \tfrac{i+2}{3} \rceil\right) - 5 = 2(i+2) - 5 = 2i - 1.
\end{aligned}
$$

Thus, the corresponding edge weights are $1, 3, \dots, n$ and

$$
\begin{aligned}
wt_f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) =& f(x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) + f(x_{\frac{n+5}{2}})\\
=& 2\left(\left\lceil \frac{\frac{n+3}{2}+1}{3} \right\rceil - 1\right) + 2\lfloor \tfrac{n}{6} \rfloor + 2\left(\left\lceil \frac{\frac{n-3}{2}-1}{3} \right\rceil\right) = 2\left(\lceil \tfrac{n+5}{6} \rceil + \lfloor \tfrac{n}{6} \rfloor + \lceil \tfrac{n-5}{6} \rceil\right) - 2\\
=& n - 1.
\end{aligned}
$$

For $i = 1, 2, \dots, (n-5)/2$ is

$$
\begin{aligned}
wt_f(x_{n-i} x_{n-i+1}) =& f(x_{n-i}) + f(x_{n-i} x_{n-i+1}) + f(x_{n-i+1})\\
=& 2\lceil \tfrac{i}{3} \rceil + 2\lceil \tfrac{i+1}{3} \rceil + 2\lceil \tfrac{i-1}{3} \rceil = 2\left(\lceil \tfrac{i-1}{3} \rceil + \lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil\right) = 2i + 2.
\end{aligned}
$$

Thus, these edge weights are $4, 6, \dots, n-3$.

Moreover,

$$wt_f(x_nx_1) = f(x_n) + f(x_nx_1) + f(x_1) = 0 + 2 + 0 = 2.$$

That means that the edge weights are distinct numbers from the set $\{1, 2, \dots, n\}$.

Case 2. Let $n \not\equiv 3 \pmod 6$. We define the total labeling f of C_n such that

$$\begin{aligned}
f(x_i) &= 2\left(\lceil \tfrac{i+1}{3} \rceil - 1\right) && i = 1, 2, \dots, \lceil \tfrac{n}{2} \rceil,\\
f(x_{n-i+1}) &= 2\lceil \tfrac{i-1}{3} \rceil && i = 1, 2, \dots, \lfloor \tfrac{n}{2} \rfloor,\\
f(x_ix_{i+1}) &= 2\lceil \tfrac{i}{3} \rceil - 1 && i = 1, 2, \dots, \lceil \tfrac{n}{2} \rceil,\\
f(x_{n-i}x_{n-i+1}) &= 2\lceil \tfrac{i+1}{3} \rceil && i = 1, 2, \dots, \lfloor \tfrac{n}{2} \rfloor - 1,\\
f(x_nx_1) &= 2.
\end{aligned}$$

Evidently the vertices of C_n are labeled with even numbers and the labels used are at most $\lceil n/3 \rceil$ if $n \not\equiv 2 \pmod 6$ or they are at most $(\lceil n/3 \rceil + 1)$ if $n \equiv 2 \pmod 6$.

The edge weights of the edges in C_n under the labeling f are the following.

For $i = 1, 2, \dots, \lceil n/2 \rceil - 1$ is

$$\begin{aligned}
wt_f(x_ix_{i+1}) =& f(x_i) + f(x_ix_{i+1}) + f(x_{i+1}) = 2\big(\lceil \tfrac{i+1}{3} \rceil - 1\big) + \big(2\lceil \tfrac{i}{3} \rceil - 1\big) + 2\big(\lceil \tfrac{i+2}{3} \rceil - 1\big)\\
=& 2\big(\lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil + \lceil \tfrac{i+2}{3} \rceil\big) - 5 = 2(i+2) - 5 = 2i - 1.
\end{aligned}$$

Thus, for n even the edge weights are $1, 3, \dots, n-3$ and for n odd the edge weights are $1, 3, \dots, n-2$.

$$\begin{aligned}
wt_f(x_{\lceil \frac{n}{2} \rceil}x_{\lceil \frac{n}{2} \rceil+1}) =& f(x_{\lceil \frac{n}{2} \rceil}) + f(x_{\lceil \frac{n}{2} \rceil}x_{\lceil \frac{n}{2} \rceil+1}) + f(x_{\lceil \frac{n}{2} \rceil+1})\\
=& f(x_{\lceil \frac{n}{2} \rceil}) + f(x_{\lceil \frac{n}{2} \rceil}x_{\lceil \frac{n}{2} \rceil+1}) + f(x_{n-\lfloor \frac{n}{2} \rfloor+1})\\
=& 2\left(\left\lceil \frac{\lceil \frac{n}{2} \rceil+1}{3} \right\rceil - 1\right) + \left(2\left\lceil \frac{\lceil \frac{n}{2} \rceil}{3} \right\rceil - 1\right) + 2\left(\left\lceil \frac{\lfloor \frac{n}{2} \rfloor-1}{3} \right\rceil\right)\\
=& 2\left(\left\lceil \frac{\lceil \frac{n}{2} \rceil-1}{3} \right\rceil + \left\lceil \frac{\lceil \frac{n}{2} \rceil}{3} \right\rceil + \left\lceil \frac{\lfloor \frac{n}{2} \rfloor+1}{3} \right\rceil\right) - 3 = 2\left(\lceil \tfrac{n}{2} \rceil + 1\right) - 3 = 2\lceil \tfrac{n}{2} \rceil - 1,
\end{aligned}$$

which is equal to $(n-1)$ for n even and is equal to n for n odd.

For $i = 1, 2, \dots, \lfloor n/2 \rfloor - 1$ is

$$\begin{aligned}
wt_f(x_{n-i}x_{n-i+1}) =& f(x_{n-i}) + f(x_{n-i}x_{n-i+1}) + f(x_{n-i+1}) = 2\lceil \tfrac{i}{3} \rceil + 2\lceil \tfrac{i+1}{3} \rceil + 2\lceil \tfrac{i-1}{3} \rceil\\
=& 2\big(\lceil \tfrac{i-1}{3} \rceil + \lceil \tfrac{i}{3} \rceil + \lceil \tfrac{i+1}{3} \rceil\big) = 2i + 2.
\end{aligned}$$

Thus, for n even these edge weights are $4, 6, \dots, n$ and for n odd these edge weights are $4, 6, \dots, n-1$.

Moreover,

$$wt_f(x_nx_1) = f(x_n) + f(x_nx_1) + f(x_1) = 0 + 2 + 0 = 2.$$

Combining the previous facts we get that weights of the edges are distinct numbers from the set $\{1, 2, \ldots, n\}$. This completes the proof. $\square$

Figures 3.6 and 3.7 demonstrate edge irregular reflexive labelings of C_{10} and C_{12} respectively.

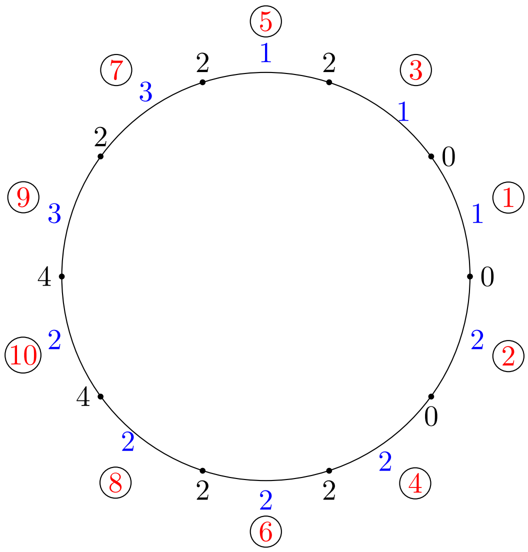

Figure 3.6: Edge irregular reflexive labeling of C_{10}

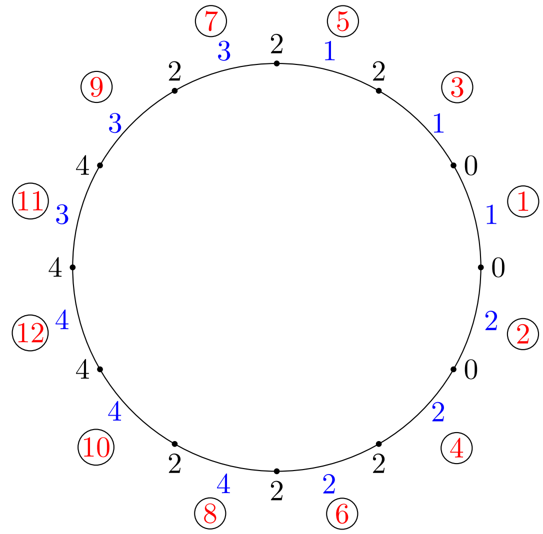

Figure 3.7: Edge irregular reflexive labeling of C_{12}

In the next theorem we give the exact values of reflexive edge strength of Cartesian product of a cycle C_n and C_3.

$\diamondsuit$ **Theorem 3.2.6.** [18] *For every positive integer n, $n \geq 3$*

$$\operatorname{res}(C_n \square C_3) = 2n.$$

Proof. We denote the vertex set and the edge set of $C_n \square C_3$, $n \geq 3$ such that

$$V(C_n \square C_3) = \{x_i, y_i, z_i : i = 1, 2, \ldots, n\},$$
$$E(C_n \square C_3) = \{x_i y_i, x_i z_i, y_i z_i, x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1} : i = 1, 2, \ldots, n\},$$

where indices are taken modulo n.

The graph $C_n \square C_3$ has $6n$ edges thus using Lemma 3.2.1 we have

$$\operatorname{res}(C_n \square C_3) \geq 2n.$$

Now we distinguish two subcases according to the parity of n.

Case 1. When n is even let us consider the total $2n$-labeling f defined in the following way

$$f(x_i) = 0 \qquad i = 1, 2, \ldots, n,$$
$$f(y_i) = n \qquad i = 1, 2, \ldots, n,$$

$$
\begin{aligned}
f(z_i) &= 2n && i = 1, 2, \ldots, n,\\
f(x_ix_{i+1}) &= i && i = 1, 2, \ldots, n-1,\\
f(x_nx_1) &= n, &&\\
f(y_iy_{i+1}) &= n+i && i = 1, 2, \ldots, n-1,\\
f(y_ny_1) &= 2n, &&\\
f(z_iz_{i+1}) &= n+i && i = 1, 2, \ldots, n-1,\\
f(z_nz_1) &= 2n, &&\\
f(x_iy_i) &= i && i = 1, 2, \ldots, n,\\
f(y_iz_i) &= n+i && i = 1, 2, \ldots, n,\\
f(x_iz_i) &= i && i = 1, 2, \ldots, n.
\end{aligned}
$$

Evidently f is a $2n$-labeling. Now we calculate the edge weights.

$$
\begin{aligned}
wt_f(x_ix_{i+1}) =& f(x_i) + f(x_ix_{i+1}) + f(x_{i+1}) = 0 + i + 0 = i\\
& \text{for } i = 1, 2, \ldots, n-1,\\
wt_f(x_1x_n) =& f(x_1) + f(x_1x_n) + f(x_n) = 0 + n + 0 = n,\\
wt_f(x_iy_i) =& f(x_i) + f(x_iy_i) + f(y_i) = 0 + i + n = n + i\\
& \text{for } i = 1, 2, \ldots, n,\\
wt_f(x_iz_i) =& f(x_i) + f(x_iz_i) + f(z_i) = 0 + i + 2n = 2n + i\\
& \text{for } i = 1, 2, \ldots, n,\\
wt_f(y_iy_{i+1}) =& f(y_i) + f(y_iy_{i+1}) + f(y_{i+1}) = n + (n+i) + n = 3n + i\\
& \text{for } i = 1, 2, \ldots, n,\\
wt_f(y_iz_i) =& f(y_i) + f(y_iz_i) + f(z_i) = n + (n+i) + 2n = 4n + i\\
& \text{for } i = 1, 2, \ldots, n,\\
wt_f(z_iz_{i+1}) =& f(z_i) + f(z_iz_{i+1}) + f(z_{i+1}) = 2n + (n+i) + 2n = 5n + i\\
& \text{for } i = 1, 2, \ldots, n.
\end{aligned}
$$

Thus the set of edge weights is $\{1, 2, \ldots, 6n\}$.

Case 2. When n is odd we define the total $2n$-labeling f $C_n \Box C_3$ such that

$$
\begin{aligned}
f(x_i) &= 0 && i = 1, 2, \ldots, n,\\
f(y_i) &= n+1 && i = 1, 2, \ldots, n-1,\\
f(y_n) &= n-1, &&\\
f(z_i) &= 2n && i = 1, 2, \ldots, n,\\
f(x_ix_{i+1}) &= i && i = 1, 2, \ldots, n-1,\\
f(x_nx_1) &= n, &&\\
f(y_iy_{i+1}) &= n+i && i = 1, 2, \ldots, n-2,\\
f(y_ny_1) &= n+1, &&\\
f(y_{n-1}y_n) &= n+2, &&
\end{aligned}
$$

$$
\begin{aligned}
f(z_i z_{i+1}) &= n+i & i &= 1,2,\dots,n-1,\\
f(z_1 z_n) &= 2n, & &\\
f(x_i y_i) &= i & i &= 1,2,\dots,n-1,\\
f(x_n y_n) &= 2, & &\\
f(y_i z_i) &= n+i & i &= 1,2,\dots,n-1,\\
f(y_n z_n) &= n+2, & &\\
f(x_i z_i) &= i & i &= 1,2,\dots,n.
\end{aligned}
$$

Also in this case the vertices are labeled with even numbers and the labels are at most $2n$. For the edge weights we have

$$
\begin{aligned}
wt_f(x_i x_{i+1}) =& f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 0 + i + 0 = i\\
& \text{for } i = 1,2,\dots,n-1,\\
wt_f(x_1 x_n) =& f(x_1) + f(x_1 x_n) + f(x_n) = 0 + n + 0 = n,\\
wt_f(x_i y_i) =& f(x_i) + f(x_i y_i) + f(y_i) = 0 + i + (n+1) = n+i+1\\
& \text{for } i = 1,2,\dots,n-1,\\
wt_f(x_n y_n) =& f(x_n) + f(x_n y_n) + f(y_n) = 0 + 2 + (n-1) = n+1,\\
wt_f(x_i z_i) =& f(x_i) + f(x_i z_i) + f(z_i) = 0 + i + 2n = 2n+i\\
& \text{for } i = 1,2,\dots,n,\\
wt_f(y_i y_{i+1}) =& f(y_i) + f(y_i y_{i+1}) + f(y_{i+1}) = (n+1) + (n+i) + (n+1) = 3n+i+2\\
& \text{for } i = 1,2,\dots,n-2,\\
wt_f(y_{n-1} y_n) =& f(y_{n-1}) + f(y_{n-1} y_n) + f(y_n) = (n+1) + (n+2) + (n-1) = 3n+2,\\
wt_f(y_n y_1) =& f(y_n) + f(y_n y_1) + f(y_1) = (n-1) + (n+1) + (n+1) = 3n+1,\\
wt_f(y_i z_i) =& f(y_i) + f(y_i z_i) + f(z_i) = (n+1) + (n+i) + 2n = 4n+i+1\\
& \text{for } i = 1,2,\dots,n-1,\\
wt_f(y_n z_n) =& f(y_n) + f(y_n z_n) + f(z_n) = (n-1) + (n+2) + 2n = 4n+1,\\
wt_f(z_i z_{i+1}) =& f(z_i) + f(z_i z_{i+1}) + f(z_{i+1}) = 2n + (n+i) + 2n = 5n+i\\
& \text{for } i = 1,2,\dots,n.
\end{aligned}
$$

Which means that also for n odd the edge weights are distinct numbers from the set $\{1, 2, \dots, 6n\}$. □

Figure 3.8 gives edge irregular reflexive labeling of $C_4 \square C_3$.

3.3 Edge Irregular Reflexive Labeling of Prisms

The prism D_n, $n \geq 3$, is a trivalent graph which can be defined as the Cartesian product $P_2 \square C_n$ of a path on two vertices with a cycle on n vertices. We denote the vertex set and the edge set of D_n such that $V(D_n) = \{x_i, y_i : i = 1, 2, \dots, n\}$ and $E(D_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i :$

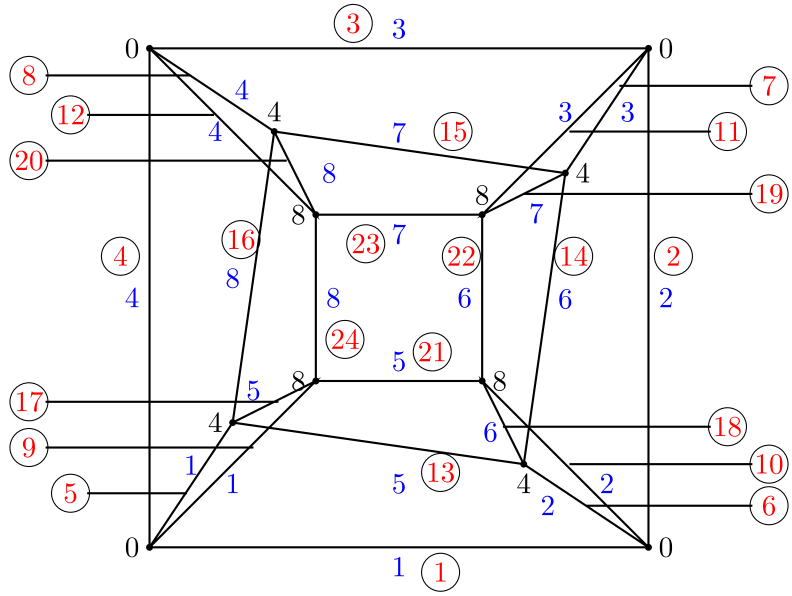

Figure 3.8: Edge irregular reflexive labeling of $C_4 \square C_3$

$i = 1, 2, \ldots, n\}$, where indices are taken modulo n. Here we have considered the outer cycle vertices as y_i and inner cycle vertices as x_i.

♢ **Theorem 3.3.1.** [116] *For $n \geq 3$,*

$$\operatorname{res}(D_n) = \begin{cases} n+1 & \textit{if } n \textit{ is odd,} \\ n & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. As the prism D_n has $3n$ edges, immediately from Lemma 3.2.1 we get that $\operatorname{res}(D_n) \geq n+1$ for n is odd and $\operatorname{res}(D_n) \geq n$ for n is even.

Let $k = n$ for n even and let $k = n+1$ for n odd. We define the labeling f of D_n such that

$$\begin{aligned} f(x_i) &= 0 && i = 1, 2, \ldots, n, \\ f(y_i) &= k && i = 1, 2, \ldots, n, \\ f(x_i x_{i+1}) &= f(y_i y_{i+1}) = i && i = 1, 2, \ldots, n-1, \\ f(x_1 x_n) &= f(y_1 y_n) = n, && \\ f(x_i y_i) &= i && i = 1, 2, \ldots, n. \end{aligned}$$

Evidently f is k-labeling. The edge weights of the edges in D_n under the labeling f are the following.

$$\begin{aligned} wt_f(x_i x_{i+1}) =& 0 + i + 0 = i && \text{for } i = 1, 2, \ldots, n-1, \\ wt_f(x_1 x_n) =& 0 + n + 0 = n, && \\ wt_f(x_i y_i) =& 0 + i + k = k + i && \text{for } i = 1, 2, \ldots, n, \end{aligned}$$

$$
\begin{aligned}
wt_f(y_i y_{i+1}) &= k + i + k = 2k + i \qquad\qquad \text{for } i = 1, 2, \dots, n-1, \\
wt_f(y_1 y_n) &= k + n + k = 2k + n.
\end{aligned}
$$

That means that for n odd $\{wt_f(e) : e \in E(D_n)\} = \{1, 2, \dots, n, n+2, n+3, \dots, 2n+1, 2n+3, 2n+4, \dots, 3n+2\}$ and for n even $\{wt_f(e) : e \in E(D_n)\} = \{1, 2, \dots, 3n\}$. Thus the edge weights are distinct, that is f is a edge irregular reflexive labeling of a prism D_n. □

Figures 3.9 and 3.10 demonstrate edge irregular reflexive labeling of D_8 and D_9 respectively.

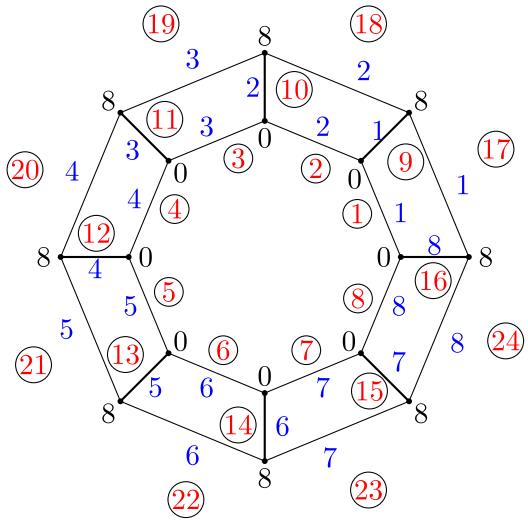

Figure 3.9: Edge irregular reflexive labeling of D_8

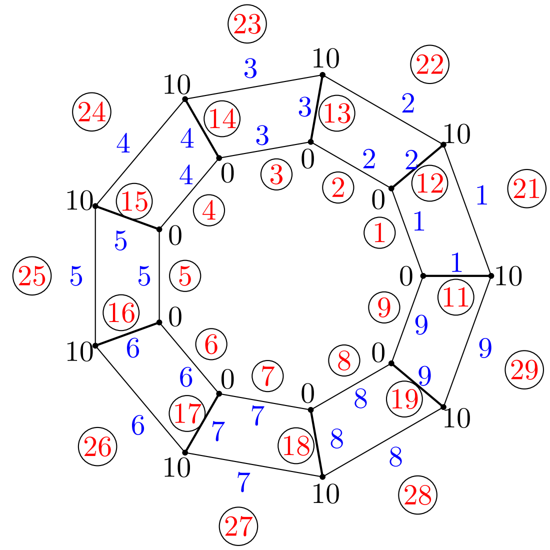

Figure 3.10: Edge irregular reflexive labeling of D_9

3.4 Edge Irregular Reflexive Labeling of Wheels

The wheel W_n, $n \geq 3$, is a graph obtained by joining all vertices of C_n to a further vertex called the centre. We denote the vertex set and the edge set of W_n such that $V(W_n) = \{x, x_i : i = 1, 2, \dots, n\}$ and $E(W_n) = \{x_i x_{i+1}, x x_i : i = 1, 2, \dots, n\}$, where indices are taken modulo n. We prove the following result for wheels.

◇ **Theorem 3.4.1.** [116] *For $n \geq 3$,*

$$
\operatorname{res}(W_n) = \begin{cases} 4 & \text{if } n = 3, \\ \lceil \frac{2n}{3} \rceil & \text{if } n \equiv 0, 2 \pmod 3 \text{ and } n \geq 5, \\ \lceil \frac{2n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3). \end{cases}
$$

Proof. According to the fact that the wheel W_n has $2n$ edges, using Lemma 3.2.1 we obtain the following lower bound for wheel $\operatorname{res}(W_n) \geq k = \lceil \frac{2n}{3} \rceil$ if $n \equiv 0, 2 \pmod 3$ and $\operatorname{res}(W_n) \geq k = \lceil \frac{2n}{3} \rceil + 1$ if $n \equiv 1 \pmod 3$. It is easy to see that k is even.

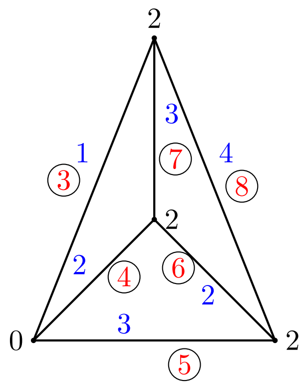

Figure 3.11: Edge irregular reflexive labeling of W_3

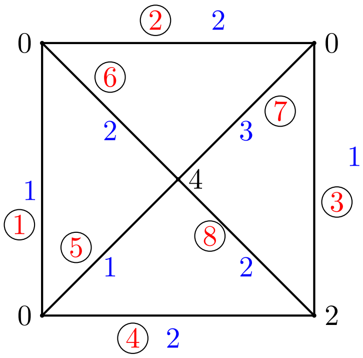

Figure 3.12: Edge irregular reflexive labeling of W_4

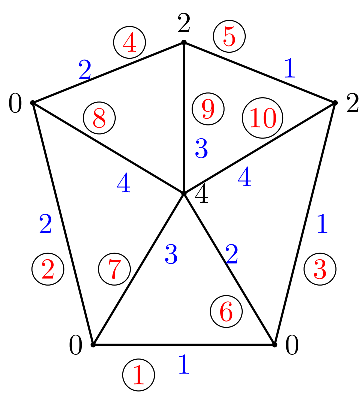

Figure 3.13: Edge irregular reflexive labeling of W_5

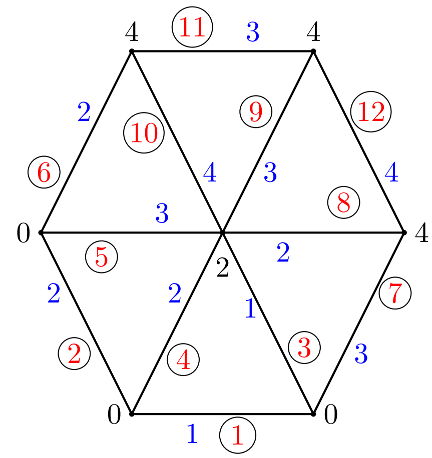

Figure 3.14: Edge irregular reflexive labeling of W_6

Thus $\text{res}(W_3) \geq 2$. If $\text{res}(W_3) \leq 3$ then the vertices of W_3 can be labeled only with 0's or 2's and the set of all possible edge weights is a subset of the set $\{1, 2, \dots, 7\}$. As the edge weights 1 and 2 could be realizable only as 1=0+1+0; 2=0+2+0 then five edges have end vertex labeled with 0. However, this is a contradiction as the edge weights 6 and 7 can be realizable only as 6=2+2+2 and 7=2+3+2. Thus $\text{res}(W_3) \geq 4$. The corresponding labelings for W_n, $n = 3, 4, 5, 6$ are illustrated on Figures 3.11, 3.12, 3.13 and 3.14 respectively.

For $n \geq 7$ we define a total labeling of W_n such that

$$f(x) = k,$$

$$f(x_i) = \begin{cases} 0 & i = 1, 2, \dots, k-1, \\ 2 & i = k, \\ k & i = k+1, k+2, \dots, n-1, \\ k-2 & i = n, \end{cases}$$

$$f(xx_i) = \begin{cases} i & i = 1, 2, \dots, k-1, \\ k-2 & i = k, \\ n-2k+1+i & i = k+1, k+2, \dots, n-1, \\ 5 & i = n, \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} i & i = 1, 2, \dots, k-2, \\ k-3 & i = k-1, \\ k-1 & i = k, \\ i-k+3 & i = k+1, k+2, \dots, n-2, \\ 4 & i = n-1, \\ 2 & i = n. \end{cases}$$

Evidently for $n \geq 7$ f is a k-labeling. Now we calculate the edge weights.

$$\begin{aligned} wt_f(x_i x_{i+1}) =& 0 + i + 0 = i && \text{for } i = 1, 2, \dots, k-2, \\ wt_f(x_{k-1} x_k) =& 0 + (k-3) + 0 = k-1, \\ wt_f(x_k x_{k+1}) =& 2 + (k-1) + k = 2k+1, \\ wt_f(x_i x_{i+1}) =& k + (i-k+3) + k = k+3+i && \text{for } i = k+1, k+2, \dots, n-2, \\ wt_f(x_{n-1} x_n) =& k + 4 + (k-2) = 2k+1, \\ wt_f(x_n x_1) =& (k-2) + 2 + 0 = k, \\ wt_f(xx_i) =& k + i + 0 = k+i && \text{for } i = 1, 2, \dots, k-1, \\ wt_f(xx_k) =& k + (k-2) + 2 = 2k, \\ wt_f(xx_i) =& k + (n-2k+1+i) + k = n+1+i && \text{for } i = k+1, k+2, \dots, n-1, \\ wt_f(xx_n) =& k + 5 + (k-2) = 2k+3. \end{aligned}$$

It is easy to check that the edge weights are from the set $\{1, 2, \dots, 2n\}$. This concludes the proof.

□

Figure 3.15 and 3.16 demonstrate edge irregular reflexive labelings of W_{10} and W_{12} respectively.

Let us receall the following definitions from Section 2.1.

A *fan graph* F_n is obtained from wheel W_n if one rim edge, say $x_1 x_n$ is deleted. A *basket* B_n is obtained by removing a spoke, say xx_n, from wheel W_n. Before we will give the exact value of reflexive edge strength of fan graphs and baskets we give the following observation.

♢ **Observation 3.4.2.** [116] *Let e be an arbitrary edge in G. Then*

$$\operatorname{res}(G - \{e\}) \leq \operatorname{res}(G).$$

Proof. The proof is trivial. □

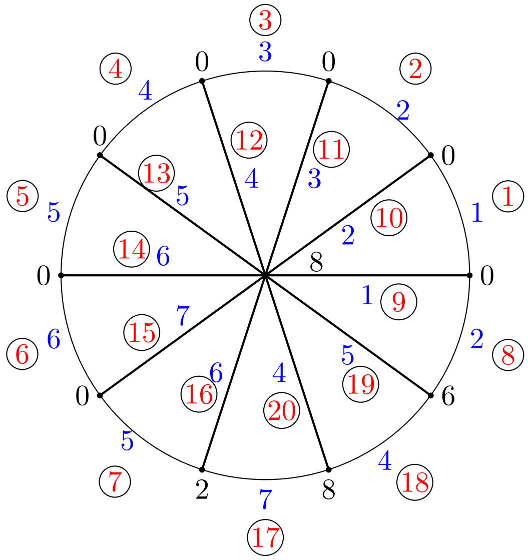

Figure 3.15: Edge irregular reflexive labeling of W_{10}

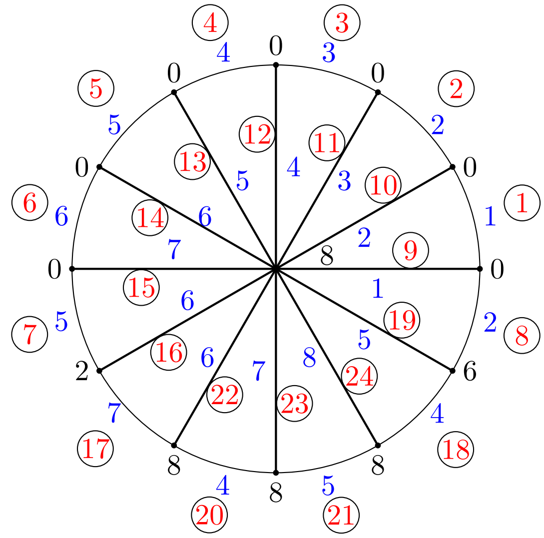

Figure 3.16: Edge irregular reflexive labeling of W_{12}

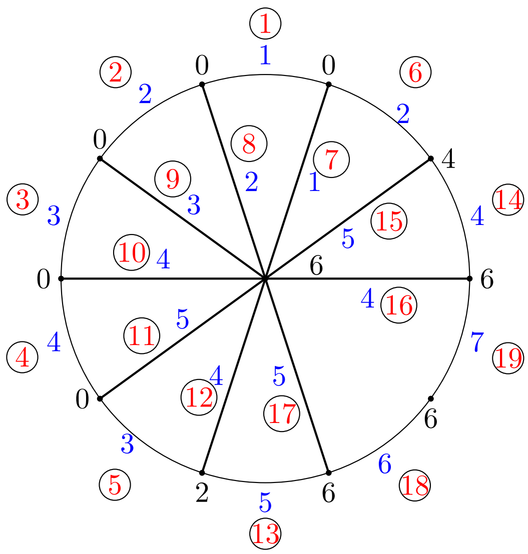

Figure 3.17: Edge irregular reflexive labeling of B_{10}

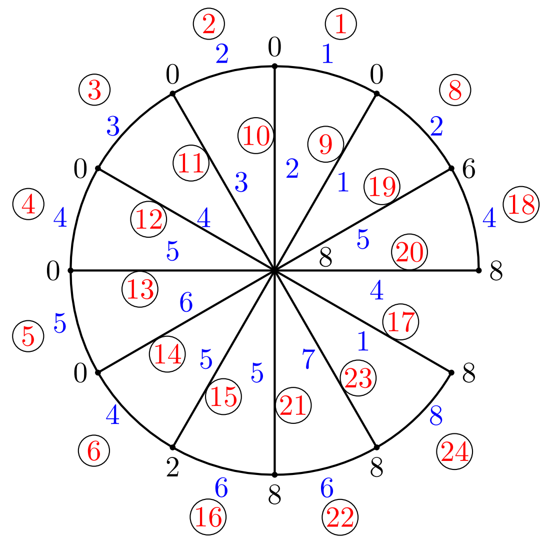

Figure 3.18: Edge irregular reflexive labeling of F_{12}

♢ **Theorem 3.4.3.** [116] *For* $n \geq 3$,

$$\operatorname{res}(F_n) = \operatorname{res}(B_n) = \begin{cases} 3 & \text{if } n = 3, \\ 4 & \text{if } n = 4, \\ \lceil \frac{2n}{3} \rceil & \text{if } n \geq 4. \end{cases}$$

Proof. Using similar arguments as in the proof of Theorem 3.4.1 we get that $\operatorname{res}(F_3) \geq 3$ and

$\text{res}(F_4) \geq 4$. The corresponding labeling for F_3 can be obtained from the labeling illustrated for W_3, see Figure 3.11, when the edge labeled with 4 is deleted. For the fan graph F_4 we can delete arbitrary rim edge from W_4.

For $n \geq 5$, combining Lemma 3.2.1 and Observation 3.4.2 we get that $\text{res}(F_n) = \text{res}(W_n) = \lceil 2n/3 \rceil$ for $n \equiv 0, 2 \pmod 3$ and $\lceil 2n/3 \rceil \leq \text{res}(F_n) \leq \lceil 2n/3 \rceil + 1$ for $n \equiv 1 \pmod 3$.

Let $n \equiv 1 \pmod 3$, $n \geq 7$. Then the number $k = \lceil 2n/3 \rceil$ is odd. We define k-labeling of F_n such that

$$f(x) = k - 1,$$

$$f(x_i) = \begin{cases} k-1 & i = 1, \\ k-3 & i = 2, \\ 0 & i = 3, 4, \dots, k, \\ 2 & i = k+1, \\ k-1 & i = k+2, k+3, \dots, n, \end{cases}$$

$$f(xx_i) = \begin{cases} 4 & i = 1, \\ 5 & i = 2, \\ i-2 & i = 3, 4, \dots, k, \\ k-3 & i = k+1, \\ 2i-2k+1 & i = k+2, k+3, \dots, n, \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} 4 & i = 1, \\ 2 & i = 2, \\ i-2 & i = 3, 4, \dots, k-1, \\ k-4 & i = k, \\ k-2 & i = k+1, \\ 2i-2k+2 & i = k+2, k+3, \dots, n-1. \end{cases}$$

It is not difficult to check that the edge weights are distinct numbers from the set $1, 2, \dots, 2n-1$.

The proof for basket B_n can be done analogously as for the fan graph. Evidently $V(B_n) = V(F_n)$ and $E(B_n) = E(F_n) \cup \{x_1 x_n\} - \{xx_n\}$. For $n \equiv 1 \pmod 3$, $n \geq 7$, the following $\lceil 2n/3 \rceil$-labeling g of B_n defined such that $g(y) = f(y)$ for $y \in V(F_n)$ or $y \in E(F_n) - \{xx_n\}$ and $g(x_1 x_n) = f(xx_n)$ has desired properties. $\square$

Figures 3.17 and 3.18 demonstrate edge irregular reflexive labelings of B_{10} and F_{12} respectively.

3.5 Edge Irregular Reflexive Labeling of Join of Graphs

The join $G \oplus H$ of the disjoint graphs G and H is the graph $G \cup H$ together with all the edges joining vertices of $V(G)$ and vertices of $V(H)$.

In the next two theorems we will deal with the join of a path or a cycle with $2K_1$.

◇ **Theorem 3.5.1.** [18] *For every positive integer n, $n \geq 2$*

$$\operatorname{res}(P_n \oplus (2K_1)) = \begin{cases} 3 & \text{if } n = 2, \\ n+1 & \text{if } n \text{ is odd, } n \geq 3, \\ n & \text{if } n \text{ is even, } n \geq 4. \end{cases}$$

Proof. We denote the vertex set and the edge set of $P_n \oplus (2K_1)$, $n \geq 2$ such that

$$V(P_n \oplus (2K_1)) = \{x_i : i = 1, 2, \dots, n\} \cup \{y, z\},$$
$$E(P_n \oplus (2K_1)) = \{x_i x_{i+1} : i = 1, 2, \dots, n-1\} \cup \{yx_i, zx_i : i = 1, 2, \dots, n\}.$$

As $|E(P_n \oplus (2K_1))| = 3n - 1$ then by Lemma 3.2.1 we have

$$\operatorname{res}(P_n \oplus (2K_1)) \geq \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

However, it is easy to see that $\operatorname{res}(P_2 \oplus (2K_1)) \geq 3$. The corresponding 3-labeling for $P_2 \oplus (2K_1)$ is illustrated in Figure 3.19.

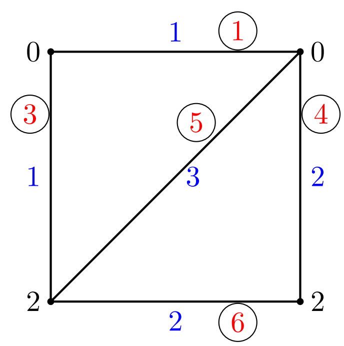

Figure 3.19: Edge irregular reflexive labeling of $P_2 \oplus (2K_1)$

For $n \geq 3$ we distinguish two subcases according to the parity of n.

Case 1. When n is even we define n-labeling f of $P_n \oplus (2K_1)$ such that

$$\begin{aligned} f(x_i) &= 0 && i = 1, 2, \dots, \tfrac{n}{2}, \\ f(x_{\frac{n}{2}+1}) &= n - 2, && \\ f(x_i) &= n && i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n, \\ f(y) &= 0, && \\ f(z) &= n, && \\ f(x_i x_{i+1}) &= \tfrac{n}{2} + i && i = 1, 2, \dots, \tfrac{n}{2} - 1, \\ f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) &= 2, && \end{aligned}$$

$$
\begin{aligned}
f(x_{\frac{n}{2}+1}x_{\frac{n}{2}+2}) &= 3, && \\
f(x_ix_{i+1}) &= i-\tfrac{n}{2} && i=\tfrac{n}{2}+2,\tfrac{n}{2}+3,\dots,n-1,\\
f(yx_i) &= i && i=1,2,\dots,\tfrac{n}{2},\\
f(yx_{\frac{n}{2}+1}) &= 3, && \\
f(yx_i) &= i-\tfrac{n}{2} && i=\tfrac{n}{2}+2,\tfrac{n}{2}+3,\dots,n,\\
f(zx_i) &= \tfrac{n}{2}+i && i=1,2,\dots,\tfrac{n}{2},\\
f(zx_{\frac{n}{2}+1}) &= \tfrac{n}{2}+2, && \\
f(zx_i) &= i-1 && i=\tfrac{n}{2}+2,\tfrac{n}{2}+3,\dots,n.
\end{aligned}
$$

For the edge weights we get

$$
\begin{aligned}
wt_f(x_ix_{i+1}) =& f(x_i)+f(x_ix_{i+1})+f(x_{i+1}) = 0+(\tfrac{n}{2}+i)+0=\tfrac{n}{2}+i\\
& \text{for } i=1,2,\dots,\tfrac{n}{2}-1,\\
wt_f(x_{\frac{n}{2}}x_{\frac{n}{2}+1}) =& f(x_{\frac{n}{2}})+f(x_{\frac{n}{2}}x_{\frac{n}{2}+1})+f(x_{\frac{n}{2}+1}) = 0+2+(n-2)=n,\\
wt_f(x_{\frac{n}{2}+1}x_{\frac{n}{2}+2}) =& f(x_{\frac{n}{2}+1})+f(x_{\frac{n}{2}+1}x_{\frac{n}{2}+2})+f(x_{\frac{n}{2}+2}) = (n-2)+3+n=2n+1,\\
wt_f(x_ix_{i+1}) =& f(x_i)+f(x_ix_{i+1})+f(x_{i+1}) = n+(i-\tfrac{n}{2})+n=\tfrac{3n}{2}+i\\
& \text{for } i=\tfrac{n}{2}+2,\tfrac{n}{2}+3,\dots,n-1,\\
wt_f(yx_i) =& f(y)+f(yx_i)+f(x_i) = 0+i+0=i\\
& \text{for } i=1,2,\dots,\tfrac{n}{2},\\
wt_f(yx_{\frac{n}{2}+1}) =& f(y)+f(yx_{\frac{n}{2}+1})+f(x_{\frac{n}{2}+1}) = 0+3+(n-2)=n+1,\\
wt_f(yx_i) =& f(y)+f(yx_i)+f(x_i) = 0+(i-\tfrac{n}{2})+n=\tfrac{n}{2}+i\\
& \text{for } i=\tfrac{n}{2}+2,\tfrac{n}{2}+3,\dots,n,\\
wt_f(zx_i) =& f(z)+f(zx_i)+f(x_i) = n+(\tfrac{n}{2}+i)+0=\tfrac{3n}{2}+i\\
& \text{for } i=1,2,\dots,\tfrac{n}{2},\\
wt_f(zx_{\frac{n}{2}+1}) =& f(z)+f(zx_{\frac{n}{2}+1})+f(x_{\frac{n}{2}+1}) = n+(\tfrac{n}{2}+2)+(n-2)=\tfrac{5n}{2},\\
wt_f(zx_i) =& f(z)+f(zx_i)+f(x_i) = n+(i-1)+n=2n+i-1\\
& \text{for } i=\tfrac{n}{2}+2,\tfrac{n}{2}+3,\dots,n.
\end{aligned}
$$

Thus the set of edge weights is $\{1,2,\dots,3n-1\}$.

Case 2. When n is odd we define $(n+1)$-labeling f in the following way

$$
\begin{aligned}
f(x_i) &= 0 && i=1,2,\dots,\tfrac{n+1}{2},\\
f(x_{\frac{n+3}{2}}) &= n-1, && \\
f(x_i) &= n+1 && i=\tfrac{n+5}{2},\tfrac{n+7}{2},\dots,n,\\
f(y) &= 0, && \\
f(z) &= n+1, &&
\end{aligned}
$$

$$
\begin{aligned}
f(x_i x_{i+1}) &= \tfrac{n+1}{2} + i && i = 1, 2, \ldots, \tfrac{n-1}{2},\\
f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) &= 2, &&\\
f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) &= 2, &&\\
f(x_i x_{i+1}) &= i - \tfrac{n+3}{2} && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \ldots, n-1,\\
f(y x_i) &= i && i = 1, 2, \ldots, \tfrac{n+1}{2},\\
f(y x_{\frac{n+3}{2}}) &= 3, &&\\
f(y x_i) &= i - \tfrac{n+1}{2} && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \ldots, n,\\
f(z x_i) &= \tfrac{n-1}{2} + i && i = 1, 2, \ldots, \tfrac{n+1}{2},\\
f(z x_{\frac{n+3}{2}}) &= \tfrac{n+1}{2}, &&\\
f(z x_i) &= i - 3 && i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \ldots, n.
\end{aligned}
$$

Evidently, the vertices are labeled with even numbers and the label of every element is at most $n+1$.

Now we will calculate the weights of the edges.

$$
\begin{aligned}
wt_f(x_i x_{i+1}) =& f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 0 + (\tfrac{n+1}{2} + i) + 0 = \tfrac{n+1}{2} + i\\
& \text{for } i = 1, 2, \ldots, \tfrac{n-1}{2},\\
wt_f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) =& f(x_{\frac{n+1}{2}}) + f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = 0 + 2 + (n-1) = n+1,\\
wt_f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) =& f(x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) + f(x_{\frac{n+5}{2}}) = (n-1) + 2 + (n+1) = 2n+2,\\
wt_f(x_i x_{i+1}) =& f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = (n+1) + (i - \tfrac{n+3}{2}) + (n+1) = \tfrac{3n+1}{2} + i\\
& \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \ldots, n-1,\\
wt_f(y x_i) =& f(y) + f(y x_i) + f(x_i) = 0 + i + 0 = i\\
& \text{for } i = 1, 2, \ldots, \tfrac{n+1}{2},\\
wt_f(y x_{\frac{n+3}{2}}) =& f(y) + f(y x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = 0 + 3 + (n-1) = n+2,\\
wt_f(y x_i) =& f(y) + f(y x_i) + f(x_i) = 0 + (i - \tfrac{n+1}{2}) + (n+1) = \tfrac{n+1}{2} + i\\
& \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \ldots, n,\\
wt_f(z x_i) =& f(z) + f(z x_i) + f(x_i) = (n+1) + (\tfrac{n-1}{2} + i) + 0 = \tfrac{3n+1}{2} + i\\
& \text{for } i = 1, 2, \ldots, \tfrac{n+1}{2},\\
wt_f(z x_{\frac{n+3}{2}}) =& f(z) + f(z x_{\frac{n+3}{2}}) + f(x_{\frac{n+3}{2}}) = (n+1) + \tfrac{n+1}{2} + (n-1) = \tfrac{5n+1}{2},\\
wt_f(z x_i) =& f(z) + f(z x_i) + f(x_i) = (n+1) + (i-3) + (n+1) = 2n + i - 1\\
& \text{for } i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \ldots, n.
\end{aligned}
$$

It is easy to check that the edge weights are distinct consecutive integers $\{1, 2, \ldots, 3n-1\}$.

This concludes the proof. $\square$

In the next theorem we give bounds for the graph $C_n \oplus (2K_1)$.

◇ **Theorem 3.5.2.** [18] *For every positive integer n, $n \geq 3$*

$$\operatorname{res}(C_n \oplus (2K_1)) = \begin{cases} 5 & \text{if } n = 4, \\ n+1 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}. \end{cases}$$

Proof. We denote the vertex set and the edge set of $C_n \oplus (2K_1)$, $n \geq 2$ such that

$$\begin{aligned} V(C_n \oplus (2K_1)) &= \{x_i : i = 1, 2, \dots, n\} \cup \{y, z\}, \\ E(C_n \oplus (2K_1)) &= \{x_i x_{i+1}, y x_i, z x_i : i = 1, 2, \dots, n\}, \end{aligned}$$

where the indices are taken modulo n. The graph $C_n \oplus (2K_1)$ has $3n$ edges thus applying Lemma 3.2.1 we obtain

$$\operatorname{res}(C_n \oplus (2K_1)) \geq \begin{cases} n & \text{if } n \text{ is even, } n \geq 6, \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Let us consider two subcases according to the parity of n.

Case 1. Let n be even number.

It is easy to see that $\operatorname{res}(C_4 \oplus (2K_1)) \geq 5$. The corresponding 5-labeling for $C_4 \oplus (2K_1)$ is illustrated in Figure 3.20.

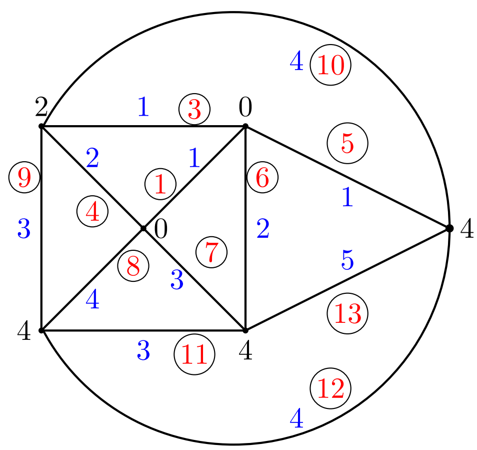

Figure 3.20: Edge irregular reflexive labeling of $C_4 \oplus (2K_1)$

For $n \geq 6$ we define n-labeling f of $C_n \oplus (2K_1)$ such that

$$\begin{aligned} f(x_i) &= 0 & i = 1, 2, \dots, \tfrac{n}{2} - 1, \\ f(x_{\frac{n}{2}}) &= n - 4, \\ f(x_{\frac{n}{2}+1}) &= n - 2, \end{aligned}$$

$$
\begin{aligned}
f(x_{\frac{n}{2}+2}) &= n-2, \\
f(x_i) &= n && i = \tfrac{n}{2}+3, \tfrac{n}{2}+4, \dots, n, \\
f(y) &= 0, \\
f(z) &= n, \\
f(x_i x_{i+1}) &= \tfrac{n}{2}+i-1 && i = 1, 2, \dots, \tfrac{n}{2}-2, \\
f(x_{\frac{n}{2}-1} x_{\frac{n}{2}}) &= 2, \\
f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) &= 6, \\
f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) &= 5, \\
f(x_{\frac{n}{2}+2} x_{\frac{n}{2}+3}) &= 4, \\
f(x_i x_{i+1}) &= i-\tfrac{n}{2} && i = \tfrac{n}{2}+3, \tfrac{n}{2}+3, \dots, n-1, \\
f(x_n x_1) &= n-1, \\
f(y x_i) &= i && i = 1, 2, \dots, \tfrac{n}{2}-1, \\
f(y x_{\frac{n}{2}}) &= 3, \\
f(y x_{\frac{n}{2}+1}) &= 2, \\
f(y x_{\frac{n}{2}+2}) &= 3, \\
f(y x_i) &= i-\tfrac{n}{2}-1 && i = \tfrac{n}{2}+3, \tfrac{n}{2}+4, \dots, n, \\
f(z x_i) &= \tfrac{n}{2}+i-1 && i = 1, 2, \dots, \tfrac{n}{2}-1, \\
f(z x_{\frac{n}{2}}) &= \tfrac{n}{2}+4, \\
f(z x_{\frac{n}{2}+1}) &= \tfrac{n}{2}+3, \\
f(z x_{\frac{n}{2}+2}) &= \tfrac{n}{2}+4, \\
f(z x_i) &= i && i = \tfrac{n}{2}+3, \tfrac{n}{2}+4, \dots, n.
\end{aligned}
$$

For the edge weights we get

$$
\begin{aligned}
wt_f(x_i x_{i+1}) =& f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = 0 + (\tfrac{n}{2}+i-1) + 0 = \tfrac{n}{2}+i-1 \\
& \text{for } i = 1, 2, \dots, \tfrac{n}{2}-2, \\
wt_f(x_{\frac{n}{2}-1} x_{\frac{n}{2}}) =& f(x_{\frac{n}{2}-1}) + f(x_{\frac{n}{2}-1} x_{\frac{n}{2}}) + f(x_{\frac{n}{2}}) = 0 + 2 + (n-4) = n-2, \\
wt_f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) =& f(x_{\frac{n}{2}}) + f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = (n-4) + 6 + (n-2) = 2n, \\
wt_f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) =& f(x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = (n-2) + 5 + (n-2) = 2n+1, \\
wt_f(x_{\frac{n}{2}+2} x_{\frac{n}{2}+3}) =& f(x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2} x_{\frac{n}{2}+3}) + f(x_{\frac{n}{2}+3}) = (n-2) + 4 + n = 2n+2, \\
wt_f(x_i x_{i+1}) =& f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) = n + (i-\tfrac{n}{2}) + n = \tfrac{3n}{2}+i \\
& \text{for } i = \tfrac{n}{2}+3, \tfrac{n}{2}+4, \dots, n-1, \\
wt_f(x_n x_1) =& f(x_n) + f(x_n x_1) + f(x_1) = n + (n-1) + 0 = 2n-1,
\end{aligned}
$$

$$
\begin{aligned}
wt_f(yx_i) =& f(y) + f(yx_i) + f(x_i) = 0 + i + 0 = i \\
& \text{for } i = 1, 2, \dots, \tfrac{n}{2} - 1, \\
wt_f(yx_{\frac{n}{2}}) =& f(y) + f(yx_{\frac{n}{2}}) + f(x_{\frac{n}{2}}) = 0 + 3 + (n-4) = n - 1, \\
wt_f(yx_{\frac{n}{2}+1}) =& f(y) + f(yx_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = 0 + 2 + (n-2) = n, \\
wt_f(yx_{\frac{n}{2}+2}) =& f(y) + f(yx_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = 0 + 3 + (n-2) = n + 1, \\
wt_f(yx_i) =& f(y) + f(yx_i) + f(x_i) = 0 + (i - \tfrac{n}{2} - 1) + n = \tfrac{n}{2} + i - 1 \\
& \text{for } i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n, \\
wt_f(zx_i) =& f(z) + f(zx_i) + f(x_i) = n + (\tfrac{n}{2} + i - 1) + 0 = \tfrac{3n}{2} + i - 1 \\
& \text{for } i = 1, 2, \dots, \tfrac{n}{2} - 1, \\
wt_f(zx_{\frac{n}{2}}) =& f(z) + f(zx_{\frac{n}{2}}) + f(x_{\frac{n}{2}}) = n + (\tfrac{n}{2} + 4) + (n-4) = \tfrac{5n}{2}, \\
wt_f(zx_{\frac{n}{2}+1}) =& f(z) + f(zx_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1}) = n + (\tfrac{n}{2} + 3) + (n-2) = \tfrac{5n}{2} + 1, \\
wt_f(zx_{\frac{n}{2}+2}) =& f(z) + f(zx_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2}) = n + (\tfrac{n}{2} + 4) + (n-2) = \tfrac{5n}{2} + 2, \\
wt_f(zx_i) =& f(z) + f(zx_i) + f(x_i) = n + i + n = 2n + i \\
& \text{for } i = \tfrac{n}{2} + 3, \tfrac{n}{2} + 4, \dots, n.
\end{aligned}
$$

It is easy to get that the edge weights are $\{1, 2, \dots, 3n\}$.

Case 2. Let n be odd. Then we define $(n+1)$-labeling f in the following way

$$
\begin{aligned}
f(x_i) &= 0 & & i = 1, 2, \dots, \tfrac{n+1}{2}, \\
f(x_{\frac{n+3}{2}}) &= n - 1, \\
f(x_i) &= n + 1 & & i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n, \\
f(y) &= 0, \\
f(z) &= n + 1, \\
f(x_i x_{i+1}) &= \tfrac{n+1}{2} + i & & i = 1, 2, \dots, \tfrac{n-1}{2}, \\
f(x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}) &= 2, \\
f(x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}) &= 3, \\
f(x_i x_{i+1}) &= i - \tfrac{n+1}{2} & & i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n - 1, \\
f(x_n x_1) &= n + 1, \\
f(yx_i) &= i & & i = 1, 2, \dots, \tfrac{n+1}{2}, \\
f(yx_{\frac{n+3}{2}}) &= 3, \\
f(yx_i) &= i - \tfrac{n+1}{2} & & i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n, \\
f(zx_i) &= \tfrac{n-1}{2} + i & & i = 1, 2, \dots, \tfrac{n+1}{2}, \\
f(zx_{\frac{n+3}{2}}) &= \tfrac{n+3}{2}, \\
f(zx_i) &= i - 2 & & i = \tfrac{n+5}{2}, \tfrac{n+7}{2}, \dots, n.
\end{aligned}
$$

Thus the vertices are labeled with even numbers 0, $n-1$ or $n+1$.

For the edge weights we get the following.

$$\begin{aligned}
wt_f(x_ix_{i+1}) =& f(x_i)+f(x_ix_{i+1})+f(x_{i+1}) = 0+(\tfrac{n+1}{2}+i)+0 = \tfrac{n+1}{2}+i \\
& \text{for } i=1,2,\dots,\tfrac{n-1}{2}, \\
wt_f(x_{\frac{n+1}{2}}x_{\frac{n+3}{2}}) =& f(x_{\frac{n+1}{2}})+f(x_{\frac{n+1}{2}}x_{\frac{n+3}{2}})+f(x_{\frac{n+3}{2}}) = 0+2+(n-1) = n+1, \\
wt_f(x_{\frac{n+3}{2}}x_{\frac{n+5}{2}}) =& f(x_{\frac{n+3}{2}})+f(x_{\frac{n+3}{2}}x_{\frac{n+5}{2}})+f(x_{\frac{n+5}{2}}) = (n-1)+3+(n+1) = 2n+3, \\
wt_f(x_ix_{i+1}) =& f(x_i)+f(x_ix_{i+1})+f(x_{i+1}) = (n+1)+(i-\tfrac{n+1}{2})+(n+1) = \tfrac{3n+3}{2}+i \\
& \text{for } i=\tfrac{n+5}{2},\tfrac{n+7}{2},\dots,n-1, \\
wt_f(x_nx_1) =& f(x_n)+f(x_nx_1)+f(x_1) = (n+1)+(n+1)+0 = 2n+2, \\
wt_f(yx_i) =& f(y)+f(yx_i)+f(x_i) = 0+i+0 = i \\
& \text{for } i=1,2,\dots,\tfrac{n+1}{2}, \\
wt_f(yx_{\frac{n+3}{2}}) =& f(y)+f(yx_{\frac{n+3}{2}})+f(x_{\frac{n+3}{2}}) = 0+3+(n-1) = n+2, \\
wt_f(yx_i) =& f(y)+f(yx_i)+f(x_i) = 0+(i-\tfrac{n+1}{2})+(n+1) = \tfrac{n+1}{2}+i \\
& \text{for } i=\tfrac{n+5}{2},\tfrac{n+7}{2},\dots,n, \\
wt_f(zx_i) =& f(z)+f(zx_i)+f(x_i) = (n+1)+(\tfrac{n-1}{2}+i)+0 = \tfrac{3n+1}{2}+i \\
& \text{for } i=1,2,\dots,\tfrac{n+1}{2}, \\
wt_f(zx_{\frac{n+3}{2}}) =& f(z)+f(zx_{\frac{n+3}{2}})+f(x_{\frac{n+3}{2}}) = (n+1)+\tfrac{n+3}{2}+(n-1) = \tfrac{5n+3}{2}, \\
wt_f(zx_i) =& f(z)+f(zx_i)+f(x_i) = (n+1)+(i-2)+(n+1) = 2n+i \\
& \text{for } i=\tfrac{n+5}{2},\tfrac{n+7}{2},\dots,n.
\end{aligned}$$

Evidently, the edge weights are distinct numbers from the set $\{1,2,\dots,3n\}$. □

3.6 Edge Irregular Reflexive Labeling for Generalised Friendship Graphs

In this section we will investigate the edge irregular reflexive labeling for generalized friendship graphs. The *friendship graph* f_m is a collection of m triangles with a common vertex. It may be also pictured as a wheel W_{2m} with every alternate rim edge removed. Let us mention that the reflexive edge strength for wheels can be found in Theorem 3.4.1. The *generalized friendship graph* $f_{n,m}$ is a collection of m cycles all of order n, meeting at a common vertex. We will refer to the friendship graph f_m as an instance of the generalized friendship graph and write it as $f_{3,m}$. For our purposes, we refer to vertices in the following way: the central vertex is named x and all other vertices addressed in the form x_i^j, where j, $1\le j\le m$ indicates which cycle contains the vertex and i, $1\le i\le n$ points to the position of the vertex within the cycle. So $x=x_0^j$ for all j. Then we denote the edge set of the generalized friendship graph such that $E(f_{n,m})=\{x_i^jx_{i+1}^j : 1\le i\le n-2, 1\le j\le m\}\cup\{xx_1^j, xx_{n-1}^j : 1\le j\le m\}$.

In this section we determine the exact value of the reflexive edge strength for the generalized friendship graphs $f_{n,m}$ for $n = 3, 4, 5$ and $m \geq 1$.

♢ **Theorem 3.6.1.** [19] *For every positive integer* $m \geq 1$

$$\operatorname{res}(f_{3,m}) = \begin{cases} 3 & \text{if } m = 2, \\ m & \text{if } m \text{ is even, } m \geq 4, \\ m+1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. The graph $f_{3,m}$ has $3m$ edges thus by Lemma 3.2.1 we have

$$\operatorname{res}(f_{3,m}) \geq \begin{cases} m & \text{if } m \text{ is even,} \\ m+1 & \text{if } m \text{ is odd.} \end{cases}$$

As $f_{3,1}$ is isomorphic to C_3 thus according to Theorem 3.2.5 we get $\operatorname{res}(C_3) = 2$.

It is easy to see that $\operatorname{res}(f_{3,2}) \geq 3$. The corresponding edge irregular reflexive labeling for $f_{3,2}$ is illustrated in Figure 3.21. This graph is also known as a *bowtie graph*.

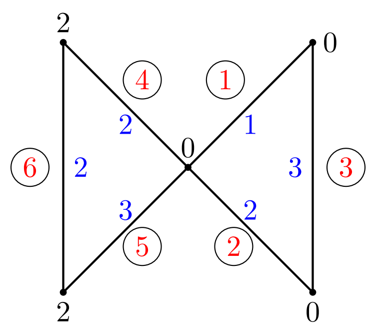

Figure 3.21: Edge irregular reflexive labeling of $f_{3,2}$

For $m \geq 3$ we distinguish two cases.

Case 1. When m is even we define a m-labeling f of $f_{3,m}$ such that

$$\begin{aligned}
f(x) &= m-2, && \\
f(x_1^j) &= 0 && j = 1, 2, \dots, m-2, \\
f(x_1^j) &= m && j = m-1, m, \\
f(x_2^j) &= 2\lceil \tfrac{i}{2} \rceil - 2 && j = 1, 2, \dots, m-2, \\
f(x_2^j) &= m && j = m-1, m, \\
f(xx_1^j) &= i && j = 1, 2, \dots, m-2, \\
f(xx_1^{m-1}) &= m-3, && \\
f(xx_1^m) &= m-1, &&
\end{aligned}$$

$$
\begin{aligned}
f(xx_2^j) &= m-1 && j=1,3,\dots,m-3,\\
f(xx_2^j) &= m && j=2,4,\dots,m,\\
f(xx_2^{m-1}) &= m-2, &&\\
f(x_1^jx_2^j) &= 1 && j=1,3,\dots,m-3,\\
f(x_1^jx_2^j) &= 2 && j=2,4,\dots,m-2,\\
f(x_1^{m-1}x_2^{m-1}) &= m-1, &&\\
f(x_1^mx_2^m) &= m. &&
\end{aligned}
$$

Then we get

$$
\begin{aligned}
wt_f(xx_1^j) =& f(x)+f(xx_1^j)+f(x_1^j) = (m-2)+j+0 = m+j-2\\
&\text{for } j=1,2,\dots,m-2,\\
wt_f(xx_1^{m-1}) =& f(x)+f(xx_1^{m-1})+f(x_1^{m-1}) = (m-2)+(m-3)+m = 3m-5,\\
wt_f(xx_1^m) =& f(x)+f(xx_1^m)+f(x_1^m) = (m-2)+(m-1)+m = 3m-3,\\
wt_f(xx_2^j) =& f(x)+f(xx_2^j)+f(x_2^j) = (m-2)+(m-1)+(2\lceil\tfrac{j}{2}\rceil-2) = 2m-4+j\\
&\text{for } j=1,3,\dots,m-3,\\
wt_f(xx_2^j) =& f(x)+f(xx_2^j)+f(x_2^j) = (m-2)+m+(2\lceil\tfrac{j}{2}\rceil-2) = 2m-4+j\\
&\text{for } j=2,4,\dots,m-2,\\
wt_f(xx_2^{m-1}) =& f(x)+f(xx_2^{m-1})+f(x_2^{m-1}) = (m-2)+(m-2)+m = 3m-4,\\
wt_f(xx_2^m) =& f(x)+f(xx_2^m)+f(x_2^m) = (m-2)+m+m = 3m-2,\\
wt_f(x_1^jx_2^j) =& f(x_1^j)+f(x_1^jx_2^j)+f(x_2^j) = 0+1+(2\lceil\tfrac{j}{2}\rceil-2) = j\\
&\text{for } j=1,3,\dots,m-3,\\
wt_f(x_1^jx_2^j) =& f(x_1^j)+f(x_1^jx_2^j)+f(x_2^j) = 0+2+(2\lceil\tfrac{j}{2}\rceil-2) = j\\
&\text{for } j=2,4,\dots,m-2,\\
wt_f(x_1^{m-1}x_2^{m-1}) =& f(x_1^{m-1})+f(x_1^{m-1}x_2^{m-1})+f(x_2^{m-1}) = m+(m-1)+m = 3m-1,\\
wt_f(x_1^mx_2^m) =& f(x_1^m)+f(x_1^mx_2^m)+f(x_2^m) = m+m+m = 3m.
\end{aligned}
$$

Its is not difficult to see that the edge weights are $\{1,2,\dots,3m\}$. This shows that f is a edge irregular reflexive labeling of $f_{3,m}$ for $m \geq 4$ even.

Case 2. When m is odd we define a $(m+1)$-labeling f of $f_{3,m}$ such that

$$
\begin{aligned}
f(x) &= m-1, &&\\
f(x_1^j) &= 0 && j=1,2,\dots,m-1,\\
f(x_1^m) &= m-1, &&\\
f(x_2^j) &= 2\lceil\tfrac{j}{2}\rceil-2 && j=1,2,\dots,m-1,\\
f(x_2^m) &= m+1, &&\\
f(xx_1^j) &= j && j=1,2,\dots,m,
\end{aligned}
$$

$$
\begin{aligned}
f(xx_2^j) &= m && j = 1,3,\dots,m-2,\\
f(xx_2^j) &= m+1 && j = 2,4,\dots,m-1,\\
f(xx_2^m) &= m-1, &&\\
f(x_1^j x_2^j) &= 1 && j = 1,3,\dots,m-2,\\
f(x_1^j x_2^j) &= 2 && j = 2,4,\dots,m-1,\\
f(x_1^m x_2^m) &= m. &&
\end{aligned}
$$

Thus the vertices are labeled with even numbers and the edge weights are

$$
\begin{aligned}
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = (m-1) + j + 0 = m + j - 1\\
& \text{for } j = 1,2,\dots,m-1,\\
wt_f(xx_1^m) =& f(x) + f(xx_1^m) + f(x_1^m) = (m-1) + m + (m-1) = 3m-2,\\
wt_f(xx_2^j) =& f(x) + f(xx_2^j) + f(x_2^j) = (m-1) + m + (2\lceil \tfrac{j}{2} \rceil - 2) = 2m - 2 + j\\
& \text{for } j = 1,3,\dots,m-2,\\
wt_f(xx_2^j) =& f(x) + f(xx_2^j) + f(x_2^j) = (m-1) + (m+1) + (2\lceil \tfrac{j}{2} \rceil - 2) = 2m - 2 + j\\
& \text{for } j = 2,4,\dots,m-1,\\
wt_f(xx_2^m) =& f(x) + f(xx_2^m) + f(x_2^m) = (m-1) + (m-1) + (m+1) = 3m-1,\\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = 0 + 1 + (2\lceil \tfrac{j}{2} \rceil - 2) = j\\
& \text{for } j = 1,3,\dots,m-2,\\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = 0 + 2 + (2\lceil \tfrac{j}{2} \rceil - 2) = j\\
& \text{for } j = 2,4,\dots,m-1,\\
wt_f(x_1^m x_2^m) =& f(x_1^m) + f(x_1^m x_2^m) + f(x_2^m) = (m-1) + m + (m+1) = 3.
\end{aligned}
$$

Thus also for n odd, $m \geq 3$, the edge weights are distinct numbers from the set $\{1, 2, \dots, 3m\}$. This concludes the proof. $\square$

◇ **Theorem 3.6.2.** [19] *For every positive integer $m \geq 1$*

$$
\operatorname{res}(f_{4,m}) = \begin{cases} \lceil \frac{4m}{3} \rceil & \text{if } m \equiv 0,1 \pmod 3,\\ \lceil \frac{4m}{3} \rceil + 1 & \text{if } m \equiv 2 \pmod 3. \end{cases}
$$

Proof. Let us denote $m = 3r + t$, where $r \geq 0$ and $t \in \{0, 1, 2\}$.

As $|E(f_{4,3r+t})| = 4(3r + t) = 12r + 4t$ then according to Lemma 3.2.1 we get

$$
\operatorname{res}(f_{4,3r+t}) \geq \begin{cases} \lceil \frac{12r+4t}{3} \rceil & \text{if } t \neq 2,\\ \lceil \frac{12r+4t}{3} \rceil + 1 & \text{if } t = 2 \end{cases}
$$

that is

$$
\operatorname{res}(f_{4,3r+t}) \geq 4r + 2t
$$

for every $r \geq 0$ and $t \in \{0,1,2\}$.

We define a $(4r+2t)$-labeling f of $f_{4,3r+t}$ such that

$$
\begin{aligned}
f(x) &= 0, && \\
f(x_i^j) &= 0 && j = 1,2,\dots,r+t,\ i = 1,3, \\
f(x_i^j) &= 2j-2 && j = r+t+1, r+t+2,\dots,2r+t,\ i = 1,3, \\
f(x_i^j) &= 4r+2t && j = 2r+t+1, 2r+t+2,\dots,3r+t,\ i = 1,3, \\
f(x_2^j) &= 4r+2t && j = 1,2,\dots,3r+t, \\
f(xx_1^j) &= 2j-1 && j = 1,2,\dots,r+t, \\
f(xx_1^j) &= 1 && j = r+t+1, r+t+2,\dots,2r+t, \\
f(xx_1^j) &= 6r+2t+2-2j && j = 2r+t+1, 2r+t+2,\dots,3r+t, \\
f(xx_3^j) &= 2j && j = 1,2,\dots,r+t, \\
f(xx_3^j) &= 2 && j = r+t+1, r+t+2,\dots,2r+t, \\
f(xx_3^j) &= 6r+2t+1-2j && j = 2r+t+1, 2r+t+2,\dots,3r+t, \\
f(x_1^j x_2^j) &= 2r-1+2j && j = 1,2,\dots,r+t, \\
f(x_1^j x_2^j) &= 2r+1 && j = r+t+1, r+t+2,\dots,2r+t, \\
f(x_1^j x_2^j) &= 8r+2t+2-2j && j = 2r+t+1, 2r+t+2,\dots,3r+t, \\
f(x_3^j x_2^j) &= 2r+2j && j = 1,2,\dots,r+t, \\
f(x_3^j x_2^j) &= 2r+2 && j = r+t+1, r+t+2,\dots,2r+t, \\
f(x_3^j x_2^j) &= 8r+2t+1-2j && j = 2r+t+1, 2r+t+2,\dots,3r+t.
\end{aligned}
$$

Evidently, the vertices are labeled by even numbers and all the labels are not greater then $4r+2t$. Moreover, for the edge weights we get the following.

$$
\begin{aligned}
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + (2j-1) + 0 = 2j-1 \\
& \text{for } j = 1,2,\dots,r+t, \\
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + 1 + (2j-2) = 2j-1 \\
& \text{for } j = r+t+1, r+t+2,\dots,2r+t, \\
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + (6r+2t+2-2j) + (4r+2t) = 10r+4t+2-2j \\
& \text{for } j = 2r+t+1, 2r+t+2,\dots,3r+t \\
wt_f(xx_3^j) =& f(x) + f(xx_3^j) + f(x_3^j) = 0 + 2j + 0 = 2j \\
& \text{for } j = 1,2,\dots,r+t \\
wt_f(xx_3^j) =& f(x) + f(xx_3^j) + f(x_3^j) = 0 + 2 + (2j-2) = 2j \\
& \text{for } j = r+t+1, r+t+2,\dots,2r+t, \\
wt_f(xx_3^j) =& f(x) + f(xx_3^j) + f(x_3^j) = 0 + (6r+2t+1-2j) + (4r+2t) = 10r+4t+1-2j \\
& \text{for } j = 2r+t+1, 2r+t+2,\dots,3r+t,
\end{aligned}
$$

$$
\begin{aligned}
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = 0 + (2r - 1 + 2j) + (4r + 2t) = 6r + 2t - 1 + 2j \\
& \qquad\qquad \text{for } j = 1, 2, \dots, r + t, \\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = (2j - 2) + (2r + 1) + (4r + 2t) = 6r + 2t - 1 + 2j \\
& \qquad\qquad \text{for } j = r + t + 1, r + t + 2, \dots, 2r + t, \\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = (4r + 2t) + (8r + 2t + 2 - 2j) + (4r + 2t) \\
=& 16r + 6t + 2 - 2j \\
& \qquad\qquad \text{for } j = 2r + t + 1, 2r + t + 2, \dots, 3r + t, \\
wt_f(x_3^j x_2^j) =& f(x_3^j) + f(x_3^j x_2^j) + f(x_2^j) = 0 + (2r + 2j) + (4r + 2t) = 6r + 2t + 2j \\
& \qquad\qquad \text{for } j = 1, 2, \dots, r + t, \\
wt_f(x_3^j x_2^j) =& f(x_3^j) + f(x_3^j x_2^j) + f(x_2^j) = (2j - 2) + (2r + 2) + (4r + 2t) = 6r + 2t + 2j \\
& \qquad\qquad \text{for } j = r + t + 1, r + t + 2, \dots, 2r + t, \\
wt_f(x_3^j x_2^j) =& f(x_3^j) + f(x_3^j x_2^j) + f(x_2^j) = (4r + 2t) + (8r + 2t + 1 - 2j) + (4r + 2t) \\
=& 16r + 6t + 1 - 2j \\
& \qquad\qquad \text{for } j = 2r + t + 1, 2r + t + 2, \dots, 3r + t.
\end{aligned}
$$

Its is not difficult to see that the edge weights $\{1, 2, \dots, 12r + 4t\}$. This shows that f is a edge irregular reflexive labeling of $f_{4,m}$ for $m \geq 1$. □

♢ **Theorem 3.6.3.** [19] *For every positive integer $m \geq 1$*

$$
\operatorname{res}(f_{5,m}) = \begin{cases} \lceil \frac{5m}{3} \rceil & \text{if } m \not\equiv 3, 4 \pmod 6, \\ \lceil \frac{5m}{3} \rceil + 1 & \text{if } m \equiv 3, 4 \pmod 6. \end{cases}
$$

Proof. As the number of edges of $f_{5,m}$ is $5m$ then using Lemma 3.2.1 we obtain

$$
\operatorname{res}(f_{5,m}) \geq k = \begin{cases} \lceil \frac{5m}{3} \rceil & \text{if } m \not\equiv 3, 4 \pmod 6, \\ \lceil \frac{5m}{3} \rceil + 1 & \text{if } m \equiv 3, 4 \pmod 6. \end{cases}
$$

It is easy to see that for $m \equiv 5 \pmod 6$ the number k is odd and otherwise k is even.

As the graph $f_{5,1}$ is isomorphic to C_5 thus according to Theorem 3.2.5 we get $\operatorname{res}(C_5) = 2$.

From the lower bound for $\operatorname{res}(f_{5,m})$ we get that $\operatorname{res}(f_{5,2}) \geq 4$ and $\operatorname{res}(f_{5,3}) \geq 6$. A corresponding edge irregular reflexive labeling for $f_{5,2}$ and an edge irregular reflexive labeling for $f_{5,3}$ are illustrated in Figures 3.22 and 3.23 respectively.

We distinguish two cases according to the parity of k.

Let $m \geq 4$.

Case 1. When $m \not\equiv 5 \pmod 6$, that is when k is an even number, we define a k-labeling f of $f_{5,m}$ in the following way.

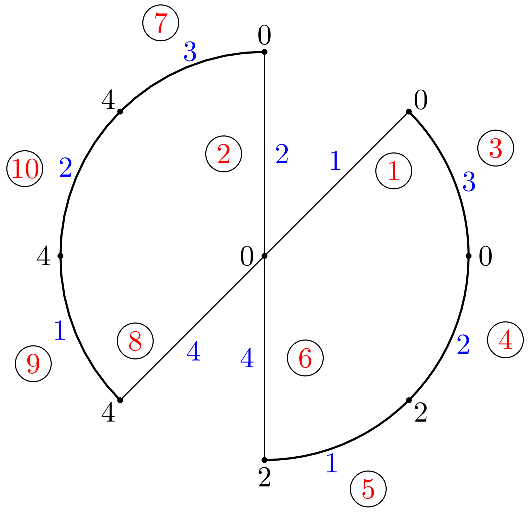

Figure 3.22: Edge irregular reflexive labeling of $f_{5,2}$

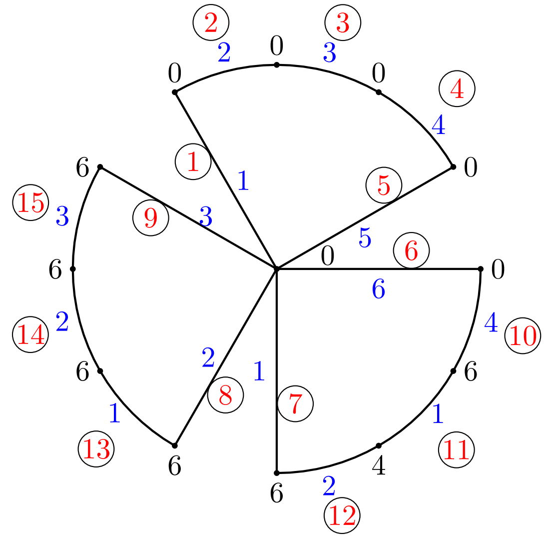

Figure 3.23: Edge irregular reflexive labeling of $f_{5,3}$

$$
\begin{aligned}
f(x) &= 0, & & \\
f(x_i^j) &= 0 & & j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil,\ i = 1, 4, \\
f(x_i^j) &= k - 2\lceil \tfrac{m-4}{6} \rceil & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m,\ i = 1, 4, \\
f(x_i^j) &= k & & j = 1, 2, \dots, m,\ i = 2, 3, \\
f(xx_1^j) &= 2j - 1 & & j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(xx_1^j) &= 2m + 2 - 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
f(xx_4^j) &= 2j & & j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(xx_4^j) &= 2m + 1 - 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
f(x_1^j x_2^j) &= k + 2 - 2j & & j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(x_1^j x_2^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil - 1 + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
& & & m \equiv 0 \pmod 6, \\
f(x_1^j x_2^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil - 3 + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
& & & m \equiv 1, 2 \pmod 6, \\
f(x_1^j x_2^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil - 7 + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
& & & m \equiv 3 \pmod 6, \\
f(x_1^j x_2^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil - 9 + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
& & & m \equiv 4 \pmod 6, \\
f(x_3^j x_4^j) &= k + 1 - 2j & & j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(x_3^j x_4^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m,
\end{aligned}
$$

$$
\begin{array}{ll}
 & m \equiv 0 \pmod 6, \\
f(x_3^j x_4^j) = k - 2m - 4\lceil \frac{m-4}{6} \rceil - 2 + 2j & j = m - 2\lceil \frac{m-4}{6} \rceil + 1, m - 2\lceil \frac{m-4}{6} \rceil + 2, \ldots, m, \\
 & m \equiv 1, 2 \pmod 6, \\
f(x_3^j x_4^j) = k - 2m - 4\lceil \frac{m-4}{6} \rceil - 6 + 2j & j = m - 2\lceil \frac{m-4}{6} \rceil + 1, m - 2\lceil \frac{m-4}{6} \rceil + 2, \ldots, m, \\
 & m \equiv 3 \pmod 6, \\
f(x_3^j x_4^j) = k - 2m - 4\lceil \frac{m-4}{6} \rceil - 8 + 2j & j = m - 2\lceil \frac{m-4}{6} \rceil + 1, m - 2\lceil \frac{m-4}{6} \rceil + 2, \ldots, m, \\
 & m \equiv 4 \pmod 6, \\
f(x_2^j x_3^j) = k + 1 - j & j = 1, 2, \ldots, m,\ m \equiv 0 \pmod 6, \\
f(x_2^j x_3^j) = k - j & j = 1, 2, \ldots, m,\ m \equiv 1 \pmod 6, \\
f(x_2^j x_3^j) = k - 1 - j & j = 1, 2, \ldots, m,\ m \equiv 2 \pmod 6, \\
f(x_2^j x_3^j) = k - 2 - j & j = 1, 2, \ldots, m,\ m \equiv 3 \pmod 6, \\
f(x_2^j x_3^j) = k - 3 - j & j = 1, 2, \ldots, m,\ m \equiv 4 \pmod 6.
\end{array}
$$

For the edge weights we get

$$
\begin{aligned}
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + (2j-1) + 0 = 2j - 1 \\
& \text{for } j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + (2m + 2 - 2j) + (k - 2\lceil \tfrac{m-4}{6} \rceil) \\
=& 2m + k - 2\lceil \tfrac{m-4}{6} \rceil + 2 - 2j \\
& \text{for } j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m, \\
wt_f(xx_4^j) =& f(x) + f(xx_4^j) + f(x_4^j) = 0 + 2j + 0 = 2j \\
& \text{for } j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
wt_f(xx_4^j) =& f(x) + f(xx_4^j) + f(x_4^j) = 0 + (2m + 1 - 2j) + (k - 2\lceil \tfrac{m-4}{6} \rceil) \\
=& 2m + k - 2\lceil \tfrac{m-4}{6} \rceil + 1 - 2j \\
& \text{for } j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m, \\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = 0 + (k + 2 - 2j) + k = 2k + 2 - 2j \\
& \text{for } j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil.
\end{aligned}
$$

For $j = m - 2\lceil \frac{m-4}{6} \rceil + 1, m - 2\lceil \frac{m-4}{6} \rceil + 2, \ldots, m$ we have

$$
\begin{aligned}
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = (k - 2\lceil \tfrac{m-4}{6} \rceil) + f(x_1^j x_2^j) + k \\
=& \begin{cases}
3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 1 + 2j, & m \equiv 0 \pmod 6, \\
3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 3 + 2j, & m \equiv 1, 2 \pmod 6, \\
3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 7 + 2j, & m \equiv 3 \pmod 6, \\
3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 9 + 2j, & m \equiv 4 \pmod 6.
\end{cases}
\end{aligned}
$$

Furthermore, the weights of edges $x_3^j x_4^j$ for $j = 1, 2, \ldots, m - 2\lceil \frac{m-4}{6} \rceil$ are

$$wt_f(x_3^j x_4^j) = f(x_3^j) + f(x_3^j x_4^j) + f(x_4^j) = k + (k + 1 - 2j) + 0 = 2k + 1 - 2j,$$

and for $j = m - 2\lceil \frac{m-4}{6} \rceil + 1, m - 2\lceil \frac{m-4}{6} \rceil + 2, \ldots, m$ the weights are

$$\begin{aligned} wt_f(x_3^j x_4^j) =& f(x_3^j) + f(x_3^j x_4^j) + f(x_4^j) = k + f(x_3^j x_4^j) + (k - 2\lceil \tfrac{m-4}{6} \rceil) \\ =& \begin{cases} 3k - 2m - 6\lceil \frac{m-4}{6} \rceil + 2j, & m \equiv 0 \pmod 6, \\ 3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 2 + 2j, & m \equiv 1, 2 \pmod 6, \\ 3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 6 + 2j, & m \equiv 3 \pmod 6, \\ 3k - 2m - 6\lceil \frac{m-4}{6} \rceil - 8 + 2j, & m \equiv 4 \pmod 6. \end{cases} \end{aligned}$$

And for $j = 1, 2, \ldots, m$ we get

$$\begin{aligned} wt_f(x_2^j x_3^j) =& f(x_2^j) + f(x_2^j x_3^j) + f(x_3^j) = k + f(x_2^j x_3^j) + k \\ =& \begin{cases} 3k + 1 - j, & m \equiv 0 \pmod 6, \\ 3k - j, & m \equiv 1 \pmod 6, \\ 3k - 1 - j, & m \equiv 2 \pmod 6, \\ 3k - 2 - j, & m \equiv 3 \pmod 6, \\ 3k - 3 - j, & m \equiv 4 \pmod 6. \end{cases} \end{aligned}$$

Its is not difficult to see that the edge weights are distinct numbers from the set $\{1, 2, \ldots, 5m\}$.

Case 2. When $m \equiv 5 \pmod 6$, that is when $k = \lceil \frac{5m}{3} \rceil$ is odd. Then we define a k-labeling f of $f_{5,m}$ such that.

$$\begin{aligned}
f(x) &= 0, & & \\
f(x_i^j) &= 0 & & j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil,\ i = 1, 4, \\
f(x_i^j) &= k - 1 - 2\lceil \tfrac{m-4}{6} \rceil & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m,\ i = 1, 4, \\
f(x_i^j) &= k - 1 & & j = 1, 2, \ldots, m,\ i = 2, 3, \\
f(xx_1^j) &= 2j - 1 & & j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(xx_1^j) &= 2m + 2 - 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m, \\
f(xx_4^j) &= 2j & & j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(xx_4^j) &= 2m + 1 - 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m, \\
f(x_1^j x_2^j) &= k + 1 - 2j & & j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(x_1^j x_2^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m, \\
f(x_3^j x_4^j) &= k - 2j & & j = 1, 2, \ldots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
f(x_3^j x_4^j) &= k - 2m - 4\lceil \tfrac{m-4}{6} \rceil + 1 + 2j & & j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \ldots, m, \\
f(x_2^j x_3^j) &= k + 1 - j & & j = 1, 2, \ldots, m.
\end{aligned}$$

The edge weights are

$$
\begin{aligned}
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + (2j-1) + 0 = 2j-1 \\
& \qquad\qquad \text{for } j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
wt_f(xx_1^j) =& f(x) + f(xx_1^j) + f(x_1^j) = 0 + (2m+2-2j) + (k-1-2\lceil \tfrac{m-4}{6} \rceil) \\
=& 2m + k - 2\lceil \tfrac{m-4}{6} \rceil + 1 - 2j \\
& \qquad\qquad \text{for } j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
wt_f(xx_4^j) =& f(x) + f(xx_4^j) + f(x_4^j) = 0 + 2j + 0 = 2j \\
& \qquad\qquad \text{for } j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
wt_f(xx_4^j) =& f(x) + f(xx_4^j) + f(x_4^j) = 0 + (2m+1-2j) + (k-1-2\lceil \tfrac{m-4}{6} \rceil) \\
=& 2m + k - 2\lceil \tfrac{m-4}{6} \rceil - 2j \\
& \qquad\qquad \text{for } j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = 0 + (k+1-2j) + (k-1) = 2k - 2j \\
& \qquad\qquad \text{for } j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
wt_f(x_1^j x_2^j) =& f(x_1^j) + f(x_1^j x_2^j) + f(x_2^j) = (k-1-2\lceil \tfrac{m-4}{6} \rceil) + (k - 2m - 4\lceil \tfrac{m-4}{6} \rceil + 2j) + (k-1) \\
=& 3k - 2m - 6\lceil \tfrac{m-4}{6} \rceil - 2 + 2j \\
& \qquad\qquad \text{for } j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
wt_f(x_3^j x_4^j) =& f(x_3^j) + f(x_3^j x_4^j) + f(x_4^j) = (k-1) + (k-2j) + 0 = 2k - 1 - 2j \\
& \qquad\qquad \text{for } j = 1, 2, \dots, m - 2\lceil \tfrac{m-4}{6} \rceil, \\
wt_f(x_3^j x_4^j) =& f(x_3^j) + f(x_3^j x_4^j) + f(x_4^j) = k + (k - 2m - 4\lceil \tfrac{m-4}{6} \rceil + 1 + 2j) + (k - 2\lceil \tfrac{m-4}{6} \rceil) \\
=& 3k - 2m - 6\lceil \tfrac{m-4}{6} \rceil - 1 + 2j \\
& \qquad\qquad \text{for } j = m - 2\lceil \tfrac{m-4}{6} \rceil + 1, m - 2\lceil \tfrac{m-4}{6} \rceil + 2, \dots, m, \\
wt_f(x_2^j x_3^j) =& f(x_2^j) + f(x_2^j x_3^j) + f(x_3^j) = (k-1) + (k+1-j) + (k-1) = 3k - 1 - j \\
& \qquad\qquad \text{for } j = 1, 2, \dots, m.
\end{aligned}
$$

Also in this case the edge weights are $\{1, 2, \dots, 5m\}$.

Thus we constructed a edge irregular reflexive labeling of $f_{5,m}$ for $m \geq 1$. $\square$

3.7 Conclusion

In this chapter we defined reflexive vertex strength and reflexive edge strength. We showed that all graphs have reflexive edge strength and produced the reflexive edge strength for several classes of graphs. For further investigation we state to solve the corresponding problem for the reflexive vertex strength of graphs. In next chapter, we will recall the definition of reflexive vertex strength and calculate this parameter for several graphs.

Chapter 4

Vertex Irregular Reflexive Labeling

4.1 Introduction

Since the irregularity strength of a graph G, $\mathrm{s}(G)$ is based on an edge labeling only, we can visualise the same labeling scheme as a vertex irregular reflexive labeling with every vertex having label 0. So we have the following theorem.

$\diamondsuit$ **Theorem 4.1.1.** [100] *For all graphs having* $\mathrm{s}(G)$ *is*

$$\mathrm{rvs}(G) \leq \mathrm{s}(G).$$

Nierhoff proved

Theorem 4.1.2. [92] *Let G be a graph with no component of order ≤ 2, and $G \neq K_3$, then*

$$\mathrm{s}(G) \leq |V(G)| - 1.$$

Using Nierhoff's result and Theorem 4.1.1 we obtain

$\diamondsuit$ **Theorem 4.1.3.** [100] *Let G be a simple graph with no isolated vertices then,*

$$\mathrm{rvs}(G) \leq |V(G)| - 1.$$

Proof. By Theorem 4.1.2, the theorem holds for all graphs with no component isomorphic to K_2 and not itself isomorphic to K_3.

For K_3 label the edges $1, 1, 2$ and the vertices with 0 except one vertex incident to edges with different labels which receives the label 2. This graph is clearly vertex irregular reflexive with $\mathrm{rvs}(K_3) = 2 = |V(G)| - 1$.

We will now consider graphs with connected components of order 2. Let V represent the vertices in the entire graph and let V' represent the set of vertices in those components of the graph not isomorphic to K_2.

If there is a K_2 component, label the edge with 1 and the vertices with 0 and 2. Label the vertices of the remainder of the graph with 2, then by Nierhoff, the graph is still irregular and all vertex weights are increased by 2. For r components isomorphic to K_2, label the r edges with integers $1, 2, \ldots, r$ and for the i^{th} K_2, label the vertices $i-1, i+1, i$ odd and $i-2, i, i$ even. The maximum weight of the K_2 components is $2r+1$ and occurs when r is odd. Adding $2r+2$ to the vertices of the remainder of the graph ensures that the graph remains vertex irregular.

The edge labels have not been changed so if $2r+2 \leq |V|-1$ then the theorem holds. If $2r+2$ is the new reflexive vertex strength then, since we have added $2r$ vertices, $2r+2 \leq |V|-1$ whenever $|V'| \geq 3$. If the graph is isomorphic to rK_2, then the same labeling gives a reflexive vertex strength of r, r even or $r+1, r$ odd. In either case $\operatorname{rvs}(rK_2) \leq 2r-1$. $\square$

♢ **Corollary 4.1.4.** [100] *Let G be a simple graph with r isolated vertices, then*

$$\operatorname{rvs}(G) \leq \max\{2r, |V(G)| - r - 1\}.$$

Proof. The component(s) of the graph with minimum degree greater than 0 can, by Theorem 4.1.3, be labeled with $\operatorname{rvs}(G - \overline{K_r}) \leq |V(G)| - r - 1$. Label the r isolated vertices with even integers from 0 to $2r$, and add $2r$ to each remaining vertex. In this way the graph is vertex irregular and the largest label must be no more than $\max\{2r, |V(G)| - r - 1\}$. $\square$

♢ **Observation 4.1.5.** *For any graph on n vertices with minimum degree δ the least possible vertex weight is δ and the greatest vertex weight is at least $\delta + n - 1$.*

We now give the following lemma.

♢ **Lemma 4.1.6.** [117] *The largest vertex weight of a graph G of order p and the minimum degree δ under any vertex irregular reflexive labeling is at least*

1. *$p + \delta - 1$ if $p \equiv 0 \pmod 4$ or $p \equiv 1 \pmod 4$ and $\delta \equiv 0 \pmod 2$ or $p \equiv 3 \pmod 4$ and $\delta \equiv 1 \pmod 2$,*

2. *$p + \delta$ otherwise.*

Proof. Let f be a vertex irregular reflexive labeling of a graph G of order p and the minimum degree δ. Let us denote the vertices of G by the symbols $v_1, v_2, \ldots, v_p$ such that $wt_f(v_i) < wt_f(v_{i+1})$ for $i = 1, 2, \ldots, p-1$.

Then the vertex weight of a vertex v_1 is

$$wt_f(v_1) = f(v_1) + \sum_{uv_1 \in E(G)} f(uv_1) \geq 0 + \sum_{uv_1 \in E(G)} 1 \geq \delta.$$

As the vertex weights are distinct we get

$$wt_f(v_p) \geq wt_f(v_1) + p - 1 \geq p + \delta - 1.$$

Let us consider that $wt_f(v_p) = p + \delta - 1$ which means that

$$\{wt_f(v_i) : i = 1, 2, \ldots, p\} = \{\delta, \delta + 1, \ldots, p + \delta - 1\}.$$

Thus the sum of all vertex weights is

$$\sum_{i=1}^{p} wt_f(v_i) = \sum_{i=1}^{p} (\delta + i - 1) = \frac{p(p + 2\delta - 1)}{2}.$$

Evidently, this sum must be an even integer as

$$\sum_{i=1}^{p} wt_f(v_i) = \sum_{i=1}^{p} f(v_i) + 2 \sum_{e \in E(G)} f(e)$$

and every vertex label is even. Thus

$$p(p + 2\delta - 1) \equiv 0 \pmod 4$$

but it is not possible if $p \equiv 1 \pmod 4$ and $\delta \equiv 1 \pmod 2$ or $p \equiv 2 \pmod 4$ or $p \equiv 3 \pmod 4$ and $\delta \equiv 0 \pmod 2$. □

For regular graphs we immediately get

♢ **Corollary 4.1.7.** [117] *Let G an r-regular graph of order p. Then*

$$\mathrm{rvs}(G) = \begin{cases} \left\lceil \dfrac{p + r - 1}{r + 1} \right\rceil & \text{if } p \equiv 0, 1 \pmod 4, \\ \left\lceil \dfrac{p + r}{r + 1} \right\rceil & \text{if } p \equiv 2, 3 \pmod 4. \end{cases}$$

4.2 Vertex Irregular Reflexive Labeling for Some Graphs

Having shown that all graphs can bear a vertex irregular reflexive labeling, we now look at the reflexive vertex strength of some classes of graphs. The following theorem gives $\mathrm{rvs}(K_{1,n})$.

♢ **Theorem 4.2.1.** [100] *The reflexive vertex strength for the star $K_{1,n}$ is*

$$\mathrm{rvs}(K_{1,n}) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

Proof. The vertex weight for $v \in V(K_{1,n})$ is calculated by adding the label of v and its incident edge label. This is the same as the edge weight for the corresponding edge as given in Theorem 3.2.3. The bound and construction of the labeling scheme to prove this theorem is exactly the same as that offered in the proof of Theorem 3.2.3. □

Figures 4.1 and 4.2 provide $\mathrm{rvs}(K_{1,6})$ and $\mathrm{rvs}(K_{1,5})$ respectively.

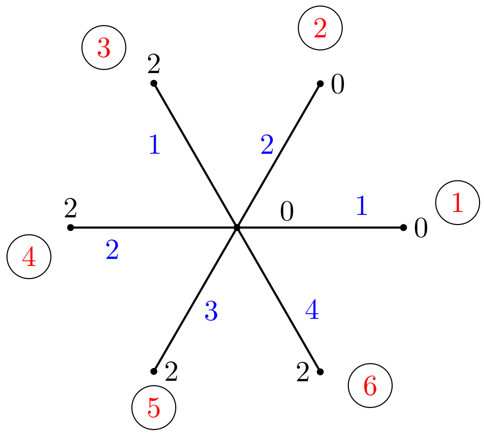

Figure 4.1: Vertex irregular reflexive labeling of $K_{1,6}$

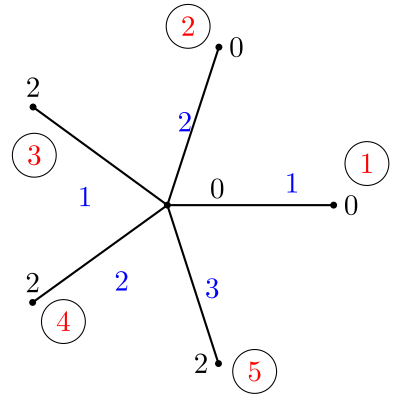

Figure 4.2: Vertex irregular reflexive labeling of $K_{1,5}$

Lemma 4.2.2. [100] ***(Missing Value Lemma)*** *The sequence of vertex weights in a vertex irregular reflexive graph cannot include an odd number of odd weights.*

Proof. The proof follows immediate from Handshaking Lemma, which states that every finite undirected graph has even number of vertices of odd degree.

$$\begin{aligned}
\sum_{v\in V(G)} \deg(v) &= 2|E(G)|, \\
\sum_{odd} \deg(v) + \sum_{even} \deg(v) &= 2|E(G)|, \\
\sum_{odd} \deg(v) &= 2|E(G)| - \sum_{even} \deg(v).
\end{aligned}$$

Since $2|E(G)| - \sum\limits_{even} deg(v)$ is an even number so $\sum\limits_{odd} deg(v)$ has to be even. Thus the number of odd weights has to be even. This concludes the proof. $\square$

In next two theorems we establish rvs of path and cycle.

$\diamondsuit$ **Theorem 4.2.3.** [100] *The reflexive vertex strength for the path P_n is*

$$\text{rvs}(P_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil & n \equiv 3, 6, 9 \pmod{12}, \\ \lceil \frac{n+2}{3} \rceil & n \equiv 2 \pmod{12}, \\ \lceil \frac{n}{3} \rceil & \textit{otherwise}. \end{cases}$$

Proof. We first show that the values mentioned in the theorem are tight lower bounds, and then provide labeling schemes that achieve these bounds.

The least maximum weight for a path on n vertices is n. So the least maximum label must be $\lceil n/3 \rceil$.

For $n \equiv 6, 9 \pmod{12}$, note that the sequence $1, 2, \ldots, n$, contains an odd number of odd numbers and so, by the Missing Value Lemma, the least maximum weight must be at least $n+1$.

For $n \equiv 3 \pmod{12}$ note that $\lceil n/3 \rceil$ is odd however we require an even integer label for the vertex resulting in a least maximum weight of at least $n+1$.

For $n \equiv 2 \pmod{12}$, both conditions are involved. We need a least maximum weight of $n+1$ to have an even number of odd weights but $n+1 \equiv 3 \pmod{12}$ so we need to consider a least maximum weight of at least $n+2$.

We now provide a labeling scheme that achieves these least minimum maximum labels, called strengths, which we will refer to as s. We identify all vertices in the path P_n as $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$. Note here that for $n \equiv 3 \pmod{12}$ has different strength than $n \equiv 5, 7 \pmod{12}$ but the same labeling scheme is applied.

Define the labeling scheme ψ as follows.

$$\psi(v_i) = 0 \qquad\qquad 1 \leq i \leq 3.$$

Case 1. When $n \equiv 0, 4 \pmod{12}$

$$\begin{aligned}
\psi(v_i) &= \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \leq i \leq \lceil \frac{n}{2} \rceil + 1, \end{cases} \\
\psi(v_{n-i}) &= 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \leq i \leq \lceil \tfrac{n}{2} \rceil - 2, \\
\psi(e_i) &= \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases} \\
\psi(e_{n-i}) &= 2\lceil \tfrac{i}{3} \rceil \qquad 1 \leq i \leq \lceil \tfrac{n}{2} \rceil - 1.
\end{aligned}$$

Case 2. When $n \equiv 3, 5, 7 \pmod{12}$

$$\begin{aligned}
\psi(v_i) &= \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \leq i \leq \lfloor \frac{n}{2} \rfloor, \end{cases} \\
\psi(v_{n-i}) &= 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \leq i \leq \lfloor \tfrac{n}{2} \rfloor, \\
\psi(e_i) &= \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \end{cases}
\end{aligned}$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lfloor \tfrac{n}{2} \rfloor.$$

Case 3. When $n \equiv 1 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_i) = \lceil \tfrac{n}{3} \rceil - 3 \qquad i = \lceil \tfrac{n}{2} \rceil + 1,$$

$$\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 2,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1, \end{cases}$$

$$\psi(e_i) = \lceil \tfrac{n}{3} \rceil \qquad i = \lceil \tfrac{n}{2} \rceil, \lfloor \tfrac{n}{2} \rfloor,$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1.$$

Case 4. When $n \equiv 2 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_i) = \lceil \tfrac{n+2}{3} \rceil \qquad i = \lceil \tfrac{n}{2} \rceil + 1,$$

$$\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lceil \tfrac{n}{2} \rceil - 2,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lceil \tfrac{n}{2} \rceil - 1.$$

Case 5. When $n \equiv 6 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lceil \tfrac{n}{2} \rceil - 1,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2,4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3,5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lceil \tfrac{n}{2} \rceil - 1.$$

Case 6. When $n \equiv 8 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5,0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1,2,3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lceil \tfrac{n}{2} \rceil - 1,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2,4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3,5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil - 1, \end{cases}$$

$$\psi(e_i) = \lceil \tfrac{n}{3} \rceil \qquad i = \lceil \tfrac{n}{2} \rceil,$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lceil \tfrac{n}{2} \rceil - 1.$$

Case 7. When $n \equiv 9 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5,0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1,2,3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lceil \tfrac{n}{2} \rceil - 1,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2,4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3,5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor, \end{cases}$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lfloor \tfrac{n}{2} \rfloor.$$

Case 8. When $n \equiv 10 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5,0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1,2,3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil + 1, \end{cases}$$

$$\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lceil \tfrac{n}{2} \rceil - 2,$$

$$
\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6, \ 1 \le i \le \lceil \frac{n}{2} \rceil + 1, \end{cases}
$$
$$
\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lceil \tfrac{n}{2} \rceil - 1.
$$

Case 9. When $n \equiv 11 \pmod{12}$

$$
\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6, \ 4 \le i \le \lfloor \frac{n}{2} \rfloor, \end{cases}
$$
$$
\psi(v_{n-i}) = 2\lfloor \tfrac{i+1}{3} \rfloor \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor,
$$
$$
\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6, \ 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}
$$
$$
\psi(e_{n-i}) = 2\lceil \tfrac{i}{3} \rceil \qquad 1 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1.
$$

It is simple to check that the lower numbered vertices bear odd weights while the higher numbers have even weights. The weights of the central vertices vary depending on their residue modulo 12 but the strengths are the values presented as lower bounds. $\square$

Figure 4.3 and Figure 4.4 provide vertex irregular reflexive labelings of P_7 and P_8, respectively.

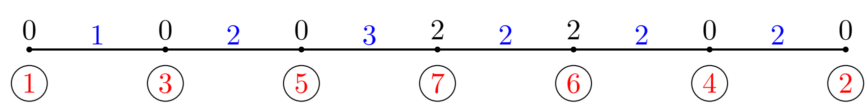

Figure 4.3: Vertex irregular reflexive labeling of P_7

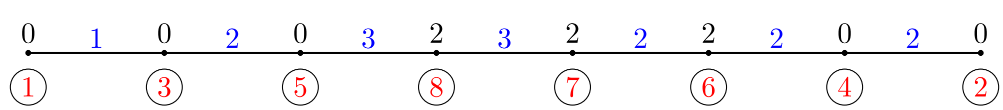

Figure 4.4: Vertex irregular reflexive labeling of P_8

$\diamondsuit$ **Theorem 4.2.4.** [100] *The reflexive vertex strength for the cycle C_n is*

$$\mathrm{rvs}(C_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil & n \equiv 0, 1, 4, 5, 9 \pmod{12}, \\ \lceil \frac{n+2}{3} \rceil & n \equiv 2, 3, 6, 10, 11 \pmod{12}, \\ \lceil \frac{n+3}{3} \rceil & n \equiv 7, 8 \pmod{12}. \end{cases}$$

Proof. We follow the convention of the previous theorem by showing that the values mentioned in the theorem are lower bounds of for maximum weight and then offer a labeling scheme that achieves these bounds.

The smallest possible vertex weight for C_n is 2, so the least maximum weight must be greater than or equal to $n+1$ yielding $rvs(C_n) \geq \lceil (n+1)/3 \rceil$. This value is the bound for $n \equiv 0, 1, 4, 5, 9 \pmod{12}$. For $n \equiv 2, 3, 6, 10, 11 \pmod{12}$ simple counting shows that the sequence contains an odd number of odd numbers and so the Missing Value Lemma provides for this bound to be $\lceil (n+2)/3 \rceil$. For $n = 7, 8$ the least maximum weight gives a residue of $9 \pmod{12}$. When this number is divided by 3, the result is three equal odd numbers which does not allow for a labeling. In these cases the lower bound for the strength is $\lceil (n+3)/3 \rceil$.

We now provide a labeling scheme that achieves these strengths, which again we will refer to as s. We have numbered all vertices as $v_1, v_2, \ldots v_n$ and edges $e_1, e_2, \ldots, e_n$. One can begin at any arbitrary vertex v_1 and number sequentially. We have numbered clockwise for convenience.

Here is how we will label vertices of C_n for different cases $n \geq 6$.

$$\psi(v_i) = 0 \qquad\qquad 1 \leq i \leq 3.$$

Case 1. When $n \equiv 1, 2, 3, 4, 5, 8, 9 \pmod{12}$

$$\begin{aligned}
\psi(v_i) &= \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod{6}, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod{6}, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod{6},\ 4 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases} \\
\psi(v_{n-i}) &= 2\lceil \tfrac{i+1}{3} \rceil \qquad 0 \leq i \leq \lfloor \tfrac{n}{2} \rfloor - 1, \\
\psi(e_i) &= \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod{6}, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod{6}, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod{6}, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod{6},\ 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases} \\
\psi(e_{n-i}) &= 2\lceil \tfrac{i+2}{3} \rceil - 1 \qquad 0 \leq i \leq \lfloor \tfrac{n}{2} \rfloor - 1.
\end{aligned}$$

Case 2. When $n \equiv 5 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod{6}, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod{6}, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod{6},\ 4 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_{n-i}) = 2\lceil \tfrac{i+1}{3} \rceil \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2,4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3,5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor, \end{cases}$$

$$\psi(e_i) = \lceil \tfrac{n+1}{3} \rceil \qquad i = \lceil \tfrac{n}{2} \rceil, \lceil \tfrac{n}{2} \rceil + 1,$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i+2}{3} \rceil - 1 \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 2.$$

Case 3. when $n \equiv 7 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5,0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1,2,3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil + 1, \end{cases}$$

$$\psi(v_{n-i}) = 2\lceil \tfrac{i+1}{3} \rceil \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 2,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2,4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3,5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i+2}{3} \rceil - 1 \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1.$$

Case 4. When $n \equiv 10 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5,0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1,2,3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$

$$\psi(v_{n-i}) = 2\lceil \tfrac{i+1}{3} \rceil \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1,$$

$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2,4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3,5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil - 1, \end{cases}$$

$$\psi(e_i) = \lceil \tfrac{n+2}{3} \rceil \qquad i = \lceil \tfrac{n}{2} \rceil, \lceil \tfrac{n}{2} \rceil + 1,$$

$$\psi(e_{n-i}) = 2\lceil \tfrac{i+2}{3} \rceil - 1 \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 2.$$

Case 5. When $n \equiv 11 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$
$$\psi(v_{n-i}) = 2\lceil \tfrac{i+1}{3} \rceil \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1,$$
$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$
$$\psi(e_i) = \lceil \tfrac{n+2}{3} \rceil \qquad i = \lceil \tfrac{n}{2} \rceil + 1,$$
$$\psi(e_{n-i}) = 2\lceil \tfrac{i+2}{3} \rceil - 1 \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 2.$$

Case 6. When $n \equiv 0 \pmod{12}$

$$\psi(v_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil & i \equiv 5, 0 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 1, 2, 3 \pmod 6,\ 4 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$
$$\psi(v_{n-i}) = 2\lceil \tfrac{i+1}{3} \rceil \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1,$$
$$\psi(e_i) = \begin{cases} 4\lceil \frac{i}{6} \rceil - 3 & i \equiv 1 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 2 & i \equiv 2, 4 \pmod 6, \\ 4\lceil \frac{i}{6} \rceil - 1 & i \equiv 3, 5 \pmod 6, \\ 4\lfloor \frac{i}{6} \rfloor & i \equiv 0 \pmod 6,\ 1 \le i \le \lceil \frac{n}{2} \rceil, \end{cases}$$
$$\psi(e_{n-i}) = 2\lceil \tfrac{i+2}{3} \rceil - 1 \qquad 0 \le i \le \lfloor \tfrac{n}{2} \rfloor - 1.$$

It will be easy to verify that the weights are distinct. □

Figures 4.5 and 4.6 demonstrate vertex irregular reflexive labelings of C_{10} and C_{12} respectively.

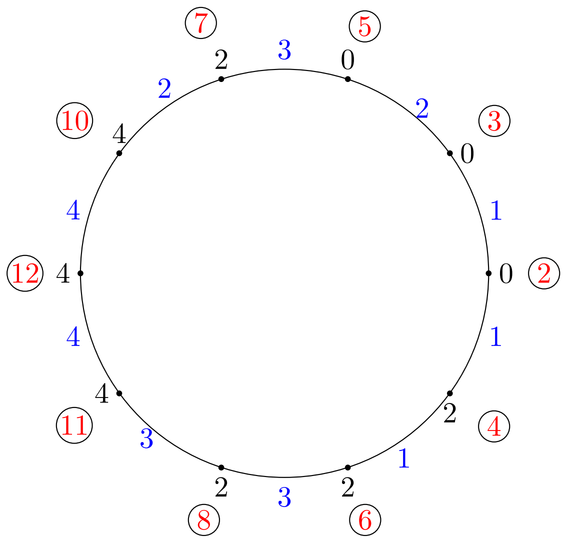

Figure 4.5: Vertex irregular reflexive labeling of C_{10}

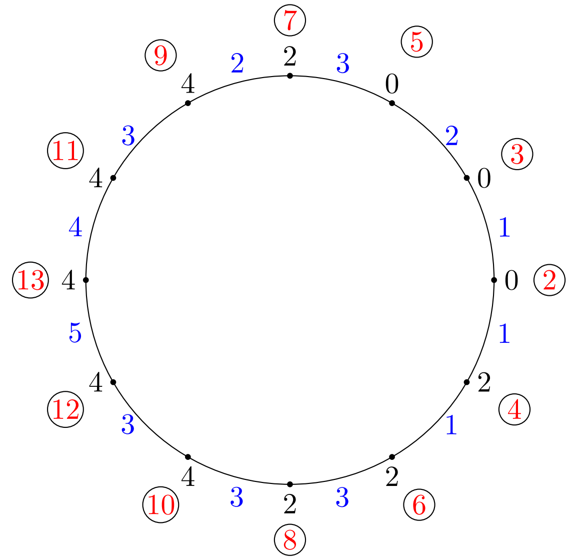

Figure 4.6: Vertex irregular reflexive labeling of C_{12}

♢ **Theorem 4.2.5.** [100] *For $n \geq 2$ we have*

$$\operatorname{rvs}(K_n) = 2.$$

Proof. Trivially, $\operatorname{rvs}(K_n) \geq 2$ for any $n \geq 2$.

To show equality, we define a suitable vertex irregular reflexive labeling ψ as follows. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$.

$$\psi(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\ 2 & \text{otherwise.} \end{cases}$$

and

$$\psi(v_i v_j) = \begin{cases} 1 & \text{for } i \leq j \leq n - i + 1, i \neq j, \\ 2 & \text{otherwise.} \end{cases}$$

This labeling yields weights from the sequence $\{n-1, n, \dots, 2n-1\}$ with one weight from the sequence missing. The missing weight is $\lceil (3n-2)/2 \rceil$. □

Figure 4.7 provides vertex irregular reflexive labeling of K_6.

4.3 Vertex Irregular Reflexive Labeling of Prisms

In this section we will deal with the reflexive vertex strength of prism. First we recall the definition of prism. The prism D_n, $n \geq 3$, is a trivalent graph which can be defined

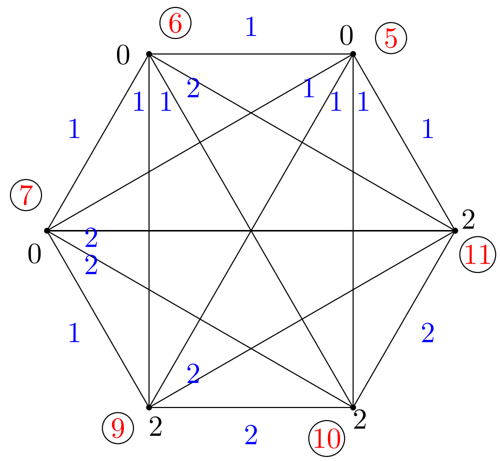

Figure 4.7: Vertex irregular reflexive labeling of K_6

as the Cartesian product $P_2 \square C_n$ of a path on two vertices with a cycle on n vertices. We denote the vertex set and the edge set of D_n such that $V(D_n) = \{x_i, y_i : i = 1, 2, \ldots, n\}$ and $E(D_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i : i = 1, 2, \ldots, n\}$, where indices are taken modulo n. The next theorem gives rvs for prisms.

$\diamondsuit$ **Theorem 4.3.1.** [117] *For $n \geq 3$,*

$$\mathrm{rvs}(D_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Proof. As the prism D_n is 3-regular graph of order $2n$, using from Corollary 4.1.7 we get that

$$\mathrm{rvs}(D_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1.$$

We define the total labeling f of D_n in the following way

$$\begin{aligned}
f(x_i) = f(y_i) &= 0 && i = 1, 2, \ldots, \lceil \tfrac{n}{2} \rceil + 1,\\
f(x_i) = f(y_i) &= \lceil \tfrac{n}{2} \rceil && i = \lceil \tfrac{n}{2} \rceil + 2, \lceil \tfrac{n}{2} \rceil + 3, \ldots, n,\\
& && \quad \text{and } n \equiv 0, 3 \pmod 4,\\
f(x_i) = f(y_i) &= \lceil \tfrac{n}{2} \rceil + 1 && i = \lceil \tfrac{n}{2} \rceil + 2, \lceil \tfrac{n}{2} \rceil + 3, \ldots, n,\\
& && \quad \text{and } n \equiv 1, 2 \pmod 4,\\
f(x_i x_{i+1}) &= 1 && i = 1, 2, \ldots, n-1,\\
f(x_1 x_n) &= 1, &&\\
f(y_i y_{i+1}) &= \lceil \tfrac{n}{2} \rceil + 1 && i = 1, 2, \ldots, n-1,\\
f(y_1 y_n) &= \lceil \tfrac{n}{2} \rceil + 1, &&\\
f(x_i y_i) &= i && i = 1, 2, \ldots, \lceil \tfrac{n}{2} \rceil + 1,\\
f(x_i y_i) &= i - \lceil \tfrac{n}{2} \rceil && i = \lceil \tfrac{n}{2} \rceil + 2, \lceil \tfrac{n}{2} \rceil + 3, \ldots, n,
\end{aligned}$$

$$
\begin{aligned}
& & \text{and } n \equiv 0,3 \pmod 4,\\
f(x_i y_i) &= i - 1 - \left\lceil \tfrac{n}{2} \right\rceil & i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n,\\
& & \text{and } n \equiv 1,2 \pmod 4.
\end{aligned}
$$

Evidently f is $(\lceil n/2 \rceil + 1)$-labeling and the vertices are labeled with even numbers.

For the vertex weights of the vertices x_i, $i = 1, 2, \dots, \lceil n/2 \rceil + 1$ in D_n under the labeling f we get

$$wt_f(x_i) = 0 + 1 + 1 + i = i + 2.$$

If $i = \lceil n/2 \rceil + 2, \lceil n/2 \rceil + 3, \dots, n$ and $n \equiv 0,3 \pmod 4$ then

$$wt_f(x_i) = \left\lceil \tfrac{n}{2} \right\rceil + 1 + 1 + \left(i - \left\lceil \tfrac{n}{2} \right\rceil\right) = i + 2$$

and for $i = \lceil n/2 \rceil + 2, \lceil n/2 \rceil + 3, \dots, n$ and $n \equiv 1,2 \pmod 4$

$$wt_f(x_i) = \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + 1 + 1 + \left(i - 1 - \left\lceil \tfrac{n}{2} \right\rceil\right) = i + 2.$$

Thus $\{wt_f(x_i) : i = 1, 2, \dots, n\} = \{3, 4, \dots, n + 2\}$.

For the vertex weights of the vertices y_i, $i = 1, 2, \dots, n$ in D_n under the labeling f we have the following

$$
\begin{aligned}
wt_f(y_i) =& 0 + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + i = i + 2 + 2\left\lceil \tfrac{n}{2} \right\rceil\\
& \qquad \text{for } i = 1, 2, \dots, \left\lceil \tfrac{n}{2} \right\rceil + 1,\\
wt_f(y_i) =& \left\lceil \tfrac{n}{2} \right\rceil + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(i - \left\lceil \tfrac{n}{2} \right\rceil\right) = i + 2 + 2\left\lceil \tfrac{n}{2} \right\rceil\\
& \qquad \text{for } i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n, \text{ and } n \equiv 0,3 \pmod 4,\\
wt_f(y_i) =& \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(i - 1 - \left\lceil \tfrac{n}{2} \right\rceil\right) = i + 2 + 2\left\lceil \tfrac{n}{2} \right\rceil\\
& \qquad \text{for } i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n, \text{ and } n \equiv 1,2 \pmod 4.
\end{aligned}
$$

Which means

$$
\begin{aligned}
\{wt_f(y_i) : i = 1, 2, \dots, n\} =& \left\{2\left\lceil \tfrac{n}{2} \right\rceil + 3, 2\left\lceil \tfrac{n}{2} \right\rceil + 4, \dots, n + 2\left\lceil \tfrac{n}{2} \right\rceil + 2\right\}\\
=& \begin{cases} \{n + 3, n + 4, \dots, 2n + 2\} & \text{for } n \text{ even},\\ \{n + 4, n + 5, \dots, 2n + 3\} & \text{for } n \text{ odd}. \end{cases}
\end{aligned}
$$

Thus the vertex weights are all distinct, that is f is a vertex irregular reflexive labeling of a prism D_n. $\square$

Figures 4.8 and 4.9 provide vertex irregular reflexive labelings of D_9 and D_{12} respectively.

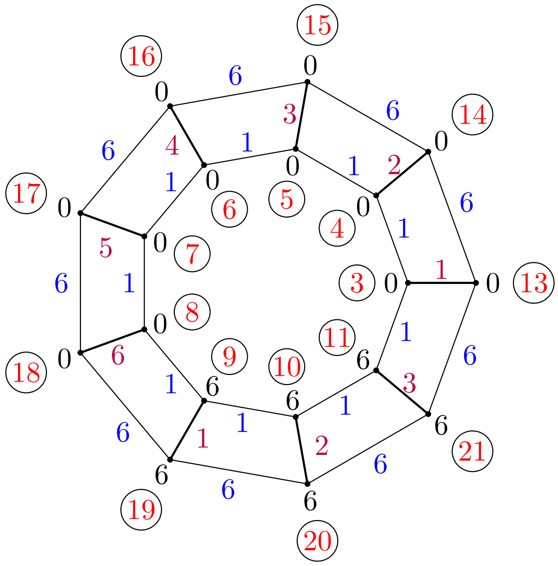

Figure 4.8: Vertex irregular reflexive labeling of D_9

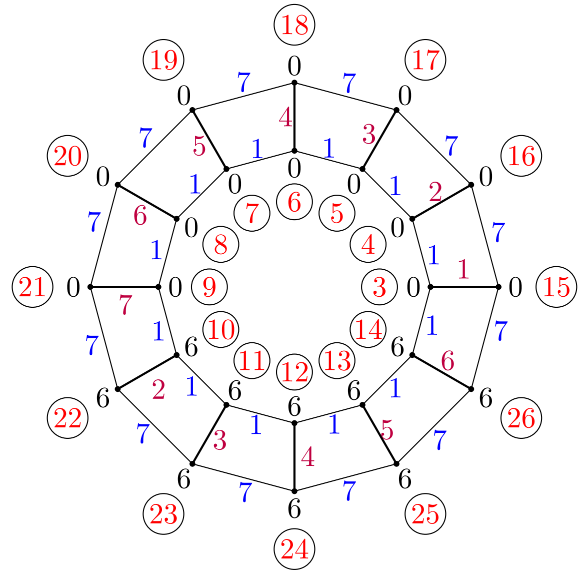

Figure 4.9: Vertex irregular reflexive labeling of D_{12}

4.4 Vertex Irregular Reflexive Labeling of Wheels

As defined in Section 3.4, the wheel W_n, $n \geq 3$, is a graph obtained by joining all vertices of C_n to a further vertex called the centre. We denote the vertex set and the edge set of W_n such that $V(W_n) = \{x, x_i : i = 1, 2, \dots, n\}$ and $E(W_n) = \{x_i x_{i+1}, x x_i : i = 1, 2, \dots, n\}$, where indices are taken modulo n. The wheel is of order $n+1$ and size $2n$. We prove the following result for wheels.

◇ **Theorem 4.4.1.** [117] *For $n \geq 3$,*

$$\operatorname{rvs}(W_n) = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Proof. Let $n \geq 3$. As $\delta(W_n) = 3$ then the smallest vertex weight is at least 3. The wheel W_n contains n vertices of degree 3 thus the largest weight over all vertices of degree 3 is at least $n+2$. Every vertex weight of a vertex of degree 3 is the sum of four labels from which at least one is even thus we have

$$\operatorname{rvs}(W_n) \geq \left\lceil \frac{n+2}{4} \right\rceil.$$

However, if $n = 8t+2$, $t \geq 1$, we get that the fraction

$$\lceil \tfrac{n+2}{4} \rceil = \lceil \tfrac{8t+4}{4} \rceil = 2t+1$$

is odd. The number $n+2 = 8t+4$ can be realizable as the sum of four labels not greater that

$2t+1$ only in the following way

$$n+2 = 8t+4 = (2t+1)+(2t+1)+(2t+1)+(2t+1),$$

but it is a contradiction as the vertex label must be even. Thus, for $n \equiv 2 \pmod 8$ we get

$$\operatorname{rvs}(W_n) \geq \left\lceil \tfrac{n+2}{4} \right\rceil + 1.$$

Let

$$R = \begin{cases} \left\lceil \frac{n+2}{4} \right\rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \left\lceil \frac{n+2}{4} \right\rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Let us denote by K the largest even number not greater than R. Thus

$$K = \begin{cases} R & \text{if } n \equiv 2,3,4,5,6 \pmod 8, \\ R-1 & \text{if } n \equiv 0,1,7 \pmod 8. \end{cases}$$

For $n = 3, 4$ we get that $\operatorname{rvs}(W_n) \geq 2$. The corresponding vertex irregular reflexive labelings for W_3 and W_4 are illustrated on Figures 4.10 and 4.11 respectively.

For $n \geq 5$ we define the total R-labeling f of W_n such that

$$\begin{aligned}
f(x) &= K, & & \\
f(x_i) &= 0 & & i = 1,2,\dots,2R+K-2,\ i \leq n-1, \\
f(x_i) &= K & & i = 2R+K-1, 2R+K, \dots, n, \\
f(x_i x) &= \left\lceil \tfrac{i}{3} \right\rceil & & i = 1,2,\dots,2R+K-2,\ i \leq n-1, \\
f(x_i x) &= i+3-2R-K & & i = 2R+K-1, 2R+K, \dots, n-1, \\
f(x_n x) &= R, & & \\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-1}{3} \right\rceil + 1 & & i = 1,2,\dots,2R+K-2,\ i \leq n-1, \\
f(x_i x_{i+1}) &= R & & i = 2R+K-1, 2R+K, \dots, n-1, \\
f(x_1 x_n) &= 1. & &
\end{aligned}$$

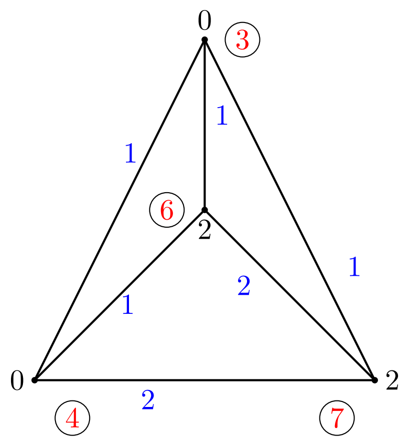

Figure 4.10: Vertex irregular reflexive labeling of W_3

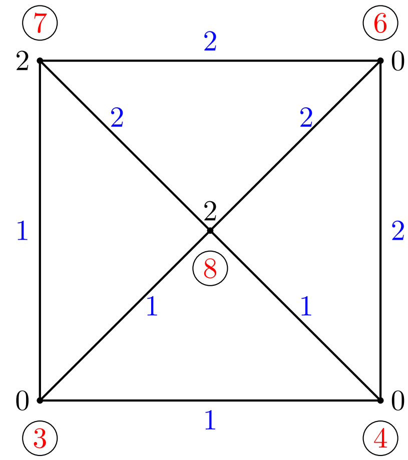

Figure 4.11: Vertex irregular reflexive labeling of W_4

For the vertex weights of the vertices of degree 3 under the labeling f we get

$$\begin{aligned}
wt_f(x_1) =& 0+1+1+1=3,\\
wt_f(x_i) =& 0+\left\lceil \tfrac{i}{3}\right\rceil+\left(\left\lceil \tfrac{i-2}{3}\right\rceil+1\right)+\left(\left\lceil \tfrac{i-2}{3}\right\rceil+1\right)=i+2\\
&\qquad\qquad\text{for } i=2,3,\dots,2R+K-2,\ i\le n-1,\\
wt_f(x_{2R+K-1}) =& K+2+\left(\left\lceil \tfrac{2R+K-3}{3}\right\rceil+1\right)+R=2R+K+2,\\
wt_f(x_i) =& K+(i+3-2R-K)+R+R=i+3\\
&\qquad\qquad\text{for } i=2R+K,2R+K+1,\dots,n-1,\\
wt_f(x_n) =& K+1+R+R=2R+K+1.
\end{aligned}$$

It is easy to get that the weights of vertices x_i, $i=1,2,\dots,n$, $n\ge 5$ and $n\ne 10$ are distinct numbers from the set $\{3,4,\dots,n+2\}$. For $n=10$ we get $\{wt_f(x_i): i=1,2,\dots,10\}=\{3,4,\dots,11,13\}$.

The weight of the vertex x is

$$wt_f(x)=f(x)+\sum_{i=1}^{n} f(x_ix)=K+\sum_{i=1}^{n} f(x_ix)=K+R+\sum_{i=1}^{n-1} f(x_ix)>K+R+n-1.$$

Evidently, for $n\ge 5$, the vertex weights are distinct. □

Figure 4.12 provides vertex irregular reflexive labeling of wheel W_{12}.

Let us recall from Section 2.1 that a fan graph graph F_n is obtained from wheel W_n if one rim edge, say x_1x_n is deleted. A basket B_n is obtained by removing a spoke, say xx_n, from wheel W_n. Before we will give the exact value of reflexive vertex strength of fan graph graphs and baskets we give the following observation.

◇ **Observation 4.4.2.** [117] *Let f be a vertex irregular reflexive k-labeling of a graph G. If*

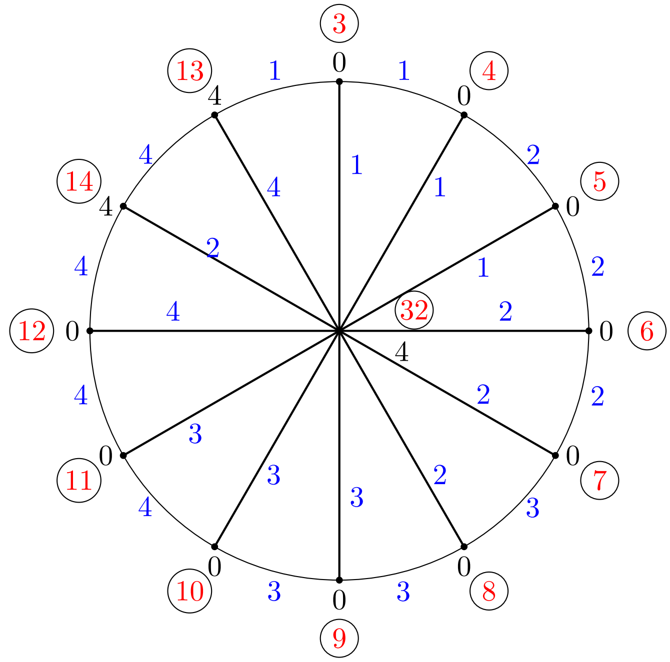

Figure 4.12: Vertex irregular reflexive labeling of W_{12}

there exists an edge uv in G such that

$$wt_f(u) - f(uv) \notin \{wt_f(x) : x \in V(G) - \{v\}\}$$
$$wt_f(v) - f(uv) \notin \{wt_f(x) : x \in V(G) - \{u\}\}$$

then f is a vertex irregular reflexive k-labeling of a graph $G - \{uv\}$. It should be noted here that removing an edge yields no repetition.

Proof. The proof is trivial. □

Immediately from this observation we get the corollary.

♢ **Corollary 4.4.3.** [117] *Let* $\mathrm{rvs}(G) = k$ *and let f be the corresponding vertex irregular reflexive k-labeling of a graph G. If the vertex weights of vertices u, v are the smallest over all vertex weights under the labeling f and $uv \in E(G)$ then*

$$\mathrm{rvs}(G - \{uv\}) \le \mathrm{rvs}(G).$$

For the fan graph F_n we prove the following theorem.

♢ **Theorem 4.4.4.** [117] *For $n \ge 3$,*

$$\mathrm{rvs}(F_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil & \text{if } n \not\equiv 3 \pmod 8, \\ \lceil \frac{n+1}{4} \rceil + 1 & \text{if } n \equiv 3 \pmod 8. \end{cases}$$

Proof. The fan graph F_n contains two vertices of degree 2, thus the smallest vertex weight is at least 2. The fan graph F_n contains $n-2$ vertices of degree 3, thus the largest weight of a vertex of degree 3 is at least n.

If all vertex weight of vertices of degree 3 are at most n, then one of the vertices of degree 2 has to have weight at least $n+1$ and thus $\text{rvs}(F_n) \geq \lceil (n+1)/3 \rceil$. If a vertex of degree 3 has weight greater than n then $\text{rvs}(F_n) \geq \lceil (n+1)/4 \rceil$. As we are trying to minimize the parameter k for which there exists vertex irregular reflexive k-labeling of F_n we get

$$\text{rvs}(F_n) \geq \lceil \tfrac{n+1}{4} \rceil$$

which can be obtain when both vertices of degree 2 in F_n will have weights less than n.

According to the proof of Theorem 4.4.1 and Corollary 4.4.3, for $n \geq 5$, we get

$$\text{rvs}(F_n) \leq \text{rvs}(W_n) = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Moreover, we can derive vertex irregular reflexive $\text{rvs}(W_n)$-labeling of F_n from vertex irregular reflexive $\text{rvs}(W_n)$-labeling of W_n.

Combining the previous facts we get that for $n \not\equiv 2, 3, 7 \pmod 8$

$$\text{rvs}(F_n) = \lceil \tfrac{n+1}{4} \rceil$$

and $n \equiv 2, 3, 7 \pmod 8$

$$\lceil \tfrac{n+1}{4} \rceil \leq \text{rvs}(F_n) \leq \lceil \tfrac{n+1}{4} \rceil + 1.$$

For $n = 3, 4$ we get that $\text{rvs}(F_n) \geq 2$. The corresponding reflexive vertex irregular 2-labelings for F_3 and F_4 are illustrated on Figure 4.13 and Figure 4.14 respectively.

Let $n = 8t + 3$, $t \geq 1$, then $\lceil (n+1)/4 \rceil = 2t + 1$. As this is odd we can not get the number $n + 1 = 8t + 4$ as the sum of four labels less or equal to $2t + 1$ from which at least one is even. Thus $\text{rvs}(F_n) = \lceil (n+1)/4 \rceil + 1$ but in this case

$$\text{rvs}(W_n) = \lceil \tfrac{n+2}{4} \rceil + 1 = \lceil \tfrac{n+1}{4} \rceil + 1$$

and we are done.

We denote the vertex set and the edge set of F_n such that $V(F_n) = \{x, x_i : i = 1, 2, \ldots, n\}$ and $E(F_n) = \{x_i x_{i+1}, x x_i : i = 1, 2, \ldots, n-1\}$.

If $n = 8t + 2$, $t \geq 1$ then $\lceil (n+1)/4 \rceil = 2t + 1$. We define $(2t+1)$-labeling of F_n such that

$$\begin{aligned}
f(x) &= 2t, & \\
f(x_i) &= 0 & i &= 1, 2, \ldots, 6t, \\
f(x_i) &= 2t & i &= 6t+1, 6t+2, \ldots, 8t+2, \\
f(x_1 x) &= 1, & \\
f(x_i x) &= \lceil \tfrac{i-1}{3} \rceil & i &= 2, 3, \ldots, 6t,
\end{aligned}$$

$$
\begin{aligned}
f(x_i x) &= i - 6t && i = 6t+1, 6t+2, \ldots, 8t+1,\\
f(x_{8t+2} x) &= 2t+1, &&\\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 && i = 1, 2, \ldots, 6t,\\
f(x_i x_{i+1}) &= 2t+1 && i = 6t+1, 6t+2, \ldots, 8t+1.
\end{aligned}
$$

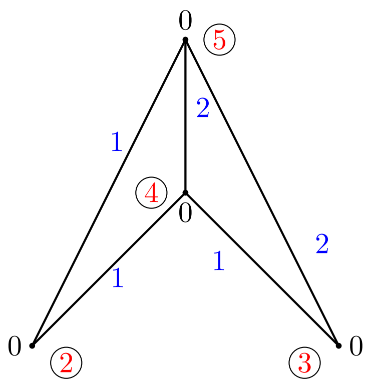

Figure 4.13: Vertex irregular reflexive labeling of F_3

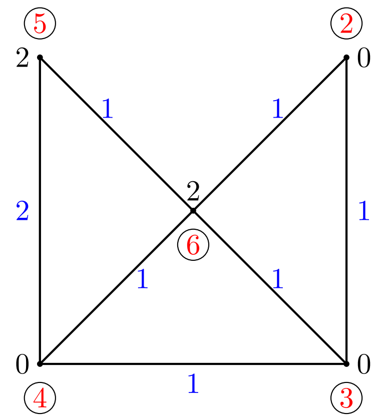

Figure 4.14: Vertex irregular reflexive labeling of F_4

It is easy to get that the set of all vertex weights is $\{2, 3, \ldots, 8t+3, 8t^2+8t+3\}$.

If $n = 8t+7$, $t \geq 0$ then $\lceil (n+1)/4 \rceil = 2t+2$. We define $(2t+2)$-labeling of F_n such in the following way

$$
\begin{aligned}
f(x) &= 2t+2, &&\\
f(x_i) &= 0 && i = 1, 2, \ldots, 6t+4,\\
f(x_i) &= 2t+2 && i = 6t+5, 6t+6, \ldots, 8t+7,\\
f(x_1 x) &= 1, &&\\
f(x_i x) &= \left\lceil \tfrac{i-1}{3} \right\rceil && i = 2, 3, \ldots, 6t+4,\\
f(x_i x) &= i - 6t - 4 && i = 6t+5, 6t+6, \ldots, 8t+6,\\
f(x_{8t+7} x) &= 2t+2, &&\\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 && i = 1, 2, \ldots, 6t+4,\\
f(x_i x_{i+1}) &= 2t+2 && i = 6t+5, 6t+6, \ldots, 8t+6.
\end{aligned}
$$

Evidently the vertex weights are distinct. □

◇ **Theorem 4.4.5.** [117] *For $n \geq 3$,*

$$
\operatorname{rvs}(B_n) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil & \text{if } n \not\equiv 3 \pmod 8,\\ \left\lceil \frac{n+1}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 8. \end{cases}
$$

Proof. The basket B_n contains one vertex of degree 2, thus the smallest vertex weight is at least 2 and it contains $n-1$ vertices of degree 3, thus the largest weight of a vertex of degree 3 is at least $n+1$. Thus

$$\mathrm{rvs}(B_n) \geq \lceil \tfrac{n+1}{4} \rceil.$$

Analogously as in the proof of the previous theorem, using Theorem 4.4.1 and Lemma 4.4.2, for $n \geq 5$, we have,

$$\mathrm{rvs}(B_n) \leq \mathrm{rvs}(W_n) = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Moreover, we can derive a vertex irregular reflexive $\mathrm{rvs}(W_n)$-labeling of B_n from the vertex irregular reflexive $\mathrm{rvs}(W_n)$-labeling of W_n defined in the proof of Theorem 4.4.1 by deleting the spoke x_1x in W_n.

Combining the previous facts we get that for $n \not\equiv 2, 3, 7 \pmod 8$

$$\mathrm{rvs}(B_n) = \lceil \tfrac{n+1}{4} \rceil$$

and $n \equiv 2, 3, 7 \pmod 8$

$$\lceil \tfrac{n+1}{4} \rceil \leq \mathrm{rvs}(B_n) \leq \lceil \tfrac{n+1}{4} \rceil + 1.$$

For $n = 3, 4$ we get that $\mathrm{rvs}(B_n) \geq 2$. The basket B_3 is isomorphic to the fan graph F_3. The vertex irregular reflexive 2-labelings for B_4 is illustrated on Figure 4.15.

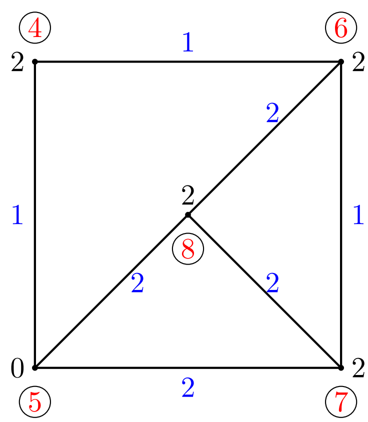

Figure 4.15: Vertex irregular reflexive labeling of B_4

Let $n = 8t+3$, $t \geq 1$, then $\lceil (n+1)/4 \rceil = 2t+1$. As this is odd we can not get the number $n+1 = 8t+4$ as the sum of four labels less or equal to $2t+1$ from which at least one is even. Thus $\mathrm{rvs}(B_n) = \lceil (n+1)/4 \rceil + 1$ but in this case

$$\mathrm{rvs}(W_n) = \lceil \tfrac{n+2}{4} \rceil + 1 = \lceil \tfrac{n+1}{4} \rceil + 1$$

and we are done.

Let us denote the vertex set and the edge set of the basket B_n such that $V(B_n) = \{x, x_i :$

$i = 1, 2, \ldots, n\}$ and $E(B_n) = \{xx_i : i = 2, 3, \ldots, n\} \cup \{x_i x_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_n x_1\}$.

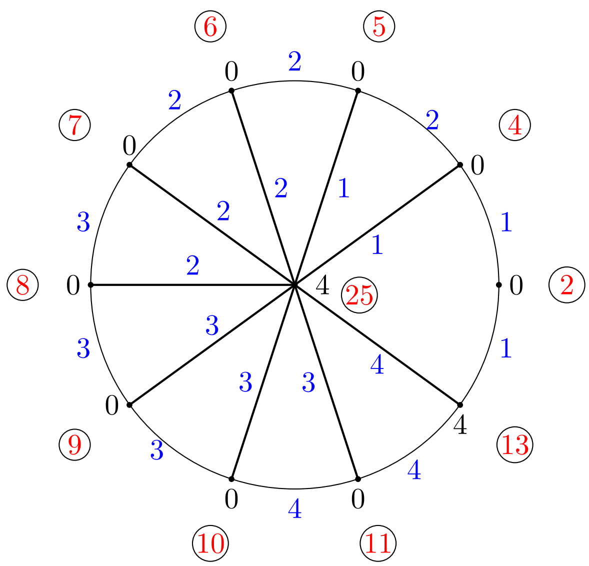

Figure 4.16: Vertex irregular reflexive labeling of B_{10}

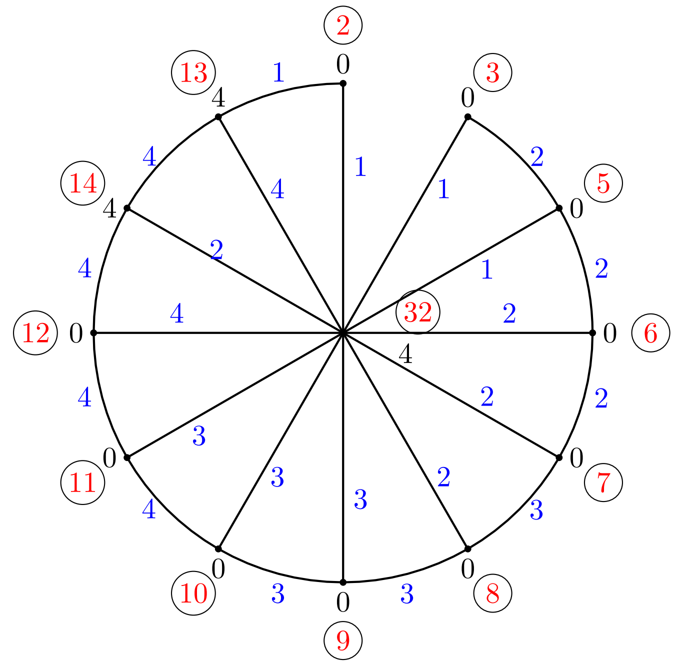

Figure 4.17: Vertex irregular reflexive labeling of F_{12}

If $n = 8t + 2$, $t \geq 1$ then $\lceil (n+1)/4 \rceil = 2t + 1$. We define $(2t+1)$-labeling of B_n such that

$$
\begin{aligned}
f(x) &= 2t, & & \\
f(x_i) &= 0 & & i = 1, 2, \ldots, 6t+1, \\
f(x_i) &= 2t & & i = 6t+2, 6t+3, \ldots, 8t+2, \\
f(x_i x) &= \left\lceil \tfrac{i-1}{3} \right\rceil & & i = 2, 3, \ldots, 6t+1, \\
f(x_i x) &= i - 6t & & i = 6t+2, 6t+3, \ldots, 8t+1, \\
f(x_{8t+2} x) &= 2t+1, & & \\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 & & i = 1, 2, \ldots, 6t, \\
f(x_i x_{i+1}) &= 2t+1 & & i = 6t+1, 6t+2, \ldots, 8t+1, \\
f(x_1 x_{8t+2}) &= 1. & &
\end{aligned}
$$

If $n = 8t + 7$, $t \geq 0$ then $\lceil (n+1)/4 \rceil = 2t + 2$. We define $(2t+2)$-labeling of B_n such in the following way

$$
\begin{aligned}
f(x) &= 2t+2, & & \\
f(x_i) &= 0 & & i = 1, 2, \ldots, 6t+5, \\
f(x_i) &= 2t+2 & & i = 6t+6, 6t+7, \ldots, 8t+7, \\
f(x_i x) &= \left\lceil \tfrac{i-1}{3} \right\rceil & & i = 2, 3, \ldots, 6t+5,
\end{aligned}
$$

$$
\begin{aligned}
f(x_i x) &= i - 6t - 4 && i = 6t+6, 6t+7, \dots, 8t+6,\\
f(x_{8t+7} x) &= 2t + 2, &&\\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 && i = 1, 2, \dots, 6t+5,\\
f(x_i x_{i+1}) &= 2t + 2 && i = 6t+6, 6t+7, \dots, 8t+6,\\
f(x_1 x_{8t+7}) &= 1. &&
\end{aligned}
$$

It is not difficult to show that in both cases the described labelings have desired properties. □

Figures 4.4 and 4.4 demonstrate vertex irregular reflexive labelings of B_{10} and F_{12}, respectively.

4.5 Conclusion

In this chapter we have described vertex irregular reflexive labeling and gave results for the reflexive vertex strength of several graphs and established their labeling patterns.

In next chapter, we will describe another type of labeling that is H-antimagic labeling of graphs.

Chapter 5

H-antimagic Labeling

5.1 Introduction

An *edge-covering* of a graph G is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. Then it is said that G admits an $(H_1, H_2, \ldots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*. Note that in this case, every subgraph isomorphic to H must be in the H-covering.

Gutiérrez and Lladó [52] defined an H-magic labeling as follows.

Definition 5.1.1. *The graph G admitting an H-covering is called* H-magic *if there exists a total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that, for each subgraph H' isomorphic to H,*

$$\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k$$

, where k is constant.

When $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, we say that G is H-supermagic.

. The H-(super)magic labelings are an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [73] and Enomoto et al., [38], respectively. In [52], star-(super)magic and path-(super)magic labelings of some connected graphs were considered and proved that the path P_n and the cycle C_n are P_h-supermagic for some value of h. Lladó and Moragas [75] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h-magic for some value of h. Maryati et al., [85] and also Salman et al., [101] proved that certain families of trees are path-supermagic. Ngurah et al., [91] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati et al., [84] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h.

Combining the (a, d)-edge-antimagic total labeling, defined in Definition 2.2.11 and H-antimagic labeling, Inayah et al., [60] introduced an (a, d)-H-*antimagic labeling* of a graph G

admitting an H-covering as a bijective function $f: V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H, the H'-weights

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

form an arithmetic progression $a, a+d, \ldots, a+(t-1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Such a labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a,d)-H-antimagic labeling is called *(super) (a,d)-H-antimagic.*

In [61], super (a,d)-H-antimagic labelings for some shackles of a connected graph H are investigated. In [104] was proved that wheels are cycle-antimagic. The existence of super $(a,1)$-tree-antimagic labelings for disconnected graphs are studied in [21].

The (super) (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [76]. Further information on super d-antimagic labelings can be found in [17, 26].

Let G be an arbitrary graph and H be a connected graph of order at least 2. We define a graph operation G^H in the following way.

1. Denote the edges in G arbitrarily by $e_1, e_2, \ldots, e_{|E(G)|}$;
2. Take $|E(G)|$ copies of H, say $H_1, H_2, \ldots, H_{|E(G)|}$;
3. In every H_i, $i = 1, 2, \ldots, |E(G)|$, choose two distinct adjacent vertices, say x_i, y_i;
4. Replace every edge e_i in $E(G)$ by subgraph H_i in such a way that its end vertices and $x_i, y_i \in V(H_i)$ are identified.

The resulting graph G^H is of order $(|V(H)| - 2)|E(G)| + |V(G)|$ and size $|E(H)||E(G)|$. Note that the graph G^H is not defined uniquely. It means for graphs G and H there may be many non-isomorphic graphs obtained by using this construction.

In this chapter, we investigate the existence of super (a,d)-H-antimagic labelings for G^H. We show connection between H-antimagic labelings and edge-antimagic total labelings and describe a construction how to obtain the H-antimagic graph from a smaller edge-antimagic total graph G.

5.2 Partitions with Determined Differences

For construction H-antimagic labelings of graphs we will use the partitions of a set of integers with determined differences. This concept was introduced in [16].

Let n, k, d and i be positive integers. We will consider the partition $\mathcal{P}^n_{k,d}$ of the set $\{1, 2, \ldots, kn\}$ into n, $n \geq 2$, k-tuples such that the difference between the sum of the numbers in the $(i+1)$th k-tuple and the sum of the numbers in the ith k-tuple is always equal to the constant d, where

$i = 1, 2, \ldots, n-1$. Thus these sums form an arithmetic sequence with the difference d. By the symbol $\mathcal{P}^n_{k,d}(i)$ we denote the ith k-tuple in the partition with the difference d, where $i = 1, 2, \ldots, n$.

Let $\sum \mathcal{P}^n_{k,d}(i)$ be the sum of the numbers in $\mathcal{P}^n_{k,d}(i)$. Evidently, from the definition, $\sum \mathcal{P}^n_{k,d}(i+1) - \sum \mathcal{P}^n_{k,d}(i) = d$. It is obvious that if there exists a partition of the set $\{1, 2, \ldots, kn\}$ with the difference d, there also exists a partition with the difference $-d$. By the notation $\mathcal{P}^n_{k,d}(i) \oplus c$ we mean that we add the constant c to every number in $\mathcal{P}^n_{k,d}(i)$.

If $k = 1$ then only the following partition of the set $\{1, 2, \ldots, n\}$ is possible

$$\mathcal{P}^n_{1,1}(i) = \{i\} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

If $k = 2$ then we have several partitions of the set $\{1, 2, \ldots, 2n\}$. Let us define the partitions into 2-tuples in the following way:

$$\begin{aligned} \mathcal{P}^n_{2,0}(i) &= \{i, 2n+1-i\}, \\ \sum \mathcal{P}^n_{2,0}(i) &= 2n+1, & \text{for} \quad i = 1, 2, \ldots, n. \\ \mathcal{P}^n_{2,2}(i) &= \{i, n+i\}, \\ \sum \mathcal{P}^n_{2,2}(i) &= n+2i, & \text{for} \quad i = 1, 2, \ldots, n. \\ \mathcal{P}^n_{2,4}(i) &= \{2i-1, 2i\}, \\ \sum \mathcal{P}^n_{2,4}(i) &= 4i-1, & \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Moreover, for $3 \leq n \equiv 1 \pmod 2$

$$\mathcal{P}^n_{2,1}(i) = \begin{cases} \{\frac{n+1}{2} + \frac{i-1}{2}, n+1+\frac{i-1}{2}\} & \text{for} \quad i \equiv 1 \pmod 2, \\ \{\frac{i}{2}, n+\frac{n+1}{2}+\frac{i}{2}\} & \text{for} \quad i \equiv 0 \pmod 2, \end{cases}$$
$$\sum \mathcal{P}^n_{2,1}(i) = n + \frac{n+1}{2} + i, \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Note that we are able to obtain the partitions into 2-tuples $\mathcal{P}^n_{2,0}(i)$ and $\mathcal{P}^n_{2,2}(i)$ as $\mathcal{P}^n_{1,s}(i) \cup \left(\mathcal{P}^n_{1,t}(i) \oplus n\right)$, where $s, t = \pm 1$. We use this idea to construct the other partitions. More precisely,

$$\mathcal{P}^n_{k,d}(i) = \mathcal{P}^n_{l,s}(i) \cup \left(\mathcal{P}^n_{m,t}(i) \oplus ln\right),$$

where $k = l + m$.

For example, we are able to obtain $\mathcal{P}^n_{3,d}(i)$ from the partitions $\mathcal{P}^n_{1,s}(i)$, $s = \pm 1$ and $\mathcal{P}^n_{2,t}(i)$, $t = 0, \pm 2, \pm 4$ and also $t = \pm 1$ for n odd. This means that $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5$ and if $n \equiv 1 \pmod 2$ also for $d = 0, \pm 2$. Moreover, we are able to construct $\mathcal{P}^n_{3,9}$ in the following way

$$\begin{aligned} \mathcal{P}^n_{3,9}(i) &= \{3(i-1)+1, 3(i-1)+2, 3(i-1)+3\}, \\ \sum \mathcal{P}^n_{3,9}(i) &= 9i-3, \quad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Thus $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5, \pm 9$. Note that if $n \equiv 1 \pmod 2$ then also the differences $d = 0, \pm 2$ are realizable.

Summarizing the previous fact we get the following theorem.

$\diamondsuit$ **Theorem 5.2.1.** [34] *Let n, k, d and i be positive integers. There exists a partition $\mathcal{P}^n_{k,d}$ of the set $\{1, 2, \ldots, kn\}$ into n, $n \geq 2$, k-tuples such that the difference between the sum of the numbers in the $(i+1)^{th}$ k-tuple and the sum of the numbers in the i^{th} k-tuple is d, $i = 1, 2, \ldots, n-1$ for*

$$d = k^2$$

or

$$d = s + t,$$

where s and t are realizable differences in partitions $\mathcal{P}^n_{l,s}$ and $\mathcal{P}^n_{m,t}$, $k = l + m$.

Moreover, the corresponding i^{th} k-tuple in the partition $\mathcal{P}^n_{k,d}$ can be obtained such that

$$\mathcal{P}^n_{k,k^2}(i) = \{k(i-1)+1, k(i-1)+2, \ldots, k(i-1)+k\}$$

or

$$\mathcal{P}^n_{k,d}(i) = \mathcal{P}^n_{l,s}(i) \cup \left(\mathcal{P}^n_{m,t}(i) \oplus ln\right),$$

where $k = l + m$, respectively.

Let us note that each of the defined partition $\mathcal{P}^n_{k,d}$ has the property that

$$\sum \mathcal{P}^n_{k,d}(i) = C^n_{k,d} + di,$$

where $C^n_{k,d}$ is a constant depending on the parameters k and d. Table 5.1 gives the values of feasible differences for partition $\mathcal{P}^n_{k,d}$ for $k \leq 7$.

It indicates that, for a given k, the number of feasible values of d is quite big. However, for $k \geq 6$ not every number from the set $\pm\big((k-1)^2+1\big), \pm\big((k-1)^2-1\big), \ldots, \pm 1$ for k odd (or $\pm\big((k-1)^2+1\big), \pm\big((k-1)^2-1\big), \ldots, 0$ for k even) can be realizable as a difference d in the partition $\mathcal{P}^n_{k,d}$. However, it is a simple observation that for $k \geq 6$ all numbers from the set

$$\begin{aligned} &\pm 1, \pm 3, \ldots, \pm(k+14) && \text{for } k \text{ odd} \\ &0, \pm 2, \ldots, \pm(k+14) && \text{for } k \text{ even} \end{aligned} \tag{5.1}$$

are feasible as a difference d in the partition $\mathcal{P}^n_{k,d}$.

5.3 Counting the Upper Bound of the Difference d

The next theorem gives the upper bound of the difference d if the graph G^H is super (a,d)-H-antimagic.

$\diamondsuit$ **Theorem 5.3.1.** [34] *Let G be a (p_G, q_G)-graph and let H be a connected (p_H, q_H)-graph. If G^H admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in*

k	d	
	for every n	moreover for n odd
1	± 1	
2	$0, \pm 2, \pm 4$	± 1
3	$\pm 1, \pm 3, \pm 5, \pm 9$	$0, \pm 2$
4	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$	$\pm 1, \pm 3, \pm 5$
5	$\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17, \pm 25$	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10$
6	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 16, \pm 18, \pm 20, \pm 24, \pm 26, \pm 36$	$\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 15, \pm 17$
7	$\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17, \pm 19, \pm 21, \pm 23, \pm 25, \pm 27, \pm 29, \pm 35, \pm 37, \pm 49$	$0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 16, \pm 18, \pm 24, \pm 26$

Table 5.1: The feasible differences d for partition $\mathcal{P}^n_{k,d}$, $k \le 7$.

G^H is q_G then

$$d \le p_H^2 + q_H^2 - 2p_H + \frac{p_H(p_G - 2)}{q_G - 1}.$$

Proof. Let G be an arbitrary (p_G, q_G)-graph and let H be a connected (p_H, q_H)-graph. Let G^H contains exactly q_G subgraphs isomorphic to H. Let G^H admits a super (a, d)-H-antimagic labeling f,

$$f : V(G^H) \cup E(G^H) \to \{1, 2, \dots, p + q\},$$

where $p = |V(G^H)| = (p_H - 2)q_G + p_G$ and $q = |E(G^H)| = q_H q_G$.

The smallest possible weight of a subgraph isomorphic to H can be obtained when the smallest possible numbers are used to label its elements. It means, when the numbers $1, 2, \dots, p_H$ are used as the vertex labels and the numbers $p+1, p+2, \dots, p+q_H$ are used as the edge labels. Thus

$$\begin{aligned} a \ge & 1 + 2 + \cdots + p_H + (p+1) + (p+2) + \cdots + (p + q_H) \\ = & \frac{(p_H+1)p_H}{2} + pq_H + \frac{(q_H+1)q_H}{2}. \end{aligned} \tag{5.2}$$

The largest possible weight of a subgraph isomorphic to H can be realizable if the largest possible numbers are used to label vertices as well the edges of this subgraph. Thus

$$\begin{aligned} a + (q_G - 1)d \leq & p + (p-1) + \cdots + (p - p_H + 1) \\ & + (p+q) + (p+q-1) + \cdots + (p+q-q_H+1) \\ = & \frac{(2p - p_H + 1)p_H}{2} + \frac{(2p + 2q - q_H + 1)q_H}{2}. \end{aligned} \tag{5.3}$$

Combining Inequalities (5.2) and (5.3) and after some mathematical manipulations we get the upper bound for the difference d in the following form.

$$d \leq p_H^2 + q_H^2 - 2p_H + \frac{p_H(p_G - 2)}{q_G - 1}.$$

□

If G is a tree, that is, $p_G = q_G + 1$, then from Theorem 5.3.1 it follows that $d \leq p_H^2 + q_H^2 - p_H$.

♢ **Corollary 5.3.2.** [34] *Let G be a tree of order p_G and let H be a connected (p_H, q_H)-graph. If G^H admits a super (a, d)-H-antimagic labeling then*

$$d \leq p_H^2 + q_H^2 - p_H.$$

Carlson [32] defines an amalgamation of graphs as follows. Let $G_1, G_2, \ldots, G_k$ be a finite collection of graphs and let each G_i have a fixed vertex v_i called the *terminal*. The *amalgamation* Amal$\{G_i, v_i\}$ is formed by taking all the G_i's and identifying their terminals. By $amal(H, k)$ we denote a graph, where the amalgamation is constructed from k copies of connected graph H.

If the graph G is isomorphic to a star $K_{1,n}$, $n \geq 2$, then the graph $K_{1,n}^H$ is isomorphic to the amalgamation $amal(H, n)$. Using Corollary 5.3.2 we immediately obtain the following result.

♢ **Corollary 5.3.3.** [34] *Let H be a connected (p_H, q_H)-graph. If the amalgamation $amal(H, n)$ admits a super (a, d)-H-antimagic labeling and number of subgraphs isomorphic to H in $amal(H, n)$ is n then*

$$d \leq p_H^2 + q_H^2 - p_H.$$

A *shackle* of $G_1, G_2, \ldots, G_k$, denoted by $shack(G_1, G_2, \ldots, G_k)$, is a graph constructed from non-trivial connected graphs $G_1, G_2, \ldots, G_k$ such that for every $1 \leq i, j \leq k$ with $|i - j| \geq 2$, G_i and G_j have no common vertex, and for every $1 \leq i \leq k - 1$, G_i and G_{i+1} share exactly one common vertex, called a *linkage vertex*, where the $k - 1$ linkage vertices are all distinct. In the case when all G_i's are isomorphic to a connected graph H, we call the resulting graph as a shackle of H denoted by $shack(H, k)$.

If the graph G is isomorphic to a path P_n, $n \geq 2$, then the graph P_n^H is isomorphic to the shackle $shack(H, n-1)$ and by Corollary 5.3.2 the upper bound for the difference d is as follows.

♢ **Corollary 5.3.4.** [34] *Let H be a connected (p_H, q_H)-graph. If the shackle $shack(H, n-1)$ admits a super (a, d)-H-antimagic labeling and number of subgraphs isomorphic to H in $shack(H, n-1)$ is $n - 1$ then*

$$d \leq p_H^2 + q_H^2 - p_H.$$

Note that this upper bound was proved by Lemma 6 in [61].

On the other hand if the graph H is isomorphic to K_2 then from Theorem 5.3.1 it follows.

$\diamondsuit$ **Corollary 5.3.5.** [34] *If H is isomorphic to K_2 and G^H admits a super (a,d)-EAT labeling then*

$$d \leq 1 + \frac{2p_G - 4}{q_G - 1}.$$

This upper bound for the difference d was proved in [22].

5.4 Main Result

In this section we show connection between H-antimagic labelings and edge-antimagic total labelings. We describe a construction how to obtain the H-antimagic graph from a smaller edge-antimagic total graph G. Note that if $H \cong K_2$ then $G^H \cong G$ and the result trivially holds.

The following theorem gives the main result.

$\diamondsuit$ **Theorem 5.4.1.** [34] *Let G be a (b,d^*)-EAT graph and H be a connected graph of order at least 3. If G^H contains exactly q_G subgraphs isomorphic to H then G^H is super (a,d)-H-antimagic and $d = d^* + d_v + d_e$, where d_v and d_e are feasible values of differences in the partitions $\mathcal{P}^{q_G}_{p_H-3,d_v}$ and $\mathcal{P}^{q_G}_{q_H,d_e}$, respectively.*

Proof. Let g be a (b,d^*)-EAT labeling of G. The set of all edge-weights of the edges of G under the labeling g is

$$\{wt_g(e) : e \in E(G)\} = \{b, b+d^*, \dots, b+(q_G-1)d^*\}.$$

Denote the edges of G by the symbols $e_1, e_2, \dots, e_{q_G}$ such that

$$wt_g(e_i) = b + (i-1)d^*,$$

where $i = 1, 2, \dots, q_G$.

Let H be a connected (p_H, q_H)-graph, $p_H \geq 3$.

Let G^H contains exactly q_G subgraphs isomorphic to H, say $H_1, H_2, \dots, H_{q_G}$, where the subgraph H_i replaces the edge e_i in G, $i = 1, 2, \dots, q_G$.

Construct a total labeling f, $f : V(G^H) \cup E(G^H) \to \{1, 2, \dots, q_G(p_H + q_H - 2) + p_G\}$ in the following way:

- $f(v) = g(v)$, if there exist integers t, s, $1 \leq t < s \leq q_G$ such that $v \in V(H_t) \cap V(H_s)$.
- As $p_H \geq 3$ then there exists a vertex x, $x \in V(H_i)$ and $x \neq v$. Then for $i = 1, 2, \dots, q_G$ let
$$f(x) = g(e_i).$$
- For $i = 1, 2, \dots, q_G$ let
$$\{f(y) : y \in V(H_i), y \neq v \text{ and } y \neq x\} = \mathcal{P}^{q_G}_{p_H-3,d_v}(i) \oplus (p_G + q_G).$$

- For $i = 1, 2, \dots, q_G$ let

$$\{f(e) : e \in E(H_i)\} = \mathcal{P}^{q_G}_{q_H,d_e}(i) \oplus \big((p_H - 2)q_G + p_G\big),$$

where d_v depends on p_H and d_e depends on q_H.

It is not difficult to check that the vertices are labeled with the smallest possible numbers $1, 2, \dots, (p_H - 2)q_G + p_G$.

Moreover, for the weight of the subgraph H_i, $i = 1, 2, \dots, q_G$, we obtain

$$\begin{aligned}
wt_f(H_i) &= \sum_{u \in V(H_i)} f(u) + \sum_{e \in E(H_i)} f(e)\\
&= \sum_{\substack{v \sim e_i\\ e_i \in E(G)}} f(v) + f(x) + \sum_{u \in V(H_i)\setminus\{v,x\}} f(u) + \sum_{e \in E(H_i)} f(e)\\
&= \sum_{\substack{v \sim e_i\\ e_i \in E(G)}} g(v) + g(e_i) + \sum \Big(\mathcal{P}^{q_G}_{p_H-3,d_v}(i) \oplus (p_G + q_G)\Big)\\
&\quad + \sum \Big(\mathcal{P}^{q_G}_{q_H,d_e}(i) \oplus \big((p_H - 2)q_G + p_G\big)\Big)\\
&= \Big(b + (i-1)d^*\Big) + \Big(C^{q_G}_{p_H-3,d_v} + d_v i + (p_H - 3)(p_G + q_G)\Big)\Big)\\
&\quad + \Big(C^{q_G}_{q_H,d_e} + d_e i + q_H\big((p_H - 2)q_G + p_G\big)\Big)\\
&= C^{q_G}_{p_H-3,d_v} + C^{q_G}_{q_H,d_e} + b - d^* + (p_H - 3)(p_G + q_G)\\
&\quad + q_H\big((p_H - 2)q_G + p_G\big) + (d^* + d_v + d_e)i.
\end{aligned}$$

This concludes the proof. □

The largest feasible value of the difference d for a super (a, d)-H-antimagic labeling of G^H is given by the following corollary.

♢ **Corollary 5.4.2.** [34] *Let G be a (super) (b, d^*)-EAT graph and H be a connected graph of order at least 3. If G^H contains exactly q_G subgraphs isomorphic to H then G^H is super $(a, d^* + (p_H - 3)^2 + q_H^2)$-$H$-antimagic graph.*

Proof. From Theorem 5.2.1 it follows that the largest possible value of the difference in the partition $\mathcal{P}^{q_G}_{p_H-3,d_v}$ is $(p_H - 3)^2$ and the largest possible value of the difference in the partition $\mathcal{P}^{q_G}_{q_H,d_e}$ is q_H^2. According to Theorem 5.4.1 the result follows. □

Next corollary gives the formula for other feasible differences of d as a function of p_H and q_H.

♢ **Corollary 5.4.3.** [34] *Let G be a (super) (b, d^*)-EAT graph and H be a connected graph of order at least 3. If G^H contains exactly q_G subgraphs isomorphic to H then G^H is super*

(a,d)-H-antimagic, where

$$d = d^* + (p_H - 3 - t)^2 + (q_H - s)^2 \pm t \pm s$$

for every $t = 0, 1, \dots, p_H - 3$ and $s = 0, 1, \dots, q_H$.

5.5 Special Families of Graphs

In this section we consider two special families of graphs, namely amalgamation of graphs and shackle of graphs.

If the graph $G \cong K_{1,n}$, $n \geq 2$, then the graph $K_{1,n}^H$ is known as amalgamation of H. According to Corollary 5.3.3, if $K_{1,n}^H$ admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in $K_{1,n}^H$ is n then $d \leq p_H^2 + q_H^2 - p_H$.

In [114] authors proved the following result.

Theorem 5.5.1. [114] *The star $K_{1,n}$, $n \geq 2$, admits a super (a,d)-EAT labeling for $d = 0, 1, 2$.*

Then we have the following theorem, in [34].

$\diamondsuit$ **Theorem 5.5.2.** [34] *Let H be a connected (p_H, q_H)-graph, $p_H \geq 9$ and let n be an integer, $n \geq 2$. If $K_{1,n}^H$ contains exactly n subgraphs isomorphic to H then $K_{1,n}^H$ admits a super (a,d)-H-antimagic labeling for*

$$0 \leq d \leq p_H + q_H + 27.$$

Proof. It follows from Theorem 5.4.1, Theorem 5.5.1 and Expression (5.1) for partition of numbers. $\square$

Note that Theorem 5.4.1 gives much more feasible values of the difference d for super (a,d)-H-antimagic labeling of $K_{1,n}^H$. Furthermore there exist several feasible differences d which is not possible to obtain from the proof of Theorem 5.4.1. For these values of difference d we propose the following.

Open Problem 1. *Determine for which values of differences d, $0 \leq d \leq p_H^2 + q_H^2 - p_H$, not covered by Theorem 5.4.1, there exists a super (a,d)-H-antimagic labeling of $K_{1,n}^H$.*

As we mentioned before, if the graph $G \cong P_n$, $n \geq 2$, then the graph P_n^H is known as shackle of H. According to Corollary 5.3.4, if P_n^H admits a super (a,d)-H-antimagic labeling and number of subgraphs isomorphic to H in P_n^H is $n-1$ then $d \leq p_H^2 + q_H^2 - p_H$.

For edge-antimagicness of paths in [23] is proved the following.

Theorem 5.5.3. [23] The path P_n, $n \geq 2$, admits a super (a,d)-EAT labeling if and only if $d = 0, 1, 2, 3$.

Then we get.

$\diamondsuit$ **Theorem 5.5.4.** *Let H be a connected (p_H, q_H)-graph, $p_H \geq 9$ and let n be an integer, $n \geq 3$. If P_n^H contains exactly $n-1$ subgraphs isomorphic to H then P_n^H admits a super (a,d)-H-antimagic labeling for*

$$0 \leq d \leq p_H + q_H + 28.$$

Proof. Using Theorem 5.4.1, Theorem 5.5.3 and Expression (5.1) for partition of numbers we immediately obtain that $0 \leq d \leq p_H + q_H + 28$. $\square$

By the same way as for amalgamation we can formulate analogous open problem for shackle of H.

Open Problem 2. *Determine for which values of differences d, $0 \leq d \leq p_H^2 + q_H^2 - p_H$, not covered by Theorem 5.4.1, there exists a super (a,d)-H-antimagic labeling of P_n^H.*

Inayah et al., [61] studied the existence of H-antimagic labeling of shackle of H by using a different method. Their different approach gives different sets of differences obtained by desired constructions.

A *block* of a graph is a maximal subgraph with no cut-vertex. Ngurah et al., [91] defined a *blockcut-vertex graph* of a graph G as a graph H where vertices of H are blocks and cut-vertices in G and two vertices are adjacent in H if and only if one vertex is a block in G and the other is a cut-vertex in G belonging to the block. Barrientos [28] defined a *chain graph* as a graph with blocks $B_1, B_2, \dots, B_k$ such that for every i, $i = 1, 2, \dots, k-1$, the blocks B_i and B_{i+1} have a common vertex in such a way that the blockcut-vertex graph is a path.

In [91] Ngurah et al., investigated H-supermagicness of chain graphs consisting of k blocks where each block is identical and isomorphic to a given cycle. Inspired by their work we decided to deal with the (super) H-antimagicness of chain graphs in general. Moreover, we also extended the results for more general cases – the graphs forming the chain need not necessarily to be blocks.

Let H be a connected graph of order at least 2. We denote by k_H-path a chain graph with k graphs $H_1, H_2, \dots, H_k$ where each graph is identical and isomorphic to the given graph H. Note, that H need not to be a block. Moreover, when H is not isomorphic to K_2 then k_H-path s not defined uniquely. There exists many nonisomorphic k_H-paths for a given graph H and a given k.

$\diamondsuit$ **Theorem 5.5.5.** *Let G be a k_H-path, $k \geq 2$ containing exactly k subgraphs isomorphic to H. Then G is a super (a,d)-H-antimagic graph for $d \in \{0, 1, 2, 3\}$.*

Proof. Let H be a connected graph of order $p \geq 2$ and size q. Let G be a k_H-path, $k \geq 2$, containing exactly k subgraphs isomorphic to H, say $H_1, H_2, \dots, H_k$.

Let $x_2, x_3, \dots, x_k$ be the vertices of G such that $x_i \in V(H_{i-1}) \cap V(H_i)$ for every $i = 2, 3, \dots, k$. Let x_1 be a vertex of H_1 such that $x_1 \neq x_2$ and let x_{k+1} be a vertex of H_k such that $x_{k+1} \neq x_k$.

Let e_i, $i = 1, 2, \dots, k$ be an edge of H_i such that e_i is adjacent to x_i.

Bača et al., [23] proved the path P_k, $k \geq 2$, has a super (a, d^*)-edge-antimagic total labeling if and only if $d^* \in \{0, 1, 2, 3\}$. By $f_{d^*}(P_k)$ we denote the super (a, d^*)-edge-antimagic total labeling of a path P_k. For purposes of this proof we denote the elements of a path P_k such that $P_k = x_1e_1x_2e_2x_3 \dots x_{k-1}e_{k-1}x_k$.

Now we define a labeling g of G in the following way.

1. First label the vertices x_i, $i = 1, 2, \dots, k+1$ of G such that
$$g(x_i) = f_{d^*}(x_i).$$

2. Then label the edges e_i, $i = 1, 2, \dots, k$ of G such that
$$g(e_i) = f_{d^*}(e_i) + k(p+q-3).$$
As f_{d^*} is an (a,d)-edge-antimagic total labeling then the partial sums wt of the subgraphs $H_1, H_2, \dots, H_k$ are at this moment
$$b, b+d^*, \dots, b+(k-1)d^*,$$
where $b = a + k(p+q-3)$. Now we rename the subgraphs H_i, $i = 1, 2, \dots, k$ by the symbols $\mathcal{H}_i$, $i = 1, 2, \dots, k$ such that the partial sums are now such that
$$wt(\mathcal{H}_i) = b + (i-1)d^*. \tag{5.4}$$

3. Note that at this moment in every subgraph $\mathcal{H}_i$, $i = 1, 2, \dots, k$, exactly $p+q-3$ are not labeled. Let $U(\mathcal{H}_i)$ denote the set of all unlabeled vertices and edges in $\mathcal{H}_i$. We label these elements such that
$$\begin{aligned} g(U(\mathcal{H}_i)) = &\left\{ i + jk : j = 1, 2, \dots, \left\lceil \tfrac{p+q-3}{2} \right\rceil \right\} \\ &\cup \left\{ mk + 1 - i : m = \left\lceil \tfrac{p+q-3}{2} \right\rceil + 2, \left\lceil \tfrac{p+q-3}{2} \right\rceil + 3, \dots, p+q-3 \right\}, \end{aligned}$$
moreover, use the smallest numbers from this set to label the unlabeled vertices of $\mathcal{H}_i$.

It is easy to see that the labeling g uses the numbers $1, 2, \dots, k(p-1)+1$ to label the vertices and the numbers $k(p-1)+2, k(p-1)+3, \dots, k(p+q-1)+1$ to label edges of G. Moreover, every number is used once as a label.

Now let us consider the weights of the subgraphs $\mathcal{H}_i$, $i = 1, 2, \dots, k$, under the labeling g. First we count the partial sums of $\mathcal{H}_i$ when only the elements of $U(\mathcal{H}_i)$ are considered. It is easy to get that
$$wt(U(\mathcal{H}_i)) = \begin{cases} \frac{k(p+q-3)(p+q-1)}{2} & \text{if } p+q \text{ is odd,} \\ \frac{k((p+q-3)^2+2(p+q-3)-1}{2} + i & \text{if } p+q \text{ is even.} \end{cases} \tag{5.5}$$
It is easy to get, see (5.4) and (5.5), that the labeling g is super (a,d)-H-antimagic for $d \in \{0, 1, 2, 3\}$. Moreover, when $p+q$ is even then also $d = 4$ is feasible. □

If a graph H does not contain an articulation we immediately obtain the following corollary.

$\diamondsuit$ **Corollary 5.5.6.** [34] *If H is a block then k_H-path, $k \geq 2$, is a super (a,d)-H-antimagic graph for $d \in \{0,1,2,3\}$.*

Note that as the difference $d = 0$ is feasible, Corollary 5.5.6 includes the result of Ngurah et al., [91] about H-supermagicness of chain graphs.

In [16] Bača et al., defined the partitions of a set of integers with determined differences. We would like to use this concept for construction of H-antimagic labelings of chain graphs also for another differences. Our goal is also to find an upper bound for feasible values of the difference d as a function of order and a size of the graph H.

5.6 Conclusion

In this chapter, we examined the existence of super (a,d)-H-antimagic labeling for graph operation G^H, where G is a (b,d^*)-edge-antimagic total graph and H is a connected graph of order at least 3. We have found super (a,d)-H-antimagic labeling for all differences $d = d^* + d_v + d_e$, where d^* is the feasible value of difference in super edge-antimagic graph G and d_v (respectively, d_e) are feasible values of differences in the partitions $\mathcal{P}^{q_G}_{p_H-3,d_v}$ (respectively, $\mathcal{P}^{q_G}_{q_H,d_e}$). Additionally, we showed that for a connected (p_H, q_H)-graph H the graph $K^H_{1,n}$ (respectively, P^H_n) admits a super (a,d)-H-antimagic labeling for every difference $0 \leq d \leq p_H + q_H + 27$ (respectively, $0 \leq d \leq p_H + q_H + 28$).

Chapter 6

Applications of Graph Labelings

The field of graph theory has had much influence in various fields because of its applications. One of the important areas in graph theory is graph labeling whose influence extends to such applications in the areas as coding theory, X-ray crystallography, radar systems, astronomy, circuit design, social networking, network security, communication network addressing, solution of linear congruence systems and database management.

In this chapter we will briefly make references to Bloom and Golomb's paper [31] on graph labeling application and follow with some new relevance.

6.1 Application of Graph Labeling

Graph labelings have interesting applications in variety of fields, [31, 43, 96].

Bloom and Golomb explained how graph labeling is applicable to coding theory, ambiguity in X-ray crystallography, communication network labeling, circuit layouts and finite additive number theory and ruler problems, [31].

While dealing with X-ray crystallography some times more than one crystal structure has the same diffraction information. Bloom and Golomb, [31] showed that this is equivalent to determining all labelings of appropriate graphs which produce a pre-specified set of edge labels. Similarly in coding theory, the design of certain important classes of good non periodic codes for pulse radar and missile guidance is equivalent to labeling the complete graph in such a way that all the edge labels are distinct. The vertex labels then determine the time positions at which pulses are transmitted.

Graceful labeling plays a significant role in communication networks. A communication network is composed devices represented as vertices. Every vertex has computing power and can exchange messages over communication links (edges). It is useful for identification to assign each user terminal (vertex) a vertex label, subject to the constraint that all connecting communication links receive distinct labels. In this way, the numbers of any two communicating terminals automatically specify (by simple subtraction) the edge label of the connecting edge.

6.1.1 Linear Congruence

An equation of the form $ax \equiv b \ (mod\ m)$ where $0 \leq x \leq m$ holds true for some x is said to be linear congruence equation. In this section we will address the problem of solving a special type of linear congruence equation.

Given $n, k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{Z}_n$, it is known classically [Uspensky and Heaslet 1939; Vandiver 1924], [120, 121] that the linear congruence $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 1(in\ \mathbb{Z}_n)$ has a solution if and only if $\gcd(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_n^X$, the group of units of $\mathbb{Z}_n$. Adams and Ponomarenko, [2] asked when such a solution exists with distinct $x_i \in \mathbb{Z}_n$, a question that appears to have been overlooked in the literature. In general, some additional conditions are necessary; for example, $1x_1 + 1x_2 + 1x_3 = 1$ does not have a solution with distinct $x_i \in \mathbb{Z}_3$.

Their partial solution has a stronger coefficient condition and another restriction involving $\phi(n)$, the *Euler's totient function*. The general case remains open.

Given a composite n, $k < n$, and $\gcd(a_1, a_2, \ldots, a_n) \in \mathbb{Z}_n^X$ (group of units in $\mathbb{Z}_n$), then there exist distinct $x_i \in \mathbb{Z}_n$ $(x_i \neq x_j, 1 \leq i, j \leq k)$ such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n \equiv 1$ has a solution in $\mathbb{Z}_n$.

Adams and Ponomarenko [2] proved that if $k \leq \phi(n)$ and $a_i \in \mathbb{Z}_n^X$ for $1 \leq i \leq k$, where Euler's totient function, $\phi(n)$ represents number of positive integers less than or equal to n which are relatively prime to n, then there exists distinct $x_i \in \mathbb{Z}_n$ satisfying

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \equiv 1.$$

In [55], the authors study a special case of finding the solution of linear congruence when

$$a_1 + a_2 + \cdots + a_k = n - 1$$

using super edge-antimagic labeling of trees.

In [112], Sugeng and Miller proved that every caterpillar has a b-edge consecutive edge magic graph for every b and super edge magic labeling is a special case of b-edge consecutive edge magic labeling when $b = n$.

Consider the sequence of length $n+1$ as $(a_1, a_2, \ldots, a_k, \underbrace{0, 0, \ldots, 0}_{n+1-k})$ such that

$$a_1 + a_2 + \cdots + a_k = n - 1.$$

Adding 1 to each element of the sequence we get another sequence $(a_1 + 1, a_2 + 1, \ldots, a_k + 1, \underbrace{1, 1, \ldots, 1}_{n+1-k})$. When this sequence is considered as the degree sequence of a graph, the summation satisfies the Handshaking Lemma (every finite undirected graph has an even number of vertices with odd degree consequently there is even number of odd degree vertices), that is

$$\sum_{i=1}^{k}(a_i + 1) + \underbrace{1, 1, \ldots, 1}_{n+1-k} = 2n.$$

Although it is quite easy to determine the degree sequence of a given graph, the converse procedure that is to determine whether a given degree sequence is graphical (that is if there exists some finite simple graph having this degree sequence) is not straight forward. Authors in [30], have developed algorithm to determine whether a degree sequence is graphic.

As explained in [55] we construct a caterpillar of order $n+1$ and size n with the sequence $(a_1+1, a_2+1, \ldots, a_k+1, \underbrace{1, 1, \ldots, 1}_{n+1-k})$ as follows.

Let the vertex set $V = A \cup B$, $v_{2i-1} \in A$ and $\deg(v_{2i-1}) = a_{2i-1}+1$, $v_{2i} \in B$ and $deg(v_{2i}) = a_{2i+1}$ when $2i \leq k$, where $1 \leq i \leq (k+1)/2$ when k is odd, $1 \leq i \leq k/2$ when k is even.

Let u_i^j denote the vertices of degree 1 in A and B.

Then edge set $E = \{v_i v_{i+1} : 1 \leq i \leq k\} \cup \{v_i u_i^j : 1 \leq i < k, 1 \leq j \leq a_i\} \cup \{v_i u_i^j : 1 < i < k, 1 \leq j \leq a_i - 1\}$ and $|E| = n$.

For convenience, we denote plane representation of caterpillar by $C_{a,b}$ with bipartition A', B' of its vertex set $V(C_{a,b})$. The a-vertices of A' are labeled u_i, $1 \leq i \leq a$ and the b-vertices of B' are labeled w_i, $1 \leq i \leq b$. The super edge-magic labeling of the caterpillar $C_{a,b}$ is defined by function f as shown below

Define

$$f : V(C_{a,b}) \cup E(C_{a,b}) \to \{1, 2, \ldots, p+q\}$$

in such a way that

$$\begin{aligned} f(u_i) =& i && i \in \{1, 2, \ldots, a\}, \\ f(w_j) =& a + j && j \in 1, 2, \ldots, b, \\ f(u_i w_j) =& 2(a+b) - (i+j-1) && \text{for all } i, j, \text{ where } u_i w_j \text{ is an edge.} \end{aligned}$$

Thus,

$$f(u_i w_j) + f(u_i) + f(w_j) = 3a + 2b + 1, \ u_i w_j \in E(C_{a,b}),$$

the above labeling is super edge-magic labeling.

We know that,

$$\sum_{v \in V(G)} f(v)\deg(v) + \sum_{e \in E(G)} f(e) = qc(f)$$

Since f is a super edge-magic labeling, we get the following congruences.

$$\sum_{j=1}^{a}((\deg(u_j)-1)f(u_j) + \sum_{i=1}^{b}((\deg(w_i)-1)f(w_i) + \sum_{m=1}^{2n+1} m = 0 \text{ (in } \mathbb{Z}_n),$$

$$\sum_{j=1}^{a}((\deg(u_j)-1)f(u_j) + \sum_{i=1}^{b}((\deg(w_i)-1)f(w_i) + \frac{(2n+1)(2n+2)}{2} = 0 \text{ (in } \mathbb{Z}_n),$$

$$\sum_{j=1}^{a}((\deg(u_j)-1)f(u_j) + \sum_{i=1}^{b}((\deg(w_i)-1)f(w_i) + 1 = 0 \text{ (in } \mathbb{Z}_n).$$

Let k be the number of vertices in $V(C_{a,b})$ whose degree is greater than one. For brevity, we write the above equation as

$$\sum_{i=1}^{k}(d_i - 1)f(v_i) + 1 = 0 \text{ (in } \mathbb{Z}_n), \ d_i - 1 > 0,$$

$$\sum_{i=1}^{k}(d_i - 1)f(v_i) = -1 \text{ (in } \mathbb{Z}_n),$$

$$\sum_{i=1}^{k}(d_i - 1)f(v_i) = n - 1 \text{ (in } \mathbb{Z}_n).$$

Therefore, $f(v_i) \in \{1, 2, \ldots, n\}, 1 \leq i \leq k$ and are distinct in $\mathbb{Z}_n$.

Now, $a_i = d_i - 1$ for $1 \leq i \leq k$ so

$$\sum_{i=1}^{k} a_i f(v_i) = n - 1 \text{ (in } \mathbb{Z}_n).$$

Let $x_i = n - f(v_i)$, $1 \leq i \leq k$.

We can observe that $x_i = n - f(v_i)$ for $(1 \leq i \leq k)$ will be less than 0 when $n < f(v_i)$.

To get a distinct solution, we will have to label one vertex of degree 1 with label $(a+b)$ while constructing the super edge-magic labeling.

Finally, we will verify that $x_i = n - f(v_i)$ is a solution of

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k \equiv 1.$$

So,

$$\begin{aligned}
a_1x_1 + a_2x_2 + \cdots + a_kx_k =& a_1(n - f(v_1)) + a_2(n - f(v_2)) + \cdots + a_k(n - f(v_k)) \\
=& a_1n + a_2n + \cdots + a_kn - (a_1f(v_1) + a_2f(v_2) + \cdots + a_kf(v_k)) \\
=& 0 - (n - 1) = 1 \text{ in } \mathbb{Z}_n.
\end{aligned}$$

Thus, the solution $x_i = n - f(v_i)$ are distinct in $\mathbb{Z}_n$.

The process is explained with the help of following example.

Example 1: Find a solution to $3x_1 + 2x_2 \equiv 1$ in $\mathbb{Z}_6$.

Here $n = 6$ and $k = 2$. In order to find a solution we need to check that is $2 \leq \phi(n) = \phi(6)$? Obviously $2 \leq \phi(6) = 2$. So we can find solution of above equation.

Form a $(n + 1)$ degree sequence $(3, 2, 0, 0, 0, 0, 0)$. Now add 1 to all of them, so we have $(4, 3, 1, 1, 1, 1, 1)$. We can form a caterpillar with this degree sequence since it satisfies the Handshaking Lemma. We will form a caterpillar with order 7 and size 6 whose degree sequence is $(4, 3, 1, 1, 1, 1, 1)$ and it has super edge magic labeling. One such possible way is shown in Figure 6.1.

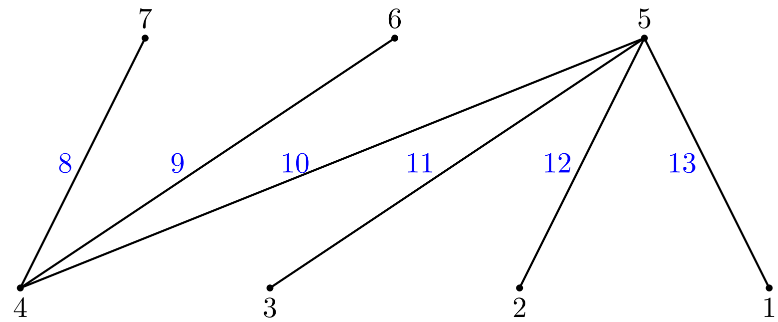

Figure 6.1: Supermagic labeling of caterpillar of order 7 and size 6

Now as explained in [55], to find a solution we will find the non pendant vertices that is with label 5 and 4 in our case.

Finally the solution will be $x_1 = 6 - 5 = 1$ and $x_2 = 6 - 4 = 2$.

Check:

$$3 \times 1 + 2 \times 2 = 7 \equiv 1 \; in \; \mathbb{Z}_6.$$

So solution $x_1 = 1$ and $x_2 = 2$ is possible with the help of super edge magic labeling of graphs.

Example 2: Find a solution to $5x_1 + 2x_2 + 2x_3 \equiv 1$ in $\mathbb{Z}_{10}$.

Here $n = 12$ and $k = 3$. In order to find a solution we need to check that is $3 \leq \phi(n) = \phi(12)$? Obviously $3 < \phi(12) = 4$. So we can find solution of above equation.

Form a $(n+1)$ degree sequence $(6, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0)$. Now add 1 to all of them, so we have $(6, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1)$. We need to form a caterpillar graph with order 11 and size 10 whose degree sequence is $(6, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1)$ and it posses a super edge magic labeling. One such possible way is shown in Figure 6.2.

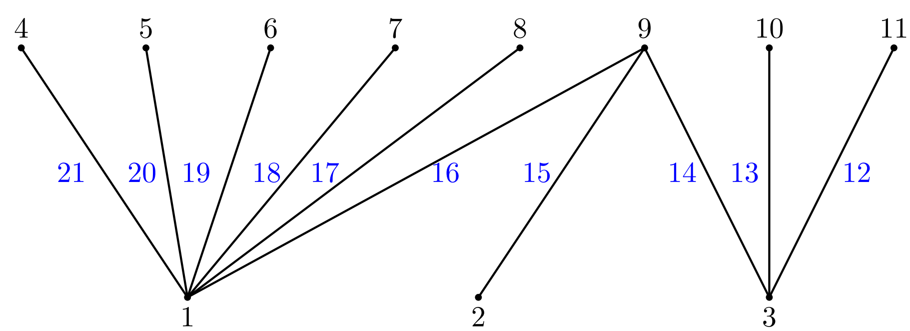

Figure 6.2: Supermagic labeling of caterpillar of order 11 and size 10

Now to find a solution we will find the non pendant vertices that is with label 1, 9 and 3 in our case.

Finally the solution will be $x_1 = 10 - 1 = 9$, $x_2 = 10 - 9 = 1$, $x_3 = 10 - 3 = 7$.

Check:

$$5 \times 9 + 2 \times 1 + 2 \times 7 = 61 \equiv 1 \ in \ \mathbb{Z}_{10}.$$

So solution $x_1 = 9$, $x_2 = 1$ and $x_3 = 7$ is possible with the help of super edge magic labeling of graphs.

6.1.2 Irregular Reflexive Labeling

As mentioned in Section 3.1, irregular reflexive labeling is the return to the original spirit of irregular labeling, restricting the vertex labels to be even numbers to indicate the presence of loops and allowing 0 as a possible vertex label, indicating a vertex with no loop.

In this section we will explain how irregular reflexive labeling can be applied to network analysis.

In real world, there is a variety of network. One side of real world networks is regular networks (same weights of graph elements) while the other side of networks is totally irregular (different weights of graph elements). Most of the networks in real world are combinations of both where some part of network is regular and other is irregular.

The irregular (or irregular reflexive) networks are mostly used in the analysis of networks. What happens when networks are completely irregular or what happens when they are completely regular? One can study and analyse networks in this regard. Since most of real world networks are in between so having identified both extremes - one can use this to estimate what happens within real world networks.

It will be very difficult to apply irregular reflexive labeling as direct application to some networks which are very dynamic like social networks. By the time we set up a network, the network configuration would have changed and it will be hard to accommodate those changes. Rather it would be easy to form application of irregular reflexive labeling to communication networks. It could be constructed in a way to make sure that it is an irregular network. This identification helps to uniquely identifying each terminals and thus making sure that no two vertices have same weights. Also if the network is not irregular then it could have happened that there is some fault in network or could be that a link is broken between two vertices.

Some recent applications of graph labeling can be found in secret sharing [109] and relational databases [115].

Graph labeling presents a common context for many applied and theoretical problems. This has been illustrated in the previous section, in which we narrated some brief applications in variety of areas of science. The reward of such efforts is obvious and often immediate. This principle is apparent in the many examples which are drawn from these labeling techniques. Thus there could be possible more application of labeling. Readers are invited to explore more applications.

In the following section we will state some conjectures and open problems.

6.2 Conjectures and Open Problems

Graph labeling field has many interesting open problems and conjectures. Some of them are open for many years and only partial success has been achieved.

Following conjecture by Ringel and Kotzig, also known as Graceful Tree Conjecture has been the focus of many papers. The motivation for this conjecture was the actual conjecture by Ringel that complete graph K_{2n+1} can be decomposed into $2n+1$ subgraphs that are isomorphic to a given tree of order n.

Conjecture 6.2.1. [99] *All trees are graceful.*

Among the trees known to be graceful are: caterpillars [99], trees with at most 4 end-vertices [58], [125] and [69]; trees with diameter at most 5, see [125] and [57]; trees with $n(V) \leq 35$ are graceful, symmetrical trees [29], [95] and many more, refer [47].

We now describe a fundamental conjecture about harmonious labeling, which has been open since its origin in 1980.

Conjecture 6.2.2. [51] *All trees are harmonious.*

Many authors including Graham and Sloane tried to prove this conjecture, see [47]. The recent success is "All trees with at most 31 vertices are harmonious". This result is proved by Fang [41] in 2011 using probabilistic backtracking, tabu search, two-stage constraint solving and hybrid algorithm.

Hartsfield and Ringel gave the following conjecture.

Conjecture 6.2.3. [54] *Every connected graph except K_2 is antimagic.*

As discussed in Chapter 2, this conjecture is still open for graphs in general. The weaker version of this conjecture is that every tree except K_2 is antimagic.

Kotzig, Rosa [74] and Ringel, Lladó [98] posed the following conjecture about edge magic total labeling.

Conjecture 6.2.4. [74, 79, 98] *Every tree admits edge magic total labeling.*

The notion of super edge-magic total labeling was introduced by Enomoto et al., [38]. They proposed following conjecture.

Conjecture 6.2.5. [38] *Every tree admits super edge magic total labeling.*

Sugeng et al., [113] posed following conjecture for ladders graphs.

Conjecture 6.2.6. [113] *The ladder $L_2 = P_n \square P_2$ is super (a, d)-EAT if n is even and $d \in \{0, 2\}$.*

Tanna et al., [118] gave the following conjecture for vertex irregular reflexive labeling for generalised Petersen graph.

Conjecture 6.2.7. [118] *Prove that* $RVS(GP(n,k)) = \lceil (n/2) \rceil + 1$.

The following are some of the open problems in graph labeling.

Open Problem 6.2.1. [27] *Find* (a,d)*-EAT labelings for even cycles with* $d \in \{4,5\}$ *and for odd cycle with* $d = 5$.

Open Problem 6.2.2. [27] *Find* $(a,5)$*-EAT labelings for paths* P_n*, for the feasible values of* a.

Many researchers would solve existing labeling conjectures or open problems for some graphs but so far there is no great progress in generalising a set of graphs that adheres a property of particular labeling since it has certain characteristics so Tanna et al., gave following open problem. Solving such a categorization would be great achievement in graph labeling and will lead to further classification of any new labeling scheme.

Further, it could be extended that why a particular graph G will admit labeling scheme x and will not adhere labeling scheme y since it does not fall in that category.

This categorization of graphs admitting a set of labeling may generate big results in graph labeling.

Open Problem 6.2.3. [116] *Categorise graphs in general that admits a set of labeling.*

There is great potential of developing applications of irregular labeling or reflexive irregular labeling to network analysis or social network analysis.

Open Problem 6.2.4. [116] *Develop an application of irregular labeling in Social Networks.*

Many times a graphs is very close to attend a particular labeling but only a few or one element does not match the labeling pattern. For example, when trying to prove that the graph is magic, it might happen that only one element of graph does not adhere the magic constant but it has some different magic constant that is bimagic labeling. So this might be of interest to know the type of graphs that misses the magicness by just one.

Open Problem 6.2.5. [82] *Develop a classification of graphs admitting a bimagic labeling with two constants* k_1 *and* k_2 *but one of them being used only once.*

Similar open problem could be developed for irregular reflexive labeling.

Open Problem 6.2.6. [118] *Categorize graphs admitting irregular reflexive labeling with strength* s *but using strength* s *only once.*

6.2.1 Future Work

Irregular reflexive labeling is a recent concept and only a little work has been done in this field. So there a vast development scope in this area. In particular, once we have labeled graph G using irregular reflexive labeling, a big question to be address here "Is such a labeling unique?". If not, how many ways a graph G could be labeled differently.

Further, if there is more than one way of labeling a graph G, which of these labeling patterns is the best? For example G has $\text{rvs}(G) = s$, once s is achieved it really does not matter how many times you have used s in the labeling of G.

One possible answer could be usage of s optimal times, that is using in such a way that the total weight is as least as possible. If someone is using labeling to write up a code or algorithm, it occupies the least store space. Thus least time to compilation and run time. Consider the case when weights represent cost of vertices in a network then the lower the cost, the better the efficiency. So we must use the maximum strength in such a way that the cost is minimized.

Another future work could be development of an algorithm, where the input is one vertex configuration at a time and its incident edges. Next input will have another one vertex and again incident edges. Every stage one should make sure that the graph is irregular reflexive. This could be very difficult since there is no way to predict the next input or its degree. Or it could be impossible but either way, it will be a great work.

One more dimension to future work would be distance reflexive labeling. This concept will be similar to those of distance magic labeling, [10, 44, 45, 72].

Miller et al., in [89], introduced an new labeling labeling technique based on neighbourhood of a vertex. They defined function $f : V(G) \to \{1, 2, \dots, v\}$ to be 1-vertex magic labeling of a graph with v vertices as a bijective function with the property that there is a constant k such that for any vertex x, the sum of labels of all vertices in the neighbourhood of x is a k, that is,

$$\sum_{y \in N(x)} f(y) = k,$$

where $N(x)$ is the set of all vertices adjacent to x.

One could develop existence of distance reflexive labeling for regular graphs.

6.3 Conclusion

In this chapter, we discussed various application of graph labeling and posed some conjectures and open problems. We have also discussed some possibilities of future work.

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