A Bayesian Analysis of a Regime Switching Volatility Model

A dissertation presented for the degree of Doctor of Philosophy

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Declaration of Originality

“This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to the final version of my thesis being made available worldwide when deposited un the Universities Digital Repository, subject to the provisions of the Copyright Act 1968.”

Glen Livingston Jr
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For Jae.
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Abstract

Non-linear time series data is often generated by complex systems. While linear models provide a good first approximation of a system, often a more sophisticated non-linear model is required to properly account for the features of such data. Correctly accounting for these features should lead to the fitting of a more appropriate model.

Determining the features exhibited by a particular data set is a difficult task, particularly for inexperienced modellers. Therefore, it is important to move towards a modelling paradigm where little to no user input is required, in order to open statistical modelling to users less experienced in MCMC. This sort of modelling process requires a general class of models that is able to account for the features found in most linear and non-linear data sets. One such class is the STAR-GARCH class of models. These are reasonably general models that permit regime changes in the conditional mean and allow for changes in the conditional covariance.

In this thesis, we develop original algorithms that combine the tasks of parameter estimation and model selection for univariate and multivariate STAR-GARCH models. The model order of the conditional mean and the model index of the conditional covariance equation are included as parameters for the model requiring estimation.

Combining the tasks of parameter estimation and model selection is facilitated through the Reversible Jump MCMC methodology. Other MCMC algorithms employed for the posterior distribution simulators are the Gibbs sampler, Metropolis-Hastings, Multiple-Try Metropolis and Delayed Rejection Metropolis-Hastings algorithms. The posterior simulation algorithms are successfully implemented in the statistical software program R, and their performance is tested in both extensive simulation studies and practical applications to real world data.

The current literature on multivariate extensions of STAR, GARCH, and STAR-GARCH models is quite limited from a Bayesian perspective. The implementation of a set of estimation algorithms that not only provide parameter estimates but is also able to automatically fit the model with highest posterior probability is a significant and original contribution. The impact of such a contribution will hopefully be a step forward on the path towards the automation of time series modelling.
Below are some details of the notation and functions used throughout this thesis unless stated otherwise.

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<td>$x_t$</td>
<td>Univariate data point.</td>
</tr>
<tr>
<td>$x$</td>
<td>Complete univariate data set.</td>
</tr>
<tr>
<td>$x_t$</td>
<td>Multivariate data point.</td>
</tr>
<tr>
<td>$X$</td>
<td>Complete multivariate data set.</td>
</tr>
<tr>
<td>$C, \Sigma, H$</td>
<td>Upper case symbol indicates a matrix.</td>
</tr>
<tr>
<td>$H, \Phi$</td>
<td>Bold and upper case indicates a matrix made up of either a vector of vectors, or a matrix of matrices.</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$n \times n$ identity matrix.</td>
</tr>
<tr>
<td>$\Omega^T$</td>
<td>Transpose of the matrix $\Omega$.</td>
</tr>
<tr>
<td>$\text{tr}(\Omega)$</td>
<td>Trace of the square matrix $\Omega$.</td>
</tr>
<tr>
<td>$p(\Theta</td>
<td>\Omega)$</td>
</tr>
<tr>
<td>$\text{vec}(\Omega)$</td>
<td>Stacking the columns of the matrix $\Omega$ into a column vector.</td>
</tr>
<tr>
<td>$\text{vech}(\Omega)$</td>
<td>Stacking the lower triangle of the matrix $\Omega$ into a column vector.</td>
</tr>
<tr>
<td>$\exp[\omega]$</td>
<td>$e^\omega$.</td>
</tr>
<tr>
<td>$A \odot B$</td>
<td>The component-wise multiplication of the matrices $A$ and $B$.</td>
</tr>
<tr>
<td>$A \otimes B$</td>
<td>The Kronecker product of the matrices $A$ and $B$.</td>
</tr>
<tr>
<td>$\Gamma_p(n)$</td>
<td>The multivariate gamma function.</td>
</tr>
<tr>
<td>$\mathcal{N}(\mu, \sigma^2)$</td>
<td>Univariate normal distribution with mean $\mu$ and variance $\sigma^2$. The density of the distribution is given by $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{1}{2\sigma^2} (x - \mu)^2 \right]$.</td>
</tr>
<tr>
<td>$\mathcal{N}_p(\mu, \Sigma)$</td>
<td>Multivariate normal distribution with mean $\mu$ and covariance matrix $\Sigma$ for $x$, a column vector of length $p$. The density of the distribution is given by $f(x) = (2\pi)^{-\frac{p}{2}}</td>
</tr>
<tr>
<td>$\mathcal{N}_{n,p}(M,U,V)$</td>
<td>Matrix normal distribution with $n \times p$ location matrix $M$, $n \times n$ scale matrix $U$, and $p \times p$ scale matrix $V$ for $X$, an $n \times p$ matrix. The density of the distribution is given by $f(X) = \frac{\exp \left[ -\frac{1}{2} \text{tr} \left( V^{-1} (X - M)^T U^{-1} (X - M) \right) \right]}{(2\pi)^{-\frac{np}{2}}</td>
</tr>
<tr>
<td>$\mathcal{G}(\alpha, \beta)$</td>
<td>Gamma distribution with shape and rate parameters $\alpha$ and $\beta$, respectively. The density of the distribution is given by $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp (-\beta x)$.</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
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</tr>
<tr>
<td>IG ($\alpha, \beta$)</td>
<td>Inverse Gamma distribution with shape and scale parameters $\alpha$ and $\beta$, respectively. The density of the distribution is given by $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$.</td>
</tr>
<tr>
<td>$W(V, n)$</td>
<td>Wishart distribution with scale matrix $V$ and degrees of freedom $n$ for $X$, a $p \times p$ matrix. The density of the distribution is given by $f(X) = \frac{</td>
</tr>
<tr>
<td>IW($\Psi, \nu$)</td>
<td>Inverse Wishart distribution with scale matrix $\Psi$ and degrees of freedom $\nu$ for $X$, a $p \times p$ matrix. The density of the distribution is given by $f(X) = \frac{\Psi^{\frac{\nu}{2}}}{2^{\frac{n(p-1)}{2}}} \left</td>
</tr>
<tr>
<td>$T_p(\mu, \Sigma)$</td>
<td>Multivariate $t$-distribution with location vector $\mu$, degrees of freedom $\nu$, and scale matrix $\Sigma$ for $x$, a column vector of length $p$. The density of the distribution is given by $f(x) = \frac{\Gamma \left(\frac{\nu+p}{2}\right)}{\Gamma \left(\frac{\nu}{2}\right) \nu^{\frac{p}{2}} \pi^{\frac{p(p-1)}{2}}} \left</td>
</tr>
<tr>
<td>$U(a, b)$</td>
<td>Uniform distribution over the range $a$ to $b$. The density of the distribution is given by $f(x) = \begin{cases} \frac{1}{b-a} &amp; \text{for } x \in [a, b] \ 0 &amp; \text{otherwise} \end{cases}$.</td>
</tr>
</tbody>
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