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Article

Multiple q-Zeta Brackets

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Abstract: The multiple zeta values (MZVs) possess a rich algebraic structure of algebraic relations, which is conjecturally determined by two different (shuffle and stuffle) products of a certain algebra of noncommutative words. In a recent work, Bachmann constructed a q-analogue of the MZVs — the so-called bi-brackets — for which the two products are dual to each other, in a very natural way. We overview Bachmann's construction and discuss the radial asymptotics of the bi-brackets, its links to the MZVs, and related linear (in)dependence questions of the q-analogue.

Keywords: multiple zeta value; q-analogue; multiple divisor sum; double shuffle relations; linear independence; radial asymptotics

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Apart from the "standard" q-model of the multiple zeta values (MZVs),

$$\zeta_q(s_1,\dots,s_l) := (1-q)^{s_1+\dots+s_l} \sum_{n_1>\dots>n_l>0} \frac{q^{(s_1-1)n_1+\dots+(s_l-1)n_l}}{(1-q^{n_1})^{s_1}\cdots(1-q^{n_l})^{s_l}},$$

introduced in the earlier works [1,2], the different q-version

$$\mathfrak{z}_q(s_1,\dots,s_l) := (1-q)^{s_1+\dots+s_l} \sum_{n_1>\dots>n_l>0} \frac{q^{n_1}}{(1-q^{n_1})^{s_1}\cdots(1-q^{n_l})^{s_l}}$$

has received a special attention in the more recent work [3] by Castillo Medina, Ebrahimi-Fard and Manchon. One of the principal features of the latter q-MZVs is that they are well defined for any collection of integers $s_1,\dots,s_l$, so they do not require regularisation as the former q-MZVs and the MZVs themselves.

In the other recent work [4,5] Bachmann and Kühn introduced and studied a different q-analogue of the MZVs, namely,

$$[s_1,\dots,s_l] := \frac{1}{(s_1-1)!\cdots(s_l-1)!}\sum_{\substack{n_1>\dots>n_l>0\\ d_1,\dots,d_l>0}} d_1^{s_1-1}\cdots d_l^{s_l-1}q^{n_1d_1+\dots+n_ld_l}$$
$$= \frac{1}{(s_1-1)!\cdots(s_l-1)!}\sum_{\substack{m_1,\dots,m_l>0\\ d_1,\dots,d_l>0}} d_1^{s_1-1}\cdots d_l^{s_l-1}q^{(m_1+\dots+m_l)d_1+(m_2+\dots+m_l)d_2+\dots+m_ld_l}. \tag{1}$$

The series are generating functions of multiple divisor sums, called (*mono-*)*brackets*, with the $\mathbb{Q}$-algebra spanned by them denoted by $\mathcal{MD}$. Note that the q-series (1) can be alternatively written

$$[s_1,\dots,s_l] = \frac{1}{(s_1-1)!\cdots(s_l-1)!}\sum_{n_1>\dots>n_l>0}\frac{P_{s_1-1}(q^{n_1})\cdots P_{s_l-1}(q^{n_l})}{(1-q^{n_1})^{s_1}\cdots(1-q^{n_l})^{s_l}},$$

where $P_{s-1}(q)$ are the (slightly modified) Eulerian polynomials:

$$\frac{P_{s-1}(q)}{(1-q)^s} = \left(q\,\frac{\mathrm{d}}{\mathrm{d}q}\right)^{s-1}\frac{q}{1-q} = \sum_{d=1}^{\infty} d^{s-1}q^d.$$

Since $P_{s-1}(1)=(s-1)!$ it is not hard to verify that

$$\lim_{q\to 1^-}(1-q)^{s_1+\dots+s_l}[s_1,\dots,s_l] = \zeta(s_1,\dots,s_l) := \sum_{n_1>\dots>n_l>0}\frac{1}{n_1^{s_1}\cdots n_l^{s_l}}. \tag{2}$$

More recently [6] Bachmann introduced a more general model of the brackets

$$\begin{bmatrix} s_1,\dots,s_l\\ r_1,\dots,r_l\end{bmatrix} := \frac{1}{r_1!\,(s_1-1)!\cdots r_l!\,(s_l-1)!}\sum_{\substack{n_1>\dots>n_l>0\\ d_1,\dots,d_l>0}} n_1^{r_1}d_1^{s_1-1}\cdots n_l^{r_l}d_l^{s_l-1}q^{n_1d_1+\dots+n_ld_l}$$
$$= \frac{1}{r_1!\,(s_1-1)!\cdots r_l!\,(s_l-1)!}\sum_{n_1>\dots>n_l>0}\frac{n_1^{r_1}P_{s_1-1}(q^{n_1})\cdots n_l^{r_l}P_{s_l-1}(q^{n_l})}{(1-q^{n_1})^{s_1}\cdots(1-q^{n_l})^{s_l}}, \tag{3}$$

which he called *bi-brackets*, in order to describe, in a natural way, the double shuffle relations of these q-analogues of MZVs. Note that the *stuffle* (also known as *harmonic* or *quasi-shuffle*) product for the both models (1) and (3) in Bachmann's work comes from the standard rearrangement of the multiple sums obtained from the term-by-term multiplication of two series. The other *shuffle* product is then interpreted for the model (3) only, as a dual product to the stuffle one via the *partition duality*. Bachmann further conjectures [6] that the $\mathbb{Q}$-algebra $\mathcal{BD}$ spanned by the bi-brackets (3) coincides with the $\mathbb{Q}$-algebra $\mathcal{MD}$.

The goal of this note is to make an algebraic setup for Bachmann's double stuffle relations as well as to demonstrate that those relations indeed reduce to the corresponding stuffle and shuffle relations in the limit as $q\to 1^-$. We also address the reduction of the bi-brackets to the mono-brackets.

1. Asymptotics

The following result allows one to control the asymptotic behaviour of the bi-brackets not only as $q \to 1^-$ but also as q approaches radially a root of unity. This produces an explicit version of the asymptotics used in [7] for proving some linear and algebraic results in the case $l = 1$.

Lemma 1. *As* $q = 1 - \varepsilon \to 1^-$,

$$\frac{1}{(s-1)!}\,\frac{P_{s-1}(q^n)}{(1-q^n)^s} = \frac{1}{n^s\varepsilon^s}\big((1-\varepsilon)F_{s-1}(\varepsilon) + \hat{\lambda}_s \cdot \varepsilon^s\big) - \hat{\lambda}_s + O(\varepsilon)$$

where the polynomials $F_k(\varepsilon) \in \mathbb{Q}[\varepsilon]$ *of degree* $\max\{0, k-1\}$ *are generated by*

$$\begin{aligned}\sum_{k=0}^{\infty} F_k(\varepsilon)x^k &= \frac{1}{1-(1-e^{-\varepsilon x})/\varepsilon} \\ &= 1 + x + \left(-\frac{1}{2}\varepsilon + 1\right)x^2 + \left(\frac{1}{6}\varepsilon^2 - \varepsilon + 1\right)x^3 \\ &\quad + \left(-\frac{1}{24}\varepsilon^3 + \frac{7}{12}\varepsilon^2 - \frac{3}{2}\varepsilon + 1\right)x^4 \\ &\quad + \left(\frac{1}{120}\varepsilon^4 - \frac{1}{4}\varepsilon^3 + \frac{5}{4}\varepsilon^2 - 2\varepsilon + 1\right)x^5 + \cdots\end{aligned}$$

and

$$\sum_{s=0}^{\infty} \hat{\lambda}_s x^s = -\frac{xe^x}{1-e^x} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}x^{2k}$$

is the generating function of Bernoulli numbers.

Proof. The proof is technical but straightforward. □

By moving the constant term $\hat{\lambda}_s$ to the right-hand side, we get

$$\begin{aligned}\frac{1}{2} + \frac{P_0(q^n)}{1-q^n} &= \frac{1}{n}\cdot\left(\varepsilon^{-1} - \frac{1}{2}\right) + O(\varepsilon) \\ \frac{1}{12} + \frac{P_1(q^n)}{(1-q^n)^2} &= \frac{1}{n^2}\cdot\left(\varepsilon^{-2} - \varepsilon^{-1} + \frac{1}{12}\right) + O(\varepsilon), \\ \frac{P_2(q^n)}{(1-q^n)^3} &= \frac{1}{n^3}\cdot\left(\varepsilon^{-3} - \frac{3}{2}\varepsilon^{-2} + \frac{1}{2}\varepsilon^{-1}\right) + O(\varepsilon), \\ -\frac{1}{720} + \frac{P_3(q^n)}{(1-q^n)^4} &= \frac{1}{n^4}\cdot\left(\varepsilon^{-4} - 2\varepsilon^{-3} + \frac{7}{6}\varepsilon^{-2} - \frac{1}{6}\varepsilon^{-1} - \frac{1}{720}\right) + O(\varepsilon),\end{aligned}$$

and so on.

Proposition 1. *Assume that* $s_1 > r_1 + 1$ *and* $s_j \ge r_j + 1$ *for* $j = 2, \dots, l$. *Then*

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \sim \frac{\zeta(s_1 - r_1, s_2 - r_2, \dots, s_l - r_l)}{r_1!\, r_2! \cdots r_l!}\,\frac{1}{(1-q)^{s_1+s_2+\cdots+s_l}} \qquad \text{as} \quad q \to 1^-,$$

where $\zeta(s_1, \dots, s_l)$ *denotes the standard MZV.*

Another way to tackle the asymptotic behaviour of the (bi-)brackets is based on the Mellin transform

$$\varphi(t) \mapsto \widetilde{\varphi}(s) = \int_0^\infty \varphi(t) t^{s-1} \mathrm{d}t$$

which maps

$$q^{n_1 d_1 + \cdots + n_l d_l}\Big|_{q=e^{-t}} \mapsto \frac{\Gamma(s)}{(n_1 d_1 + \cdots + n_l d_l)^s};$$

see [8,9]. Note that the bijective correspondence between the bi-brackets and the zeta functions

$$\frac{\Gamma(s)}{r_1!\,(s_1-1)! \cdots r_l!\,(s_l-1)!} \sum_{\substack{n_1 > \cdots > n_l > 0 \\ d_1, \ldots, d_l > 0}} \frac{n_1^{r_1} d_1^{s_1-1} \cdots n_l^{r_l} d_l^{s_l-1}}{(n_1 d_1 + \cdots + n_l d_l)^s}$$

can be potentially used for determining the linear relations of the former. A simple illustration is the linear independence of the depth 1 bi-brackets.

Theorem 1. *The bi-brackets* $\left[\begin{smallmatrix} s_1 \\ r_1 \end{smallmatrix}\right]$, *where* $0 \le r_1 < s_1 \le n$, $s_1 + r_1 \le n$, *are linearly independent over* $\mathbb{Q}$. *Therefore, the dimension* $d_n^{\mathcal{BD}}$ *of the* $\mathbb{Q}$*-space spanned by all bi-brackets of weight at most* n *is bounded from below by* $\lfloor (n+1)^2/4 \rfloor \ge n(n+2)/4$.

Proof. Indeed, the functions

$$\frac{\Gamma(s)}{r_1!\,(s_1-1)!} \sum_{n_1, d_1 > 0} \frac{n_1^{r_1} d_1^{s_1-1}}{(n_1 d_1)^s} = \Gamma(s) \frac{\zeta(s - s_1 + 1)\zeta(s - r_1)}{(s_1-1)!\, r_1!},$$
$$\text{where} \quad 0 \le r_1 < s_1 \le n, \quad s_1 + r_1 \le n,$$

are linearly independent over $\mathbb{Q}$ (because of their disjoint sets of poles at $s = s_1$ and $s = r_1 + 1$, respectively); thus the corresponding bi-brackets $\left[\begin{smallmatrix} s_1 \\ r_1 \end{smallmatrix}\right]$ are $\mathbb{Q}$-linearly independent as well. □

A similar (though more involved) analysis can be applied to describe the Mellin transform of the depth 2 bi-brackets; note that it is more easily done for another q-model we introduce further in Section 3.

2. The Stuffle Product

Consider the alphabet $Z = \{z_{s,r} : s, r = 1, 2, \ldots\}$ on the double-indexed letters $z_{s,r}$ of the pre-defined weight $s + r - 1$. On $\mathbb{Q}Z$ define the (commutative) product

$$\begin{aligned} z_{s_1,r_1} \diamond z_{s_2,r_2} := \binom{r_1+r_2-2}{r_1-1} \Bigg(& z_{s_1+s_2,r_1+r_2-1} \\ & + \sum_{j=1}^{s_1} (-1)^{s_2-1} \binom{s_1+s_2-j-1}{s_1-j} \lambda_{s_1+s_2-j} z_{j,r_1+r_2-1} \\ & + \sum_{j=1}^{s_2} (-1)^{s_1-1} \binom{s_1+s_2-j-1}{s_2-j} \lambda_{s_1+s_2-j} z_{j,r_1+r_2-1} \Bigg), \end{aligned} \tag{4}$$

where

$$\sum_{s=0}^\infty \lambda_s x^s = -\frac{x}{1-e^x} = 1 + \sum_{s=1}^\infty \frac{B_s}{s!} x^s$$

is the generating function of Bernoulli numbers. Note that $\hat{\lambda}_s = \lambda_s$ for $s \geq 2$, while $\hat{\lambda}_1 = \frac{1}{2} = -\lambda_1$ in the notation of Section 1.

As explained in [4] (after the proof of Proposition 2.9), the product $\diamond$ is also associative. With the help of (4) define the stuffle product on the $\mathbb{Q}$-algebra $\mathbb{Q}\langle Z\rangle$ recursively by $1 \sqcap\!\sqcap\, w = w \sqcap\!\sqcap\, 1 := w$ and

$$aw \sqcap\!\sqcap\, bv := a(w \sqcap\!\sqcap\, bv) + b(aw \sqcap\!\sqcap\, v) + (a \diamond b)(w \sqcap\!\sqcap\, v), \tag{5}$$

for arbitrary $w, v \in \mathbb{Q}\langle Z\rangle$ and $a, b \in Z$.

Proposition 2. *The evaluation map*

$$[\,\cdot\,]\colon z_{s_1,r_1} \dots z_{s_l,r_l} \mapsto \begin{bmatrix} s_1, \dots, s_l \\ r_1 - 1, \dots, r_l - 1 \end{bmatrix} \tag{6}$$

extended to $\mathbb{Q}\langle Z\rangle$ by linearity satisfies $[w \sqcap\!\sqcap\, v] = [w] \cdot [v]$, so that it is a homomorphism of the $\mathbb{Q}$-algebra $(\mathbb{Q}\langle Z\rangle, \sqcap\!\sqcap)$ onto $(\mathcal{BD}, \,\cdot\,)$, the latter hence being a $\mathbb{Q}$-algebra as well.

Proof. The proof follows the lines of the proof of ([4] Proposition 2.10) based on the identity

$$\begin{aligned} &\frac{n^{r_1-1} P_{s_1-1}(q^n)}{(s_1-1)!\,(r_1-1)!\,(1-q^n)^{s_1}} \cdot \frac{n^{r_2-1} P_{s_2-1}(q^n)}{(s_2-1)!\,(r_2-1)!\,(1-q^n)^{s_2}} \\ &\quad = \binom{r_1+r_2-2}{r_1-1} \frac{n^{r_1+r_2-2}}{(r_1+r_2-2)!} \Bigg(\frac{P_{s_1+s_2-1}(q^n)}{(s_1+s_2-1)!\,(1-q^n)^{s_1+s_2}} \\ &\qquad + \sum_{j=1}^{s_1} (-1)^{s_2-1} \binom{s_1+s_2-j-1}{s_1-j} \lambda_{s_1+s_2-j} \frac{P_{j-1}(q^n)}{(j-1)!\,(1-q^n)^j} \\ &\qquad + \sum_{j=1}^{s_2} (-1)^{s_1-1} \binom{s_1+s_2-j-1}{s_2-j} \lambda_{s_1+s_2-j} \frac{P_{j-1}(q^n)}{(j-1)!\,(1-q^n)^j} \Bigg). \qquad \square \end{aligned}$$

Modulo the highest weight, the commutative product (4) on Z assumes the form

$$z_{s_1,r_1} \diamond z_{s_2,r_2} \equiv \binom{r_1+r_2-2}{r_1-1} z_{s_1+s_2,r_1+r_2-1},$$

so that the stuffle product (5) reads

$$\begin{aligned} z_{s_1,r_1} w \sqcap\!\sqcap\, z_{s_2,r_2} v \equiv z_{s_1,r_1}(w \sqcap\!\sqcap\, z_{s_2,r_2} v) + z_{s_2,r_2}(z_{s_1,r_1} w \sqcap\!\sqcap\, v) \\ + \binom{r_1+r_2-2}{r_1-1} z_{s_1+s_2,r_1+r_2-1}(w \sqcap\!\sqcap\, v) \end{aligned} \tag{7}$$

for arbitrary $w, v \in \mathbb{Q}\langle Z\rangle$ and $z_{s_1,r_1}, z_{s_2,r_2} \in Z$. If we set $z_s := z_{s,1}$ and further restrict the product to the subalgebra $\mathbb{Q}\langle Z'\rangle$, where $Z' = \{z_s : s = 1, 2, \dots\}$, then Proposition 1 results in the following statement.

Theorem 2 ([4]). *For admissible words $w = z_{s_1} \dots z_{s_l}$ and $v = z_{s'_1} \cdots z_{s'_m}$ of weight $|w| = s_1 + \cdots + s_l$ and $|v| = s'_1 + \cdots + s'_m$, respectively,*

$$[w \sqcap\!\sqcap\, v] \sim (1-q)^{-|w|-|v|} \zeta(w * v) \qquad \textit{as} \quad q \to 1^-,$$

where $*$ denotes the standard stuffle (harmonic) product of MZVs on $\mathbb{Q}\langle Z'\rangle$.

Since $[w] \sim (1-q)^{-|w|}\zeta(w)$, $[v] \sim (1-q)^{-|v|}\zeta(v)$ as $q \to 1^-$ and $[w \sqcap\!\!\sqcap v] = [w]\cdot[v]$, Theorem 2 asserts that the stuffle product (5) of the algebra $\mathcal{MD}$ reduces to the stuffle product of the algebra of MZVs in the limit as $q \to 1^-$. This fact has been already established in [4].

3. The Duality

As an alternative extension of the mono-brackets (1) we introduce the *multiple q-zeta brackets*

$$\begin{aligned}
\mathfrak{Z}\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix} &= \mathfrak{Z}_q\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix} \\
&:= c \sum_{\substack{m_1,\dots,m_l>0 \\ d_1,\dots,d_l>0}} m_1^{r_1-1} d_1^{s_1-1} \cdots m_l^{r_l-1} d_l^{s_l-1} q^{(m_1+\dots+m_l)d_1+(m_2+\dots+m_l)d_2+\dots+m_l d_l} \\
&= c \sum_{m_1,\dots,m_l>0} \frac{m_1^{r_1-1} P_{s_1-1}(q^{m_1+\dots+m_l}) m_2^{r_2-1} P_{s_2-1}(q^{m_2+\dots+m_l}) \cdots m_l^{r_l-1} P_{s_l-1}(q^{m_l})}{(1-q^{m_1+\dots+m_l})^{s_1}(1-q^{m_2+\dots+m_l})^{s_2}\cdots(1-q^{m_l})^{s_l}}
\end{aligned} \tag{8}$$

where

$$c = \frac{1}{(r_1-1)!\,(s_1-1)!\cdots(r_l-1)!\,(s_l-1)!}.$$

Then

$$\begin{bmatrix} s_1 \\ r_1-1 \end{bmatrix} = \mathfrak{Z}\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \qquad \text{and} \qquad [s_1,\dots,s_l] = \begin{bmatrix} s_1,\dots,s_l \\ 0,\dots,0 \end{bmatrix} = \mathfrak{Z}\begin{bmatrix} s_1,\dots,s_l \\ 1,\dots,1 \end{bmatrix}.$$

By applying iteratively the binomial theorem in the forms

$$\frac{(m+n)^{r_1-1}}{(r_1-1)!}\,\frac{n^{r_2-1}}{(r_2-1)!} = \sum_{j=1}^{r_1+r_2-1} \binom{j-1}{r_2-1} \frac{m^{r_1+r_2-j-1}}{(r_1+r_2-j-1)!}\,\frac{n^{j-1}}{(j-1)!}$$

and

$$\frac{(n-m)^{r-1}}{(r-1)!} = \sum_{i=1}^{r} (-1)^{r+i} \frac{n^{i-1}}{(i-1)!}\,\frac{m^{r-i}}{(r-i)!}$$

we see that the $\mathbb{Q}$-algebras spanned by either (3) or (8) coincide. More precisely, the following formulae link the two versions of brackets.

Proposition 3. *We have*

$$\begin{aligned}
\begin{bmatrix} s_1, & s_2, & \dots, & s_l \\ r_1-1, & r_2-1, & \dots, & r_l-1 \end{bmatrix} &= \sum_{j_2=1}^{r_1+r_2-1} \binom{j_2-1}{r_2-1} \sum_{j_3=1}^{j_2+r_3-1} \binom{j_3-1}{r_3-1} \cdots \sum_{j_l=1}^{j_{l-1}+r_l-1} \binom{j_l-1}{r_l-1} \\
&\quad \times \mathfrak{Z}\begin{bmatrix} s_1, & s_2, & \dots, & s_{l-1}, & s_l \\ r_1+r_2-j_2, & j_2+r_3-j_3, & \dots, & j_{l-1}+r_l-j_l, & j_l \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{Z}\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix} &= \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_{l-1}=1}^{r_{l-1}} (-1)^{r_1+\dots+r_{l-1}-i_1-\dots-i_{l-1}} \\
&\quad \times \binom{r_1-i_1+i_2-1}{r_1-i_1} \cdots \binom{r_{l-2}-i_{l-2}+i_{l-1}-1}{r_{l-2}-i_{l-2}} \binom{r_{l-1}-i_{l-1}+r_l-1}{r_{l-1}-i_{l-1}} \\
&\quad \times \begin{bmatrix} s_1, & s_2, & \dots, & s_{l-1}, & s_l \\ i_1-1, & r_1-i_1+i_2-1, & \dots, & r_{l-2}-i_{l-2}+i_{l-1}-1, & r_{l-1}-i_{l-1}+r_l-1 \end{bmatrix}.
\end{aligned}$$

Proposition 3 allows us to construct an isomorphism φ of the two $\mathbb{Q}$-algebras $\mathbb{Q}\langle Z\rangle$ with two evaluation maps $[\,\cdot\,]$ and $\mathfrak{Z}[\,\cdot\,]$,

$$\mathfrak{Z}[z_{s_1,r_1}\dots z_{s_l,r_l}] = \mathfrak{Z}\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l, \end{bmatrix}$$

such that

$$[w] = \mathfrak{Z}[\varphi w] \qquad \text{and} \qquad \mathfrak{Z}[w] = [\varphi^{-1}w].$$

Note however that the isomorphism breaks the simplicity of defining the stuffle product $\sqcap\!\sqcap$ from Section 2.

Another algebraic setup can be used for the $\mathbb{Q}$-algebra $\mathbb{Q}\langle Z\rangle$ with evaluation $\mathfrak{Z}$. We can recast it as the $\mathbb{Q}$-subalgebra $\mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y$ of the $\mathbb{Q}$-algebra $\mathfrak{H} := \mathbb{Q}\langle x,y\rangle$ by setting $\mathfrak{Z}[1] = 1$ and

$$\mathfrak{Z}[x^{s_1}y^{r_1}\dots x^{s_l}y^{r_l}] = \mathfrak{Z}\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix}.$$

The depth (or length) is defined as the number of appearances of the subword xy, while the weight is the number of letters minus the length.

Proposition 4 (Duality)**.**

$$\mathfrak{Z}\begin{bmatrix} s_1,s_2,\dots,s_l \\ r_1,r_2,\dots,r_l \end{bmatrix} = \mathfrak{Z}\begin{bmatrix} r_l,r_{l-1},\dots,r_1 \\ s_l,s_{l-1},\dots,s_1 \end{bmatrix}.$$

Proof. This follows from the rearrangement of the summation indices:

$$\sum_{i=1}^{l} d_i \sum_{j=i}^{l} m_j = \sum_{i=1}^{l} d'_i \sum_{j=i}^{l} m'_j$$

where $d'_i = m_{l+1-i}$ and $m'_j = d_{l+1-j}$. $\square$

Denote by τ the anti-automorphism of the algebra $\mathfrak{H}$, interchanging x and y; for example, $\tau(x^2yxy) = xyxy^2$. Clearly, τ is an involution preserving both the weight and depth, and it is also an automorphism of the subalgebra $\mathfrak{H}^0$. The duality can be then stated as

$$\mathfrak{Z}[\tau w] = \mathfrak{Z}[w] \qquad \text{for any} \quad w \in \mathfrak{H}^0. \tag{9}$$

We also extend τ to $\mathbb{Q}\langle Z\rangle$ by linearity.

The duality in Proposition 4 is exactly the partition duality given earlier by Bachmann for the model (3).

4. The Dual Stuffle Product

We can now introduce the product which is dual to the stuffle one. Namely, it is the duality composed with the stuffle product and, again, with the duality:

$$w \,\overline{\sqcap\!\sqcap}\, v := \varphi^{-1}\tau(\tau\varphi w \sqcap\!\sqcap \tau\varphi v) \qquad \text{for} \quad w,v \in \mathbb{Q}\langle Z\rangle. \tag{10}$$

It follows then from Propositions 2 and 4 that

Proposition 5. *The evaluation map* (6) *on* $\mathbb{Q}\langle Z\rangle$ *satisfies* $[w \,\overline{\sqcap\!\sqcap}\, v] = [w]\cdot[v]$, *so that it is also a homomorphism of the* $\mathbb{Q}$*-algebra* $(\mathbb{Q}\langle Z\rangle, \overline{\sqcap\!\sqcap})$ *onto* $(\mathcal{BD}, \cdot)$.

Note that (7) is also equivalent to the expansion from the right ([10] Theorem 9):

$$\begin{aligned} wz_{s_1,r_1} \sqcap\!\sqcap\, vz_{s_2,r_2} \equiv (w \sqcap\!\sqcap\, vz_{s_2,r_2})z_{s_1,r_1} + (wz_{s_1,r_1} \sqcap\!\sqcap\, v)z_{s_2,r_2} \\ + \binom{r_1+r_2-2}{r_1-1}(w \sqcap\!\sqcap\, v)z_{s_1+s_2,r_1+r_2-1}. \end{aligned} \tag{11}$$

The next statement addresses the structure of the dual stuffle product (10) for the words over the sub-alphabet $Z' = \{z_s = z_{s,1} : s = 1,2,\ldots\} \subset Z$. Note that the words from $\mathbb{Q}\langle Z'\rangle$ can be also presented as the words from $\mathbb{Q}\langle x, xy\rangle$ necessarily ending with xy.

Proposition 6. *Modulo the highest weight and depth,*

$$aw \,\overline{\sqcap\!\sqcap}\, bv \equiv a(w \,\overline{\sqcap\!\sqcap}\, bv) + b(aw \,\overline{\sqcap\!\sqcap}\, v) \tag{12}$$

for arbitrary words $w, v \in \mathbb{Q} + \mathbb{Q}\langle x, xy\rangle xy$ *and* $a, b \in \{x, xy\}$.

Proof. First note that restricting (11) further modulo the highest depth implies

$$wz_{s_1,r_1} \sqcap\!\sqcap\, vz_{s_2,r_2} \equiv (w \sqcap\!\sqcap\, vz_{s_2,r_2})z_{s_1,r_1} + (wz_{s_1,r_1} \sqcap\!\sqcap\, v)z_{s_2,r_2},$$

and that we also have

$$\begin{aligned} wz_{s_1,r_1+1} \sqcap\!\sqcap\, vz_{s_2,r_2} &\equiv (wz_{s_1,r_1} \sqcap\!\sqcap\, vz_{s_2,r_2})y + (wz_{s_1,r_1+1} \sqcap\!\sqcap\, v)z_{s_2,r_2} \\ wz_{s_1,r_1+1} \sqcap\!\sqcap\, vz_{s_2,r_2+1} &\equiv (wz_{s_1,r_1} \sqcap\!\sqcap\, vz_{s_2,r_2+1})y + (wz_{s_1,r_1+1} \sqcap\!\sqcap\, vz_{s_2,r_2})y. \end{aligned}$$

The relations already show that

$$wa' \sqcap\!\sqcap\, vb' \equiv (w \sqcap\!\sqcap\, vb')a' + (wa' \sqcap\!\sqcap\, v)b' \tag{13}$$

for arbitrary words $w, v \in \mathbb{Q} + \mathbb{Q}\langle Z\rangle$ and $a', b' \in Z \cup \{y\}$, where

$$z_{s_1,r_1} \ldots z_{s_{l-1},r_{l-1}} z_{s_l,r_l} y = z_{s_1,r_1} \ldots z_{s_{l-1},r_{l-1}} z_{s_l,r_l+1}.$$

Secondly note that the isomorphism φ of Proposition 3 acts trivially on the words from $\mathbb{Q}\langle Z'\rangle$. Therefore, applying $\tau\varphi$ to the both sides of (10) and extracting the homogeneous part of the result corresponding to the highest weight and depth we arrive at

$$\tau(w \,\overline{\sqcap\!\sqcap}\, v) \equiv \tau w \sqcap\!\sqcap\, \tau v \qquad \text{for all} \quad w, v \in \mathbb{Q}\langle Z'\rangle.$$

Denoting

$$\bar{a} = \tau a = \begin{cases} y & \text{if } a = x, \\ xy & \text{if } a = xy, \end{cases}$$

and using (13) we find out that

$$\begin{aligned}\tau(aw\,\overline{\sqcap\!\sqcap}\,bv) &\equiv \tau(aw)\sqcap\!\sqcap\,\tau(bv) \equiv (\tau w)\bar{a}\sqcap\!\sqcap(\tau v)\bar{b} \\ &\equiv (\tau w\sqcap\!\sqcap(\tau v)\bar{b})\bar{a} + ((\tau w)\bar{a}\sqcap\!\sqcap\,\tau v)\bar{b} \\ &\equiv (\tau w\sqcap\!\sqcap\,\tau(bv))\bar{a} + (\tau(aw)\sqcap\!\sqcap\,\tau v)\bar{b} \equiv (\tau(w\,\overline{\sqcap\!\sqcap}\,bv))\bar{a} + (\tau(aw\,\overline{\sqcap\!\sqcap}\,v))\bar{b} \\ &\equiv \tau(a(w\,\overline{\sqcap\!\sqcap}\,bv) + b(aw\,\overline{\sqcap\!\sqcap}\,v)),\end{aligned}$$

which implies the desired result. □

Theorem 3. *For admissible words* $w = z_{s_1}\dots z_{s_l}$ *and* $v = z_{s'_1}\cdots z_{s'_m}$ *of weight* $|w| = s_1 + \dots + s_l$ *and* $|v| = s'_1 + \dots + s'_m$, *respectively,*

$$[w\,\overline{\sqcap\!\sqcap}\,v] \sim (1-q)^{-|w|-|v|}\zeta(w \sqcup\!\sqcup v) \qquad \text{as} \quad q \to 1^-,$$

where $\sqcup\!\sqcup$ *denotes the standard shuffle product of MZVs on* $\mathbb{Q}\langle Z'\rangle$.

Proof. Because both φ and τ respect the weight, Proposition 6 shows that the only terms that can potentially interfere with the asymptotic behaviour as $q \to 1^-$ correspond to the same weight but lower depth. However, according to (10) and (11), the 'shorter' terms do not belong to $\mathbb{Q}\langle Z'\rangle$, that is, they are linear combinations of the monomials $z_{q_1,r_1}\dots z_{q_n,r_n}$ with $r_1 + \dots + r_n = l + m > n$, hence $r_j \geq 2$ for at least one j. The latter circumstance and Proposition 1 then imply

$$\lim_{q\to 1^-}(1-q)^{|w|+|v|}[z_{q_1,r_1}\dots z_{q_n,r_n}] = 0. \quad \square$$

Theorem 3 asserts that the dual stuffle product (10) restricted from $\mathcal{BD}$ to the subalgebra $\mathcal{MD}$ reduces to the shuffle product of the algebra of MZVs in the limit as $q \to 1^-$. This result is implicitly stated in [6]. More is true: using (7) and Proposition 6 we obtain

Theorem 4. *For two words* $w = z_{s_1}\dots z_{s_l}$ *and* $v = z_{s'_1}\cdots z_{s'_m}$, *not necessarily admissible,*

$$[w\sqcap\!\sqcap\,v - w\,\overline{\sqcap\!\sqcap}\,v] \sim (1-q)^{-|w|-|v|}\zeta(w * v - w \sqcup\!\sqcup v) \qquad \text{as} \quad q \to 1^-,$$

whenever the MZV on the right-hand side makes sense.

In other words, the q-zeta model of bi-brackets provides us with a (far reaching) regularisation of the MZVs: the former includes the extended double shuffle relations as the limiting $q \to 1^-$ case.

Conjecture 1 (Bachmann [6]). The resulting double stuffle (that is, stuffle and dual stuffle) relations exhaust all the relations between the bi-brackets. Equivalently (and simpler), the stuffle relations and the duality exhaust all the relations between the bi-brackets.

We would like to point out that the duality τ from Section 3 also exists for the algebra of MZVs ([10] Section 6). However the two dualities are not at all related: the limiting $q \to 1^-$ process squeezes the appearances of x before y in the words $x^{s_1}yx^{s_2}y\dots x^{s_l}y$, so that they become $x^{s_1-1}yx^{s_2-1}y\dots x^{s_l-1}y$. Furthermore, the duality of MZVs respects the shuffle product: the dual shuffle product coincides with the shuffle product itself. On the other hand, the dual stuffle product of MZVs is very different from the stuffle (and shuffle) products. It may be an interesting problem to understand the double stuffle relations of the algebra of MZVs.

5. Reduction to Mono-Brackets

In this final section we present some observations towards another conjecture of Bachmann about the coincidence of the $\mathbb{Q}$-algebras of bi- and mono-brackets.

Conjecture 2 (Bachmann). $\mathcal{MD} = \mathcal{BD}$.

Based on the representation of the elements from $\mathcal{BD}$ as the polynomials from $\mathbb{Q}\langle x, y\rangle$ (see also the last paragraph of Section 4), we can loosely interpret this conjecture for the algebra of MZVs as follows: all MZVs lie in the $\mathbb{Q}$-span of:

$$\zeta(s_1, s_2, \dots, s_l) = \zeta(x^{s_1-1}yx^{s_2-1}y\dots x^{s_l-1}y)$$

with all s_j to be at least 2 (so that there is no appearance of y^r with $r \geq 2$). The latter statement is already known to be true: Brown [11] proves that one can span the $\mathbb{Q}$-algebra of MZVs by the set with all $s_j \in \{2, 3\}$.

In what follows we analyse the relations for the model (8), because it makes simpler keeping track of the duality relation. We point out from the very beginning that the linear relations given below are all experimentally found (with the check of 500 terms in the corresponding q-expansions) but we believe that it is possible to establish them rigorously using the double stuffle relations given above.

The first presence of the q-zeta brackets that are not reduced to ones from $\mathcal{MD}$ by the duality relation happens in weight 3. It is $\mathfrak{z}\big[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\big]$ and we find out that

$$\mathfrak{z}\big[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\big] = \tfrac{1}{2}\mathfrak{z}\big[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big] + \mathfrak{z}\big[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\big] - \mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,1 \end{smallmatrix}\big].$$

There are 34 totally q-zeta brackets of weight up to 4,

$$\begin{gathered}
\mathfrak{z}\big[\ \big]^*,\ \mathfrak{z}\big[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\big]^*,\ \mathfrak{z}\big[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\big]^*,\ \mathfrak{z}\big[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\big], \\
\mathfrak{z}\big[\begin{smallmatrix} 1,1 \\ 1,1 \end{smallmatrix}\big]^*,\ \mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1 \\ 1,2 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1 \\ 2,1 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 2,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 1,2 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,2 \end{smallmatrix}\big]^*,\ \mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 2,1 \end{smallmatrix}\big]^*, \\
\mathfrak{z}\big[\begin{smallmatrix} 2,2 \\ 1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1 \\ 2,2 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 3,1 \\ 1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1 \\ 1,3 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 1,3 \\ 1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1 \\ 3,1 \end{smallmatrix}\big], \\
\mathfrak{z}\big[\begin{smallmatrix} 1,1,1 \\ 1,1,1 \end{smallmatrix}\big]^*,\ \mathfrak{z}\big[\begin{smallmatrix} 2,1,1 \\ 1,1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1,1 \\ 1,1,2 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 1,2,1 \\ 1,1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1,1 \\ 1,2,1 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 1,1,2 \\ 1,1,1 \end{smallmatrix}\big] = \mathfrak{z}\big[\begin{smallmatrix} 1,1,1 \\ 2,1,1 \end{smallmatrix}\big],\ \mathfrak{z}\big[\begin{smallmatrix} 1,1,1,1 \\ 1,1,1,1 \end{smallmatrix}\big]^*,
\end{gathered}$$

where the asterisk marks the self-dual ones. Only 21 of those listed are not dual-equivalent, and only five of the latter are not reduced to the q-zeta brackets from $\mathcal{MD}$; besides the already mentioned $\mathfrak{z}\big[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\big]$ these are $\mathfrak{z}\big[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\big]$, $\mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 2,1 \end{smallmatrix}\big]$, $\mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,2 \end{smallmatrix}\big]$ and $\mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 2,1 \end{smallmatrix}\big]$. We find out that

$$\begin{aligned}
\mathfrak{z}\big[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\big] &= \tfrac{1}{4}\mathfrak{z}\big[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big] + \tfrac{3}{2}\mathfrak{z}\big[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\big] - 2\mathfrak{z}\big[\begin{smallmatrix} 2,2 \\ 1,1 \end{smallmatrix}\big], \\
\mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 2,1 \end{smallmatrix}\big] &= \mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,1 \end{smallmatrix}\big] + \tfrac{1}{2}\mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 1,1 \end{smallmatrix}\big] - \mathfrak{z}\big[\begin{smallmatrix} 2,2 \\ 1,1 \end{smallmatrix}\big] + \mathfrak{z}\big[\begin{smallmatrix} 1,3 \\ 1,1 \end{smallmatrix}\big] - \mathfrak{z}\big[\begin{smallmatrix} 2,1,1 \\ 1,1,1 \end{smallmatrix}\big] - \mathfrak{z}\big[\begin{smallmatrix} 1,2,1 \\ 1,1,1 \end{smallmatrix}\big], \\
\mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,2 \end{smallmatrix}\big] &= -\tfrac{1}{2}\mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,1 \end{smallmatrix}\big] - \tfrac{1}{2}\mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 1,1 \end{smallmatrix}\big] + 2\mathfrak{z}\big[\begin{smallmatrix} 2,2 \\ 1,1 \end{smallmatrix}\big] + \mathfrak{z}\big[\begin{smallmatrix} 3,1 \\ 1,1 \end{smallmatrix}\big] - \mathfrak{z}\big[\begin{smallmatrix} 1,3 \\ 1,1 \end{smallmatrix}\big] + \mathfrak{z}\big[\begin{smallmatrix} 1,2,1 \\ 1,1,1 \end{smallmatrix}\big] \\
\mathfrak{z}\big[\begin{smallmatrix} 1,2 \\ 2,1 \end{smallmatrix}\big] &= -\mathfrak{z}\big[\begin{smallmatrix} 2,1 \\ 1,1 \end{smallmatrix}\big] + 2\mathfrak{z}\big[\begin{smallmatrix} 2,2 \\ 1,1 \end{smallmatrix}\big] + \mathfrak{z}\big[\begin{smallmatrix} 2,1,1 \\ 1,1,1 \end{smallmatrix}\big],
\end{aligned}$$

and there is one more relation in this weight between the q-zeta brackets from $\mathcal{MD}$:

$$\tfrac{1}{3}\mathfrak{z}\big[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big] - \mathfrak{z}\big[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\big] + \mathfrak{z}\big[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\big] - 2\mathfrak{z}\big[\begin{smallmatrix} 2,2 \\ 1,1 \end{smallmatrix}\big] + 2\mathfrak{z}\big[\begin{smallmatrix} 3,1 \\ 1,1 \end{smallmatrix}\big] = 0.$$

The computation implies that the dimension $d_4^{\mathcal{BD}}$ of the $\mathbb{Q}$-space spanned by all multiple q-zeta brackets of weight not more than 4 is equal to the dimension $d_4^{\mathcal{MD}}$ of the $\mathbb{Q}$-space spanned by all such brackets from $\mathcal{MD}$ and that both are equal to 15. A similar analysis demonstrates that

$$d_5^{\mathcal{BD}} = d_5^{\mathcal{MD}} = 28 \quad \text{and} \quad d_6^{\mathcal{BD}} = d_6^{\mathcal{MD}} = 51,$$

and it seems less realistic to compute and verify that $d_n^{\mathcal{BD}} = d_n^{\mathcal{MD}}$ for $n \geq 7$ though Conjecture 2 and ([4] Conjecture (5.4)) support

$$\begin{aligned}\sum_{n=0}^{\infty} d_n^{\mathcal{MD}} x^n &\overset{?}{=} \frac{1 - x^2 + x^4}{(1-x)^2(1-2x^2-2x^3)} \\ &= 1 + 2x + 4x^2 + 8x^3 + 15x^4 + 28x^5 + 51x^6 + 92x^7 \\ &\quad + 165x^8 + 294x^9 + 523x^{10} + O(x^{11}).\end{aligned}$$

We can compare this with the count $c_n^{\mathcal{MD}}$ and $c_n^{\mathcal{BD}}$ of all mono- and bi-brackets of weight $\leq n$,

$$\sum_{n=0}^{\infty} c_n^{\mathcal{MD}} x^n = \frac{1}{1-2x} \quad \text{and} \quad \sum_{n=0}^{\infty} c_n^{\mathcal{BD}} x^n = \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n} x^n,$$

where F_n denotes the Fibonacci sequence.

In addition, we would like to point out one more expectation for the algebra of (both mono- and bi-) brackets, which is not shared by other q-models of MZVs: all linear (hence algebraic) relations between them seem to be over $\mathbb{Q}$, not over $\mathbb{C}(q)$.

Conjecture 3. A collection of (bi-)brackets is linearly dependent over $\mathbb{C}(q)$ if and only if it is linearly dependent over $\mathbb{Q}$.

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Conflicts of Interest

The author declares no conflicts of interest.

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